A Note on the Uniform Distribution of Sequences of Integers

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1. Let \( A = (a_n) \) be an infinite sequence of integers not necessarily different from each other. For any integers \( j \) and \( m \geq 2 \) we denote by \( A_N(j, m) \) the number of terms \( a_n (1 \leq n \leq N) \) satisfying the condition \( a_n \equiv j \pmod{m} \). Following Niven [2], we say that the sequence \( A \) is uniformly distributed \((\text{mod } m)\) if the limit

\[
\lim_{N \to \infty} \frac{1}{N} A_N(j, m) = \frac{1}{m}
\]

exists for all integers \( j, 1 \leq j \leq m \). If \( A \) is uniformly distributed \((\text{mod } m)\) for every integer \( m \geq 2 \), \( A \) is said to be uniformly distributed.

Niven [2] has shown that the sequence \( ([n\theta]) \) with irrational \( \theta \) is uniformly distributed. Indeed, this fact is equivalent to the well-known Bohr-Sierpiński-Weyl theorem to the effect that the sequence \( (n\theta) \) with irrational \( \theta \) is uniformly distributed \((\text{mod } 1)\).

We have proved in [3] the following theorem which is the analogue of the celebrated Weyl criterion for the uniform distribution \((\text{mod } 1)\) of sequences of real numbers.

**Theorem 1.** Let \( A = (a_n) \) be a sequence of integers. A necessary and sufficient condition for the sequence \( A \) to be uniformly distributed \((\text{mod } m), m \geq 2\), is that

\[
\lim_{N \to \infty} \frac{1}{N} S_N(A, \frac{h}{m}) = 0
\]

for all integers \( h, 1 \leq h \leq m - 1 \), where

\[
S_N(A, t) = \sum_{n=1}^{N} e(a_n t), \quad e(t) = e^{2\pi i t}.
\]

In fact, we have

\[
S_N(A, \frac{h}{m}) = \sum_{j=1}^{m} A_N(j, m) e\left(\frac{jh}{m}\right).
\]
so that

\[ \sum_{k=1}^{m-1} \left| S_N(A, \frac{h}{m}) \right|^2 = m \left[ \sum_{j=1}^{m} \left( A_N(j, m) - \frac{N}{m} \right)^2 \right]. \]

Theorem 1 follows from this relation at once.

The next result is due to Niven [2, Theorem 5.1]:

**Theorem 2.** If a sequence \( A = (a_n) \) of integers is uniformly distributed (mod \( m \)), \( m \geq 2 \), then \( A \) is uniformly distributed (mod \( d \)) for all divisors \( d \) of \( m \), \( d \geq 2 \).

In order to see this it will suffice, by virtue of Theorem 1, to observe that, for \( d \mid m \), \( d \geq 2 \), each of the fractions \( h/d \) (\( 1 \leq h \leq d - 1 \)) appears at least once among the fractions \( h/m \) (\( 1 \leq h \leq m - 1 \)).

2. On the other hand, W. Narkiewicz [1] introduced the concept of weakly uniform distribution of sequences of integers as follows. Let \( A = (a_n) \) be an infinite sequence of integers. Let \( m \) be an integer \( \geq 3 \) and suppose that the number \( S_N(m) \) of terms \( a_n \) (\( 1 \leq n \leq N \)) with \( (a_n, m) = 1 \) tends to infinity with \( N \). Then, the sequence \( A \) is said to be weakly uniformly distributed (mod \( m \)) if for every integer \( j \) with \( (j, m) = 1 \) one has

\[ \lim_{N \to \infty} \frac{A_N(j, m)}{S_N(m)} = \frac{1}{\phi(m)}, \]

where \( A_N(j, m) \) is as before the number of terms \( a_n \equiv j \) (mod \( m \)), \( 1 \leq n \leq N \), and \( \phi(m) \) is the Euler totient function. If \( A \) is weakly uniformly distributed (mod \( m \)) for every integer \( m \geq 3 \), we say that \( A \) is weakly uniformly distributed.

Thus the sequence \( (p_n) \) of all prime numbers (arranged in increasing order) is weakly uniformly distributed, as is readily seen from the prime number theorem for arithmetical progressions.

Now, let \( A = (a_n) \) be a sequence of integers and put for any residue character \( \chi \) (mod \( m \)), \( m \geq 3 \),

\[ S_N(A, \chi) = \sum_{n=1}^{N} \chi(a_n). \]

If \( \chi = \chi_0 \) is the principal character (mod \( m \)), then

\[ S_N(A, \chi_0) = \sum_{n=1}^{N} 1 = S_N(m). \]

Since we have for each \( \chi \) (mod \( m \))

\[ S_N(A, \chi) = \sum_{j=1}^{m} A_N(j, m) \chi(j), \]
we find
\[
\sum_{\chi(x) \neq 1} \left| S_N(A, \chi) \right|^2 = \phi(m) \sum_{(i, m) = 1}^{m} \left( A_N(i, m) - \frac{S_N(m)}{\phi(m)} \right)^2.
\]

Thus we have proved the following result of NARKI Ewicz [1].

**Theorem 3.** Let \( A = (a_n) \) be a sequence of integers. A necessary and sufficient condition for the sequence \( A \) to be weakly uniformly distributed \((\text{mod } m)\), where \( m \geq 3 \), is that
\[
\lim_{N \to \infty} S_N(A, \chi_0) = \infty
\]
and
\[
\lim_{N \to \infty} \frac{S_N(A, \chi)}{S_N(A, \chi_0)} = 0
\]
for all characters \( \chi \neq \chi_0 \) \((\text{mod } m)\).

There is an analogue of Theorem 2 for weakly uniform distribution of integers. In order to state our result we first introduce the notion of \( k \)-weakly uniform distribution of sequences of integers, where \( k \) is a positive integer.

Let \( A = (a_n) \) be a sequence of integers. Let \( k \) be a positive integer and let \( A^{(k)} \) be the subsequence of \( A \) consisting of all \( a_n \) in \( A \) with \((a_n, k) = 1\). The sequence \( A \) is said to be \( k \)-weakly uniformly distributed \((\text{mod } m)\), where \( m \geq 3 \), if the sequence \( A^{(k)} \) is weakly uniformly distributed \((\text{mod } m)\). If \( A \) is \( k \)-weakly uniformly distributed \((\text{mod } m)\) for every \( m \geq 3 \), \( A \) is said to be \( k \)-weakly uniformly distributed.

Again, the sequence \((p_n)\) of all prime numbers is \( k \)-weakly uniformly distributed for every fixed \( k \geq 1 \).

It will be clear that each of \( 1 \)-weakly and \( m \)-weakly uniform distributions \((\text{mod } m)\), \( m \geq 3 \), is equivalent to weakly uniform distribution \((\text{mod } m)\). Also, weakly uniform distribution \((\text{mod } \lcm[k, m])\), where \( k \geq 1 \) and \( m \geq 3 \), implies \( k \)-weakly uniform distribution \((\text{mod } m)\), but not conversely. (Here we denote by \([a, b]\) the least common multiple of two integers \( a \) and \( b \).)

For \( k \)-weakly uniform distribution of sequences of integers our Theorem 3 takes the following form:

**Theorem 3 bis.** A necessary and sufficient condition that a sequence \( A = (a_n) \) of integers be \( k \)-weakly uniformly distributed \((\text{mod } m)\), where \( k \geq 1 \) and \( m \geq 3 \), is that
\[
\lim_{N \to \infty} S_N(A^{(k)}, \chi_0) = \infty
\]
and
for all characters $\chi \neq \chi_0 \pmod m$.

We are now able to formulate the analogue of Theorem 2.

**Theorem 4.** If a sequence $A = (a_n)$ of integers is weakly uniformly distributed $\pmod m$, $m \geq 3$, then $A$ is $m$-weakly uniformly distributed $\pmod d$ for all divisors $d$ of $m$, $d \geq 3$.

Note that for $d \mid m$ a reduced residue system $\pmod m$ induces in the natural way a reduced residue system $\pmod d$ with multiplicity $\phi(m)/\phi(d)$. Theorem 4 follows from this directly, or via Theorem 3.

3. We now study a little the relation between uniform distribution and weak uniform distribution of sequences of integers.

**Theorem 5.** If a sequence of integers $(a_n)$ is uniformly distributed $\pmod m$, $m \geq 3$, then the sequence $(a_n + c)$ is weakly uniformly distributed $\pmod m$ for all integers $c$.

**Proof.** By Theorem 2, it follows from the assumption on the sequence $(a_n)$ that $(a_n)$ is uniformly distributed $\pmod d$ for all $d \mid m$, $d \geq 2$ (and also for $d = 1$ trivially), and, if we write for any integer $j$

$$A_N(j, d) = \frac{N}{d} + R_N(j, d),$$

then

$$R_N(j, d) = o(N) \text{ for } N \to \infty.$$
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\[ S_N(c; m) = \frac{N}{\phi(m)} + o(N) \quad \text{for } N \to \infty. \]

Hence, if we write \( A_N(c; j; m) \) for the number of integers \( a_n + c \equiv j \pmod{m} \), \( 1 \leq n \leq N \), then we have

\[ A_N(c; j; m) = A_N(j - c, m) = \frac{N}{m} + o(N) = \frac{S_N(c; m)}{\phi(m)} + o(S_N(c; m)) \]

for \( N \to \infty \). This last relation holds for any integer \( j \) and, a fortiori, for any integer \( j \) with \((j, m) = 1\). Thus the sequence \( (a_n + c) \) is weakly uniformly distributed \( \pmod{m} \).

A converse of Theorem 5 is the following

**Theorem 6.** Let \( m \) be an integer \( \geq 3 \) and suppose that a sequence of integers \((a_n)\) be such that the sequence \((a_n + c)\) is weakly uniformly distributed \( \pmod{m} \) for all integers \( c \), \( 0 \leq c \leq m - 1 \). Then, if \( m \) is odd, the sequence \((a_n)\) is uniformly distributed \( \pmod{m} \); whereas if \( m \) is even, the sequence \((a_n)\) is not necessarily uniformly distributed \( \pmod{m} \).

**Proof.** Case of odd \( m \). From the assumption on the sequence \((a_n)\) it follows that the sequence \((a_n + c)\) is weakly uniformly distributed \( \pmod{m} \) for any integer \( c \). Let \( c \) be an arbitrary integer and \( j \) be any integer with \((j, m) = 1\). Again, we denote by \( A_N(c; j; m) \) the number of integers \( a_n + c \equiv j \pmod{m} \), \( 1 \leq n \leq N \). Then we have, by assumption,

\[ A_N(c; j; m) = \frac{S_N(c; m)}{\phi(m)} + Q_N(c; j, m), \]

where

\[ S_N(c; m) = \sum_{n=1}^{N} 1 \leq N \quad (a_n + c, m) = 1 \]

and

\[ Q_N(c; j, m) = o(S_N(c; m)) = o(N) \quad \text{for } N \to \infty. \]

Now we have for \( N \to \infty \)

\[ A_N(c; 1, m) = \frac{S_N(c; m)}{\phi(m)} + o(N) \]
and, since \((2, m) = 1\),
\[
A_N(c + 1; 2, m) = \frac{S_N(c + 1; m)}{\phi(m)} + o(N).
\]

Hence, on noticing that \(A_N(c; 1, m) = A_N(c + 1; 2, m)\) for all \(N\) we find
\[
S_N(c; m) = S_N(c + 1; m) + o(N),
\]
so that
\[
\sum_{a=0}^{m-1} S_N(a; m) = mS_N(c; m) + o(N)
\]
for any particular \(c\). Therefore, it follows from the relation
\[
N = \sum_{a=0}^{m-1} A_N(a; m) = \frac{1}{\phi(m)} \sum_{a=0}^{m-1} S_N(a; m) + o(N),
\]
where \(j\) is any integer with \((j, m) = 1\), that
\[
\frac{S_N(c; m)}{\phi(m)} = \frac{N}{m} + o(N)
\]
or
\[
A_N(c; j, m) = \frac{N}{m} + o(N).
\]

Since the integer \(c\) is arbitrary and \(A_N(c; j, m) = A_N(j - c, m)\), we obtain for
\(N \to \infty\)
\[
A_N(j, m) = \frac{N}{m} + o(N)
\]
for all \(j, 1 \leq j \leq m\). Thus the sequence \((a_n)\) is uniformly distributed \((\text{mod} \ m)\).

*Case of even \(m\).* Consider the sequence \((a_n)\) of integers obtained by arranging in increasing order from the set of non-negative integers \(M_0 \cup M_1\), where
\[
M_0 = \{a \equiv 0, 2, 4, \ldots, m - 2 \pmod{m}\}
\]
and
\[
M_1 = \{a \equiv 1, 3, 5, \ldots, m - 1 \pmod{2m}\}.
\]

One may easily verify that the sequence \((a_n + c)\) is weakly uniformly distributed \((\text{mod} \ m)\) for all integers \(c\), but \((a_n)\) is not uniformly distributed \((\text{mod} \ m)\).
References

