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# *Coefficients of Some Trigonometric Series*

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**Introduction** : The study of special trigonometric series to characterize some properties from their coefficients has long been kept up by many authors. Their central topics are mostly concerned with *Fourier Series Problem* and *Integrability Problem* which, as is shown later, are equivalent in our cases. We consider throughout cosine and sine series :

(1) $$\sum_{n=1}^{\infty} a_n \cos nx,$$

(2) $$\sum_{n=1}^{\infty} b_n \sin nx,$$

or together,

(3) $$\sum_{n=1}^{\infty} c_n \varphi_n(x),$$

where $c_n = a_n$ or $b_n$ according as $\varphi_n(x) = \cos nx$ or $\sin nx$ respectively.

We see that in most cases cosine series (1) is more prickly than sine series (2). For example, when $b_n \downarrow 0$, (2) is a Fourier series if and only if $\sum_{n=1}^{\infty} \frac{b_n}{n} < +\infty$, but for (1) with $a_n \downarrow 0$ any effective necessary and sufficient condition has not yet been found and $\sum_{n=1}^{\infty} \frac{a_n}{n} < +\infty$ is then merely a sufficient condition. Other well known sufficient condition of W. H. Young is that $a_n \to 0$ and $\{a_n\}$ is quasi-convex. The necessity condition is much more obscure and we have little or nothing except that of R. Salem ([1] vol. 1. p. 237), i.e.,

$$(a_n - a_{n+1}) \log n \to 0$$

when $a_n \downarrow 0$. On the other hand it is known that when $c_n \downarrow 0$ or more generally $c_n \to 0$ and $\{c_n\}$ is of bounded variation, (3) is a Fourier series if and only if it represents an integrable function. G. Goes noted [3] without proof that it is so if and only if (3) is a Fourier-Stieltjes series. On the one hand, S. A. Teljakovski proved [5] that when $b_n \to 0$ and $\{b_n\}$ is quasi-convex (2) is a Fourier series or equivalently, (2) represents an integrable function if and only if $\sum_{n=1}^{\infty}\frac{|b_n|}{n}<+\infty$.

The aim of this paper is to present some additional results and make remarks concerning these theorems.

## § 1. Semi-convex null sequences.

In the first place we shall prove the following theorem which is a slight generalization of that of Young.

**Theorem A.** *If $\{a_n\}$ is a semi-convex null sequence, i.e., if $a_n \to 0$ and*

$$\sum_{n=1}^{\infty} n\,|\Delta^2 a_{n-1}+\Delta^2 a_n|<+\infty, \quad (a_0=0)$$

$$\Delta^2 a_n=\Delta a_n-\Delta a_{n+1},\ \Delta a_n=a_n-a_{n+1},$$

*then* (1) *is a Fourier series, or equivalently, it represents an integrable function.*

Before proving this, we shall remark that clearly every quasi-convex null sequence is semi-convex and require

**Lemma 1.** *If $\{a_n\}$ is a semi-convex null sequence, then*

$$n(a_{n-1}-a_{n+1})\to 0,$$

*and*

$$\sum_{n=1}^{\infty}|a_{n-1}-a_{n+1}|<+\infty.$$

**Proof.**

$$|a_{n-1}-a_{n+1}|=|\sum_{k=n}^{\infty}(\Delta^2 a_{k-1}+\Delta^2 a_k)|\leqq\sum_{k=n}^{\infty}\frac{1}{k}k\,|\Delta^2 a_{k-1}+\Delta^2 a_k|$$

$$\leqq\frac{1}{n}\sum_{k=n}^{\infty}k\,|\Delta^2 a_{k-1}+\Delta^2 a_k|=o\left(\frac{1}{n}\right),$$

and

$$\sum_{n=1}^{\infty}|a_{n-1}-a_{n+1}|\leqq\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}|\Delta^2 a_{k-1}+\Delta^2 a_k|$$

$$=\sum_{n=1}^{\infty}n\,|\Delta^2 a_{n-1}+\Delta^2 a_n|<+\infty.$$

**Proof of Theorem A.**

We have for $x\not\equiv 0 \pmod{\pi}$,

$$\sum_{n=1}^{N}a_n\cos nx=\sum_{n=1}^{N-1}(a_{n-1}-a_{n+1})\frac{\sin nx}{2\sin x}+a_{N-1}\frac{\sin Nx}{2\sin x}$$

$$+ a_N \frac{\sin(N+1)x}{2\sin x}$$

$$= \sum_{n=1}^{N-2} (\Delta a_{n-1} - \Delta a_{n+1}) \frac{\bar{D}_n(x)}{2\sin x} + (a_{N-2} - a_N) \frac{\bar{D}_{N-1}(x)}{2\sin x}$$

$$+ a_{N-1} \frac{\sin Nx}{2\sin x} + a_N \frac{\sin(N+1)x}{2\sin x},$$

where $\bar{D}_n(x) = \sum_{k=1}^{n} \sin kx$, the conjugate Dirichlet kernel.

Since $\frac{1}{n}\bar{D}_n(x) \to 0\ (n \to \infty)$ for any fixed $x \not\equiv 0 \pmod{2\pi}$, we obtain by Lemma 1,

$$\sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=1}^{\infty} (\Delta a_{n-1} - \Delta a_{n+1}) \frac{\bar{D}_n(x)}{2\sin x} \overset{df}{=} f(x).$$

And since

$$\sum_{n=1}^{\infty} | \Delta a_{n-1} - \Delta a_{n+1} | \int_{-\pi}^{\pi} \left| \frac{\bar{D}_n(x)}{2\sin x} \right| dx$$

$$= \begin{cases} \sum_{n=1}^{\infty} | \Delta^2 a_{n-1} + \Delta^2 a_n | \int_{-\pi}^{\pi} \left| \frac{\sin\frac{n}{2}x}{2\sin\frac{x}{2}} \right| \left| \frac{\sin\frac{n+1}{2}x}{2\sin x} \right| dx & (n \ \text{even}) \\ \sum_{n=1}^{\infty} | \Delta^2 a_{n-1} + \Delta^2 a_n | \int_{-\pi}^{\pi} \left| \frac{\sin\frac{n+1}{2}x}{2\sin\frac{x}{2}} \right| \left| \frac{\sin\frac{n}{2}x}{2\sin x} \right| dx & (n \ \text{odd}) \end{cases}$$

$$= O \sum_{n=1}^{\infty} | \Delta^2 a_{n-1} + \Delta^2 a_n | \left\{ \int_{-\pi}^{\pi} \left( \frac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}} \right)^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{-\pi}^{\pi} \left( \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}} \right)^2 dx \right\}^{\frac{1}{2}}$$

$$= O\left(\sum_{n=1}^{\infty} n \, | \Delta^2 a_{n-1} + \Delta^2 a_n | \right) = O(1),$$

we conclude that $f \in L$ by Lebesgue's theorem.

Thus we know that (1) should converge to $f \in L$ everywhere apart from $x \equiv 0 \pmod{\pi}$. Hence we can infer that (1) should be a Fourier series by virtue of generalized du Bois-Reymond theorem [ 1 ] [ 6 ].

**Corollary** : *If* $\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty} n \, | \lambda_n - \lambda_{n+1} | < +\infty$,

$$\sum_{n=1}^{\infty} \lambda_n \cos(2n-1)x,$$

*is a Fourier series.*

## § 2. Fourier-Stieltjes series.

In this section we give a proof of Goes' theorem and an application of it in the next.

**Theorem B.** (G. Goes [3]) *Let* $\{c_n\}$ *be of bounded variation. Then* (3) *is a Fourier-Stieltjes series if and only if it is a Fourier series, or equivalently, it represents an integrable function.*

In fact, Goes' original statement has the additional assumption $c_n \to 0$, which may be seen to be superfluous from

**Lemma 2.** *If* (1) *and* (2) *are Fourier-Stieltjes series,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = 0,$$

*and* $\sum_{n=1}^{\infty}\frac{b_n}{n}$ *should converge and therefore especially*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} b_k = 0.$$

**Proof.**

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx\, dF(x) = \frac{1}{n\pi}\int_{-\pi}^{\pi} D_n(x)\, dF(x)$$

$$= \frac{1}{n\pi}\int_{-\delta}^{\delta} D_n(x) dF + \frac{1}{n\pi}\int_{\delta<|x|<\pi} D_n(x)\, dF \overset{df}{=} I_1 + I_2.$$

Here $D_n(x)$ denotes the Dirichlet kernel and $a_0 = 0$. If we take $\delta$ so small that the total variation of $F(x)$ over $(-\delta, \delta)$ should be $< \varepsilon$,

$$I_1 = O\left(\int_{-\delta}^{\delta} |dF(x)|\right) = O(\varepsilon) = o(1),$$

$$I_2 = O\left(\frac{1}{n}\operatorname*{Max}_{\delta\leqq|x|\leqq\pi} |D_n(x)| \int_{\delta\leqq|x|\leqq\pi} |dF(x)|\right) = O\left(\frac{1}{n}\right) = o(1).$$

Hence
$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = 0.$$

On the one hand

$$\sum_{k=1}^{n}\frac{b_k}{k} = \sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin kx}{k}dG(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{k=1}^{n}\frac{\sin kx}{k}\right)dG(x).$$

But since we know that

$$s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k},$$

is uniformly bounded over $[-\pi, \pi]$, we can assert that $\sum_{n=1}^{\infty} \frac{b_n}{n}$ should converge according to Lebesgue's theorem (for Lebesgue-Stieltjes integral) and thus obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} b_k = 0.$$

**Proof of Theorem B.** Clearly it is enough to show that Fourier-Stieltjes $\longrightarrow$ Fourier. If (3) is a Fourier-Stieltjes series it is summable (C, 1) and its (C, 1) means $\sigma_n(x)$ converges almost everywhere to $\Phi'(x)$, where $\Phi(x)$ is a function of bounded variation. Hence $\Phi' \in L$. Since $\{c_n\}$ is of bounded variation, $c_n$ approaches a limit as $n \to \infty$ and it must be 0 by Lemma 2. Therefore we know that (3) should converge everywhere apart from $x \equiv 0 \pmod{2\pi}$ to a function $\phi(x)$. But since (C, 1) method is regular, we have $\sigma_n(x) \to \phi(x)$ and hence $\phi(x) = \Phi'(x)$ a. e., so $\phi(x) \in L$. Consequently we conclude that (3) is a Fourier series by operating on generalized du Bois-Reymond theorem again. Thus our proof is complete.

## § 3. An application of Theorem B.

The mere condition $a_n \downarrow 0$ is known to be insufficient even to ensure that (1) should be a Fourier-Stieltjes series from Theorem B and Salem's necessity condition. But a stronger assumption $a_n \log n \downarrow$ is then sufficient for (1) to be a Fourier series. This is a special case of the following theorem of S. Szidon [4] and we shall give a different proof of it using Theorem B.

**Theorem C.** *If $\{a_n \log n\}$ is bounded and of bounded variation, then* (1) *should be a Fourier series.*

**Lemma 3.** *If $\{a_n \log n\}$ is bounded and of bounded variation, then $\{a_n\}$ is a null sequence of bounded variation.*

**Proof.**

$$\sum_{n=2}^{\infty} |a_n - a_{n+1}| = \sum_{n=2}^{\infty} \left| \frac{a_n \log n - a_{n+1} \log(n+1)}{\log n} + \frac{a_{n+1} \log\left(1 + \frac{1}{n}\right)}{\log n} \right|$$

$$\leq \sum_{n=2}^{\infty} \frac{|a_n \log n - a_{n+1} \log(n+1)|}{\log n} + \sum_{n=2}^{\infty} \frac{|a_{n+1}| \log\left(1 + \frac{1}{n}\right)}{\log n}$$

$$\leq \sum_{n=2}^{\infty} |\Delta(a_n \log n)| + \sum_{n=2}^{\infty} \frac{|a_{n+1}| \cdot \log(n+1)}{n \log^2 n} = O(1) + O(1) = O(1).$$

**Proof of Theorem C.** We may assume that $a_1 = 0$. Then we have

$$S_n(x) = \sum_{k=1}^{n} a_k \cos kx = \sum_{k=2}^{n} (a_k \log k)\frac{\cos kx}{\log k}$$
$$= \sum_{k=1}^{n-1} (a_k \log k - a_{k+1} \log (k+1))C_k(x) + a_n \log n \cdot C_n(x),$$

where $C_1(x) = 0, \quad C_n(x) = \sum_{k=2}^{n} \frac{\cos kx}{\log k}.$

Since we know ([1]. vol. 1 p. 94) that

$$\int_{-\pi}^{\pi} |C_n(x)|\, dx = O(1),$$

we obtain

$$\int_{-\pi}^{\pi} |S_n(x)|\, dx \leq \int_{-\pi}^{\pi} \sum_{k=1}^{n-1} |\Delta(a_k \log k)|\ |C_k(x)|\, dx$$
$$+ |a_n| \log n \int_{-\pi}^{\pi} |C_n(x)|\, dx$$
$$= O\Big(\sum_{k=1}^{n-1} |\Delta(a_k \log k)|\Big) + O(|a_n| \log n) = O(1) + O(1) = O(1).$$

And we know [1] [6] that a necessary and sufficient condition for (1) to be a Fourier-Stieltjes series is

$$\int_{-\pi}^{\pi} |\sigma_n(x)|\, dx = O(1),$$

where $\sigma_n(x) = \frac{1}{n+1} \sum_{k=1}^{n} S_k(x).$

But this condition is clearly satisfied in our case since

$$\int_{-\pi}^{\pi} |S_n(x)|\, dx = O(1).$$

Therefore (1) should be a Fourier-Stieltjes series and hence a Fourier series with the help of Lemma 3 and Theorem B.

We remark that Theorem C can be proved more elementarily without depending on Theorem B but in rather lengthy fashion.

## § 4. Sine series.

S. A. Teljakovski [2] [5] has recently obtained the following

**Theorem D.** *If $\{b_n\}$ is a quasi-convex null sequence,* (2) *is a Fourier series if and only if*

$$\sum_{n=1}^{\infty}\frac{|b_n|}{n}<+\infty, \tag{4}$$

*and this cannot be replaced by the mere convergence of* $\sum_{n=1}^{\infty}\frac{b_n}{n}$.

In connection with this theorem we shall prove

**Theorem E.** *If $\{c_n\}$ is a null sequence such that*

$$\sum_{n=1}^{\infty} n^2\left|\Delta^2\left(\frac{c_n}{n}\right)\right|<\infty, \tag{5}$$

*then* (3) *is a Fourier series, or equivalently, it represents an integrable function.*

**Proof.** We prove this for (1) only since it goes in quite the same way for (2).

Let
$$S_n(x)=\sum_{k=1}^{n}a_k\cos kx \qquad (a_n\to 0).$$

Then

$$S_n(x)=\frac{d}{dx}\sum_{k=1}^{n}\frac{a_k}{k}\sin kx=\sum_{k=1}^{n-1}\Delta\left(\frac{a_k}{k}\right)\overline{D}_k'(x)+\frac{a_n}{n}\overline{D}_n'(x)$$
$$=\sum_{k=1}^{n-2}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\overline{K}_k'(x)+n\,\Delta\left(\frac{a_{n-1}}{n-1}\right)\overline{K}_{n-1}'(x)+\frac{a_n}{n}\overline{D}_n'(x),$$

where $\overline{K}_n(x)$ is the conjugat e Fejér kernel and by Zygmund's theoem ([1]vol. 1. p. 458)

$$\int_{-\pi}^{\pi}|\overline{K'}_n(x)|\,dx=O(n).$$

Moreover for any fixed $x\not\equiv 0 \pmod{2\pi}$,

$$n\Delta\left(\frac{a_{n-1}}{n-1}\right)\overline{K}_{n-1}(x)\to 0,\qquad \frac{a_n}{n}\overline{D}_n(x)\to 0,$$

when $n\to\infty$. Hence everywhere apart from $x\equiv 0 \pmod{2\pi}$,

$$f(x)=\lim_{n\to\infty}S_n(x)=\sum_{n=1}^{\infty}(n+1)\Delta^2\left(\frac{a_n}{n}\right)\overline{K'}_n(x),$$

exists and $f(x)\in L$ by Lebesgue's theorem since

$$\sum_{n=1}^{\infty}(n+1)\left|\Delta^2\left(\frac{a_n}{n}\right)\right|\cdot|\overline{K'}_n(x)|=O\left(\sum_{n=1}^{\infty}n^2\left|\Delta^2\left(\frac{a_n}{n}\right)\right|\right)=O(1).$$

Consequently we arrive at the conclusion of the theorem by generalized du Bois-Reymond theorem.

Finally we prove the following

**Theorem F.** *If* $\{c_n\}$ *is bounded and quasi-convex, the condition*

$$\sum_{n=1}^{\infty}\frac{|c_n|}{n}<+\infty, \tag{4$'$}$$

*is equivalent to* (5).

We shall demonstrate two lemmas below from which we can deduce the above theorem as a corollary.

**Lemma 4.** *If* $\{\lambda_n\}$ *is bounded and quasi-convex, then it is of bounded variation and*

$$n(\lambda_n-\lambda_{n+1})\to 0.$$

**Proof.** Since $\sum_{n=1}^{\infty}n\,|\Delta^2\lambda_n|<+\infty$, $\sum_{n=1}^{\infty}n(\Delta^2\lambda_n)$ also converges.

So,

$$\sum_{k=1}^{n-1}k(\Delta^2\lambda_k)=\lambda_1-\lambda_{n+1}-n(\lambda_n-\lambda_{n+1}),$$

has a finite limit as $n\to\infty$.
Thus we obtain

$$n(\lambda_n-\lambda_{n+1})=O(1),$$

and

$$\lambda_n-\lambda_{n+1}\to 0 \qquad (n\to\infty).$$

Hence

$$|\Delta\lambda_n|=|\sum_{k=n}^{\infty}\Delta^2\lambda_k|\leqq\sum_{k=n}^{\infty}\frac{1}{k}k\,|\Delta^2\lambda_k|\leqq\frac{1}{n}\sum_{k=n}^{\infty}k\,|\Delta^2\lambda_k|$$

$$=o\left(\frac{1}{n}\right),$$

and

$$\sum_{n=1}^{\infty}|\Delta\lambda_n|\leqq\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}|\Delta^2\lambda_k|=\sum_{n=1}^{\infty}n\,|\Delta^2\lambda_n|<+\infty.$$

**Lemma 5.** If $\sum_{k=1}^{n}\frac{\mu_k}{k}=O(1)$ or $\mu_n=O(1)$ and

$$\sum_{n=1}^{\infty} n\left|\frac{\mu_n}{n}-\frac{\mu_{n+1}}{n+1}\right|<+\infty,$$

then $\quad \mu_n \to 0, \quad \sum_{n=1}^{\infty}\frac{|\mu_n|}{n}<+\infty, \quad \sum_{n=1}^{\infty}|\mu_n-\mu_{n+1}|<+\infty,$

and conversely.

**Proof.** Let us put $\sum_{k=1}^{n}\frac{\mu_k}{k}=\lambda_{n+1}$ and suppose first that

$$\sum_{k=1}^{n}\frac{\mu_k}{k}=O(1).$$

Then $\quad \lambda_{n+1}-\lambda_n=\frac{\mu_n}{n} \quad$ and

$$\sum_{n=1}^{\infty} n\,|\varDelta^2\lambda_n|=\sum_{n=1}^{\infty} n\left|\frac{\mu_{n+1}}{n+1}-\frac{\mu_n}{n}\right|<+\infty,$$

i.e., $\{\lambda_n\}$ is bounded and quasi-convex. Therefore we have

$$\mu_n=n(\lambda_{n+1}-\lambda_n)\to 0, \qquad \sum_{n=1}^{\infty}\frac{|\mu_n|}{n}=\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<+\infty,$$

by Lemma 4. And moreover

$$\sum_{n=1}^{\infty}|\mu_n-\mu_{n+1}|=\sum_{n=1}^{\infty}|\varDelta\lambda_{n+1}-n\varDelta^2\lambda_n|\leqq\sum_{n=1}^{\infty}|\varDelta\lambda_{n+1}|+\sum_{n=1}^{\infty}n\,|\varDelta^2\lambda_n|<+\infty.$$

In case $\lambda_n=O(1)$, in fact, it follows that $\sum_{k=1}^{n}\frac{\lambda_k}{k}=O(1)$ again under the assumption

$$\sum_{n=1}^{\infty} n\left|\frac{\lambda_n}{n}-\frac{\lambda_{n+1}}{n+1}\right|<+\infty.$$

For, by Abel's lemma, we have

$$\left|\sum_{n=1}^{N}\frac{\lambda_n}{n}\right|\leqq|\lambda_N|+\sum_{n=1}^{N-1} n\left|\frac{\lambda_{n+1}}{n+1}-\frac{\lambda_n}{n}\right|=O(1)+O(1)=O(1).$$

That the converse holds is clear since

$$\sum_{n=1}^{\infty} n\left|\frac{\lambda_{n+1}}{n+1}-\frac{\lambda_n}{n}\right|\leqq\sum_{n=1}^{\infty}|\varDelta\lambda_n|+\sum_{n=1}^{\infty}\frac{|\lambda_{n+1}|}{n+1}<\infty,$$

and from the convergence of $\sum_{n=1}^{\infty}|\varDelta\lambda_n|$ and $\sum_{n=1}^{\infty}\frac{|\lambda_n|}{n}$, it follows that $\lambda_n\to 0$.

**Proof of Theorem F.**

If $\{c_n\}$ is bounded and quasi-convex so that $\sum_{n=1}^{\infty}\frac{|c_n|}{n}<\infty$, then

$$\sum_{n=1}^{\infty} n^2 \left|\Delta^2\left(\frac{c_n}{n}\right)\right| = \sum_{n=1}^{\infty}\left|n\Delta^2 c_n + 2\Delta c_{n+1} - 2\frac{c_{n+1}}{n+1} + 4\frac{c_{n+2}}{n+2}\right|$$

$$\leqq \sum_{n=1}^{\infty} n\,|\Delta^2 c_n| + 2\sum_{n=1}^{\infty}|\Delta c_{n+1}| + 2\sum_{n=1}^{\infty}\frac{|c_{n+1}|}{n+1} + 4\sum_{n=1}^{\infty}\frac{|c_{n+2}|}{n+2} < +\infty,$$

by Lemma 4. Conversely if (5) holds, then

$$\sum_{n=1}^{\infty} n^2 \left|\Delta^2\left(\frac{c_n}{n}\right)\right| = \sum_{n=1}^{\infty}\left|n\Delta^2 c_n + 2n\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|$$

$$\geqq 2\sum_{n=1}^{\infty} n\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right| - \sum_{n=1}^{\infty} n\,|\Delta^2 c_n|,$$

i.e.,

$$2\sum_{n=1}^{\infty} n\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right| \leqq \sum_{n=1}^{\infty} n^2\left|\Delta^2\left(\frac{c_n}{n}\right)\right| + \sum_{n=1}^{\infty} n\,|\Delta^2 c_n| < +\infty.$$

Hence by Lemma 5 we conclude that

$$\sum_{n=1}^{\infty}\frac{|c_n|}{n} < +\infty,$$

i.e., (4)′ holds.

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