On the Homotopy Groups of Rotation Groups $R_n$

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1 Introduction

On the homotopy groups of rotation groups $R_n$, considerably many results have been obtained: In [7] and others, MIURA determined groups $\pi_i(R_n)$ and their generators for $i \leq 14$, and in [5] KERVAIRE determined the groups $\pi_i(R_n)$ for $i \geq n + 4$.

In the present paper, we shall determine the 2-primary components of groups $\pi_i(R_n)$, $i = 15, 16$ and $17$, together with their generators. For this purpose, we consider the homotopy exact sequence

$$
i^* \rightarrow \pi_i(R_n) \xrightarrow{i_*} \pi_i(R_{n+1}) \xrightarrow{p_*} \pi_i(S^n) \xrightarrow{\partial} \pi_{i-1}(R_n) \rightarrow
$$

of bundle $(R_{n+1}, p, S^n)$, and $J$-homomorphism

$$J : \pi_i(R_n) \rightarrow \pi_{n+i}(S^n).$$

Starting with $R_3$ which is homeomorphic to real projective 3-space, we obtain our results inductively. The author wishes to thank to S. Saito for his advice throughout the preparation of the paper.

2 Preliminaries

For any fibre space $(X, p, B)$, we have the following homotopy exact sequence

$$(2.1) \quad \cdots \rightarrow \pi_i(F) \xrightarrow{i_*} \pi_i(X) \xrightarrow{p_*} \pi_i(B) \xrightarrow{\partial} \pi_{i-1}(F) \rightarrow \cdots,$$

where $F$ is the fibre $p^{-1}(x_0)$ on a base point $x_0$ of $B$, $i : F \rightarrow X$ is the inclusion map and $\partial$ is the boundary homomorphism. Homomorphisms $i_*$, $p_*$ and $\partial$ of (2.1) satisfy the following relation

$$(2.2) \quad \begin{align*}
i_*(\alpha\beta) &= i_*(\alpha)\beta \quad &\text{for } \alpha \in \pi_j(F), \beta \in \pi_i(S^j), \\
p_*(\alpha\beta) &= p_*(\alpha)\beta \quad &\text{for } \alpha \in \pi_j(X), \beta \in \pi_i(S^j), \\
\partial(\alpha E\beta) &= \partial(\alpha)\beta \quad &\text{for } \alpha \in \pi_j(B), \beta \in \pi_{i-1}(S^{j-1}).
\end{align*}$$
where \( E : \pi_i(S^l) \to \pi_{i+1}(S^{l+1}) \) is the suspension homomorphism.

Let \( R_{n+1} \) be the rotation group of euclidean \((n+1)\)-space, and \( i : R_n \to R_{n+1} \) be the inclusion map. Then \((R_{n+1}, \varphi, S^n)\) is a fibre space with fibre \( R_n \). Since the group \( R_n \) is topologically equivalent to real projective 3-space \( \text{P}^n \), \( \pi_3(R_n) \cong \mathbb{Z} \) with a generator \([\eta_2]\) (cf. [12]), and the correspondence \([\eta_2] \alpha \to \eta_2 \alpha, \alpha \in \pi_i(S^n)\), induces the homomorphism

\[
(2.3) \quad p_\# : \pi_i(R_n) \to \pi_i(S^n),
\]

which is an isomorphism for \( i \geq 2 \).

For \( n = 3 \) or 7, the bundle \((R_{n+1}, \varphi, S^n)\) is equivalent to product bundle \( S^n \times R_n \) over \( S^n \), and if we denote the homotopy class of the cross section of this bundle by \([\iota_n] \), the correspondence \((\alpha, \beta) \to i_{\#} \alpha + [\iota_n] \beta, \alpha \in \pi_i(R_n), \beta \in \pi_i(S^n)\), yields an isomorphism

\[
(2.4) \quad \pi_i(R_n) + \pi_i(S^n) \cong \pi_i(R_{n+1}).
\]

Now, the notations of this paper conform to those of [13]; in particular, \( \pi_i(X;2) \) denotes the 2-primary component of the group \( \pi_i(X) \), and a subgroup \( \pi_i^n \) of \( \pi_i(S^n) \) is defined by setting

\[
(2.5) \quad \pi_i^n = \begin{cases} 
\pi_i(S^n) & \text{if } i = n, \\
E^{-1} \pi_{2n}(S^{n+1};2) & \text{if } i = 2n - 1, \\
\pi_i(S^n;2) & \text{if } i \neq n, 2n - 1.
\end{cases}
\]

Groups \( \pi_i^n \) and their generators are given in Table 1.

Applying (2.1) to the bundle \((R_{n+1}, \varphi, S^n)\), we have

\[
(2.6) \quad i_{\#} \to \pi_i(R_n;2) \to \pi_i(R_{n+1};2) \to \pi_i^n \to \pi_{i-1}(R_n;2) \to .
\]

Let \([\alpha]\) denote an element of \( \pi_i(R_{n+1};2) \) such that \( p_{\#}([\alpha]) = \alpha \in \pi_i^n \), and let \( j : R_{n+1} \to R_m, m > n + 1 \), be the inclusion map, and define \([\alpha]_m \in \pi_i(R_m;2) \) by setting \([\alpha]_m = j_{\#}([\alpha])\).

The groups \( \pi_i(R_i;2) \) and their generators are known for \( i \leq 14 \). We need them in the subsequent calculation, so we give them in the Table 2.

For the image of the boundary homomorphism \( j : \pi_i^n \to \pi_{i-1}(R_i;2) \), we have the results given in the Table 3.

The homomorphism

\[
J : \pi_i(R_n) \to \pi_{i+d}(S^n)
\]

of G. W. Whitehead was defined as follows:
Let $f : S^i \rightarrow R_n$ be a representative of an element $\pi_1(R_n)$. Define a mapping $F : S^i \times S^{n-1} \rightarrow S^{n-1}$ by setting

$$F(x, y) = f(x)y$$

for any $x \in S^i$ and $y \in S^{n-1}$.
Let $G(F): S^{n+i} \approx S^i \times S^{n-1} \longrightarrow S^n$ be the Hopf-construction of $F$, where $A \ast B$ denotes the join of $A$ and $B$. Then, $G(F)$ represents an element $J(\alpha) \in \pi_{i+n}(S^n)$.

We have a diagram

\[
\begin{array}{ccccccc}
\cdots \cdots \cdots \pi_i(R_n;2) & \xrightarrow{i_0} & \pi_i(R_{n+1};2) & \xrightarrow{p_0} & \pi_{i+1}(R_n;2) & \longrightarrow \cdots \\
\downarrow J & & \downarrow J & & \downarrow E^{n+1} & & \downarrow J \\
\cdots \cdots \cdots \pi_{i+n} & \xrightarrow{e} & \pi_{i+n+1} & \xrightarrow{h} & \pi_{i+n+1} & \longrightarrow \cdots \\
\end{array}
\]

which is commutative up to sign, and its lower sequence is exact ([11], Proposition 4.2). Moreover, the homomorphism $J: \pi_i(R_n) \longrightarrow \pi_{i+n}(S^n)$ satisfies

\[(2.7) \quad J(\alpha \beta) = J(\alpha)J(\beta) \quad \text{for any } \alpha \in \pi_i(R_n) \text{ and } \beta \in \pi_i(S^n).
\]

Recall that

\[(2.8) \quad J([e^n]) = \alpha_n \quad \text{for } n > 8.
\]

\[(2.9) \quad J([\nu_n]) = \nu_n \quad \text{for } n > 8.
\]

3 Groups $\pi_{i+1}(R_n)$ and their generators

In this section, we shall determine the generators of the 2-primary components of $\pi_{i+1}(R_n)$.

In the sequel, we shall use the abbreviated notation $\pi_i(R_n)$ for $\pi_i(R_n;2)$.

The homotopy groups of spinor groups $Spin(n)$, $n \leq 9$, are given in [7] and [9], and we have isomorphisms

\[(3.1) \quad \pi_i(R_3) \approx \pi_i(Spin(5)) \approx \pi_i(Sp(2)),
\]

\[(3.2) \quad \pi_i(R_4) \approx \pi_i(Spin(6)) \approx \pi_i(SU(4)).
\]

The results for $\pi_{i+1}(R_n;2)$ are stated as follows:

**Proposition 2.1.** $\pi_{i+1}(R_n;2) = ([\gamma_i]_{p}^\theta) + ([\gamma_i]_{p}^\theta) \approx Z_2 + Z_2$

$\pi_{i+1}(R_3;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_4;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_5;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_6;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_7;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_8;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_9;2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_{10};2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_{11};2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_{12};2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_{13};2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_{14};2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$

$\pi_{i+1}(R_{15};2) = ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) + ([\gamma_2]_{p}^\theta) \approx Z_2 + Z_2 + Z_2 + Z_2$
\[ \pi_{15}(R_{16}): 2 = \{ [\gamma_{12}]_{16} \} + [2_{15}] \approx \mathbb{Z} + \mathbb{Z} \]
\[ \pi_{15}(R_{n}): 2 = \{ [\gamma_{12}]_{n} \} \approx \mathbb{Z} \quad \text{for } n \geq 17. \]

The following relation hold: \([8\sigma_{15}]_{15} = 2[\gamma_{12}].\]

Proof. The results for \(\pi_{15}(R_3)\) and \(\pi_{15}(R_4)\) follow directly from (2.3), (2.4) and Table 1.

\(R_3:\) Since \(\pi_1(R_3) \approx \mathbb{Z}\) and \(\pi_2(R_3)\) is finite,

\[ (3, 3) \quad i_\circ : \pi_1(R_3) \longrightarrow \pi_1(R_3) \] is trivial, i.e., \(i_\circ([\gamma_2]\nu') = i_\circ([\gamma_2]\nu') = 0.\)

Therefore \(i_\circ : \pi_{15}(R_3) \longrightarrow \pi_{15}(R_3)\) is trivial, too. From this and (2.6), we have the exact sequence

\[ 0 \longrightarrow \pi_{15}(R_3) \longrightarrow \pi_{15}(R_3) \longrightarrow \pi_{15}(R_3).\]

Using Table 2 and 3, we can prove that the kernel of \(J : \pi_{15} \longrightarrow \pi_{14}(R_4)\) is generated by \(\nu_4\epsilon'\gamma_{14}\).

In fact, from Table 1, \(\pi_1 = \{ \nu_4\epsilon'\gamma_{14} \} + \{ \nu_4\nu_4 \} + \{ \nu_4\gamma_7 \} + \{ \nu_4\nu_{12} \} + \{ E\nu'\nu_7 \} + \{ E\nu'\nu_7 \} \)
\(\approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2.\) Then we have

\[ A(\nu_4\epsilon'\gamma_{14}) = A(\nu_4\epsilon'\gamma_{14}) \quad \text{by (2.2),} \]
\[ = 2([\epsilon_5]_\nu'\gamma_{13}) \quad \text{by Table 3,} \]
\[ = 2([\epsilon_5]_\nu'\gamma_{13}) = 0. \]

\[ A(\nu_4\nu_4) = A(\nu_4)\nu_0 = [\epsilon_5]_\nu'\nu_0 + \alpha[\gamma_2]_{\nu'\nu_7}. \]

\[ A(\nu_4\gamma_7) = A(\nu_4)\gamma_7 = [\epsilon_5]_\nu'\gamma_7 + \alpha[\gamma_2]_{\nu'\nu_7} + [\gamma_2]_{\nu'\nu_7} + [\gamma_2]_{\nu'\nu_7}. \]

\[ A(E\nu'\nu_7) = A(\nu_4)\nu_0 = [\gamma_2]_{\nu'\nu_7}. \]

\[ A(E\nu'\nu_7) = A(\nu_4)\nu_0 + [\gamma_2]_{\nu'\nu_7}. \]

Thus, from the above exact sequence and by definition of \([\nu_4\epsilon'\gamma_{14}]\), we have

\[ \pi_{15}(R_3) = \{ [\nu_4\epsilon'\gamma_{14}] \} \approx \mathbb{Z}_2. \]

Let \((2.6)\_n\) denote a part of the exact sequence (2.6) starting with \(\pi_{15}\) and ending in \(\pi_{14}(R_3)\), i.e.,

\[ (2.6)\_n \quad \pi_{15} \longrightarrow \pi_{15}(R_3) \longrightarrow \pi_{15}(R_3) \longrightarrow \pi_{15}(R_3) \longrightarrow \pi_{15}(R_3). \]

\(R_4:\) Consider \((2.6)\_n\). Since \([\epsilon_5]_{3}\nu' = 4[2\nu_4\epsilon']\) in \(\pi_{15}(R_5)\), using Table 3 we have

\[ A(\gamma_{12}\nu_4) = A(\gamma_{12}\nu_4) \quad \text{by (2.2),} \]
\[ = [\epsilon_5]_{3}\gamma_3\gamma_5 \quad \text{by Table 3,} \]
\[ = 2[\epsilon_5]_{3}\nu' \quad \text{by (7.7) of [13],} \]
\[ = 8[2\nu_4\epsilon'] = 0. \]
Table 2 : $\pi_i (R_n; 2)$ for $3 \leq i \leq 14$

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On the Homotopy Groups of Rotation Groups $R_n$

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| $\pi_{12}(R_n; 2)$ | Z | 0 | 0 |
| generators | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} |

| $\pi_{13}(R_n; 2)$ | Z | Z | 0 |
| generators | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} |

| $\pi_{14}(R_n; 2)$ | Z | Z | Z |
| generators | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} | [\eta_2^3]_{e_8}, [\eta_2^4]_{e_8}, [\eta_2^5]_{e_8} |
From Table 2, there exists an element \([v_8]_a \in \pi_{18}(R_8)\) such that \(p_8([v_8]_a) = v_8a_3\).
Since \([v_8]_a\) is of order 8, \([v_8]_a\) is of order 8. On the other hand, by (3.2) and Theorem 6.1 of \([7]\), \(\pi_{18}(R_8) \approx \mathbb{Z}_8 + \mathbb{Z}_2\). Thus, from (2.6)_9, we have
\[
\pi_{18}(R_8) = \left(\left[v_8\right]_a\sigma_8\right) + \left(\left[v_8\sigma'\eta_1\right]_b\right) \approx \mathbb{Z}_8 + \mathbb{Z}_2
\]
and
\[
(3.4) \quad i_8 : \pi_{18}(R_3) \longrightarrow \pi_{18}(R_8) \text{ is a monomorphism.}
\]

\(R_7\) : Consider (2.6)_9. From Table 1 and 3, we have relations;
\[
\begin{align*}
\Delta(\eta_6v_7) &= \Delta(v_8 = \eta_6v_7) = 0 \quad \text{by Table 2,} \\
\Delta(\nu_6) &= 0 \quad \text{by Table 2.}
\end{align*}
\]

(3.5) \[\quad \Delta(\nu_6) = \Delta(v_6) = 2[v_8]_a = 2[v_8]_a \quad \text{by Table 3 and (2.2),}
\]

\[
\Delta(\eta_6v_7) = 0 \quad \text{by Table 3.}
\]

Therefore, from (2.6)_9, we have the following exact sequence
\[
0 \longrightarrow \left(\left[v_8\right]_a\sigma_8\right) + \left(\left[v_8\sigma'\eta_1\right]_b\right) \overset{i_8}{\longrightarrow} \pi_{18}(R_7) \overset{p_8}{\longrightarrow} \left[v_8\right]_a + \{\eta_6v_7\} \longrightarrow 0.
\]

From Table 2, there exist a element \([\eta_6]_e^7\) and \([\eta_6]_f^7\) in \(\pi_{18}(R_7)\) such that \(p_8([\eta_6]_e^7) = \eta_6^a\epsilon_7\) and \(p_8([\eta_6]_f^7) = \eta_6\nu_7 = v_8\) (by Lemma 6.3 of \([13]\)).
Since \(v_8\) and \(\epsilon_7\) are of order 2, \([\eta_6]_e^7\) and \([\eta_6]_f^7\) are of order 2. Thus we have
\[
\pi_{18}(R_7) = \left\{\left[\eta_6\right]_e^7 + \left[\eta_6\right]_f^7\right\} + \left\{\left[v_8\right]_a\sigma_8\right\} + \left\{\left[v_8\sigma'\eta_1\right]_b\right\} \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2.
\]

\(R_8\) : The result for \(\pi_{18}(R_8)\) follows from the result for \(\pi_{18}(R_7)\), (2.4) and Table 1.
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$R_9$ : From Table 3, we have following relations

$\begin{align*}
&\Delta(e_9) = \Delta(e_9)e_7 = [\gamma_9]e_7,
&\Delta(e_9) = \Delta(e_9)e_7 = [\gamma_9]e_7,
&\Delta(\sigma_9) = [\tau_5][\gamma_9],
&\Delta(E\sigma_9) = [\nu_9][\gamma_9].
\end{align*}$

From (2.6) and Table 3, we have the exact sequence

$$
0 \longrightarrow [\tau_5][\gamma_9] + [\tau_5][\gamma_9] + [\nu_9][\gamma_9] \longrightarrow \pi_{10}(R_9) \longrightarrow \{8\sigma_9\} \longrightarrow 0.
$$

Thus, from the fact that $8\sigma_9$ is of order infinite, we conclude

$$
\pi_{10}(R_9) = \{[8\sigma_9] \} + \{[\tau_5][\gamma_9] \} + \{[\nu_9][\gamma_9] \} \approx \mathbb{Z} + \mathbb{Z} + \mathbb{Z}.
$$

Moreover, from the exactness of the sequence (2.6), we have

$$(3.7) \quad i_9 : \pi_{10}(R_9) \longrightarrow \pi_{10}(R_9) \text{ is an epimorphism.}$$

$R_{10}$ : Consider (2.6). Using Table 3, we have

$$
\begin{align*}
&\Delta(e_{10}) = \Delta(e_9)e_7 = [\gamma_9]e_7 \quad \text{by (2.2)}
&\Delta(e_{10}) = \Delta(e_9)e_7 = [\gamma_9]e_7 \quad \text{by Table 3}
&\Delta(\sigma_{10}) = [\tau_5][\gamma_9] \quad \text{by (7.4) of [13].}
\end{align*}
$$

Thus, from (2.6), we have the exact sequence

$$
\pi_{10} = \{\sigma_9\} \longrightarrow \pi_{10}(R_{10}) \longrightarrow \pi_{10}(R_{10}) \longrightarrow 0.
$$

For the homomorphism $\Delta : \pi_{10} \longrightarrow \pi_{10}(R_{10})$, we have

$$(3.8) \quad \Delta(\sigma_9) = \Delta(e_{10}) e_7 \quad \text{by (2.2)}
&\Delta(\sigma_9) = [\tau_5][\gamma_9] \quad \text{by Table 3}
&\Delta(\sigma_9) = [\tau_5][\gamma_9] + [\nu_9][\gamma_9] \quad \text{by (7.4) of [13].}
$$

Therefore, from the above exact sequence, we have

$$
\pi_{10}(R_{10}) = \{[8\sigma_9] \} + \{[\tau_5][\gamma_9] \} + \{[\nu_9][\gamma_9] \} \approx \mathbb{Z} + \mathbb{Z} + \mathbb{Z}.
$$

$R_{11}$ : Consider the diagram (2.7)

$$
\begin{align*}
&\pi_{11} = \{v_{10}^2\} \longrightarrow \pi_{11}(R_{10}) \longrightarrow \pi_{11}(R_{11}) \longrightarrow \pi_{11} = 0
&\Delta \downarrow E^{11} \quad J \quad \downarrow E \quad J
&\pi_{11} = \{v_{11}^2\} \longrightarrow \pi_{11} \longrightarrow \pi_{11} \longrightarrow \pi_{11} = 0
\end{align*}
$$
Then we have
\[ \Delta E^1_{10} = A(\nu_{10}) = \sigma_{10} \nu_{17} \cong 0 \] by (10.20) of [13].

Therefore, from the above diagram and \( \pi_{10} \approx \mathbb{Z}_2 \), we have that \( \Delta : \pi_{10} \to \pi_{10}(R_{10}) \) must be non-trivial. On the other hand, we have
\[
J([\nu_5]_{10} + \nu_8) = J([\nu_5]_{10} + \sigma_8) = \nu_8 \sigma_8 = 0
\]
by (2.9). From the above diagram we have
\[
A(\nu_{10}) = \sigma_{10} \nu_{17} + x[\nu_5]_{10} \sigma_8
\]
where \( x = 1 \) or 0. Therefore, the exactness of the upper row sequence of the above diagram, we have
\[
\pi_{10}(R_{11}) = ([8\sigma_8]_{11}) + ([\nu_5]_{11} \sigma_8) \cong \mathbb{Z} + \mathbb{Z}_2,
\]
and from the exact sequence (2.6) we have that
\[
\pi_{10}(R_{11}) \to \pi_{10}(R_{11}) \to \mathbb{Z} + \mathbb{Z}_2,
\]
where
\[
i_n : \pi_{10}(R_{10}) \to \pi_{10}(R_{11})
\]
is an epimorphism.

R_{12} : Since \( \pi_{10}^1 = \pi_{10}^2 = 0 \), we have the result for \( \pi_{10}(R_{12}) \) from (2.6). To show the result for \( \pi_{10}(R_{13}) \), we shall need the following

**Lemma 3.10.** (SUGAWARA [11]). Let \( \alpha \) be an element of \( \pi_{r+1}(S^n) \). Then We have
\[
E^{n+3}_0 d(\alpha) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 2 E^{n+3}_0 & \text{if } n \text{ is even}, \end{cases}
\]
where \( \Delta : \pi_{r+1}(S^n) \to \pi_r(R_n) \) is boundary homomorphism and \( p : \pi_r(R_n) \to \pi_r(S^{n-1}) \) is a homomorphism induced by the bundle projection \( p : R_n \to S^{n-1} \).

R_{13} : From (3.10), \( E^{15}_0 p_0 d(\nu_{12}) = E^{15}(2\nu_{11}) \). Since \( E^{15} : \pi_1 \to \pi_2 \) is an isomorphism, we have \( p_0 d(\nu_{12}) = 2 \nu_{11} \). By definition of \( [2\nu_{11}] \), \( d(\nu_{11}) = [2\nu_{11}] \). From the fact that order of \( [2\nu_{11}] \) is equal to 4, we have that the kernel of \( \Delta : \pi_{13}^2 \to \pi_{14}(R_{12}) \) is generated by \( 4\nu_{12} = \gamma_{12}^3 \). Thus there exists an element \( [\gamma_{12}^3] \in \pi_{13}(R_{12}) \) such that \( p_0([\gamma_{12}^3]) = 4\nu_{12} \).

Consider the following diagram
\[
\begin{array}{cccccc}
0 & \to & \pi_{12}^2 & \to & \pi_{13}(R_{12}) & \to & \pi_{14}(R_{13}) & \to & \{\gamma_{12}^3\} & \to & 0 \\
& \downarrow J & & \downarrow E & & \downarrow J & & \downarrow H & & \downarrow E_{13} & & \\
0 & \to & \pi_{13}^2 & \to & \pi_{14}^3 & \to & \pi_{16}^3 & \to & \{\gamma_{15}^3\} & \to & 0
\end{array}
\]
of (2.7). The upper row sequence is exact by the above fact and the lower sequence is exact by (10.11) of [13]. We have

\[ \pi^{(2)} = \{E^* \rho' \} + \{\varepsilon_{12} \} \cong \mathbb{Z}_{16} + \mathbb{Z}_2 \]  

by Theorem 10.5 of [13],

\[ \pi^{(1)} = \{\rho_{13} \} + \{\varepsilon_{11} \} \cong \mathbb{Z}_{32} + \mathbb{Z}_2 \]  

by Theorem 10.10 of [13],

\[ E^* \rho' = 2\rho_{13} \]  

by Lemma 10.9 of [13],

and

\[ H(\rho_{13}) = 4\nu_{25} = \gamma_{35}^3 \]  

by (10.11) of [13].

Next we prove

**Lemma 3.12** \( J[[8\sigma_8]] = E^* \rho' + x\varepsilon_{12} \) for \( x = 0 \) or \( 1 \).

**Proof.** Consider the diagram

\[
\begin{array}{cccc}
\pi_{15}(R_0) & \longrightarrow & \pi_{15}(R_0) & \longrightarrow & \pi^{(1)} \\
J & \downarrow & J & \downarrow & J \\
\pi_{29} & \longrightarrow & \pi^{(2)} & \longrightarrow & \pi^{(3)} \\
\end{array}
\]

of (2.7). Then we have

\[ H(J[[8\sigma_8]]) = \pm E^*(8\sigma_8) \]

\[ = 8\sigma_{17} \]

\[ = H(\rho') \]  

by (10.2) of [13].

Thus, \( J[[8\sigma_8]] \equiv \rho' \mod E\pi_{23}^{(3)} \).

By definition of \([8\sigma_8]_{12}\) and the diagram (2.7),

\[ J[[8\sigma_8]_{12}] = J[j_8[[8\sigma_8]](\rho_8)] \]

\[ = E^*j_8[[8\sigma_8]] \]

\[ = E^*\rho' \mod E^*\pi_{23} \]

\[ = E^*\rho' + x\varepsilon_{12}, \]

where \( x = 0 \) or \( 1 \) and \( j_8 : \pi_{15}(R_0) \longrightarrow \pi_{15}(R_{12}) \) is a homomorphism induced by the inclusion map \( j : R_0 \longrightarrow R_{12} \).

From the above diagram, we have

\[ J[[\gamma_{12}]] \equiv \rho_{13} \mod E\pi_{23}^{(3)}. \]

Therefore, from (3.11),

\[ [8\sigma_8]_{13} = 2[\gamma_{13}]. \]

Thus we have, from the exactness of the upper sequence of the diagram (3.11),

\[ \pi_{15}(R_{13}) = [[\gamma_{13}]] + [[\nu_{25}]]_{12} \cong \mathbb{Z} + \mathbb{Z}_2. \]

**R_{14} and R_{15}** : Consider (2.6)_{13} and (2.6)_{14}. According to Theorem 3 of [5],
we have
\[ d(v_{13}) \approx 0, \quad 2d(v_{13}) = 0 ; \quad d(\gamma_{14}) \approx 0, \quad d(\gamma_{14}) \approx 0. \]

Therefore we conclude
\[ \pi_{16}(R_{n+1}) = \{[\gamma_{13}]_{n+1}\} \cong \mathbb{Z} \quad \text{for } n = 13 \text{ and } 14. \]

**R_{15}**: Consider (2.6)_{15}. From Theorem 23.4 of [12], we have
\[ (3.14) \quad p_{6}d(v_{16}) = 2v_{16}. \]

On the other hand \( d : \pi_{16} \cong \mathbb{Z}_{2} \rightarrow \pi_{16}(R_{15}) \cong \mathbb{Z} \) is trivial. Thus we have the exact sequence
\[ 0 \rightarrow \pi_{16}(R_{15}) \rightarrow \pi_{16}(R_{16}) \rightarrow \{2v_{16}\} \cong \mathbb{Z} \rightarrow 0. \]

Therefore we obtain that
\[ \pi_{16}(R_{16}) = \{[\gamma_{13}]_{16}\} + \{[2v_{16}]\} \cong \mathbb{Z} + \mathbb{Z}. \]

**R_{n} for \( n \geq 17 \)**: From (3.12), (2.6)_{16} and the stability of \( \pi_{16}(R_{n}) \), we have
\[ \pi_{16}(R_{n}) = \{[\gamma_{13}]_{n}\} \cong \mathbb{Z} \quad \text{for } n \geq 17. \]

4. **Groups \( \pi_{16}(R_{n}) \) and their generators**

The results for \( \pi_{16}(R_{n}:2) \) are stated as follows:

**Proposition 4.1.** \( \pi_{16}(R_{3}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} \cong \mathbb{Z}_{2} \)

\[ \pi_{16}(R_{4}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{5}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{6}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{7}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{8}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{9}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{10}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{11}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{12}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{13}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]

\[ \pi_{16}(R_{14}:2) = \{[v_{14}]^{\nu} \eta_{7} \mu_{7}\} + \{[v_{15}]^{\nu} \eta_{8} \mu_{8}\} \cong \mathbb{Z}_{2} + \mathbb{Z}_{2} \]
\[
\begin{align*}
\pi_{10}(R_5; 2) &= \{[\gamma_{16}] \} + \{[\gamma_{16}] \} \approx \mathbb{Z}_2 + \mathbb{Z}_2 \\
\pi_{10}(R_6; 2) &= \{[\gamma_{16}] \} + \{[\gamma_{16}] \} + \{[\gamma_{16}] \} \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \\
\pi_{10}(R_7; 2) &= \{[\gamma_{16}] \} + \{[\gamma_{16}] \} \approx \mathbb{Z}_2 + \mathbb{Z}_2 \\
\pi_{10}(R_n; 2) &= \{[\gamma_{16}] \} \quad \text{for } n > 17.
\end{align*}
\]

**Proof.** From (2.3), (2.4) and Table 1, we have the results for \(\pi_{10}(R_5)\) and \(\pi_{10}(R_6)\).

\(R_5\) : Consider the exact sequence

\[
i^*_6: \pi_{10}(R_6) \longrightarrow \pi_{10}(R_5) \longrightarrow \pi_{10}(R_5) \longrightarrow \pi_{10}(R_4).
\]

By the same argument as in the case of \(i^*: \pi_{10}(R_4) \longrightarrow \pi_{10}(R_5)\), we have that \(i^*: \pi_{10}(R_6) \longrightarrow \pi_{10}(R_5)\) is trivial. On the other hand from Table 3 we can prove that the kernel of \(\partial: \pi_{10}(R_5) \longrightarrow \pi_{10}(R_4)\) is generated by \(\nu_4\gamma_{16}\) and \(\nu_6\). Then, the exactness of the above sequence, we have an isomorphism

\[
p^*_6: \pi_{10}(R_5) \longrightarrow \{[\nu_4\gamma_{16}] \} + \{[\nu_6] \}.
\]

From Table 3 and the result for \(\pi_{10}(R_5)\), there exist elements \([\nu_4\gamma_{16}] \in \pi_{10}(R_5)\) and \([\nu_6] \in \pi_{10}(R_5)\) such that \(p^*_6([\nu_4\gamma_{16}]) = \nu_4\gamma_{16}\) and \(p^*_6([\nu_6]) = \nu_6\). Therefore we obtain from (2.2) that

\[
\pi_{10}(R_5) = \{[\nu_4\gamma_{16}] \} + \{[\nu_6] \} \approx \mathbb{Z}_2 + \mathbb{Z}_2.
\]

\(R_6\) : From Table 3 and (3.4), it follows that the sequence

\[
i^*_6: \pi_{10}(R_6) \longrightarrow \pi_{10}(R_5) \longrightarrow \pi_{10}(R_5) \longrightarrow 0
\]

is exact. From (3.2) and Theorem 6.1 of [7], the above sequence splits. From Table 3, there exists an element \([\nu_5] \in \pi_{10}(R_5)\) such that \(p^*_6([\nu_5]) = \nu_5\). Thus we have from (2.2)

\[
\pi_{10}(R_6) = \{[\nu_4\gamma_{16}] \} + \{[\nu_6] \} + \{[\nu_5] \} \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2.
\]

\(R_7\) : From (3.5), the exact sequence (2.6) yields the following exact sequence

\[
i^*_7: \pi_{17}(R_7) \longrightarrow \pi_{16}(R_7) \longrightarrow \pi_{16}(R_7) \longrightarrow \{4\nu_0\gamma_{16}\} + \{[\gamma_6] \} \longrightarrow 0.
\]

Now we have following relations:

\[
A(\xi_6) = 2[\xi_6] + a[\nu_6] + b[\nu_5], \quad a, b = 0 \text{ or } 1,
\]

(4.1)

\[
A(\nu_6\gamma_{16}) = 0.
\]
In fact, since $E^p R_6 \cdot A(\zeta_6) = 2E^p \zeta_6$ by (3.10), we have $A(\zeta_6) = 2[\zeta_6] + a[\nu_6] + b[\nu_6] \zeta_6$ for some integers $a, b$. And

$$A(\nu_6 \nu_{14}) = A(\nu_6 + \varepsilon_0 \nu_{14})$$
$$= A(\nu_6 + \varepsilon_0 \nu_{14})$$
$$= 0$$

By (7.10) of [13], $A(\nu_6 \sigma_9 = \eta_9 \varepsilon_9$. From Table 2, there exist elements $[\eta_9], \eta_9 \varepsilon_9$ and $[\eta_9], \eta_9 \mu_7$ such that $\rho_9([\eta_9], \eta_9 \varepsilon_9) = \eta_9 \varepsilon_9$ and $\rho_9([\eta_9], \eta_9 \mu_7) = \eta_9 \mu_7$. Since $\eta_9 \varepsilon_9$ and $\mu_7$ are of order 2, it follows that $[\eta_9], \eta_9 \varepsilon_9$ and $[\eta_9], \eta_9 \mu_7$ are of order 2. Thus, from the above exact sequence, we have

$$
\pi_{16}(R_7) = \{[\zeta_6] + [\nu_6] + [\nu_6] \varepsilon_9 + [\nu_6] \mu_7 + [\nu_6] \eta_9\varepsilon_9 + [\nu_6] \eta_9 \mu_7 \}
+ \{[\eta_9], \eta_9 \varepsilon_9 + [\eta_9], \eta_9 \mu_7 \} \cong Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2.
$$

Moreover, by Lemma 6.7 of [13], we obtain that the kernel of

$$(4.2) \quad A: \pi_7 \to \pi_{16}(R_6)$$

is generated by $4 \zeta_6 = \eta_9 \varepsilon_9$ and $\nu_6 \nu_{16}$.

$R_8$: By (2.4) and Table 1, the results for $\pi_{16}(R_6)$ are given.

$R_9$: By (3.7), (2.6) yields the following exact sequence

$$A \quad \pi_7 \to \pi_{16}(R_9) \to \pi_{16}(R_9) \to 0.$$

For the homomorphism $A: \pi_7 \to \pi_{16}(R_9)$, making use of the Table 1 and 3 and the formula (2.2), we have that

$$A(\nu_6) = [\nu_6] \zeta_6,$$
$$A(\nu_6) = [\nu_6] \eta_9 \varepsilon_9,$$

$$(4.3) \quad A(\nu_6 \eta_9 \mu_7) = [\zeta_6] \eta_9 \varepsilon_9,$$
$$A(\nu_6 \eta_9 \varepsilon_9) = [\zeta_6] \eta_9 \varepsilon_9,$$
$$A(\nu_6 \eta_9 \mu_7) = [\zeta_6] \eta_9 \varepsilon_9.$$ (3.6),

$A(\nu_6 \eta_9 \varepsilon_9) = [\zeta_6] \eta_9 \varepsilon_9.$

Thus, from the exactness of the above sequence, we have

$$\pi_{16}(R_9) = \{[\zeta_6] + [\nu_6] + [\nu_6] \varepsilon_9 + [\nu_6] \mu_7 + [\nu_6] \eta_9 \varepsilon_9 + [\nu_6] \eta_9 \mu_7 \}
+ \{[\eta_9], \eta_9 \varepsilon_9 + [\eta_9], \eta_9 \mu_7 \} \cong Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2 + Z_2.$$

From (4.3) and the exact sequence (2.6),

$$(4.4) \quad i_+: \pi_{16}(R_9) \to \pi_{16}(R_9) \text{ is an epimorphism.}$$

$R_{10}$: From (3.8) and (2.6), we obtain that the sequence
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\[ \pi^1_{12} \xrightarrow{\beta} \pi_{10}(R_9) \xrightarrow{i_6} \pi_{10}(R_{10}) \xrightarrow{p_6} \{2\sigma_0\} \longrightarrow 0 \]

is exact. By the use of (2.2), (3.8) and Tables 1 and 3, we have the following relations:

\[ \text{by Lemma 6.3 of [13]}, \]

\[ A(E_\tau) = d(\sigma_0) = [\iota_7]_9 + [\iota_7]_9 \tau_7 \tau_5, \]

(4.5) \[ A(V_\tau) = d(\sigma_0) = [\iota_7]_9 + [\iota_7]_9 \tau_7 \tau_5 \]

\[ = [\iota_7]_9 + [\iota_7]_9 \tau_7 \tau_5 \]

\[ \text{Thus, from the above exact sequence, it follows that the sequence} \]

(4.6) \[ 0 \longrightarrow \{[\iota_7]_9 \tau_7 \} + \{[\iota_7]_9 \tau_5 \} \xrightarrow{i_6} \pi_{10}(R_{10}) \xrightarrow{d} \{2\sigma_0\} \xrightarrow{j} 0 \]

is exact. Consider the diagram

\[ \begin{array}{ccc}
\pi^1_{10} &=& \{\sigma_{10}\} \\
\xrightarrow{d} \pi_{10}(R_{10}) \approx \downarrow E^{11} \xrightarrow{d} \pi_{10}(R_{10}) \approx \downarrow H \\
\pi^1_{12} &=& \{\sigma_{21}\} \\
\end{array} \]

\[ \text{of (2.7). Then we have} \]

\[ E^{13} \cdot p_6 \cdot d(\sigma_{10}) = 2E^{13} \cdot \sigma_0 \]

\[ \text{by (3.10).} \]

Since $E^{13} : \pi^0_{10} \longrightarrow \pi^0_{12}$ is an isomorphism, $p_6 \cdot d(\sigma_{10}) = 2\sigma_0$.

Therefore, by definition of $[2\sigma_0]$, we have

(4.7) \[ d(\sigma_{10}) = [2\sigma_0]. \]

By (12.19) of [13], the homomorphism $dE^{11} : \pi^0_{12} \longrightarrow \pi^0_{15}$ is a monomorphism.

On the other hand

\[ H \cdot dE^{11}(\sigma_{10}) = \pm 2\sigma_{19} \]

by Proposition 2.5 and 2.7 of [13],

\[ = \pm H \cdot d(\sigma_{10}) \]

Thus, from the exact sequence $\pi^0_{15} \xrightarrow{E} \pi^0_{18} \xrightarrow{H} \pi^0_{15}$ of [13] and (4.6), we have

(4.8) \[ J([2\sigma_0]) = \pm d(\sigma_{21}) \mod E^{0}_{25}. \]

Therefore it follows that $[2\sigma_0]$ is of order 16. We have a relation:

\[ J(2[2\sigma_0]) = Jd(2\sigma_{21}) \]

by (4.6),

\[ = \pm dE^{11}(2\sigma_{10}); \]
Thus we have $\mathcal{J}[g_{2\sigma_0}] - [\tau_7]_{10} \mu_7 = 0$. From the exact sequence (4.6),

$$8[2\sigma_0] - [\tau_7]_{10} \mu_7 = x[\zeta_6]_{10} + y[\tau_7]_{10} \mu_7^2,$$

for some integers $x, y$ ($x, y = 0$ or $1$). On the other hand,

$$\mathcal{J}([\tau_7]_{10} \mu_7^2) = \mathcal{J}([\tau_7]_{10}) E^{10} \mu_7^3 \quad \text{by (2.8)},$$

$$= \sigma_{10} \mu_7^2 = 0 \quad \text{by (7.1) of [13] and (2.9)}.$$

Therefore we have $x = 1$ and $\mathcal{J}([\zeta_5]_{10}) = \mu_{10} \sigma_{19}$. Thus we have obtained a relation

$$8[2\sigma_0] = [\tau_7]_{10} \mu_7^2 + [\zeta_6]_{10} + y[\tau_7]_{10} \mu_7^2,$$

where $y = 0$ or $1$. It follows from the exactness of the above sequence that

$$\pi_{10}(R_{10}) = \{[2\sigma_0]\} + \{[\tau_7]_{10} \mu_7^2\} + \{[\tau_7]_{10} \mu_7^3\} \approx \mathbb{Z}_1 + \mathbb{Z}_2 + \mathbb{Z}_2.$$

$R_{11}$: By (3.9), we have an exact sequence

$$\begin{array}{c}
\Delta \\
\pi_{11}^{10} \longrightarrow \pi_{10}(R_{10}) \longrightarrow \pi_{10}(R_{11}) \longrightarrow 0.
\end{array}$$

Then it follows from (4.7) that

(4.9) $\Delta : \pi_{11}^{10} \longrightarrow \pi_{10}(R_{10})$ is a monomorphism

and

$$\pi_{10}(R_{11}) = \{[\tau_7]_{11} \mu_7\} + \{[\tau_7]_{11} \mu_7^2\} \approx \mathbb{Z}_2 + \mathbb{Z}_2.$$

$R_{12}$: Consider the exact sequence

$$\begin{array}{c}
\Delta \\
\pi_{12}^{11} = \{\nu_{11}\} \longrightarrow \pi_{10}(R_{11}) \longrightarrow \pi_{10}(R_{12}) \longrightarrow \pi_{11}^{10} = 0
\end{array}$$

of (2.6). We have the relation;

$$\Delta(\nu_{11}) = \Delta(\nu_{11}) \nu_{12}^2 \quad \text{by (2.2)},$$

$$= [\tau_7]_{11} \mu_7^2 \quad \text{by Table 3}.$$

Then it follows from the exactness of the above sequence that

(4.10) $\Delta : \pi_{12}^{11} \longrightarrow \pi_{10}(R_{11})$ is a monomorphism

and
\[ \pi_{10}(R_{12}) = \{[\tau_7]_{10}/\tau_7\} \cong \mathbb{Z}_2. \]

**R_{13}:** \( \pi_7^6 = \pi_8^6 = 0 \) by Table 1. Then, from the exactness of (2.6), it follows that
\[ \pi_{10}(R_{13}) = \{[\tau_7]_{13}/\tau_7\} \cong \mathbb{Z}_2. \]

**R_{14}:** From (3.13) and (2.6), the following sequence
\[
0 = \pi_7^6 \rightarrow \pi_{10}(R_{14}) \xrightarrow{i_{10}} \pi_{10}(R_{14}) \xrightarrow{\partial_{10}} \{2\nu_{14}\} \rightarrow 0
\]
is exact. Consider the diagram
\[
\begin{array}{ccc}
\pi_7^6 = \{\nu_{14}\} & \xrightarrow{\partial} & \pi_{10}(R_{14}) \xrightarrow{i_{10}} \pi_{10}(R_{14}) = \{\nu_{13}\} \\
\cong E^{15} & \xrightarrow{d} & J \xrightarrow{H} \cong E^{14} \\
\pi_8^6 = \{\nu_{20}\} & \xrightarrow{\partial} & \pi_9^6 = \{\nu_{27}\}
\end{array}
\]
of (2.7). Then we have
\[ E^{15}p_4d\nu_{14} = 2E^{14}\nu_{14} \quad \text{by (3.10).} \]

Since \( E^{17} : \pi_7^6 \rightarrow \pi_8^6 \) is an isomorphism, we have \( p_4d\nu_{14} = 2\nu_{13} \). Then, by definition of \([2\nu_{13}]\), we have
\[ (4.11) \quad d\nu_{14} = [2\nu_{13}]. \]

On the other hand,
\[ \pi_8^6 = \{w_{14}\} + \{\sigma_{14}/\theta_2\} \cong \mathbb{Z}_8 + \mathbb{Z}_2 \quad \text{by Theorem 12.16 of [13]}, \]
\[ H(w_{14}) = \nu_{27} \quad \text{by Lemma 12.15 of [13]}, \]
and
\[ d\nu_{20} = \pm 2w_{14} \quad \text{(cf. page 159 or [13])}. \]

Thus, from the above diagram
\[ (4.12) \quad J([2\nu_{13}]) = Jd\nu_{14} \]
\[ = \pm dE^{15}\nu_{14} \]
\[ = \pm d\nu_{20} \]
\[ = \pm 2w_{14}. \]

If \([2\nu_{13}]\) is of order 8, then, from the above exact sequence, we have
\[ i_{10}([\tau_7]_{14}/\tau_7) = [\tau_7]_{14}/\tau_7 = 4[2\nu_{14}], \]
and
\[ 0 \neq \sigma_{14}/\theta_2 = J([\tau_7]_{14}/\tau_7) \quad \text{by (2.9)} \]
\[ = Jd(2\nu_{14}) \]
\[ = \pm 8w_{14} \quad \text{by (4.12)} \]
\[ = 0. \]
This is a contradiction, and hence \([2\nu_{13}]\) must be of order 4. From the exactness of the above sequence, we have

\[ \pi_{16}(R_{14}) = \{[2\nu_{13}]\} + \{[\tau_7]_{14}/\tau_7\} \approx \mathbb{Z}_4 + \mathbb{Z}_2. \]

Moreover,

\[ \text{(4.13)} \quad \text{The kernel of } d : \pi_{17}^{14} \longrightarrow \pi_{16}(R_{14}) \text{ is } \{4\nu_{14}\} = \{\eta_{18}^4\}. \]

**R_{15}**: From (2.6), (4.10) and \(\pi_{15}(R_{14}) \approx \mathbb{Z}\), it follows that the sequence

\[ 0 \longrightarrow \{[\tau_7]_{15}/\tau_7\} \longrightarrow \pi_{16}(R_{15}) \longrightarrow \pi_{16}^{14} = \{\eta_{14}^4\} \longrightarrow 0 \]

is exact. Consider the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \{[\tau_7]_{15}/\tau_7\} \\
& \downarrow J & \downarrow J \\
0 & \longrightarrow & \{[\sigma_{15}]/\sigma_{22}\} + \{\omega_{15}\} \\
& \downarrow J & \downarrow H \\
& & \{\eta_{29}\} \\
& & 0
\end{array}
\]

of (2.7), where the lower sequence is exact by Lemma 12.14 and (12.20) of [13]. We have

\[ J([\omega_{15}]) = \sigma_{15}/\sigma_{22} \quad \text{by (2.8) and (2.9).} \]

Thus, from the above diagram, we have

\[ \text{(4.14)} \quad J : \pi_{16}(R_{15}) \longrightarrow \pi_{16}^{14} \text{ is a monomorphism.} \]

On the other hand,

\[ \pi_{16}^{14} = \{\omega_{15}\} + \{\sigma_{15}/\sigma_{22}\} + \{\mu^a\} \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \quad \text{by Theorem 12.16 of [13]} \]

and

\[ H(\gamma^a) = \eta_{29} \quad \text{by Lemma 12.14 of [13].} \]

Therefore, there exists an element \([\eta_{14}^7]\) \(\in \pi_{16}(R_{15})\) of order 2 such that \(p_{16}([\eta_{14}^7]) = \eta_{14}^7\). Thus, from the exactness of the above sequence, we have

\[ \pi_{16}(R_{15}) = \{[\tau_7]_{15}/\tau_7\} + \{[\eta_{14}^7]\} \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \]

and

\[ \text{(4.15)} \quad J([\eta_{14}^7]) = \eta^a. \]

**R_{16}**: Consider the diagram
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\[ \pi_{15}^i = \{ \gamma_{15}^i \} \xrightarrow{\Delta} \pi_{16}(R_{15}) \xrightarrow{i_*} \pi_{16}(R_{16}) \xrightarrow{p_*} \pi_{16}^i = \{ \gamma_{15}^i \} \]

\[ \approx \xrightarrow{E_{16}} \xrightarrow{\Delta} \xrightarrow{J} \xrightarrow{E} \xrightarrow{J} \xrightarrow{H} \approx \xrightarrow{E_{16}} \]

\[ \pi_{16}^o = \{ \gamma_{15}^o \} \xrightarrow{\Delta} \pi_{17}(R_{15}) \xrightarrow{i_*} \pi_{17}(R_{16}) \xrightarrow{p_*} \pi_{17}^o = \{ \gamma_{15}^o \} \]

for (2.7). From (4.14) and by five lemma, we have

\[ \text{(4.16)} \quad J : \pi_{16}(R_{16}) \longrightarrow \pi_{16}^o \text { is a monomorphism.} \]

From Lemma 12.14 of [13], $\Delta : \pi_{16}^o \longrightarrow \pi_{17}^o$ is trivial. Thus, from (4.14) and the above diagram, it follows that

\[ \text{(4.17)} \quad J : \pi_{17}^o \longrightarrow \pi_{16}(R_{16}) \text { is trivial.} \]

On the other hand, from Lemma 12.14 of [13], there exists an element $\gamma_{15}^o \in \pi_{16}^o$ such that $H(\gamma_{15}^o) = \gamma_{15}$ and $2\gamma_{15} = 0$. Moreover, from the above diagram, there exists an element $[\gamma_{15}] \in \pi_{16}(R_{16})$ such that

\[ p_*[\gamma_{15}] = \gamma_{15}, \]

\[ 2[\gamma_{15}] = 0, \]

and

\[ \text{(4.17)} \quad J([\gamma_{15}]) = \gamma_{15}^o. \]

Thus, from the exact sequence

\[ 0 \longrightarrow \pi_{16}(R_{15}) \xrightarrow{i_*} \pi_{16}(R_{16}) \xrightarrow{p_*} \pi_{16}^o \longrightarrow 0, \]

we have

\[ \pi_{16}(R_{16}) = \{ [\gamma_{15}] \} + \{ [\gamma_{15}]_{16} \} + \{ [\gamma_{15}]_{16} \} \]

\[ \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2. \]

From the above diagram, we have also

\[ \text{(4.18)} \quad J([\gamma_{15}]) = E[\gamma_{15}] = E[\gamma_{15}]. \]

\[ \text{by (4.15)} \]

\[ R_{17} : \text{Consider the diagram} \]

\[ \pi_{16}^o = \{ \gamma_{15}^o \} \xrightarrow{\Delta} \pi_{16}(R_{16}) \xrightarrow{i_*} \pi_{16}(R_{17}) \xrightarrow{0} \]

\[ \approx \xrightarrow{E_{17}} \xrightarrow{\Delta} \xrightarrow{J} \xrightarrow{E} \xrightarrow{J} \xrightarrow{H} \approx \xrightarrow{E_{17}} \]

\[ \pi_{16}^o = \{ \gamma_{15}^o \} \xrightarrow{i_*} \pi_{17}^o \xrightarrow{p_*} \pi_{18}^o = \{ \gamma_{18} \} \]

of (2.7), where the upper sequence is exact by (3.14).
We have a relation
\[ d(\gamma_{30}) \equiv E_{\eta^{q_3}} \mod E_{\pi^{14}_{30}} \] (cf. page 160 of [13]).

Thus, from (4.18) and (4.16), we have
\[ d(\gamma_{16}) = [\gamma_{14}, \gamma_{16}], \]
and
\[ \pi_{16}(R_{17}) = \{[\gamma_{15}]_{17}\} + \{[\tau_{7}]_{17}\}_{17} \approx \mathbb{Z}_2 + \mathbb{Z}_2. \]

Also, we obtain
\[ \text{(4.20)} \quad J : \pi_{16}(R_{17}) \longrightarrow \pi^{14}_{17} \text{ is a monomorphism.} \]

\[ R_n \text{ for } n \geq 18 : \text{Consider the diagram} \]
\[
\begin{array}{cccccc}
\pi^{14}_{17} & \longrightarrow & \pi_{16}(R_{17}) & \longrightarrow & \pi_{16}(R_{18}) & \longrightarrow & \pi^{16}_{17} = 0 \\
& \approx & & \downarrow & & \\
\pi^{14}_{18} & \longrightarrow & \pi^{14}_{17} & \longrightarrow & \pi^{14}_{16} & \longrightarrow & \pi^{14}_{13} = 0
\end{array}
\]
of (2.7). Then
\[ JD(\tau_{17}) = d_{35} = \eta^{8}_{7} \]
\[ = J([\gamma_{15}]_{17}) \quad \text{(cf. page 160 of [13])} \]
by (4.20).

Thus, from (4.20) and the stability of \( \pi_{1}(R_n) \), it follows that
\[ \pi_{16}(R_n) = \{[\tau_{7}]_{17}\}_{n} \approx \mathbb{Z}_2 \quad \text{for } n \geq 18. \]

Reference

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