On Zeros of Polynomials and Galois Extensions of Simple Rings

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Introduction. In [4], the author introduced the notion of the polynomial simple ring extensions and studied some properties of polynomial Galois extensions.

In the present paper, we shall investigate the relationship between the zeros of polynomials and Galois extensions of simple rings. As a generalization of the commutative case, some type of a finite dimensional polynomial simple Galois extension can be considered as a simple ring in which every $\omega$-irreducible polynomial over a basic simple ring possessing a zero in the Galois extension can be factored into a product of its linear factors and conversely.

Let $S$ be a simple ring, and let $\rho$ be an automorphism in $S$, $D$ a $\rho$-derivation in $S$. Then the followings are well known.

1. $S[X; \rho, D] = \{ \sum_{i} X^i s_i ; s_i \in S \}$, the free right $S$-module with an $S$-basis $\{X^i\}$, can be regarded as a polynomial ring with an indeterminate $X$ by the multiplication rule $sX = X(s\rho) + sD$ for each $s \in S$.

2. Each two-sided ideal of $S[X; \rho, D]$ is generated by a (uniquely determined) monic polynomial, and hence, if $T$ is a two-sided ideal, then $T = \langle f(X) \rangle S[X; \rho, D]$ for some monic $f(X)$ which is called the generator of $T$.

A polynomial $f(X)$ is called non-vanishing (resp. vanishing) if $\langle f(X) \rangle$, the two-sided ideal generated by $f(X)$, is a proper ideal of $S[X; \rho, D]$ (resp. coincides with $S[X; \rho, D]$).

A non-vanishing polynomial $f(X)$ is called $\omega$-irreducible if each proper left factor $h(X)$ of it (i.e. $f(X) = h(X)g(X)$ and $\deg h(X) < \deg f(X)$) is vanishing.

3. The generator of a two-sided ideal $M$ is $\omega$-irreducible if and only if $M$ is maximal.

4. Every proper two-sided ideal ($\neq 0$) has a unique factorization as a product of maximal ideals. **

5. Every non-vanishing polynomial has an essentially unique factorization as

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* Cf. [1]
** Cf. [2], p. 38.
a product of \( w \)-irreducible polynomials and a vanishing polynomial in the sense of the factorization determined within a vanishing polynomial.

Let \( M \) be a maximal ideal of \( S[X; \rho, D] \) whose generator is \( f(X) = X^n + \sum_{i=0}^{n-1} X^i s_i \). Then

\[
R = S[y] = S \oplus yS \oplus y^2S \oplus \cdots \oplus y^{n-1}S \cong S[X; \rho, D]/M, \quad \text{where } y \text{ is the residue class of } X \bmod M, \text{ is called an } n \text{-dimensional polynomial simple ring extension over } S.
\]

\( S[X; \rho], S[X; D] \) mean the cases \( D := 0, \rho = 1 \) respectively, and finally, \( S[X] \) means the case \( D = 0 \) and \( \rho = 1 \).

For the other notations and terminologies used in this paper, we refer to \([4]\).

§1. \( w \)-irredcibility and zeros of polynomials

**Lemma 1.1** Let \( X-s \) be a polynomial in \( S[X; \rho, D] \). Then the followings are equivalent.

(a) \( X-s \) is non-vanishing.
(b) \( X-s \) is \( w \)-irreducible.
(c) \( s(s \rho) = s^2 \) and \( D \) is an inner \( \rho \)-derivation generated by \( s \).

In particular, if \( D = 0 \),

(c') \( s \) is regular and \( \rho = \tilde{s}^{-1} \) provided \( s \neq 0 \).

**Proof.** (a)\( \rightarrow \) (b). If we note that the generator of each ideal is a monic polynomial of the lowest degree which is contained in the ideal, the implication is clear.

(b)\( \rightarrow \) (c)\( \rightarrow \) (a). The first implication is a direct consequence of the fact that \( X(X-s) \in (X-s)S[X; \rho, D] \) and \( t(X-s) \in (X-s)S[X; \rho, D] \) for each \( t \in S \). Next, the conditions (c) shows that \( X(X-s) = (X-s)(X-(s-s \rho)) \) and \( t(X-s) = (X-s)(t \rho) \) for each \( t \in S \). Hence, \( X-s \) is non-vanishing. Now, let \( D = 0 \). Then (c) yields that \( ts = s(t \rho) \) for each \( t \in S \). Hence \( S = SsS = sS \) shows that the regularity of \( s \) and \( \rho = \tilde{s}^{-1} \).

**Corollary 1.1** Let \( X-s \) be \( w \)-irreducible in \( S[X; \rho, D] \).

(a) If \( X-t \) is \( w \)-irreducible for some \( t \neq s \), then \( t-s \) is regular.
(b) If \( D = 0 \), then \( X-t \) is \( w \)-irreducible if and only if \( t = sz \) for some \( z \in Z = V_s(S) \).
(c) If \( \rho = 1 \), then \( X-t \) is \( w \)-irreducible if and only if \( t = s+z \) for some \( z \in Z \).

**Proof.** (a) We have \( ut-t(u \rho) = uD, us-s(u \rho) = uD \) for each \( u \in S \) by Lemma 1.1. Hence \( u(t-s) = (t-s)(u \rho) \) shows that the regularity of \( t-s \).
(b) Let \( X-t \) be \( w \)-irreducible. If \( t = 0 \), then \( t = s \cdot 0 \). On the other hand, if \( t \neq 0 \), \( \rho = t^{-1} \) implies \( s = tz \) for some \( z \in Z \). The converse is clear.
(c) We can prove the assertion in the same way as in the proof of (a).

Let \( f(X) = \sum_{i=0}^{n} X^i s_i \) be a polynomial of \( S[X; \rho, D] \). Then an element \( t \) in \( S \)
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is called a zero of $f(X)$ if $f(t) = \sum_{i=0}^{n} t_i s_i = 0$.

**Lemma 1.2** Let $f(X)$ be a polynomial of $S[X; \rho, D]$. If $s_1, \ldots, s_k$ are distinct zeros of $f(X)$ in $S$ such that $X - s_1$ is $w$-irreducible then $f(X) = \prod_{i=1}^{k} (X - s_{\pi(i)}) h_\pi(X)$ where $\pi$ is an arbitrary permutation of $k$-letters and $h_\pi(X) \in S[X; \rho, D]$.

**Proof.** Dividing $f(X)$ by $X - s_{\pi(1)}$, we have $f(X) = (X - s_{\pi(1)}) h_1(X) + t_1$ for some $t_1 \in S$. Then $w$-irreducibility of $X - s_{\pi(1)}$ yields at once $0 = f(s_{\pi(1)}) = t_1$ in $S[X; \rho, D]/(X - s_{\pi(1)}) \cong S$. Therefore we have $f(X) = (X - s_{\pi(1)}) h_1(X)$. Next, let $h_1(X) = (X - s_{\pi(2)}) h_2(X) + t_2$ for some $t_2 \in S$. Then $f(X) = (X - s_{\pi(1)}) (X - s_{\pi(2)}) h_2(X) + (X - s_{\pi(1)}) t_2$. Hence $0 = f(s_{\pi(2)}) = (s_{\pi(2)} - s_{\pi(1)}) t_2$ in $S[X; \rho, D]/(X - s_{\pi(2)}) \cong S$. Since $s_{\pi(2)} - s_{\pi(1)}$ is regular by Corollary 1.1 (a), $t_2 = 0$. Repeating the same procedure, we have $f(X) = \prod_{i=1}^{k} (X - s_{\pi(i)}) h_\pi(X)$.

§ 2. Zeros of polynomials and Galois extensions

Throughout the present section, we assume that $R = S[y] = S \oplus yS \oplus y^2S \oplus \cdots \oplus y^{n-1} S (n > 1)$ be an $n$-dimensional polynomial simple ring extension over $S$ defined by $S[X; \rho]/(f(X))$ (resp. $S[X; D]/f(X)$), where $f(X) = X^n + \sum_{i=0}^{n-1} X^i s_i$ and $y$ is the residue class of $X$ modulo $(f(X))$. Then, $R[X; P]$ with $P = y^r$ can be considered as a polynomial ring containing $S[X; \rho]$ (resp. $R[X; E]$ with $E = I_y$) can be considered as a polynomial ring containing $S[X; D]$. Thus, if we consider $f(X)$ in $R[X; P]$ (resp. $R[X; E]$), $y$ is a zero of $f(X)$ and $X - y$ is $w$-irreducible in $R[X; P]$ (resp. $R[X; E]$). We call a zero $x$ in $R$ of $h(X) \in S[X; \rho]$ (resp. $R[X; D]$) a root in $R$ of $h(X)$ if $X - x$ is $w$-irreducible in $R[X; P]$ (resp. $R[X; E]$).

Let $\sigma$ be an $S$-automorphism of $R$. Then $\sigma y = y \sigma$ if $s y = s (\sigma p)$ and $y \sigma = y + v_\sigma$ if $s y = s y + s D$ for some $v_\sigma \in V = V(S)$ [4, p. 175].

**Theorem 2.1** Let $R = S[y] = S \oplus yS \oplus y^2 S \oplus \cdots \oplus y^{n-1} S$ with respect to $\emptyset$ with $\sigma y = y \sigma$, $c_i \neq 1 \in C = V(R)$ for each $\sigma \in \emptyset$, then every $w$-irreducible polynomial of $S[X; \rho]$ possessing a root $x$ in $R$ has an essentially unique factorization into a product of $w$-irreducible linear factors in $R[X; P]$ of order $n$.

(a) If $R$ is a weakly Galois extension over $S$ with respect to $\emptyset$ and $\sigma y = y \sigma$, $c_i \neq 1 \in C = V(R)$ for each $\sigma \in \emptyset$, then every $w$-irreducible polynomial of $S[X; \rho]$ possessing a root $x$ in $R$ has an essentially unique factorization into a product of $w$-irreducible linear factors in $R[X; P]$. Furthermore, if this is the case, the order of $\emptyset$ is $n$.

(b) If $f(X)$ has a factorization into $(X - y) (X - y c_1) \cdots (X - y c_{n-1})$ in $R[X; P]$ such that $c_i \neq c_j$ if $i \neq j$, $c_i \neq 1 \in C$, then $R$ is weakly Galois over $S$.

**Proof.** Let $x$ be a root in $R$ of a $w$-irreducible polynomial $h(X)$ of $S[X; \rho]$. Then $h(X) = (X - x) g(X)$ in $R[X; P]$. We set $x = \sum_{\alpha=0}^{n-1} \alpha u_\alpha (u_\alpha \in S)$. Then $x \sigma P = \sum_{\alpha=0}^{n-1} (\sigma u_\alpha) P = \sum_{\alpha=0}^{n-1} \sigma u_\alpha P = x P \sigma = x \sigma$ since $x P = x$ for each $\sigma \in \emptyset$ and $x (\sigma p) = (x \sigma p) \sigma = x \sigma (\sigma p)$. Consequently, $x \sigma$ is a root in $R$ of $h(X)$. Thus $h(X) = \prod_{\alpha \in \emptyset} (X - x \sigma) h(X)$ where $\emptyset = x \emptyset$. Noting here $x \sigma P = x \sigma$ and $x \sigma (X - x \sigma) = (X -
We have $x\sigma \cdot x_\tau = x_\tau \cdot x_\sigma$ for each $\sigma, \tau \in \mathcal{G}$. Hence $P_{\sigma \in \mathcal{G}}(X - x_\sigma) \in S[X; \rho]$ and is non-vanishing. Now, let $f(X) = P_{\sigma \in \mathcal{G}}(X - x_\sigma)g(X) + r(X)$ where $g(X), r(X) \in S[X; \rho]$ and $\deg r(X) < \deg P_{\sigma \in \mathcal{G}}(X - x_\sigma)$. Then $P_{\sigma \in \mathcal{G}}(X - x_\sigma) (g(X) - k(X)) = r(X)$ and $\deg P_{\sigma \in \mathcal{G}}(X - x_\sigma) (g(X) - k(X)) \geq \deg P_{\sigma \in \mathcal{G}}(X - x_\sigma)$ if $g(X) - k(X) \neq 0$. Hence $k(X) = g(X) + S[X; \rho]$. Therefore, $w$-irreducibility of $k(X)$ yields $k(X) \in S$. By making use of this fact, we find $f(X) = P_{\sigma \in \mathcal{G}}(X - y_\sigma)$, and hence, the order of $\mathcal{G}$ have to coincide with $n$ by Lemma 1.2.

(b) Let $f(X) = (X - y)(X - yc_1) \cdots (X - yc_{n-1})$ in $R[X; P]$. Then the map $\sigma_i$ defined by $\sum_{k=0}^{n-1} y^k \sigma_i = \sum_{k=0}^{n-1} y^k e_i^{k} u_k$ is an $S$-automorphism of $R$. For, since $yc_i$ is a root of $f(X)$, $\sigma_i$ is well defined, and $(sy)^i = y(sy)^i = yc_i(sy) = sa_i y \sigma_i$. Hence $\sigma_i$ is a ring monomorphism. If $(\sum_{k=0}^{n-1} y^k \sigma_i) = \sum_{k=0}^{n-1} y^k e_i^{k} u_k = 0$, then $(\sum_{k=0}^{n-1} y^k \sigma_i) = \sum_{k=0}^{n-1} y^k e_i^{k} u_k = 0$ is contained in the kernel of $\sigma_i$. Thus $\{(yc_i)^k; k = 0, 1, \ldots, n-1\}$ is an $S$-basis. Consequently, $\sigma_i$ is an $S$-automorphism.

Let $\mathcal{G}$ be the group generated by $\{1, \sigma_1, \ldots, \sigma_{n-1}\}$. If $x = \sum_{k=0}^{n-1} y^k u_k$ is an arbitrary element of $J(\mathcal{G}, R)$, then $\sum_{k=0}^{n-1} y^k \sum_{i=0}^{n-1} c_i^{j} u_k = 0$ for each $i = 1, 2, \ldots, n-1$. This means that

\[
\begin{pmatrix}
\vdots \\
\sum_{j=0}^{n-2} c_j^{i} & \sum_{j=0}^{n-2} c_j^{i+1} & \cdots & \sum_{j=0}^{n-2} c_j^{n-1} \\
\sum_{j=0}^{n-3} c_j^{i} & \sum_{j=0}^{n-3} c_j^{i+1} & \cdots & \sum_{j=0}^{n-3} c_j^{n-1} \\
\vdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & 1
\end{pmatrix} = 0
\]

On the other hand,

\[
\det \begin{pmatrix}
\sum_{j=0}^{n-2} c_j^{i} & \sum_{j=0}^{n-2} c_j^{i+1} & \cdots & \sum_{j=0}^{n-2} c_j^{n-1} \\
\sum_{j=0}^{n-3} c_j^{i} & \sum_{j=0}^{n-3} c_j^{i+1} & \cdots & \sum_{j=0}^{n-3} c_j^{n-1} \\
\vdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & 1
\end{pmatrix} \neq 0
\]

shows that $u_i = u_{i+1} = u_{i+2} = \cdots = u_{n-1} = 0$, that is $J(\mathcal{G}, R) = S$.

**Corollary 2.1** Let $S$ be a simple ring with the infinite center $Z$, and let $R = S[y] \equiv S[X; \rho]/(f(X))$.

(a) If $R/S$ is Galois and $yc_i \in yC$ for each $\sigma \in \mathcal{G}(R/S)$ where $\mathcal{G}(R/S)$ the $S$-automorphism group of $R$, then $R/S$ is outer.

(b) If $R$ is a division ring, then $R/S$ is an outer Galois extension if and only if each zero in $R$ of $f(X)$ is a root of $f(X)$. Moreover, if this is the case, every $w$-irreducible polynomial $h(X)$ of degree $m$ of $S[X; \rho]$ possessing a root $x$ in $R$ has
an essentially unique factorization into \((X - x)(X - xc_1)\cdots(X - xc_{m-1})s (s \in S)\) such that \(c_i \neq c_j\) if \(i \neq j\) and \(c_i (\neq 1) \in C\).

**Proof.** (a) The order of \(\Theta(R/S)\) is \(n\) by Theorem 2.1 (a). Hence the order of \(V = [V^e : C^e] \leq n\). Now if we note that \(V\) is a simple ring possessing infinitely many elements, \(V = C\) by a generalized Scott's Theorem [5, Lemma 1].

(b) The number of distinct zeros \(\{x_\alpha\}\) in \(R\) of \(f(X)\) such that \(x_\alpha P = x_\alpha\) is either infinite or at most \(n\) [3, Theorem 6]. Hence if each zero in \(R\) of \(f(X)\) is a root of \(f(X)\), we have \(f(X) = \Pi_{s \in \Theta(R/S)} (X - y_\sigma)\). Thus the assertion is an immediate consequence of (a). The converse is clear.

**Theorem 2.2** Let \(R = S[X] \cong S[X; D]/(f(X))\).

(a) If \(R\) is a weakly Galois extension over \(S\) with respect to \(\Theta\) with \(y_\sigma = y + c_\sigma, c_\sigma(\neq 0) \in C\), for each \(\sigma \in \Theta\), then every \(w\)-irreducible polynomial of \(S[X; D]\) possessing a root in \(R\) has a non-zero constant term and an essentially unique factorization into a product of \(w\)-irreducible linear factors in \(R[X; E]\). Furthermore, if this is the case, the order of \(\Theta\) is \(n\).

(b) Let \(\chi(S) \geq n\) or \(\chi(S) = 0\) and \(V \neq C\). Then \(R/S\) is an outer Galois extension if and only if \(w\)-irreducible polynomial \(h(X)\) of degree \(m\) of \(S[X; D]\) possessing a root \(x\) in \(R\) has an essentially unique factorization \((X - x)(X - (x + c_1))\cdots(X - (x + c_{n-1}))s (s \in S)\) such that \(c_i \neq c_j\) if \(i \neq j\), \(c_i (\neq 0) \in C\).

**Proof.** (a) The proof is quite similar to that of Theorem 2.1 (a).

(b) If \(\chi(S) \geq n\) or \(\chi(S) = 0\) and \(V \neq C\), then \(V = C\) by [4, Theorem 2.2]. Thus there exists an element \(t \in S\) such that \(y - t \in C\) and \([y - t]k; k = 0, 1, \ldots; n - 1\] is an \(S\)-basis for \(R\). If \((y - t)^n + (y - t)^{n-1} u_{n-1} + \cdots + u_0 = 0 (u_k \in S)\), then \(g(X) = (X - t)^n + \sum_{k=0}^{n-1} (X - t)u_k = X^n + \sum_{k=0}^{n-1} X^w_k (w_k \in S)\) is \(w\)-irreducible in \(S[X; D]\) and it possesses a root \(y\) in \(R\). Hence \(g(X) = (X - y)(X - (y + c_1))\cdots(X - (y + c_{n-1}))\) in \(R[X; E]\). Then the map \(\sigma_i\) defined by \(\sum_{k=0}^{n-1} (y + c_i)^k a_k (a_k \in S)\) is an \(S\)-automorphism of \(R\). Hence \((y - t)\sigma_i = (y - t)d_i\) for some \(d_i \in C\) since \(y - t\) is contained in \(C\). Hence we can see that \(J(\Theta, R) = S\) by similar methods to that of Theorem 2.1 (b) if \(\Theta\) is the group generated by \(\{1, \sigma_1, \ldots, \sigma_{n-1}\}\). Therefore \(R = S[C]\) is an outer Galois extension over \(S\). Conversely, if \(R/S\) is outer Galois, \(y_\sigma = y + c_\sigma, c_\sigma \in C\) for each \(\sigma \in \Theta(R/S)\). Hence the assertion is clear from (a).
References


