Exceptional Lie Group $F_4$ and its Representation Rings

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Introduction

The aim of this paper is to determine the real and complex representation rings $RO(F_4)$ and $R(F_4)$ of $F_4$, which is a simply connected compact Lie group of exceptional type $F$. Let $\mathcal{J}$ denote the Jordan algebra of all 3-hermitian matrices over the division ring of Cayley numbers. We know that the group $F_4$ is obtained as the automorphism group of $\mathcal{J}$. In Chapter I, we shall arrange some properties of $F_4$: the subgroups $Spin(8)$, $Spin(9)$, maximal torus $T$, the Weyl group $W$ and the Lie algebra $\mathfrak{F}_4$. The origin of the results of Chapter I are found in H. Freudenthal [1], however we rewrite them with some modifications. In Chapter II, we shall determine the ring structures of $RO(F_4)$ and $R(F_4)$. Let $\mathfrak{S}_0$ be the set of all elements of $\mathcal{J}$ with zero trace and let $\mathfrak{F}_4$ be the Lie algebra of $F_4$. Then $\mathfrak{S}_0$ and $\mathfrak{F}_4$ are $F_4$-R-modules in the natural way. The results are follows: $RO(F_4)$ is
a polynomial ring \( \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \kappa] \) with 4 variables \( \lambda_1, \lambda_2, \lambda_3, \kappa \), where \( \lambda_i \) is the class of the exterior \( F_i \)-module \( \Lambda_i(\mathbb{F}_4) \) in \( RO(F_4) \) for \( i = 1, 2, 3 \), and \( \kappa \) is the class of \( \mathbb{F}_4 \) in \( RO(F_4) \). \( R(F_4) \) is also a polynomial ring \( \mathbb{Z}[\lambda_1^C, \lambda_2^C, \lambda_3^C, \kappa^C] \), where \( \lambda_1^C, \lambda_2^C, \lambda_3^C, \kappa^C \) are the complexification of \( \lambda_1, \lambda_2, \lambda_3, \kappa \) respectively. In the final section, we consider the relationship between \( R(F_4) \) and \( R(\text{Spin}(9)), R(\text{Spin}(8)) \).

**Chapter I**

1. **Jordan algebra \( \mathfrak{J} \)**

Let \( \mathfrak{C} \) be the division ring of Cayley numbers. \( \mathfrak{C} \) is an 8-dimensional \( \mathbb{R} \)-module with a base \( e_0, \ldots, e_7 \) and the multiplications among them are given as follows:

- \( e_0 \) is the unit of \( \mathfrak{C} \) (which is often denoted by 1)
- \( e_i^2 = -e_0 \) for \( i \neq 0 \)
- \( e_i e_j = -e_j e_i \) for \( i, j \neq 0, \ i \neq j \)

and

\[
\begin{align*}
  e_1 e_2 &= e_3, \\
  e_2 e_3 &= -e_7, \\
  e_2 e_4 &= e_6.
\end{align*}
\]

The conjugation \( \bar{u} \) of \( u \in \mathfrak{C} \) is defined by \( \bar{u} = e_0 u_0 + \sum_{i=1}^{7} e_i u_i \) and the real part \( \text{Re}(u) \) of \( u \) by \( \frac{1}{2}(u + \bar{u}) \). We define the inner product \( \langle u, v \rangle \) of \( u = \sum_{i=0}^{7} e_i u_i, \ v = \sum_{i=0}^{7} e_i v_i \) by \( \sum_{i=0}^{7} u_i v_i \) and the length of \( u \) by \( |u| = \sqrt{\langle u, u \rangle} \).

We describe here some formulae in \( \mathfrak{C} \) used in later.

1.1 For \( u, v, a, b \in \mathfrak{C} \), we have

\[
\begin{align*}
  (1) & \quad \bar{u} = u, \quad \bar{uv} = \bar{v}u, \\
  (2) & \quad \text{Re}(uv) = \text{Re}(vu), \quad \text{Re}(u(vw)) = \text{Re}(uvw), \\
  (3) & \quad 2(u, v) = uv + vu = \bar{u}v + vu, \quad |u|^2 = u\bar{u} = \bar{u}u, \\
  (4) & \quad \bar{u}(bn) + \bar{b}(au) = 2(a, b)u, \\
  (5) & \quad a(\bar{a}u) = (a\bar{a})u, \quad a(\bar{a}u) = (u\bar{a})a, \quad u(\bar{a}) = (u\bar{a})a.
\end{align*}
\]

1) \( \mathbb{R} \) is the field of real numbers.
(6) \((au)v + u(va) = a(uv) + (uv)a\).

Let \(\mathfrak{S}\) denote the space of 3-hermitian matrix \(X\) over \(\mathbb{C}\)
\[
X = \begin{pmatrix}
\xi_1 & u_2 & \bar{u}_1 \\
\bar{u}_2 & \xi_2 & u_3 \\
u_3 & \bar{u}_1 & \xi_3
\end{pmatrix}, \quad \xi_i \in \mathbb{R}, \; u_i \in \mathbb{C}.
\]
Such \(X\) is often denoted by \(X(\xi, u)\). We define the Jordan product in \(\mathfrak{S}\) by

\[X \circ Y = \frac{1}{2}(XY + YX)\]

where the product \(XY\) is the usual matrix product. Then \(\mathfrak{S}\) is a non-associative commutative 27-dimensional \(\mathbb{R}\)-algebra.

We shall adopt the following notations;

\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = E_1 + E_2 + E_3,
\]

\[
F_1^u = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & \bar{u} \\ u & \bar{u} & 0 \end{pmatrix}, \quad F_2^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{u} \\ \bar{u} & 0 & 0 \end{pmatrix}, \quad F_3^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}.
\]

Then \(E_i, F_i^u\) for \(i = 1, 2, 3, \; u \in \mathbb{C}\) generate \(\mathfrak{S}\) additively and we have

\[
\begin{align*}
E_i \circ E_i &= E_i, \quad E_i \circ E_j = 0 \quad \text{for} \; i \neq j, \\
E_i \circ F_i^u &= 0, \quad 2E_i \circ F_j^u = F_j^u \quad \text{for} \; i \neq j, \\
F_i^u \circ F_i^v &= (E - E_0)(u, v), \quad 2F_i^u \circ F_i^v = F_i^{uv},
\end{align*}
\]

for \(i, j = 1, 2, 3\) and suffixes are modulo 3.

In \(\mathfrak{S}\), we define the trace, the inner product and triple inner product by

\[
\begin{align*}
\text{tr}(X) &= \xi_1 + \xi_2 + \xi_3, \\
(X, Y) &= \text{tr}(X \circ Y), \\
\text{tr}(X, Y, Z) &= (X \circ Y, Z)
\end{align*}
\]

respectively for \(X = X(\xi, u), \; Y, Z \in \mathfrak{S}\).

1.2 Lemma. \((XY, A) = (YA, X)\) for \(X, Y, A \in \mathfrak{S}\).

Proof. Let \(X = (x_{ij}), \; Y = (y_{ij}), \; A = (a_{ij})\), Then \((XY, A) = \text{Re}(XY, A) = \frac{1}{2} \text{Re}(tr\)

2) The notation \(\text{tr}(X, Y, Z)\) is differ from that of [1] where \((X, Y, Z)\) is used for this. We avoid here the notation \((X, Y, Z)\) because this is used in another sense in almost every H. Freudenthal's papers (for example, Zur ebenen Oktavengeometrie, Indag. Math. 15, 1953).
\[(XY)A + A(XY)] = \frac{1}{2} \text{Re} (\sum \sum \left[ (x_i a y_k) a_{i} + a_{ik}(x_k y_i) \right]) = \frac{1}{2} \text{Re} (\sum \sum \left[ (y_k a_i) x_{ik} + x_{ik}(y_i a_k) \right])
\]
\[= \frac{1}{2} \text{Re} (\text{tr}((YA)X + X(YA))) = (YA, X).
\]

1.3 Lemma. For \( X, X', Y, Z \in \mathfrak{S} \), we have

1. \( (X, Y) = (Y, X) \),
2. \( (X + X', Y) = (X, Y) + (X', Y), \quad (X, Y') = \xi(X, Y) \text{ for } \xi \in \mathbb{R} \),
3. \( (X, E) = \text{tr}(X) \),
4. \( (\cdot, \cdot, \cdot) \) is regular, i.e. if \( (X, Y) = 0 \text{ for all } Y \in \mathfrak{S} \), then we have \( X = 0 \).

1.4 Lemma. For \( X, X', Y, Z \in \mathfrak{S} \), we have

1. \( \text{tr}(X, Y, Z) = \text{tr}(Y, Z, X) = \text{tr}(Z, X, Y) = \text{tr}(Z, Y, X) = \text{tr}(Y, Z, X) \),
2. \( \text{tr}(X - X', Y, Z) = \text{tr}(X, Y, Z) + \text{tr}(X', Y, Z) \),
3. \( \text{tr}(X \xi, Y, Z) = \xi \text{tr}(X, Y, Z) \text{ for } \xi \in \mathbb{R} \),
4. \( \text{tr}(X, Y, E) = (X, Y) \).

Proof. (1) \( \text{tr}(X, Y, Z) = (X \circ Y, Z) = \frac{1}{2} ((XY + YX), Z) = \frac{1}{2} ((XY, Z) + (YX, Z)) = \frac{1}{2} ((YZ, X) + (ZY, X)) = (Y \circ Z, X) = \text{tr}(Y, Z, X) \). (2), (3) are easily seen.

2. Definition of group \( F_4 \)

2.1 Definition. Let \( F_4 \) denote the group of all automorphisms of \( \mathfrak{S} \), that is, each \( x \in F_4 \) satisfies

1. \( x(X + Y) = xX + xY, \quad x(X) = (xX) \xi \)
2. \( x \) is non-singular
3. \( x(X \circ Y) = xX \circ xY \)

for \( X, Y \in \mathfrak{S} \), \( \xi \in \mathbb{R} \).

Let \( F'_4 \) denote the group of \( \mathbb{R} \)-homomorphisms \( x : \mathfrak{S} \rightarrow \mathfrak{S} \) under which \( (X, Y) \) and \( \text{tr}(X, Y, Z) \) are invariant, that is, each \( x \in F'_4 \) satisfies besides 2.1 (1),

4. \( x(X, Y) = (xX, Y) \quad \text{for } X, Y \in \mathfrak{S} \).
5. \( \text{tr}(xX, xY, xZ) = \text{tr}(X, Y, Z) \quad \text{for } X, Y, Z \in \mathfrak{S} \).

2.2 Lemma. \( F'_4 \) is a subgroup of \( F_4 : F'_4 \subset F_4 \).

Proof. If \( x \in F'_4 \), \( X, Y \in \mathfrak{S} \), then \( (x(X \circ Y), xZ) = (X \circ Y, Z) = \text{tr}(X, Y, Z) = \text{tr}(xX, xY, xZ) = (xX \circ xY, xZ) \) for all \( Z \in \mathfrak{S} \). This implies \( x(X \circ Y) = xX \circ xY \), that is \( x \in F_4 \).

2.3 Lemma. (1) \( xE = E \text{ for } x \in F_4 \)
(2) \( \text{tr}(xX) = \text{tr}(X) \text{ for } x \in F'_4 \), \( X \in \mathfrak{S} \).

Proof. (1) We have \( E \circ X = X \) for any \( X \in \mathfrak{S} \). Operating \( x \) on \( E \circ X = X \), then \( xE \circ xX = xX \). Here put \( X = x^{-1}E \), then \( xE \circ E = E \). This implies \( xE = E \).
(2) \( \text{tr}(xX) = (xX, E) = (xX, xE) = (X, E) = \text{tr}(X) \).

2.4 Lemma. \( F'_4 \) is the subgroup of \( F_4 \) consisting of all \( x \in F_4 \) under which the trace of every \( X \in \mathfrak{S} \) is invariant.
Proof. If the trace of every $X \in \mathfrak{F}$ is invariant under $x \in F_4$, then $(xX, xY) = \text{tr}(xX xY) = \text{tr}(xX Y) = (X, Y)$ and $(xX, xY, xZ) = (xX xY, xZ) = (xX Y, Z) = \text{tr}(X, Y, Z)$. Hence $x \in F_4$. The converse follows from Lemmas 2.2, 2.3 (2).

We shall see that $F_4 = F_4'$ in Theorem 4.2, in particular, that the trace of every $X \in \mathfrak{F}$ is invariant under $x \in F_4$.

3. Deformation to diagonal form

3.1 Lemma. $F_4'$ is a compact group.

Proof. Since the inner product $(X, Y) = \sum_{i=1}^{3}(\xi_i \eta_i + 2(u_i, v_i))$ is invariant under each element of $F_4'$, where $X = X(\xi, u)$, $Y = Y(\eta, v) \in \mathfrak{F}$, $F_4'$ is a closed subgroup of the orthogonal group $O(27)$, which is compact. Therefore $F_4'$ is compact.

3.2 Lemma. For $a \in \mathfrak{G}$ $(a \neq 0)$, define an $R$-homomorphism $x : \mathfrak{G} \rightarrow \mathfrak{G}$ by $xX = Y(\eta, v)$, where

$$
\begin{align*}
\gamma_1 &= \xi_1, \\
\gamma_2 &= \frac{a}{|a|} \sin 2|a| + \frac{\xi_2 - \xi_3}{2} \cos 2|a| + \frac{\xi_3 + \xi_3}{2}, \\
\gamma_3 &= -\frac{a}{|a|} \sin 2|a| - \frac{\xi_2 - \xi_3}{2} \cos 2|a| + \frac{\xi_3 + \xi_3}{2}, \\
v_1 &= u_1 - \frac{\xi_2 - \xi_3}{2} \sin 2|a| - \frac{2(a, \eta) a}{|a|^2} \sin^2 |a|, \\
v_2 &= u_2 \cos |a| - \frac{\hat{a} u_3}{|a|} \sin |a|, \\
v_3 &= u_3 \sin |a| + \frac{\hat{a} u_3}{|a|} \sin |a|,
\end{align*}
$$

then we have $x \in F_4'$.

Proof. We shall show first that $x(X \circ X) = xX \circ xX$ by the direct computation.

$$
x \circ X = \begin{pmatrix} 6 \xi_1^2 + |u_2|^2 + |u_3|^2 & (\xi_1 + \xi_2) u_4 + \bar{u}_4 u_2 \\ (\xi_1 + \xi_2) u_4 + \bar{u}_4 u_2 & 6 \xi_2^2 + |u_3|^2 + |u_1|^2 & (\xi_2 + \xi_3) u_1 + \bar{u}_1 u_3 \\ (\xi_2 + \xi_3) u_1 + \bar{u}_1 u_3 & 6 \xi_3^2 + |u_1|^2 + |u_2|^2 \end{pmatrix}
$$

The $(1, 1)$-component of $xX \circ xX = \gamma_1^2 + |u_2|^2 + |u_3|^2 = \xi_1^2 + |u_2| \cos |a| - \frac{\hat{a} u_3}{|a|} \sin |a| |^2 + |u_4| \cos |a| + \frac{\hat{a} u_3}{|a|} \sin |a| |^2 = \xi_1^2 + |u_2|^2 + |u_3|^2 = \text{the (1, 1)-component of} x(X \circ X)$. The $(2, 2)$-component of $xX \circ xX = \gamma_2^2 + |u_3|^2 + |v_1|^2 = \xi_2^2 + |u_2| \cos |a| = \cdots = \frac{1}{|a|} |(a, (\xi_2 + \xi_3) u_1 + \bar{u}_1 u_3) \sin 2|a| |

+ \frac{1}{2} |\xi_2^2 - \xi_3^2 + |u_1|^2 - |u_4|^2| \cos 2|a| + \frac{1}{2} (\xi_2^2 + \xi_3^2) + 2 |u_4|^2 + |u_2|^2 + |u_3|^2 = \cdots = \text{the (2, 2)-component of} x(X \circ X)$. About the $(3, 3)$-component, the computation is similar.
to the (2, 2)-component. The (2, 3)-component of \( x(X_0X) = (\xi_2 + \xi_3)u_1 + u_2u_3 - \frac{1}{2} (\xi_2^2 - \xi_3^2 + |u_4|^2 - |u_5|^2)|a| \sin^2 |a| - |a, (\xi_2 + \xi_3)u_1 + u_2u_3| |a| \frac{2a}{|a|} \sin^2 |a| = \cdots \) (we shall use the formula \( 2a(\xi_2 + \xi_3)u_1 + u_2u_3 = (\xi_2 + \xi_3)u_1 + u_2u_3 \) (cf. 1.1 (3))) \( \cdots = (\gamma_2 + \gamma_3)v_1 + v_2v_3 \) is the (2, 3)-component of \( xX_0X \). The (3, 1)-component of \( x(X_0X) = \cdots \) (we shall use \( \tilde{a}(u_4u_5) + \tilde{a}(u_4u_5) = 2(a, u_4u_5) \) (cf. 1.1 (4))) \( \cdots = \) the (3, 1)-component of \( xX_0X \). The (1, 2)-component is similar to the (3, 1)-component. Thus we have \( x(X_0X) = xX_0X \) for any \( X \in \mathcal{I} \). By the polarization \( X \to X + Y \), we have

\[
x(X_0Y) = xX_0Y \quad \text{for } X, Y \in \mathcal{I}.
\]

Hence \( x \in F_4 \). Finally, it is easily seen that \( \text{tr}(xX) = \gamma_1 + \gamma_2 + \gamma_3 = \xi_1 + \xi_2 + \xi_3 = \text{tr}(X) \). Therefore, by Lemma 2.4, we have \( x \in F_4 \).

3.3 Theorem. [1]. For \( X_0 \in \mathcal{I} \), there exists \( x \in F_4 \) such that \( xX_0 \) is of a diagonal form.

Proof. For a fixed \( X_0 \in \mathcal{I} \), \( \mathcal{X}_0 = \{ xX_0 \mid x \in F_4 \} \) is a compact subset in \( \mathcal{I} \). Let \( X_1 \) be an element in \( \mathcal{X}_0 \) which attains the maximum value of \( \xi_1^2 + \xi_2^2 + \xi_3^2 \) for \( X = X(\xi, u) \in \mathcal{X}_0 \), then we shall show that \( X_1 \) is diagonal. Assume that \( X_1 = X_1(\xi, u) \) is not diagonal, for example, \( u_1 \neq 0 \). Put \( a(t) = \frac{u_1}{|u_1|} t \) for \( t \in \mathbb{R} (t \neq 0) \) and construct an element \( x(t) \in F_4 \) as in Lemma 3.2. Then the value of \( \gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t) \) in \( x(t)X_1 \) is \( \xi_1^2 + \frac{2(\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t))}{|a(t)|} \sin^2 |a(t)| + 2 \left( \frac{\xi_2 - \xi_3}{2} \right)^2 \cos^2 |a(t)| + 2 \left( \frac{\xi_2 + \xi_3}{2} \right)^2 + A(\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t)) \sin^2 |a(t)| \cos^2 |a(t)| \left( \xi_1^2 + \frac{(\xi_2 + \xi_3)^2}{2} \right) + 2 \left( \frac{\xi_2 - \xi_3}{2} \right)^2 \cos^2 |a(t)| \left( \xi_1^2 + \frac{(\xi_2 - \xi_3)^2}{2} \right) + 2 |u_1|^2 \) (its maximum value).

This contradicts to the fact that \( \xi_1^2 + \xi_2^2 + \xi_3^2 \) in \( X_1 \) attains the maximum value.

4. Cayley projective plane \( \mathcal{P}_2 \)

4.1 Proposition. [1]. For \( X \in \mathcal{I} \), the following five statements are equivalent.

1. \( X \neq 0 \) and \( X \) is an irreducible idempotent, i.e. \( X^2 = X \) and \( X = X_1 + X_2 \), \( X_0X_1 = X_0 \), \( X_0X_1 = X_0 \) (\( i = 1, 2 \)) imply \( X_1 = 0 \) or \( X_2 = 0 \).
2. \( X^2 = X \) and \( \text{tr}(X) = 1 \).
3. \( \text{tr}(X) = \text{tr}(X, X) = 1 \).
4. \( X = xE_i \) for some \( x \in F_4 \) and for some \( i = 1, 2, 3 \).
5. \( X = xE_1 \) for some \( x \in F_4 \).

Proof. (1)\( \rightarrow \) (2). For \( X \), there exists \( x \in F_4 \subset F_4 \) such that \( xX = \sum_{i=1}^{3} E_i \xi_i \). The
idempotency of $X$ induces that of $xX$, so that we have $\xi_i^2 = \xi_i$, hence $\xi_i = 0$ or 1 for $i = 1, 2, 3$. We see that the only one $\xi_i$ of them is 1 and the others are 0. In fact, if not, $xX$ is reducible. Since the reducibility is invariant under $x \in F_4$, $X$ is reducible. This contradicts to the hypothesis of $X$. Now, by Lemma 2.3 (2), $\text{tr}(X) = \text{tr}(xX) = \text{tr}(E_i) = 1$. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (4). For $X$, there exists $x \in F_4$ such that $xX$ is of a diagonal from $\sum_{i=1}^{3} E_i \xi_i$. The condition (3) means $\xi_1 + \xi_2 + \xi_3 = \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$. Hence we have the only one $\xi_i = 1$ and the others are 0, i.e. $xX = E_i$ for some $i$. Therefore $X = x^{-1} E_i$ where $x^{-1} \in F_4$. (4) $\Rightarrow$ (5). It suffices to show that $E_2$ (and $E_3$) can be deformed to $E_1$ by some $a \in F_4$. For the matrix $A = E_3 + E_2$, we have $AE_2A = E_1$. Since $A$ is a real matrix, $aX = aX = aX$ is an element of $F_4$ and we have $aE_2 = E_1$. (5) $\Rightarrow$ (1). We shall show first that $E_1$ is an irreducible idempotent. Assume $E_1 = X_1 + X_2$, $X_1X_2 = X_i$, $X_i \in S$ ($i = 1, 2$). Then $X_1 + X_2 = E_1 = E_1X_1 = X_1 + X_2 + 2X_1X_2$, hence $X_1X_2 = 0$. Multiply $X_1$ on $E_1 = X_1 + X_2$, then we have $E_1X_2 = X_1$. This shows that $X_1$ is of the form $E_1 \xi_1 + E_2 \xi_2 + E_3 \xi_3 + F_1^{u_1}$. From $E_1 = X_1 + X_2$, we have $X_2 = E_1 \xi_1 - E_2 \xi_2 - E_3 \xi_3 - F_1^{u_1}$ where $\xi_1 = 1$. Since $X_1X_2 = E_1 \xi_1 - E_2 (\xi_2^2 + |u_1|^2) - E_3 (\xi_3^2 + |u_1|^2)$, $X_1X_2 = 0$ implies that $\xi_2 = \xi_3 = u_1 = 0$ and $\xi_1 = 0$. Thus we have $X_1 = 0$ or $X_2 = 0$. Now, since the irreducibility and idempotency are invariant under $x \in F_4$, we see that $xE_1$ is an irreducible idempotent in $S$.

Let $\mathbb{G}P_2$ denote the space of $X \in S$ satisfying one of the five conditions of Proposition 4.1. Then we remember that $\mathbb{G}P_2$ is the projective plane over $\mathbb{G}$ [1], [5].

4.2 Theorem. $F_4 = F_4$, that is, the trace of every $X \in S$ is invariant under each $x \in F_4$.

Proof. Note that the trace of an element of the form $zE_i$ ($z \in F_4$, $i = 1, 2, 3$) is 1 by (4) $\Rightarrow$ (2) of Proposition 4.1. Now, let $x \in F_4$ and $X \in S$. For this $X$, choose $y \in F_4$ such that $yX$ is of a diagonal form $\sum_{i=1}^{3} E_i \xi_i = X_i$. Then we have $xX =$ $xy^{-1}X_i = zX_i$ (where $z = xy^{-1} \in F_4$) $= \sum_{i=1}^{3} [zE_i] \xi_i$, whence $\text{tr}(xX) = \sum_{i=1}^{3} \text{tr}(zE_i) \xi_i = \sum_{i=1}^{3} \xi_i = \text{tr}(xX) = \text{tr}(yX) = \text{tr}(X)$.

5. Principle of triality in $SO(8)$ and $Spin(8)$

For the results of this section, we refer to [1], [3], however we rewrite them with proofs.

Let $SO(8)$ denote the rotation group in $\mathbb{G}$. Let $\mathfrak{h}_4$ be the Lie algebra of $SO(8)$, that is, the $\mathbb{R}$-module consisting of $\mathbb{R}$-homomorphisms $D : \mathbb{G} \rightarrow \mathbb{G}$ such that
(Du, v) + (u, Dv) = 0 \quad \text{for } u, v \in \mathfrak{g}.

5.1 Proposition. [1] (Principle of infinitesimal triality in $\mathfrak{d}_4$)
For every $D_1 \in \mathfrak{d}_4$, there exist $D_2, D_3 \in \mathfrak{d}_4$ such that
\[(D_1u)v + u(D_2v) = D_3(uv) \quad \text{for } u, v \in \mathfrak{g},\]
and for $D_1$, such $D_2, D_3$ are unique.

5.2 Proposition [1]. (Principle of triality in $SO(8)$)
For every $d_1 \in SO(8)$, there exist $d_2, d_3 \in SO(8)$ such that
\[(d_1u)(d_2v) = d_3(uv) \quad \text{for } u, v \in \mathfrak{g},\]
and for $d_1$, such $d_2, d_3$ are unique up to the sign.

Proof. As is well known, for $d_1 \in SO(8)$, there exists $D_u \in \mathfrak{d}_4$ such that $d_1 = \exp D_u$. By Proposition 5.1, there are $D_2, D_3 \in \mathfrak{d}_4$ such that $(D_1u)v + u(D_2v) = D_3(uv)$ for $u, v \in \mathfrak{g}$. Put $d_2 = \exp D_2$ and $d_3 = \exp D_3$, then $d_3(uv) = \exp D_3(uv) = D_3(uv)$ for $u, v \in \mathfrak{g}$. Put $u = 1$, then $d_2v = d_3v$ for all $v \in \mathfrak{g}$. Therefore $u(d_2v) = d_3(uv)$. Put $v = 1$ and denote $d_21 = c$, then $uc = d_31$. This implies $u(c) = (uv)c$ for $u \in \mathfrak{g}$. From this associativity we have $c \in \mathbb{R}$, whence $c = \pm 1$. Therefore $d_2u = \pm u$ for all $u \in \mathfrak{g}$. Thus, for $d_1 = e$, only two cases $d_3 = d_3 = e$ and $d_3 = d_3 = -e$ occur.

5.3 Lemma. [3]. Let $O(8)$ be the orthogonal group in $\mathfrak{g}$. Assume that for $d_1, d_2, d_3 \in O(8)$
\[(d_1u)(d_2v) = d_3(uv) \quad \text{for all } u, v \in \mathfrak{g},\]
then we have
\[
(d_3u)(d_3v) = d_3(uv)
\]
\[
(d_3v)(d_3v) = d_3(uv)
\]

Proof. Multiply $d_3u$ on the left side and $d_3(uv)$ on the right side of the given formula, then we have $|u|^2(d_3v)(d_3(uv)) = d_3|u|uv|^2$, hence $(d_3v)(d_3(uv)) = d_3|u|uv|^2$. Replace $u$ by $uv$, then $(d_3v)(d_3|u|uv) = d_3|uv|uv|^2$, hence we have $(d_3v)(d_3(uv)) = d_3|uv|uv|^2$.

5.4 Lemma. [3]. Assume that for $d_1, d_2, d_3 \in O(8)$,
\[(d_1u)(d_2v) = d_3(uv) \quad \text{for all } u, v \in \mathfrak{g},\]
then we have $d_1, d_2, d_3 \in SO(8)$.

Proof. If $d_1 \in SO(8)$, then there exists $a_1 \in SO(8)$ such that $a_1d_1u = u$ for $u \in \mathfrak{g}$. Using the triality, for this $a_1$, there exist $a_2, a_3 \in SO(8)$ such that $(a_1d_1u)(a_2d_2v) =$
\[ a_3[(d_1u)(d_2v)] = a_2d_3(uv). \] Denote \( a_2d_2 = b_2, \) \( a_3d_3 = b_3, \) then \( \bar{u}(b_2v) = b_3(uv). \) Put \( u = 1, \) then \( b_2v = b_2\bar{v}. \) Thus we have \( \bar{u}(b_2v) = b_2(uv). \) Put \( v = 1 \) and \( b_2\bar{v} = c. \) then \( \bar{u}c = b_2u, \) hence \( \bar{u}(bc) = (uv)c. \) Put \( v = c, \) \( \bar{u}c = c\bar{u}, \) hence \( \bar{u}c = \bar{u}c \) for all \( u \in \mathbb{C}, \) hence \( c \in \mathbb{R}. \) Therefore \( \bar{u}v = uv \) for all \( u, v \in \mathbb{C}. \) This is a contradiction. Hence we have \( d_i \in SO(8). \) \( d_2, \) \( d_3 \in SO(8) \) follows from Lemma 5.3 and the above.

Let \( \mathfrak{Z}_i \) denote the space of \( F_i^\rho \) where \( u \in \mathbb{C} \) for \( i = 1, 2, 3. \) \( \mathfrak{Z}_i \) is an 8-dimensional \( \mathbb{R}-\) module and \( X \in \mathfrak{Z}_i \) is characterized by

5.5
\[ 2E_{i+1}X = X, \quad 2E_{i+2}X = X. \]

And we have

5.6
\[ 2XOY = (E - E)(X, Y) \quad \text{for } X, Y \in \mathfrak{Z}_i. \]

Let \( Spin(8) \) be the subgroup of \( F_i \) consisting of \( x \) such that \( xE_i = E_i \) for \( i = 1, 2, 3. \) Moreover it is convenient to define the following group \( Spin(8) : spin(8) \) is the subgroup of \( SO(8) \times SO(8) \times SO(8) \) which consists of \( (d_1, d_2, d_3) \) such that \( (d_1u)(d_2v) = d_3(uv) \) for \( u, v \in \mathbb{C}. \)

5.7 Proposition. [3]. \( Spin(8) \) and \( Spin(8) \) are isomorphic as group by the correspondence \( (d_1, d_2, d_3) \rightarrow d; \)

5.8
\[ d \begin{pmatrix} \xi_1 & u_3 & u_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & d_3u_3 & d_2u_2 \\ \bar{d}_3u_3 & \xi_2 & d_1u_1 \\ \bar{d}_2u_2 & \bar{d}_1u_1 & \xi_3 \end{pmatrix}. \]

Proof. Let \( d \in Spin(8), \) then for \( X, Y \in \mathfrak{Z}_i, \) we have \( dX \in \mathfrak{Z}_i \) and \( (dX, dY) = (X, Y) \) for \( i = 1, 2, 3 \) by 5.5, 5.6, hence \( d \) induces an orthogonal \( \mathbb{R} \)-homomorphism \( d_i \) in \( \mathfrak{Z}_i \) such that \( dF_i^\rho = F_i^{d\rho} \) for \( i = 1, 2, 3. \) \( 2F_i^\rho \circ F_i^\rho = F_i^{2\rho} \) implies \( (d_1u)(d_2v) = d_3(uv) \) for \( u, v \in \mathbb{C}. \) By Lemma 5.4 \( d_1, d_2, d_3 \in SO(8), \) that is, \( (d_1, d_2, d_3) \in spin(8). \)

In the sequel, we shall identify \( Spin(8) \) and \( Spin(8) \) by the correspondence 5.8.

Spin(8) has a sequence of subgroups

\[ Spin(8) \supset Spin(7) \supset G_2 \supset SU(3) \]

where \( Spin(7) \) is the subgroup of \( SO(8) \) consisting of \( \hat{a} \) such that for some \( a \in SO(7), \)
\[ (au)(\bar{a}v) = \bar{a}(uv) \] for \( u, v \in \mathbb{C}. \) (The projection \( p : Spin(7) \rightarrow SO(7) \) is defined by \( p(\hat{a}) = a). \) \( G_2 \) is the group of automorphisms in \( \mathbb{E}, \) that is, the subgroup of \( SO(7) \) consisting of \( a \) such that \( (au)(av) = a(uv) \) for \( u, v \in \mathbb{E}. \) \( SU(3) \) is the subgroup of \( G_2 \) consisting of \( a \) such that \( ae_i = e_i. \)

5.9 Proposition. \( Spin(8) \) is a simply connected covering group of \( SO(8). \)

Proof. We identify \( \mathfrak{Z}_i \) with \( \mathbb{E} \) by \( F_i^\rho \rightarrow u \) and let \( S^7 \) be the unit sphere in \( \mathbb{E}. \) \( spin(8) \) operates on \( S^7 \) by \( (d_1, d_2, d_3)u = d_iu. \) This operation is transitive by the
principle of triality and its isotropy group of $e_0$ is Spin(7). Thus we have spin(8)/Spin(7) = $S^7$. The fiberings $G_2/SU(3) = S^7$, Spin(7)/$G_2 = S^7$, spin(8)/Spin(7) = $S^7$ yield the connectivity of spin(8). Now, define $p: spin(8) \rightarrow SO(8)$ by $p(d_1, d_2, d_3) = d_1$, then $p$ is an epimorphism and its kernel is $(e, e, e), (e, -e, -e)$ by the principle of triality. Hence $p: spin(8) \rightarrow SO(8)$ is a twofold covering of $SO(8)$.

6. Spin(9) and construction lemma

Let $\mathfrak{s}_{31}$ denote the subspace of $\mathfrak{s}$ consisting of $X$ such that $E_1 \circ X = 0$ and $tr(X) = 0$. Such $X$ is of the form $(E_2 - E_3)\xi + F_i^u$ for $\xi \in \mathbb{R}, u \in \mathfrak{e}$. Hence $\mathfrak{s}_{31}$ is a 9-dimensional $\mathbb{R}$-module, and $(X, X') = 2(\xi \xi' + (u, u'))$ and $X \circ X' = (E_2 + E_3)(X, X')$ for $X, X' \in \mathfrak{s}_{31}$.

Let $\mathfrak{s}_{33}$ denote the subspace of $\mathfrak{s}$ consisting of $Y$ such that $2E_1 \circ Y = Y$. Such $Y$ is of the form $F_2^u + F_3^v$ for $u, v \in \mathfrak{e}$. Hence $\mathfrak{s}_{33}$ is a 16-dimensional $\mathbb{R}$-module and $(Y, Y') = 2((u, u') + (v, v'))$ for $Y, Y' \in \mathfrak{s}_{33}$.

Let $SO(9)$ denote the rotation group in $\mathfrak{s}_{31}$, i.e. $a \in SO(9)$ is an $\mathbb{R}$-homomorphism of $\mathfrak{s}_{31}$ such that $(aX, aY) = (X, Y)$ for $X, Y \in \mathfrak{s}_{31}$. Let Spin(9) be the subgroup of $F_4$ consisting of $x$ such that $x E_1 = E_1$.

The following lemma is sometimes convenient to construct an element of Spin(9) satisfying the given conditions.

6.1 Lemma. (construction lemma) For any given element $A \in \mathfrak{s}_{31}$ such that $(A, A) = 2$, choose any element $X_0 \in \mathfrak{s}_{31}$ such that $(A, X_0) = 0, (X_0, X_0) = 2$, choose any element $Y_0 \in \mathfrak{s}_{33}$ such that $2A \circ Y_0 = -Y_0, (Y_0, Y_0) = 2$ and put $Z_0 = 2X_0 \circ Y_0$.

Next choose any $X_1 \in \mathfrak{s}_{31}$ such that $(A, X_1) = (X_0, X_1) = 0, (X_1, X_1) = 2$, choose any $X_2 \in \mathfrak{s}_{31}$ such that $(A, X_2) = (X_0, X_2) = (X_1, X_2) = 0, (X_2, X_2) = 2$ and put $Y_1 = -2Z_0 \circ X_1, Z_2 = -2X_0 \circ Y_0, X_3 = -2Y_1 \circ Z_2$. Choose any $X_4 \in \mathfrak{s}_{31}$ such that $(A, X_4) = (X_0, X_4) = (X_1, X_4) = (X_2, X_4) = (X_3, X_4) = 0, (X_4, X_4) = 2$

and put $Z_4 = -2X_0 \circ Y_0, Y_2 = -2Z_0 \circ X_2, Y_3 = -2Z_0 \circ X_3, X_5 = -2Y_1 \circ Z_4, X_6 = 2Y_2 \circ Z_4, X_7 = -2Y_3 \circ Z_4$ and then put $Y_i = -2Z_0 \circ X_i$ for $i = 4, 5, 6, 7$, $Z_i = -2X_0 \circ Y_0$ for $i = 1, 3, 5, 6, 7$.

Now, let $a: \mathfrak{s} \rightarrow \mathfrak{s}$ be the $\mathbb{R}$-homomorphism satisfying

$$aE = E, \ aE_1 = E_1, \ a(E_2 - E_3) = A, \ aF_i^{\xi_i} = X_i, \ aF_2^{e_0} = Y_i, \ aF_3^{e_0} = Z_i \ for \ i = 0, 1, \ldots, 7,$$
then we have $a \in F_4$ (apriori $a \in \text{Spin}(9)$).

The proof is not trivial, we don’t however give the proof, because its calculation may be independent from the consideration of the present paper. It will appear in a forthcoming paper [7].

6.2 **Proposition.** Spin(9) is a simply connected covering group of SO(9).

**Proof.** Let $a \in \text{Spin}(9)$ and $X \in \mathfrak{so}_1$. Operate $a$ on $E_1 \circ X = 0$ and $\text{tr}(X) = 0$, then $E_1 \circ aX = 0$ and $\text{tr}(aX) = 0$, hence $aX \in \mathfrak{so}_1$. And $(E - E_i)(aX, aX') = a((E - E_i)(X, X')) = (E - E_i)(X, X')$, hence we have $(aX, aX') = (X, X')$. Thus $a$ induces an orthogonal $R$-homomorphism $\alpha$ in $\mathfrak{so}_1$. Let $S^8$ be the unit sphere in $\mathfrak{so}_1$, that is $S^8 = \{X \in \mathfrak{so}_1 | (X, X) = 2\}$. Spin(9) operates on $S^8$ transitively; this transitivity follows from the construction lemma 6.1. We show that its isotropy group $G = \{a \in \text{Spin}(9) | a(E_2 - E_3) = E_2 - E_3\}$ is Spin(8). For, since always $a(E_2 + E_3) = E_2 + E_3$ for $a \in \text{Spin}(9)$, we have $aE_i = E_i$ $(i = 1, 2, 3)$ for any $a \in G$. Therefore $G = \text{Spin}(8)$. Thus we have Spin(9)/Spin(8) = $S^8$, and this implies that Spin(9) is simply connected. Define the projection $\rho : \text{Spin}(9) \rightarrow \text{SO}(9)$ by $\rho(a) = \alpha$, then $\rho$ is a homomorphism and its kernel is $\{e, e, e\}$ and $\{e, -e, -e\}$. In fact, let $a \in \text{Spin}(9)$ satisfy $aX = X$ for all $X \in \mathfrak{so}_1$. First we shall see $a \in \text{Spin}(8)$. Denote $a$ by $(a_1, a_2, a_3) \in \text{spin}(8)$. Since $F_i^\alpha \in \mathfrak{so}_1$ we have $aF_i^\alpha = F_i^\alpha$. Hence, operating $a$ on $2F_1^\alpha + F_2^\alpha = F_3^\alpha$, then we have $u(a_2v) = a_3(uv)$. By the principle of triality, we have $a = (e, e, e)$ or $(e, -e, -e)$. Hence $\rho : \text{Spin}(9) \rightarrow \text{SO}(9)$ is the twofold covering of SO(9).

6.3 **Remark.** Let $S^{15}$ be the unit sphere in $\mathfrak{so}_3$, that is $S^{15} = \{Y \in \mathfrak{so}_3 | (Y, Y) = 2\}$. Spin(9) operates on $S^{15}$ transitively. The proof of the transitivity is as follows. Give a fixed element $F_2^\alpha$ and any element $Y_0 \in \mathfrak{so}_3$. Choose any $A \in \mathfrak{so}_1$ such that $2A \circ Y_0 = -Y_0$, $(A, A) = 2$ and then take $X_0, Y_0, Z_i$ for $i = 0, 1, \ldots, 7$ and construct $a \in \text{Spin}(9)$ as well as in Lemma 6.1. Then $aF_2^\alpha = Y_0$ for this $a$. Next it is easily verified that its isotropy group $\{a \in \text{Spin}(9) | aF_2^\alpha = F_2^\alpha\}$ is Spin(7). Thus we have the well known fact

$\text{Spin}(9)/\text{Spin}(7) = S^{15}$.

$F_4$ operates on the Cayley projective plane $\mathbb{O}P_2$ transitively by Proposition 4.1 (5) and its isotropy group of $E_1$ is Spin(9). Thus we have

$F_4/\text{Spin}(9) = \mathbb{O}P_2$.

Therefore, we have the following

6.4 **Theorem.** $F_4$ is a 52-dimensional simply connected compact group.

6.5 **Remark.** $F_4$ has 3 subgroups of type Spin(9); Spin(1)(9) = Spin(9), Spin(2)(9) and Spin(3)(9), where Spin(1)(9) = $\{a \in F_4 | aE_i = E_i\}$. And we have

$\text{Spin}(8) = \text{Spin}(1)(9) \cap \text{Spin}(2)(9) \cap \text{Spin}(3)(9)$. 

7. Maximal torus $T$ and Weyl group $W$

7.1 Definition. Let $G$ be a (connected) topological group. A subgroup $T$ of $G$ is a maximal torus in $G$ provided $T$ is a torus with $G = \bigcup_{x \in G} xTx^{-1}$.

It is easy to see that maximal tori are conjugate to each other in $G$. The dimension of a maximal torus $T$ is called the rank of $G$.

7.2 Theorem. The rank of $F_4$ is 4.

Proof. Let $x \in F_4$. Since the Cayley projective plane $\mathbb{C}P_2$ is a homogeneous space $F_4/\text{Spin}(9)$, $x$ induces a homeomorphism $f^x$ of $\mathbb{C}P_2$ in the natural way ($X \rightarrow xX$, $X \in \mathbb{C}P_2$). Hence $f^x$ induces an isomorphism $f^*_x : H_i(\mathbb{C}P_2) \rightarrow H_i(\mathbb{C}P_2)$ for all $i \geq 0$. We shall calculate the Lefschetz number $L(f^x) = \sum_{i=0}^\infty (-1)^i \text{tr}(f^*_x)$. For this, we recall that $\mathbb{C}P_2$ is a CW-complex with 0, 8, 16-dimensional cells [5], so that its homology groups are $H_0(\mathbb{C}P_2) = H_1(\mathbb{C}P_2) = H_6(\mathbb{C}P_2) = \mathbb{Z}$ and $H_i(\mathbb{C}P_2) = 0$ otherwise. Hence we have $L(f^x) = \text{tr}(f^*_0) + \text{tr}(f^*_8) + \text{tr}(f^*_16) = e_0 + e_8 + e_{16}$ (where $e_i$ is $-1$ or 1). Therefore, by the fixed point theorem, there exists a point $Y \in \mathbb{C}P_2$ such that $xY = Y$. For this $Y$, we can find $y \in F_4$ such that $Y = yE_1$ by Proposition 4.1 (5), $xyE_1 = yE_1$, so $y^{-1}xyE_1 = E_1$, and so that $y^{-1}xy \in \text{Spin}(9)$. As is well known, the rank of $\text{Spin}(9)$ is 4. Hence for a maximal torus $T$ (dim $T = 4$) in $\text{Spin}(9)$, there exists $z \in \text{Spin}(9)$ such that $z^{-1}(y^{-1}xy)z \in T$, so that $x \in (yz)T(yz)^{-1}$ where $yz \in F_4$. Hence we have $F_4 = \bigcup_{y \in F_4} yTy^{-1}$, Thus the proof is completed.

We shall choose a maximal torus in $\text{Spin}(8) = \text{spin}(8)$ as follows. Define a homomorphism $t : \mathbb{R}^8 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \text{spin}(8)$ ($t(\theta) = (t_1(\theta), t_2(\theta), t_3(\theta))$ where $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ is denoted by $t = (t_1, t_2, t_3, t_4)$ briefly) by

\[
\begin{align*}
& t_1e_0 = e_0 \cos \theta_1 + e_1 \sin \theta_1, & t_1e_1 = -e_0 \sin \theta_1 + e_1 \cos \theta_1, \\
& t_1e_2 = e_0 \cos \theta_2 + e_2 \sin \theta_2, & t_1e_3 = -e_0 \sin \theta_2 + e_2 \cos \theta_2, \\
& t_1e_4 = e_0 \cos \theta_4 + e_4 \sin \theta_4, & t_1e_5 = -e_0 \sin \theta_4 + e_4 \cos \theta_4, \\
& t_1e_6 = e_0 \cos \theta_1 + e_1 \sin \theta_1, & t_1e_7 = -e_0 \sin \theta_1 + e_1 \cos \theta_1, \\
& t_2e_0 = e_0 \cos(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2 + e_1 \sin(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2, & t_2e_1 = -e_0 \sin(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2 + e_1 \cos(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2, \\
& t_2e_2 = e_0 \cos(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2, & t_2e_3 = -e_0 \sin(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2, \\
& t_2e_4 = e_0 \cos(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2, & t_2e_5 = -e_0 \sin(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2, \\
& t_2e_6 = e_0 \cos(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2, & t_2e_7 = -e_0 \sin(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2, \\
& t_3e_0 = e_0 \cos(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2, & t_3e_1 = -e_0 \sin(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2, \\
& t_3e_2 = e_0 \cos(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2, & t_3e_3 = -e_0 \sin(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2, \\
& t_3e_4 = e_0 \cos(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2, & t_3e_5 = -e_0 \sin(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 - \theta_3 - \theta_4)/2, \\
& t_3e_6 = e_0 \cos(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_1 \sin(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2, & t_3e_7 = -e_0 \sin(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_1 \cos(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2.
\end{align*}
\]
7.5 \[
\begin{align*}
t_3e_3 &= -e_3\sin(\theta_1 + \theta_2 - \theta_3 - \theta_4)/2 + e_3\cos(\theta_1 + \theta_2 - \theta_3 - \theta_4)/2 \\
t_3e_4 &= e_4\cos(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_4\sin(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 \\
t_3e_5 &= -e_5\sin(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_5\cos(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 \\
t_3e_6 &= e_6\cos(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_6\sin(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 \\
t_3e_7 &= -e_7\sin(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2 + e_7\cos(\theta_1 - \theta_2 + \theta_3 - \theta_4)/2.
\end{align*}
\]

Then we can verify that
\[
(t_1u)(t_2v) = t_3(t_1v')^{-1} \quad \text{for } u, v \in \mathfrak{g}.
\]

Hence the image \(t(\mathfrak{r}^e) = T\) is a maximal torus in Spin(8) (also in \(F_4\)).

7.6 **Definition.** Let \(G\) be a topological group with a maximal torus \(T\). The Weyl group \(W(G)\) of \(G\) is \(N_T(G)/T\), where \(N_T(G)\) is the normalizer of \(T\) in \(G\).

7.7 **Lemma.** If \(x \in N_T(F_4)\), then \(xE_1 = E_{i_1}\), \(xE_2 = E_{i_2}\), \(xE_3 = E_{i_3}\) where \((i_1, i_2, i_3)\) is a substitution of \((1, 2, 3)\).

**Proof.** Let \(x \in N_T(F_4)\), then \(x^{-1}tx \in T \subset \text{Spin}(8)\) for all \(t \in T\). So that we have \(x^{-1}txE_1 = E_{i_1}\), hence \(t(xE_1) = xE_1\). Put \(xE_1 = \sum_{i=1}^{3}(E_{\xi_1} + F_{i_1}^{\pi_1})\), then \(t(xE_1) = xE_1\)

shows \(\sum_{i=1}^{3}(E_{\xi_1} + F_{i_1}^{\pi_1}) = \sum_{i=1}^{3}(E_{\xi_2} + F_{i_2}^{\pi_2})\), therefore \(t_1u_1 = u_1\), \(t_2u_2 = u_2\), \(t_3u_3 = u_3\) for all \(t = (t_1, t_2, t_3) \in T\). By the formulae 7.3–7.4, these imply \(u_1 = u_2 = u_3 = 0\).

Therefore \(xE_1 = \sum_{i=1}^{3}E_{\xi_1}\). By Proposition 4.1 (1), \(xE_1\) is an irreducible idempotent in \(\mathfrak{s}\), hence \(xE_1\) is \(E_1, E_2\) or \(E_3\). Similarly \(xE_2\) and \(xE_3\) are one of \(E_1, E_2, E_3\) respectively. Obviously \(xE_1, xE_2, xE_3\) are different to each other. Thus the proof is completed.

By Lemma 7.7, each \(w \in N_T(F_4)/T\) induces a substitution among \(E_1, E_2, E_3\). Thus we have a homomorphism

\[h : W(F_4) \longrightarrow \mathfrak{s}\]

where \(\mathfrak{s}\) is the symmetric group of all permutations of \(E_1, E_2, E_3\). We shall show that \(h\) is epimorphic. Since \(\mathfrak{s}\) is generated by \(\sigma=(1, 2, 3)\) and \(\tau=(2, 3)\), it suffices to construct elements \(x, y \in F_4\) which induce \(\sigma, \tau\) respectively. Define \(x = x(\sigma)\) by the \(R\)-homomorphism of \(\mathfrak{s}\) satisfying \(xE_i = E_{i+1}\), \(xF_{i}^{\pi} = F_{i+1}^{\pi}\) for \(u \in \mathfrak{s}, i = 1, 2, 3\). Since \(x^{-1}txE_i = E_{i+1}\), \(x^{-1}txF_{i}^{\pi} = x^{-1}F_{i+1}^{\pi}\) for \(i = 1, 2, 3\), we have \(x^{-1}(t_1, t_2, t_3)x = (t_2, t_3, t_1)\) (cf. Lemma 5.3), so that \(x \in N_T(F_4)\) and \(x\) obviously induces \(\sigma\). Next, let \(y = y(\tau)\) be the \(R\)-homomorphism given by

\[
\begin{align*}
yE_1 &= E_1 \\
yF_{i_0}^{\pi_0} &= F_{i_0}^{\pi_0} \\
yF_{i_1}^{\pi_1} &= -F_{i_1}^{\pi_1} \\
yF_{i_2}^{\pi_2} &= F_{i_2}^{\pi_2} \\
yE_2 &= E_2 \\
yF_{i_0}^{\pi_1} &= F_{i_0}^{\pi_1} \\
yF_{i_1}^{\pi_1} &= -F_{i_1}^{\pi_1} \\
yF_{i_2}^{\pi_2} &= -F_{i_2}^{\pi_2} \\
yE_3 &= E_3 \\
yF_{i_0}^{\pi_0} &= -F_{i_0}^{\pi_0} \\
yF_{i_1}^{\pi_1} &= -F_{i_1}^{\pi_1} \\
yF_{i_2}^{\pi_2} &= F_{i_2}^{\pi_2}
\end{align*}
\]
then $y$ is in $\text{Spin}(9)$, because it is easily verified that $y$ satisfies the construction conditions of Lemma 6.1. And $y$ induces $\tau$ obviously.

The kernel of $h$ is $W(\text{Spin}(8))$ which is the Weyl group of $\text{Spin}(8)$. In fact, suppose $h(w) = 1$ where $w \in W(F_4)$, then any representative $x \in N_T(F_4)$ of $w$ satisfies $xE_i = F_i$ ($i = 1, 2, 3$) so that we have $x \in \text{Spin}(8)$ (apriori, $x \in N_T(\text{Spin}(8))$). Therefore $w \in W(\text{Spin}(8))$. Thus we have an exact sequence

$$1 \longrightarrow W(\text{Spin}(8)) \longrightarrow W(F_4) \longrightarrow \mathbb{S}_3 \longrightarrow 1.$$ 

And it splits by $\sigma \mapsto x(\sigma)$, $\tau \mapsto y(\tau)$. Thus we have the following

7.8 Theorem. The Weyl group $W(F_4)$ of $F_4$ is a semidirect product of $\mathbb{S}_3$ and $W(\text{Spin}(8))$. That is,

$$W(F_4) = \mathbb{S}_3 W(\text{Spin}(8)), \mathbb{S}_3 \cap W(\text{Spin}(8)) = 1.$$ 

We remember that $W(\text{Spin}(8))$ consists of $2^3! = 192$ permutations of 4 variables $(\theta_1, \theta_2, \theta_3, \theta_4)$ composed with substitutions $(\theta_1, \theta_2, \theta_3, \theta_4) \rightarrow (\varepsilon_1 \theta_1, \varepsilon_2 \theta_2, \varepsilon_3 \theta_3, \varepsilon_4 \theta_4)$ with $\varepsilon_i = \pm 1$ and $\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = 1$.

7.9 Remark. Let $Z_3$ denote the subgroup of $\mathbb{S}_3$ generated by $\sigma$. Then we have a splitting exact sequence

$$1 \longrightarrow W(\text{Spin}(9)) \longrightarrow W(F_4) \longrightarrow Z_3 \longrightarrow 1.$$ 

7.10 Since it is easy to see that $x(\tau)Z_3 = Z_3$ for $i = 1, 2, 3$ and $y(\tau)Z_3 = Z_3$, $y(\tau)Z_3 = Z_3$, $y(\tau)Z_3 = Z_3$ by 5.5, any element of $\mathbb{S}_3$ induces a substitution among $Z_3, Z_3, Z_3$.

8. Lie algebra $\mathfrak{g}_1$

Let $\mathcal{M}$ denote the space of 3-matrices over $\mathbb{C}$ and $\mathcal{M}^-$ denote the space of 3-skew-hermitian matrices over $\mathbb{C}$ (skew-hermitian matrix $X$ is meant by $X^* = -X$). We extend the inner product of $\mathfrak{g}$ to $\mathcal{M}$ by

$$(X, Y) = \frac{1}{2} \text{tr}(XY + Y^*X^*).$$

8.1 Lemma. $(XY, A) = (YA, X)$ for $X, Y, A \in \mathcal{M}$. 

The proof is the same as Lemma 1.2.

We define the bracket product by

3) $X^*$ is $^tX$. 

$[A, X] = AX - XA$ for $A, X \in \mathfrak{g}$.

If $A \in \mathfrak{g}^-$, $X \in \mathfrak{g}$, then $[A, X]$ is often denoted by $\tilde{A}X$. Obviously we have

8.2 **Lemma.** $[\mathfrak{g}^-, \mathfrak{g}] \subset \mathfrak{g}$, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}^-$.

8.3 **Lemma.** $[\mathfrak{g}]$. For $X \in \mathfrak{g}$, there exists a pure imaginary Cayley number $u$ such that

$[X, XX] = Eu$.

**Proof.** Let $X = (u_{ij})$ where $\tilde{u}_{ij} = u_{ij}$ and $v_{ij}$ be the $(i, j)$-component of $[X, XX]$, namely $v_{ij} = \sum_{k, l=1}^{3} (x_{ik}x_{kl}u_{lj}) - (x_{ik}x_{kl})u_{lj}$. Note that the parenthesis containing a $u_{ij}$ (with double suffixes) is zero. If $i \neq j$, using 1.1 (5) we have $v_{ij} = 0$. For the case $i = j$, $v_{jj} = v_{22} = v_{33}$ by 1.1 (6). Thus we have $[X, XX] = Eu$ for some $u \in \mathbb{C}$. Since $X, XX \in \mathfrak{g}$, we have $[X, XX] \in \mathfrak{g}^-$, hence $\tilde{u} = -u$, whence $Reu = 0$.

8.4 **Lemma.** (1) For $A, X, Y \in \mathfrak{g}$, we have

$([A, X], Y) + (X, [A, Y]) = 0$.

(2) For $A \in \mathfrak{g}$ such that $tr(A) = 0$, we have

$tr([A, X], Y, Z) + tr(X, [A, Y], Z) + tr(X, Y, [A, Z]) = 0$ for $X, Y, Z \in \mathfrak{g}$.

**Proof.** (1) is obvious by Lemma 8.1. (2) $(A, [X, XX]) = (A, Eu)$ (where $\tilde{u} = -u) = \frac{1}{2} tr(Au + \tilde{u}A^*) = \frac{1}{2} tr(Au + uA) = 0$. Hence $(A, X(XX)) = (A, (XX)X)$. Thus $(AX, XX) = (XA, XX)$ by Lemma 8.1. Hence $([A, X], XX) = 0$. By the polarization $X \rightarrow Y + Z, ([A, X], YZ + ZY) + ([A, Y], XZ + ZX) + ([A, Z], XY + YX) = 0$. This means (2).

8.5 **Definition.** Let $\mathfrak{g}_1$ denote the set of $\mathbb{R}$-homomorphisms $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$\varphi(XoY) = \varphi XoY + Xo\varphi Y$.

Let $F_4'$ denote the set of $\mathbb{R}$-homomorphisms $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

$[\varphi X, Y] + (X, \varphi Y) = 0,$

$tr(\varphi X, Y, Z) + tr(X, \varphi Y, Z) + tr(X, Y, \varphi Z) = 0$.

$\mathfrak{g}_1$ and $\mathfrak{g}_4'$ are Lie $\mathbb{R}$-algebra by the bracket multiplication

$[\varphi, \psi]X = \varphi(\psi X) - \psi(\varphi X)$ for $X \in \mathfrak{g}$.

8.6 **Lemma.** $\mathfrak{g}_4'$ is a Lie subalgebra of $\mathfrak{g}_1$; $\mathfrak{g}_4' \subset \mathfrak{g}_4$.

**Proof.** For $\varphi \in \mathfrak{g}_4'$, $X, Y, Z \in \mathfrak{g}$, $(\varphi XoY, Z) + (Xo\varphi Y, Z) = tr(\varphi X, Y, Z) + tr(X, \varphi Y, Z) = -tr(X, Y, \varphi Z) = -(\varphi (XoY), \varphi Z) = (\varphi (XoY), Z)$. Hence we have

$\varphi XoY + Xo\varphi Y = \varphi (XoY)$.
The lemma 8.4 shows that for $A \in \mathfrak{g}$ such that $\text{tr}(A) = 0$, $\tilde{A} \in \mathfrak{g}_i$.

8.7 Remark. We see that $\mathfrak{g}_4 = \mathfrak{g}_i$. It will be remained to the readers.

We shall use the following notations; for $a \in \mathfrak{g}$

$$
A_i^a = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & a \\
0 & -a & 0
\end{pmatrix},
A_s^a = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
a & 0 & 0
\end{pmatrix},
A_0^a = \begin{pmatrix}
0 & 0 & 0 \\
0 & -a & 0 \\
a & 0 & 0
\end{pmatrix}.
$$

Then we have

$$
\begin{cases}
\tilde{A}_i^a E_i = 0, & \tilde{A}_i^a F_i^a = (E_{i+1} - E_{i+2})2(a, u), \\
\tilde{A}_s^a E_i = -F_i^a, & \tilde{A}_s^a F_i^a = F_{i+2}^a, \\
\tilde{A}_0^a E_i = F_i^a, & \tilde{A}_0^a F_i^a = -F_{i+1}^a.
\end{cases}
$$

Let $\mathfrak{d}_4$ denote the Lie subalgebra of $\mathfrak{g}_4$ consisting of $D$ such that $DE_i = 0$ for $i = 1, 2, 3$.

Let $D \in \mathfrak{d}_4$. $E_i F_i^a = 0, 2E_i F_i^a = F_i^a (i \neq j)$ imply $E_i D F_i^a = 0, 2E_i D F_i^a = DF_i^a$. Thus we can set $DF_i^a = F_i^a D_i^a$. And $F_i^a F_i^a = (E_{i+1} + E_{i+2})(u, v)$ and $2F_i^a F_i^a$ imply $(D_i u, v) + (u, D_i v) = 0$ and $(D_i u)v + u(D_i v) = D_{i+2}(uv)$. Hence we have

8.8 Proposition. $\mathfrak{d}_4$ and $\mathfrak{d}_4$ are isomorphic as Lie algebra by the correspondence $D_1 \in \mathfrak{d}_4 \mapsto D \in \mathfrak{d}_4$;

$$
D \begin{pmatrix}
\xi_1 & u_3 & \bar{u}_3 \\
\bar{u}_3 & \xi_2 & u_1 \\
u_2 & \bar{u}_1 & \xi_3
\end{pmatrix} = \begin{pmatrix}
0 & D_3 u_3 & \bar{D}_3 u_3 \\
D_3 u_3 & 0 & D_1 u_1 \\
\bar{D}_3 u_3 & D_1 u_1 & 0
\end{pmatrix}
$$

where $D_3, D_3 \in \mathfrak{d}_4$ are given by the infinitesimal triality for $D_1$.

We shall identify $\mathfrak{d}_4$ and $\mathfrak{d}_4$ by the above correspondence in later.

**Chapter II**

9. Representation rings

Let $G$ be a topological group. By a $G$-$K$-module ($K = \mathbb{R}$ or $\mathbb{C}$) is meant a finite dimensional right $K$-module $V$ together with a left action of $G$. That is for each $x \in G$, $u \in V$, there should be defined an element $xu \in V$ depending continuously on $x$ and $u$ so that

$$
\begin{cases}
x(u + v) = xu + xv, & x(u\xi) = (xu)\xi, \\
x(uy) = x(uy), & eu = u
\end{cases}
$$

for $x, y \in G$, $u, v \in V$, $\xi \in K$ and $e$ denotes the identity of $G$.

4) $\mathbb{C}$ is the field of complex numbers.
Two $G$-$K$-modules $V_1$ and $V_2$ are $G$-$K$-isomorphic if there exists a $G$-$K$-isomorphism $f : V_1 \to V_2$, that is, $f$ is a $K$-isomorphism such that $f(xu) = xf(u)$ for $x \in G$, $u \in V_1$.

Let $M_K(G)$ denote the set of $G$-$K$-isomorphism classes $[V]$ of $G$-$K$-modules $V$. $[V]$ will be denoted by $V$ simply.

The direct sum $V_1 \oplus V_2$, the tensor product $V_1 \otimes V_2$ of two $G$-$K$-modules $V_1$, $V_2$ and the exterior $G$-$K$-modules $\Lambda^i(V)$ ($0 \leq i \leq \dim V$) for a $G$-$K$-module $V$ define a $\lambda$-semiring structure on $M_K(G)$. That is, $A^i : M_K(G) \to M_K(G)$ for $i \geq 0$ satisfy

$$A^i(V_1 \oplus V_2) = \bigoplus_{i+j=k} (A^i(V_1) \otimes A^j(V_2)).$$

In particular, we have

9.3 Lemma. Let $V_1, \ldots, V_n$ be 1-dimensional $G$-$K$-modules. Then $A^i(V_1 \oplus \ldots \oplus V_n)$ and $\bigoplus_{i_1 < \ldots < i_n} V_{i_1} \otimes \ldots \otimes V_{i_n}$ are $G$-$K$-isomorphic.

The representation ring $R_K(G) = (R_K(G), \phi_G)$ is the universal $\lambda$-ring associated with the $\lambda$-semiring $M_K(G)$. The $\lambda$-ring $R_K(G)$ is meant a commutative ring with the unit 1 and functions $\lambda^i : R_K(G) \to R_K(G)$ for $i \geq 0$ satisfying the following properties

$$\lambda^0(\alpha) = 1, \quad \lambda^1(\alpha) = \alpha,$$

$$\lambda^i(\alpha + \beta) = \sum_{i+j=k} \lambda^i(\alpha) \lambda^j(\beta).$$

The universality is as follows : $\phi_G : M_K(G) \to R_K(G)$ is a $\lambda$-semiring homomorphism and for any $\lambda$-ring $A$ and any semiring homomorphism $\varphi : M_K(G) \to A$, there exists a unique $\lambda$-ring homomorphism $\hat{\varphi} : R_K(G) \to A$ such that $\varphi = \hat{\varphi} \phi_G$.

$M_K(G)$ has one more operation so called conjugation : for each $G$-$K$-module $V$, there corresponds the dual $G$-$K$-module $\hat{V}$ ($\hat{V}$ is $\text{Hom}_K(V, K)$ as $K$-module and group action is $(x \omega)u = \omega(x^{-1}u)$ for $x \in G$, $\omega \in \text{Hom}_K(V, K)$, $u \in V$). If $W$ is a 1-dimensional $G$-$K$-module, then we have $W \otimes \hat{W} = K$, so that $\hat{W}$ is often denoted by $W^{-1}$.

Let $H$ and $G$ be topological groups and $h : H \to G$ be a continuous homomorphism. Then to every $G$-$K$-module $V$, there corresponds an $H$-$K$-module $h^*(V)$ by the rule of group action

$$yu = h(y)u \quad \text{for } y \in H, \ u \in V.$$  

The correspondence $V \to h^*(V)$ gives rise to a $\lambda$-ring homomorphism $h^* : R_K(G) \to R_K(H)$ such that the following diagram is commutative
$M_R(G) \xrightarrow{\phi_G} M_R(H)$
$\downarrow \phi_G \quad \downarrow \phi_H$
$R_R(G) \xrightarrow{h^v} R_R(H)$

$M_R(G), R_R(G)$ are denoted by $MO(G), RO(G)$ and $M_C(G), R_C(G)$ are denoted by $M(G), R(G)$ respectively.

10. Spin(8)-C-module $\mathfrak{Z}^C_i$ and Spin(9)-C-modules $\mathfrak{Z}^C_{21}, \mathfrak{Z}^C_{23}$

Since for $d \in \text{Spin}(8), X \in \mathfrak{Z}_i$ we have $dX \in \mathfrak{Z}_i$ by 5.5, each $d \in \text{Spin}(8)$ induces a $R$-homomorphism of $\mathfrak{Z}_i$. Hence $\mathfrak{Z}_i$ is a Spin(8)-R-module and $\mathfrak{Z}^C_i = \mathfrak{Z}_i \otimes_R \mathbb{C}$ is a Spin(8)-C-module for $i = 1, 2, 3$.

Let $T$ be the maximal torus in Spin(8) which is indicated in the section 7 and let $j_1: T \rightarrow \text{Spin}(8)$ be the inclusion.

10.1 Lemma. In $j_2: M(\text{Spin}(8)) \rightarrow M(T)$, we have

\[
j_2(\mathfrak{Z}^C_1) = \bigoplus_{j=1}^4 (W_j \oplus W_j^{-1}),
\]
\[
j_2(\mathfrak{Z}^C_2) = \bigoplus_{e_1 \leq e_2 \leq e_3 \leq e_4 = -1} W_{e_1}^{e_2} \otimes W_{e_1}^{e_2} \otimes W_{e_1}^{e_2} \otimes W_{e_1}^{e_2},
\]
\[
j_2(\mathfrak{Z}^C_3) = \bigoplus_{e_1 \leq e_2 \leq e_3 \leq e_4 = -1} W_{e_1}^{e_2} \otimes W_{e_1}^{e_2} \otimes W_{e_1}^{e_2} \otimes W_{e_1}^{e_2}
\]

where $W_j^{1/2}$ is a 1-dimensional $T$-C-module, $W_j^{-1/2}$ is the dual $T$-C-module of $W_j^{1/2}$ and $W_j^{e}$ is $W_j^{e/2} \otimes W_j^{e/2}$ for $j = 1, 2, 3, 4$ ($e_j, e = \pm 1$).

Proof. Choose an additive base in $\mathfrak{Z}^C_3$ as follows:

10.2 $X_j = F_1^{e_{2j-2} - e_{2j-1} \sqrt{-1}}, \quad \bar{X}_j = F_1^{e_{2j-2} + e_{2j-1} \sqrt{-1}}$ in $\mathfrak{Z}^C_1$,

10.3 $Y_j = F_2^{e_{2j-2} - e_{2j-1} \sqrt{-1}}, \quad \bar{Y}_j = F_2^{e_{2j-2} + e_{2j-1} \sqrt{-1}}$ in $\mathfrak{Z}^C_2$,

10.4 $Z_j = F_3^{e_{2j-2} - e_{2j-1} \sqrt{-1}}, \quad \bar{Z}_j = F_3^{e_{2j-2} + e_{2j-1} \sqrt{-1}}$ in $\mathfrak{Z}^C_3$

for $j = 1, 2, 3, 4$. For $t = t(\theta) = (t_1(\theta), t_2(\theta), t_3(\theta), t_4(\theta)) \in T$ where $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4$, then we have

10.5 $tX_j = X_j \exp(\sqrt{-1} \theta_j), \quad t\bar{X}_j = \bar{X}_j \exp(-\sqrt{-1} \theta_j)$

for $j = 1, 2, 3, 4$. In fact, $t_1(e_{2j-2} - e_{2j-1} \sqrt{-1}) = t_1(e_{2j-2} - e_{2j-1} \sqrt{-1}) = (e_{2j-2} \cos \theta_j + e_{2j-1} \sin \theta_j) - (-e_{2j-2} \sin \theta_j + e_{2j-1} \cos \theta_j) \sqrt{-1} = (e_{2j-2} - e_{2j-1} \sqrt{-1})(\cos \theta_j + \sqrt{-1} \sin \theta_j) = (e_{2j-2} - e_{2j-1} \sqrt{-1}) \exp(\sqrt{-1} \theta_j)$, and $t_1(e_{2j-2} + e_{2j-1} \sqrt{-1}) = (e_{2j-2} + e_{2j-1} \sqrt{-1}) \exp(-\sqrt{-1} \theta_j)$. Similarly we have by 7.4, 7.5,
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$$tY_1 = Y_1 \exp(-\sqrt{-1}(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2),$$
$$t\hat{Y}_1 = \hat{Y}_1 \exp(-\sqrt{-1}(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2),$$
$$tY_2 = Y_2 \exp(-\sqrt{-1}(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2),$$
$$t\hat{Y}_2 = \hat{Y}_2 \exp(-\sqrt{-1}(-\theta_1 - \theta_2 + \theta_3 - \theta_4)/2),$$
$$tY_3 = Y_3 \exp(-\sqrt{-1}(-\theta_1 + \theta_2 - \theta_3 + \theta_4)/2),$$
$$t\hat{Y}_3 = \hat{Y}_3 \exp(-\sqrt{-1}(-\theta_1 + \theta_2 - \theta_3 + \theta_4)/2),$$
$$tY_4 = Y_4 \exp(-\sqrt{-1}(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2),$$
$$t\hat{Y}_4 = \hat{Y}_4 \exp(-\sqrt{-1}(-\theta_1 - \theta_2 - \theta_3 + \theta_4)/2).$$

These formulae 10.5–10.7 give the proof of the lemma.

Putting $\phi_f(W^{J^2}) = \alpha_j^{J^2}$ for $j = 1, 2, 3, 4$, then we have (cf. [2], [4])

$$R(T) = \mathbb{Z}[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8].$$

Put $\nu^C = \phi_{\text{Spin}(8)}(\nu^C)$, $\nu^C_+ = \phi_{\text{Spin}(8)}(\nu^C_+)$, $\nu^C_+ = \phi_{\text{Spin}(8)}(\nu^C_+)$ in $R(\text{Spin}(8))$ and denote $a = j_2^*(\nu^C)$, $b = j_2^*(\nu^C_+)$, $c = j_2^*(\nu^C_+)$, $d = j_2^*(\nu^C)$ in $R(T)$.

10. 8 Lemma.

$$a = j_2^*(\nu^C) = \sum_{j=1}^{4} (\alpha_j + \alpha_j^{-1}),$$
$$b = j_2^*(\nu^C_+) = \sum_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = -1} \alpha_1^{\epsilon_1/2} \alpha_2^{\epsilon_2/2} \alpha_3^{\epsilon_3/2} \alpha_4^{\epsilon_4/2},$$
$$c = j_2^*(\nu^C_+) = \sum_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = -1} \alpha_1^{\epsilon_1/2} \alpha_2^{\epsilon_2/2} \alpha_3^{\epsilon_3/2} \alpha_4^{\epsilon_4/2},$$
$$d = j_2^*(\nu^C) = j_2^*(\nu^C_+) = j_2^*(\nu^C_+) = 4 + \sum_{i \neq j} \alpha_{i} \alpha_{j}^{-1}.$$

Proof. The first three formulae are the direct consequences of Lemma 10.1.

To prove the last formula, we shall use Lemma 9.3. Pick up two different monomials from 8 monomials in $a = \sum_{j=1}^{4} (\alpha_j + \alpha_j^{-1})$, multiply them and sum up $8 \cdot 28$ monomials (the result polynomial is denoted by $a_9$). Then we have $a_9 = 4 + \sum_{i \neq j} \alpha_{i} \alpha_{j}^{-1}$. Similarly we have $b_9 = c_9 = d_9$. These show the last of the lemma.
Recall that we have (cf. [2], [4]) by using Lemma 10.8

\[ R(\text{Spin}(8)) = \mathbb{Z}[\nu_C^2, \nu_C^3, \Delta^C, \Delta_C^C]. \]

We have seen that \( \mathcal{Z}_1 \) and \( \mathcal{Z}_2 \) are \( \text{Spin}(9) \)-modules (cf. 6). Hence we have two \( \text{Spin}(9) \)-modules \( \mathcal{Z}_1^0 = \mathcal{Z}_1 \otimes \mathcal{R} \) and \( \mathcal{Z}_2^0 = \mathcal{Z}_2 \otimes \mathcal{R} \). Put \( \mu_C^1 = \phi_{\text{Spin}(9)}(\mathcal{Z}_1^0) \), \( \mu_C^2 = \phi_{\text{Spin}(9)}(\mathcal{Z}_2^0) \) and \( \Delta^C = \phi_{\text{Spin}(9)}(\Delta^C) \). And let \( j_1 : T \rightarrow \text{Spin}(9) \) be the inclusion. Then we have easily the following

10.10 Lemma. As a \( \text{Spin}(8) \)-module,

\[ \mathcal{Z}_1 = \mathcal{R} \oplus \mathcal{Z}_1, \quad \mathcal{Z}_2 = \mathcal{R} \oplus \mathcal{Z}_2. \]

Hence we have in \( R(T) \)

\[ j_1^* (\mu_C^1) = 1 + a = 1 + \sum_{j=1}^{4} (\alpha_j + \alpha_j^{-1}), \]
\[ j_1^* (\Delta^C) = b + c = \prod_{j=1}^{4} (\alpha_j^{1/2} + \alpha_j^{-1/2}), \]
\[ (j_1^* (\mu_C^2)) = a + d, \quad j_1^* (\mu_C^2) = -a + d + bc. \]

Therefore we see (cf. [2], [4]) that

10.11 \[ R(\text{Spin}(9)) = \mathbb{Z}[\mu_C^1, \mu_C^2, \Delta^C, \Delta_C^C]. \]

11. \( F_4 \)-module \( \mathcal{Z}_1^0 \)

Since \( F_4 \) is the automorphism group of \( \mathcal{Z}_1 \), \( \mathcal{Z}_1 \) is obviously an \( F_4 \)-module. Remember that the trace of every \( X \in \mathcal{Z}_1 \) is invariant under the operation of \( F_4 \) by Theorem 4.2. Let \( \mathcal{Z}_3 \) denote the set of \( X \in \mathcal{Z}_1 \) such that \( \text{tr}(X) = 0 \). Then \( \mathcal{Z}_3 \) is invariant under \( F_4 \), so that \( \mathcal{Z}_3 \) is an \( F_4 \)-module and \( \mathcal{Z}_1 \) is decomposable into the direct sum of two \( F_4 \)-modules \( \mathcal{Z}_3 \) (which is spanned by \( E \) with the trivial group action) and \( \mathcal{Z}_3 \); \( \mathcal{Z}_1 = \mathcal{Z}_3 \oplus \mathcal{Z}_3 \) by

\[ X = E \frac{1}{3} \text{tr}(X) + (X - E \frac{1}{3} \text{tr}(X)). \]

And we have an \( F_4 \)-module \( \mathcal{Z}_1^0 = \mathcal{Z}_3 \otimes \mathcal{R} \).

Let \( T \) be the same maximal torus in \( F_4 \) as in the sections 7, 10 and let \( j : T \rightarrow F_4 \) be the inclusion.

11.1 Lemma. As a \( \text{Spin}(8) \)-module, we have

\[ \mathcal{Z}_3 = \mathcal{R} \oplus \mathcal{R} \oplus \mathcal{Z}_1 \oplus \mathcal{Z}_2 \oplus \mathcal{Z}_3. \]

Putting \( \mu_C^1 = \phi_{F_4}(\mathcal{Z}_3^0), \quad \mu_C^2 = \phi_{F_4}(\Delta^C(\mathcal{Z}_3^0)) \) and \( \Delta^C = \phi_{F_4}(\Delta^C(\mathcal{Z}_3^0)) \), then we have the
11.2 Proposition.
\[ j^*(\mathcal{X}) = 2 + (a + b + c), \]
\[ j^*(\mathcal{X}) = 1 + 2(a + b + c) + (ab + bc + ca) + 3d, \]
\[ j^*(\mathcal{X}) = 3(ab + bc + ca) + abc + 6d + 2(a + b + c) + 3d. \]

Proof. The first formula is the direct consequence of Lemma 11.1. To prove the second formula, we shall use the result \( a^2 = b^2 = c^2 = d \) in Lemma 10.8. Now,
\[ j^*(\mathcal{X}) = 1 + 2(a + b + c) + (ab + bc + ca) + a_2 + b_2 + c_2 \]
\[ = 1 + 2(a + b + c) + (ab + bc + ca) + 3d. \]

To prove the last, we shall apply the same technique as the above. Pick up 3 different monomials from \( a \), multiply and sum up them (the result polynomial is denoted by \( a_3 \)). Then we have \( a_3 = bc - a \) by the direct calculation. Similarly we have \( b_3 = ca - b \) and \( c_3 = ab - c \). Hence
\[ j^*(\mathcal{X}) = (a + b + c) + 2(a_2 + b_2 + c_2) + 2(ab + bc + ca) \]
\[ + (a_3 + b_3 + c_3) + abc + 6d \]
\[ = (a + b + c) + 6d + 2(ab + bc + ca) + 2(a + b + c) + 3d. \]

11.3 Remark. \( (1 + t)^2 \prod_{j=1}^{4} (1 + \alpha_j t)(1 + \alpha_j^{-1} t) \prod_{\xi = \pm 1} (1 + \xi_1 \xi_2 \xi_3 \xi_4) = 1 + j^*(\mathcal{X}) t + j^*(\mathcal{X}) t^2 + j^*(\mathcal{X}) t^3 + \ldots \)

12. \( F_4 \)-C-module \( \mathfrak{g}^C_4 \)

The group \( F_4 \) operates on its Lie algebra \( \mathfrak{g}_4 \), in the natural way, that is, for \( x \in F_4 \) and \( \varphi \in \mathfrak{g}_4 \), \( \varphi \varphi \in \mathfrak{g}_4 \) is defined by
\[ (x\varphi)X = x(\varphi(x^{-1}X)) \quad \text{for} \quad X \in \mathfrak{g}. \]
Thus \( \mathfrak{g}_4 \) is an \( F_4 \)-R-module, whence its complex form \( \mathfrak{g}_4^C = \mathfrak{g}_4 \otimes \mathbb{C} \) is an \( F_4 \)-C-module.

To decompose \( j^*(\mathfrak{g}_4^C) \), we shall extend the operation of \( \text{Spin}(8) \). Let \( \mathfrak{M}^r \) denote the space of \( X \in \mathfrak{M} \) with real diagonal elements. For \( d = (d_1, d_2, d_3) \in \text{spin}(8) \) and \( X \in \mathfrak{M}^r \), we define \( dX \) by
\[ d \begin{pmatrix} \xi_1 & u_{12} & u_{13} \\ u_{21} & \xi_2 & u_{23} \\ u_{31} & u_{32} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & d_3 u_{12} & d_2 u_{13} \\ d_3 u_{21} & \xi_2 & d_1 u_{23} \\ d_2 u_{31} & d_1 u_{32} & \xi_3 \end{pmatrix}. \]
12.2 Lemma. For $d \in \text{Spin}(8)$, we have
\[ d(X \circ Y) = dX \circ dY \quad \text{for } X, Y \in \mathfrak{m}' \]
where $X \circ Y = \frac{1}{2}(XY + YX)$.

Proof. We shall show $d(X \circ X) = dX \circ dX$ for $X \in \mathfrak{m}'$. The $(1,1)$-component of $dX \circ dX = \frac{1}{2}(\xi_1^2 + (d_3u_{12})(d_3u_{21}) + (d_2u_{13})(d_2u_{31}) + \xi_1^2 + (d_3u_{21})(d_3u_{12}) + (d_2u_{31})(d_2u_{13})) = (use 1, 1 (3)) = \xi_1^2 + (d_3u_{12}, d_3u_{21}) + (d_3u_{13}, d_3u_{21}) = \xi_2^2 + (u_{12}, u_{21}) + (u_{13}, u_{21}) = \frac{1}{2}((\xi_1^2 + u_{12}u_{21} + u_{13}u_{21} + \xi_1^2 + u_{21}u_{12} + u_{13}u_{21} = (1,1)-component of d(X \circ X))$. The $(2,3)$-component of $2dX \circ dX = (d_3u_{12})(d_3u_{21}) + \xi_2^2 + (d_3u_{12}, d_3u_{21}) + (d_3u_{13}, d_3u_{21}) + \xi_2^2 + (d_3u_{12}, d_3u_{21}) + \xi_2^2 + (d_3u_{13}, d_3u_{21}) + \xi_2^2 + (u_{12}, u_{21}) + (d_3u_{12})(d_3u_{21}) + \xi_2^2 + (d_3u_{13})(d_3u_{21}) + \xi_2^2 + (u_{12}, u_{21}) + \xi_2^2 + (u_{13}, u_{21}) = (2,3)-component of 2d(X \circ X)$. About the other components the calculations are similar. Thus we have $d(X \circ X) = dX \circ dX$. By the polarization $X \rightarrow X + Y$, we have $d(X \circ Y) = dX \circ dY$.

12.3 Lemma. For $d \in \text{Spin}(8)$ and $A \in \mathfrak{m}' \cap \mathfrak{m}'$, we have
\[ d\tilde{A} = \tilde{d}A. \]

Proof. By Lemma 12.2, $d(A \circ X) = dA \circ dX$ for any $X \in \mathfrak{m}$. This shows that $d(AX - XA) = (dA)(dX) - (dX)(dA)$, i.e. $d\tilde{AX} = \tilde{d}A(dX)$. Replacing $X$ by $d^{-1}X$, then we have $d\tilde{A}(d^{-1}X) = \tilde{d}A$ for all $X \in \mathfrak{m}$. This proves the lemma.

Now, in $j^* : M(F_4) \rightarrow M(T)$, we have
\[ j^*(\mathfrak{g}_C^*) = C^* \oplus C^* \oplus C^* \oplus C^* \oplus (W_j + W_{j^{-1}}) \oplus \bigoplus_{i \neq j} W_{x_i}^{x_j} \oplus W_{x_i}^{x_j} \oplus W_{x_i}^{x_j} \oplus W_{x_i}^{x_j}. \]

Proof. We shall use the following notations $G_{ij}$ for $0 \leq i < j \leq 7 : G_{ij}$ is the $R$-homomorphism of $\mathfrak{g}$ satisfying
\[
\begin{align*}
G_{ij}(e_i) & = e_i, \\
G_{ij}(e_i) & = -e_j, \\
G_{ij}(e_k) & = 0 \quad \text{for } k \neq i, j.
\end{align*}
\]
(These $G_{ij}$ form an additive base of $\mathfrak{g}$). We choose now an additive base in $\mathfrak{g}_C^*$ as follows:
\[
\begin{align*}
H_1 & = G_{01}, \quad H_2 = G_{23}, \quad H_3 = G_{45}, \quad H_4 = G_{67}, \\
X_j & = \widetilde{\alpha}_1^{e_{12}j-1} e_{i-1}, \quad \tilde{X}_j = \tilde{\alpha}_1^{e_{12}j-1} e_{i-1}, \\
Y_j & = \widetilde{\alpha}_2^{e_{12}j-1} e_{i-1}, \quad \tilde{Y}_j = \tilde{\alpha}_2^{e_{12}j-1} e_{i-1}, \\
Z_j & = \widetilde{\alpha}_3^{e_{12}j-1} e_{i-1}, \quad \tilde{Z}_j = \tilde{\alpha}_3^{e_{12}j-1} e_{i-1},
\end{align*}
\]
for $j = 1, 2, 3, 4$ and
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$$S_{12} = G_{05} - G_{19} - (G_{08} + G_{16})\sqrt{-1}, \quad \tilde{S}_{12} = G_{09} - G_{19} + (G_{08} + G_{16})\sqrt{-1},$$

$$S_{13} = G_{04} - G_{15} - (G_{05} + G_{14})\sqrt{-1}, \quad \tilde{S}_{13} = G_{04} - G_{15} + (G_{05} + G_{14})\sqrt{-1},$$

$$S_{14} = G_{06} - G_{17} - (G_{07} + G_{16})\sqrt{-1}, \quad \tilde{S}_{14} = G_{06} - G_{17} + (G_{07} + G_{16})\sqrt{-1},$$

$$S_{28} = G_{24} - G_{35} - (G_{25} + G_{34})\sqrt{-1}, \quad \tilde{S}_{28} = G_{24} - G_{35} + (G_{25} + G_{34})\sqrt{-1},$$

$$S_{24} = G_{26} - G_{37} - (G_{27} + G_{36})\sqrt{-1}, \quad \tilde{S}_{24} = G_{26} - G_{37} + (G_{27} + G_{36})\sqrt{-1},$$

$$S_{22} = G_{46} - G_{57} - (G_{47} + G_{56})\sqrt{-1}, \quad \tilde{S}_{22} = G_{46} - G_{57} + (G_{47} + G_{56})\sqrt{-1},$$

$$T_{12} = G_{04} + G_{13} - G_{12}\sqrt{-1}, \quad \tilde{T}_{12} = G_{04} + G_{13} - G_{12}\sqrt{-1},$$

$$T_{13} = G_{04} + G_{15} - G_{14}\sqrt{-1}, \quad \tilde{T}_{13} = G_{04} + G_{15} - G_{14}\sqrt{-1},$$

$$T_{14} = G_{06} + G_{17} - G_{16}\sqrt{-1}, \quad \tilde{T}_{14} = G_{06} + G_{17} - G_{16}\sqrt{-1},$$

$$T_{28} = G_{24} + G_{35} - G_{34}\sqrt{-1}, \quad \tilde{T}_{28} = G_{24} + G_{35} - G_{34}\sqrt{-1},$$

$$T_{24} = G_{26} + G_{37} - G_{36}\sqrt{-1}, \quad \tilde{T}_{24} = G_{26} + G_{37} - G_{36}\sqrt{-1},$$

$$T_{34} = G_{46} + G_{57} - G_{47}\sqrt{-1}, \quad \tilde{T}_{34} = G_{46} + G_{57} - G_{47}\sqrt{-1}.$$

Then, for $t = t(\theta_1, \theta_2, \theta_3, \theta_4) \in T$, we have

$$tH_j = H_j \quad \text{for } j = 1, 2, 3, 4.$$

As for $X_j, \tilde{X}_j, Y_j, \tilde{Y}_j, Z_j, \tilde{Z}_j$, we have the same formulae as 10.5–10.7 for $j = 1, 2, 3, 4$ and

$$tS_{ij} = S_{ij} \exp(-\sqrt{-1}(\theta_1 + \theta_2)), \quad \tilde{tS}_{ij} = S_{ij} \exp(-\sqrt{-1}(\theta_1 + \theta_2)),
$$

$$tT_{ij} = T_{ij} \exp(-\sqrt{-1}(\theta_1 - \theta_2)), \quad \tilde{tT}_{ij} = T_{ij} \exp(-\sqrt{-1}(\theta_1 - \theta_2)).$$

Some of them will be proved. For example, $tY_1 = i\tilde{A}_2 - e^{-\sqrt{-1}} = i\tilde{A}_2 - e^{-\sqrt{-1}} = \tilde{A}_2^{t2}$

$$(e_0 - e_1 - \sqrt{-1}) = \tilde{A}_2^{t2} = A_2^{t2} \exp(-\sqrt{-1}(\theta_1 + \theta_2 + \theta_3 + \theta_4)/2) = Y_1 \exp(-\sqrt{-1}(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2).$$

An another example $tT_{12} = T_{12} \exp(-\sqrt{-1}(\theta_1 - \theta_2))$ will be proved. To do so, it is sufficient to show that $(t_1 T_{12} e_i = T_{12} e_i) \exp(-\sqrt{-1}(\theta_1 - \theta_2))$ for $i = 0, 1, \ldots, 7$. For $i = 0,

$$t(T_{12} e_0) = T_{12} e_0, \quad t(T_{12} e_1) = T_{12} e_1, \quad t(T_{12} e_2) = T_{12} e_2, \quad t(T_{12} e_3) = T_{12} e_3.$$

Thus the proposition is proved.

Putting $\phi^C = \phi_{F_4}(\phi^C)$, then we have by Lemma 12.4

12.5 Proposition. $j^a(e^C) = a + b + c + d$.

13. Complex representation ring $R(F_4)$

Each element $w : T \to T$ in the Weyl group $W(F_4)$ induces an automorphism $w^* : R(T) \to R(T)$. Let $R(T)^W$ denote the subring of $R(T)$ which is invariant elementwise under these operation $w^*$. Since $j^a : R(F_4) \to R(T)$ is a ring monomo-
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In $R(T)^W$, we will regard $R(F_4)$ as a subring of $R(T)^W$. From Propositions 11.2 and 12.5, we have

13.1. Lemma. $a + b + c, ab + bc + ca, abc$ and $d$ are polynomials in $x_1^2, x_2^2, x_3^2,$ and $x^C$. In fact,

\[
\begin{align*}
a + b + c &= x_1^2 - 2, \\
ab + bc + ca &= x_1^2 + x_2^2 - 3x^C - 3, \\
abc &= -5x_1^2 - 3x_2^2 + x_3^2 + 2(x_1^2)^3 - 2x_1^2x^C + 5, \\
d &= -x_1^2 + x^C - 2.
\end{align*}
\]

Let $f \in R(T)^W$, that is, $f$ be a $W(F_4)$-invariant polynomial. We know that any $W(Spin(8))$-invariant polynomial is representable as a polynomials in $x_1^2, x_2^2,$ $x_3^2,$ and $x^C$ (cf. 10) namely as a polynomial in $a, b, c, d$. Recall that the Weyl group $W(F_4)$ is the semidirect product of $W(Spin(8))$ and $\mathfrak{S}_3$, and each element of $\mathfrak{S}_3$ induces a substitution of 3 factors $a, b, c$ (cf. 7.10). Hence, $f \in R(T)^W$ is a polynomial in the elementary symmetric functions $a + b + c, ab + bc + ca, abc$ and $d$. Thus, from Lemma 13.1, $f$ can be represented as a polynomial in $x_1^2, x_2^2, x_3^2,$ and $x^C$.

Next we have to show that $x_1^2, x_2^2, x_3^2,$ and $x^C$ are algebraically independent. In fact, we know that $a, b, c,$ and $d$ algebraically independent because $R(Spin(8)) = \mathbb{Z}[a, b, c, d]$. Hence $a + b + c, ab + bc + ca, abc$ and $d$ are also algebraically independent. Using propositions 11.2, 12.5, a non-trivial algebraic relation among $x_1^2, x_2^2, x_3^2$ and $x^C$ yields a non-trivial algebraic relation among $a + b + c, ab + bc + ca, abc$ and $d$. Therefore $x_1^2, x_2^2, x_3^2$ and $x^C$ are algebraically independent. And we have $\mathbb{Z}[x_1^2, x_2^2, x_3^2, x^C] \subset R(F_4) \subset R(T)^W \subset \mathbb{Z}[x_1^2, x_2^2, x_3^2, x^C]$. Thus we can proved the following

13.2 Theorem. The complex representation ring $R(F_4)$ of $F_4$ is a polynomial ring $\mathbb{Z}[x_1^2, x_2^2, x_3^2, x^C]$, where $x_i$ is the class of the $F_4$-$C$-module $\Lambda^i(\mathfrak{g}^C)$ for $i=1, 2, 3$ and $x^C$ is the class of the Lie $C$-algebra $\mathfrak{g}_C^C$ in $R(F_4)$.

14. Real representation ring $RO(F_4)$

For a topological group $G$, we have two correspondence:

\[ c : RO(G) \to R(G), \quad r : R(G) \to RO(G), \]

where $c$ is a ring homomorphism induced by the tensoring $c'$ with $C$ (that is,
c' : MO(G) → M(G) is defined by c'(V) = V ⊗_C R and r is a homomorphism defined by restricting scalars from C to R. As is well known, the relation rc = 2 holds. If G is a compact group, then RO(G) is a free module generated by the classes of irreducible G-R-modules, so that, the relation rc = 2 implies that c is a ring monomorphism.

Let λ₁, λ₂, λ₃ and ε be the classes of F₄-R-modules Ξ₁, Ξ₂(Ξ₁), Ξ₃(Ξ₂) and Ξ₄, respectively. Since we have obviously c(ξₙ) = ξₙ for i = 1, 2, 3 and c(ε) = εₗ, c is an epimorphism, so that c is an isomorphism. Thus we have the following

14.1 Theorem. The real representation ring RO(F₄) is a polynomial ring \( Z[λ₁, λ₂, λ₃, ε] \) with 4 variables λ₁, λ₂, λ₃ and ε.

As for RO(Spin(9)) and RO(Spin(8)), we can discuss in the real range. Using the fact that c is an isomorphism, then we have by 10.9, 10.11.

14.2 RO(Spin(9)) = Z[μ₁, μ₂, μ₃, d]

where μᵢ is the class of \( Ξᵢ(Ξ₁) \) for i = 1, 2, 3, and d is the class of Ξ₂₃.

14.3 RO(Spin(8)) = Z[v₁, v₂, d⁻, d⁺]

where vᵢ is the class of \( Ξᵢ(Ξ₁) \) for i = 1, 2 and d⁻, d⁺ are the classes of Ξ₂, Ξ₃ respectively.

15. Relations of R(F₄) to R(Spin(9)) and R(Spin(8))

Let

\[
\begin{array}{ccc}
\text{Spin}(8) & \xrightarrow{k} & \text{Spin}(9) \\
F₄ & \xrightarrow{i} & \text{Spin}(9)
\end{array}
\]

be the inclusions.

15.1 Theorem. In the diagram

\[
\begin{array}{ccc}
RO(F₄) & \xrightarrow{i^*} & RO(Spin(9)) \\
\xrightarrow{k^*} & & \xrightarrow{k^*} \\
RO(Spin(9)) & \xrightarrow{i^*} & RO(Spin(8))
\end{array}
\]

namely, in

\[
\begin{array}{ccc}
Z[λ₁, λ₂, λ₃, ε] & \xrightarrow{i^*} & Z[v₁, v₂, d⁻, d⁺]
\end{array}
\]

we have

\[
\begin{align*}
l^*(λ₁) &= 1 + μ₁ + d \\
l^*(λ₂) &= μ₁ + 2μ₂ + μ₃ + d + μ₁d \\
l^*(λ₃) &= 2μ₁ + 2μ₂ - d + μ₁μ₂ + μ₁μ₃ + μ₁d + 2μ₂d \\
l^*(ε) &= μ₂ + d,
\end{align*}
\]
\[
\begin{align*}
15.3 \quad k^\mu(\nu_1) &= 1 + \nu_1 \\
k^\mu(\nu_2) &= \nu_1 + \nu_2 \\
k^\mu(\nu_3) &= -\nu_1 + \nu_2 + d_- d_+ \\
k^\mu(d) &= d_- + d_+
\end{align*}
\]

\[
\begin{align*}
15.4 \quad i^\nu(\lambda_1) &= 2 + \nu_1 + d_- + d_+ \\
i^\nu(\lambda_2) &= 1 + 2(\nu_1 + d_- + d_+) + \nu_1 d_+ + d_- d_+ + d_1 \nu_1 + 3\nu_2 \\
i^\nu(\lambda_3) &= 3(\nu_1 d_- + d_- d_+ + d_1 \nu_1) + \nu_1 d_- d_+ + 6\nu_2 + 2(\nu_1 + d_- + d_+)\nu_2 \\
i^\nu(\omega) &= \nu_1 + \nu_2 + d_- + d_+.
\end{align*}
\]

In the complex case, the relations between \(R(F_4)\), \(R(\text{Spin}(9))\) and \(R(\text{Spin}(8))\) are quite analogous to the real case (add the upper suffix \(C\)).

Proof. It suffices to show in the complex case. Since from Lemmas 11.1, 10.10 we have \(G_0 = R \oplus \mathcal{Z}_{30} \oplus \mathcal{Z}_{32}\) as a \(\text{Spin}(8) \cdot R\)-module we have obviously the first of 15.2. Using 9.2, the 2nd and 3rd of 15.2 are obtained from the first of 15.2. Since the \(j_1^*\)-image of two sides of the last of 15.2 are both \(a + b + c + d\) in \(R(T)\) and \(j_1^*\) is an isomorphism, we see that the last of 15.2 is true. 15.3 and 15.4 are the direct consequences of Lemma 10.10 and Propositions 11.2, 12.5 respectively.

References

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