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On Contraction of Fourier Series

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The purpose of this paper is to prove the generalization of the theorem due to Boas [2] on the contraction of Fourier series by using absolute summability in place of absolute convergence.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$.

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \cdots + p_n\,;\ P_{-k} = p_{-k} = 0, \text{ for } k \geq 1.$$

The sequence $\{t_n\}$, given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \frac{1}{P_n}\sum_{k=0}^{n} P_k a_{n-k},\ (P_n \neq 0),$$

defines the Nörlund means of the sequence $\{s_n\}$ generated by the sequence of constants $\{p_n\}$.

If the series

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}|$$

converges, then the series $\sum a_n$ is said to be summable $|N, p_n|$. In the special cases in which $p_n = A_n^{\alpha-1} = \binom{n+\alpha-1}{n}$ and $p_n = 1/(n+1)$, the summability $|N, p_n|$ is the same as the summability $|C, \alpha|$ and the absolute harmonic summability, respectively. Also, the summability $|C, 0|$ is the absolute convergence.

It is well-known [6] that

$$|C, 0| \subset |N, 1/n+1| \subset |C, \alpha| \subset |C, \beta| \text{ for } \beta > \alpha > 0,$$

where, if every series summable $|A|$ is also summable $|B|$, we write $|A| \subset |B|$.

A denotes a positive absolute constant that is not always the same.

2. Orthogonal series

Let $\{\psi_n(x)\}$ be an orthonormal system defined in the interval (a, b). For a function $f(x) \in L^2(a, b)$ such that

$$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x),$$

we denote by $E_n^{(2)}(f)$ the best approximation to $f(x)$ in the metric of L^2 by means of polynomials $\psi_0(x), \ldots, \psi_{n-1}(x)$. Then we have

$$E_n^{(2)}(f)=\{\sum_{k=n}^{\infty}|a_k|^2\}^{1/2}.$$

Ul'yanov [11] and Okuyama [7] proved the following theorem on the absolute summability of orthogonal series.

Theorem A. (i) *If the series* $\sum_{n=n_0}^{\infty}|a_n|^2 n\log n(\log\log n)^{1+\varepsilon}$ *converges for some* $\varepsilon>0$, *then the series* $\sum_{n=0}^{\infty}|a_n|$ *converges.*

(ii) *If the series* $\sum_{n=n_0}^{\infty}|a_n|^2 n(\log n)^{-1}(\log\log n)^{1+\varepsilon}$ *converges for some* $\varepsilon>0$, *then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|N, 1/n+1|$ *almost everywhere.*

(iii) *If* $0\leqq\alpha<1/2$ *and the series* $\sum_{n=n_0}^{\infty}|a_n|^2 n^{1-2\alpha}\log n(\log\log n)^{1+\varepsilon}$ *converges for some* $\varepsilon>0$, *then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|C, \alpha|$ *almost everywhere.*

(iv) *If the series* $\sum_{n=n_0}^{\infty}|a_n|^2(\log n)^2(\log\log n)^{1+\varepsilon}$ *converges for some* $\varepsilon>0$, *then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|C, 1/2|$ *almost everywhere.*

(v) *If* $1/2<\alpha\leqq 1$ *and the series* $\sum_{n=n_0}^{\infty}|a_n|^2\log n(\log\log n)^{1+\varepsilon}$ *converges for some* $\varepsilon>0$, *then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|C, \alpha|$ *almost everywhere.*

The case (ii) is due to Okuyama [7], and the cases (iii), (iv) and (v) are due to Ul'yanov [11]. Also, see Okuyama [8].

Leindler [5] proved the equivalent theorem.

Theorem B. *Let* $\{\lambda_n\}$ *be a positive sequence and* $\Lambda_n=\sum_{k=1}^{n}\lambda_k^{-1}$. *Then the convergence of the series* $\sum_{n=1}^{\infty}\lambda_n^{-1}E_n^{(2)}(f)$ *is equivalent to the fact that there exists a positive monotone nondecreasing sequence* $\{\mu_n\}$ *such that the series*

$$\sum_{n=1}^{\infty}|a_n|^2\mu_n$$

and

$$\sum_{n=1}^{\infty}\Lambda_n\lambda_n^{-1}\mu_n^{-1}$$

converge.

Applying Theorem B, we can obtain the following theorem form Theorem A.

Theorem 1. (i) *If the series* $\sum_{n=1}^{\infty}n^{-1/2}E_n^{(2)}(f)$ *converges, then the series* $\sum_{n=0}^{\infty}|a_n|$ *converges.*

(ii) *If the series* $\sum_{n=2}^{\infty}n^{-1/2}(\log n)^{-1}E_n^{(2)}(f)$ *converges, then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|N, 1/n+1|$ *almost everywhere.*

(iii) *If* $0\leqq\alpha<1/2$ *and the series* $\sum_{n=1}^{\infty}n^{-\alpha-1/2}E_n^{(2)}(f)$ *converges, then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|C, \alpha|$ *almost everywhere.*

(iv) *If the series* $\sum_{n=1}^{\infty}n^{-1}E_n^{(2)}(f)$ *converges, then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|C, 1/2|$ *almost everywhere.*

(v) *If* $1/2<\alpha\leqq 1$ *and the series* $\sum_{n=2}^{\infty}n^{-1}(\log n)^{-1/2}E_n^{(2)}(f)$ *converges, then the series* $\sum_{n=0}^{\infty}a_n\phi_n(x)$ *is summable* $|C, \alpha|$ *almost everywhere.*

The case (i) is due to Stechkin [9], and the cases (iii), (iv) and (v) are due to Leindler [5].

Proof of Theorem 1. We treat only the case (ii), because the other cases can be shown similarly.

For this purpose, we put $\lambda_n = n^{1/2}\log n$ and $\mu_n = n(\log n)^{-1}(\log\log n)^{1+\varepsilon}$. Then we have

$$\Lambda_n = \sum_{k=2}^{n} \lambda_k^{-1} = \sum_{k=2}^{n} k^{-1/2}(\log k)^{-1} \leq A n^{1/2}(\log n)^{-1}$$

and

$$\sum_{n=2}^{\infty} \Lambda_n \lambda_n^{-1} \mu_n^{-1} \leq A \sum_{n=2}^{\infty} n^{-1}(\log n)^{-1}(\log\log n)^{-1-\varepsilon} < \infty.$$

Thus we can establish the case (ii) by Theorem A (ii) and Theorem B.

3. Equivalence relations

Let $\{\gamma_n\}$ be a sequence of non-negative numbers. Then

Theorem 2. (i) *The convergence of two series*

$$\sum_{n=2}^{\infty} n^{-1/2}(\log n)^{-1}\{\sum_{k=n}^{\infty} \gamma_k^2\}^{1/2} \text{ and } \sum_{n=2}^{\infty} n^{-3/2}(\log n)^{-1}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}$$

is mutually equivalent.

(ii) *For* $0 \leqq \alpha < 1/2$, *the convergence of two series*

$$\sum_{n=1}^{\infty} n^{-1/2-\alpha}\{\sum_{k=n}^{\infty} \gamma_k^2\}^{1/2} \textit{ and } \sum_{n=1}^{\infty} n^{-3/2-\alpha}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}$$

is mutually equivalent.

Proof. (i) The equivalence between

$$\sum_{n=2}^{\infty} n^{-1/2}(\log n)^{-1}\{\sum_{k=n}^{\infty} \gamma_k^2\}^{1/2} < \infty$$

and

$$\sum_{n=1}^{\infty} n^{-1} 2^{n/2}\{\sum_{k=2^n}^{\infty} \gamma_k^2\}^{1/2} < \infty$$

is nothing but Cauchy's condensation theorem. On the other hand,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^{-3/2}(\log n)^{-1}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2} \\
&= \sum_{j=1}^{\infty} \sum_{n=2^{j-1}+1}^{2^j} n^{-3/2}(\log n)^{-1}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2} \\
&\leq A\sum_{j=1}^{\infty} j^{-1}2^{-3j/2} \sum_{n=2^{j-1}+1}^{2^j} \{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2} \\
&\leq A\sum_{j=1}^{\infty} j^{-1}2^{-j/2}\{\sum_{k=1}^{2^j} k^2\gamma_k^2\}^{1/2}
\end{aligned}$$

and

$$\sum_{n=2}^{\infty} n^{-3/2}(\log\ n)^{-1}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}$$

$$=\sum_{j=1}^{\infty}\sum_{n=2^{j-1}+1}^{2^j} n^{-3/2}(\log\ n)^{-1}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}$$

$$\geq A\sum_{j=1}^{\infty} j^{-1}2^{-3j/2}\sum_{n=2^{j-1}+1}^{2^j}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}$$

$$\geq A\sum_{j=1}^{\infty} j^{-1}2^{-j/2}\{\sum_{k=1}^{2^{j-1}} k^2\gamma_k^2\}^{1/2}.$$

Therefore it is sufficient to prove that the convergence of two series $\sum_{n=1}^{\infty} n^{-1}2^{n/2}\{\sum_{k=2^n}^{\infty}\gamma_k^2\}^{1/2}$ and $\sum_{n=1}^{\infty} n^{-1}2^{-n/2}\{\sum_{k=1}^{2^n} k^2\gamma_k^2\}^{1/2}$ is mutually equivalent.

By Jensen's inequality, we have

$$\sum_{n=1}^{\infty} n^{-1}2^{-n/2}\{\sum_{k=1}^{2^n} k^2\gamma_k^2\}^{1/2}$$

$$\leq \sum_{n=1}^{\infty} n^{-1}2^{-n/2}\{\sum_{j=0}^{n-1}\sum_{k=2^j}^{2^{j+1}} k^2\gamma_k^2\}^{1/2}$$

$$\leq A\sum_{n=1}^{\infty} n^{-1}2^{-n/2}(\sum_{j=0}^{n-1} 2^j\{\sum_{k=2^j}^{2^{j+1}}\gamma_k^2\}^{1/2})$$

$$= A\sum_{j=0}^{\infty} 2^j\{\sum_{k=2^j}^{2^{j+1}}\gamma_k^2\}^{1/2}\sum_{n=j+1}^{\infty} n^{-1}2^{-n/2}$$

$$\leq A\sum_{j=1}^{\infty} j^{-1}2^{j/2}\{\sum_{k=2^j}^{2^{j+1}}\gamma_k^2\}^{1/2}$$

$$= A\sum_{j=1}^{\infty} j^{-1}2^{j/2}\{\sum_{k=2^j}^{\infty}\gamma_k^2\}^{1/2}.$$

Concerning the converse part, we proceed the with the same method. Then we have by Jensen's inequality

$$\sum_{n=1}^{\infty} n^{-1}2^{n/2}\{\sum_{k=2^n}^{\infty}\gamma_k^2\}^{1/2}$$

$$= \sum_{n=1}^{\infty} n^{-1}2^{n/2}\{\sum_{j=n+1}^{\infty}\sum_{k=2^{j-1}}^{2^j-1}\gamma_k^2\}^{1/2}$$

$$\leq A\sum_{n=1}^{\infty} n^{-1}2^{n/2}\{\sum_{j=n+1}^{\infty} 2^{-2j}\sum_{k=2^{j-1}}^{2^j-1} k^2\gamma_k^2\}^{1/2}$$

$$\leq A\sum_{n=1}^{\infty} n^{-1}2^{n/2}(\sum_{j=n+1}^{\infty} 2^{-j}\{\sum_{k=2^{j-1}}^{2^j-1} k^2\gamma_k^2\}^{1/2})$$

$$\leq A\sum_{j=2}^{\infty} 2^{-j}\{\sum_{k=2^{j-1}}^{2^j-1} k^2\gamma_k^2\}^{1/2}\sum_{n=1}^{j-1} n^{-1}2^{n/2}$$

$$\leq A\sum_{j=1}^{\infty} j^{-1}2^{-j/2}\{\sum_{k=2^{j-1}}^{2^j-1} k^2\gamma_k^2\}^{1/2}$$

$$\leq A\sum_{j=1}^{\infty} j^{-1}2^{-j/2}\{\sum_{k=1}^{2^j} k^2\gamma_k^2\}^{1/2}.$$

Thus the proof of the case (i) is completely proved. The proof of the case (ii) is proved similarly.

4. Contraction theorems

We say, with Beurling [1], that f is a *contraction* of g if $|f(x)-f(y)\leqq|g(x)-g(y)|$.

As an extention of the theorem due to Beurling [1], Boas [2] proved the following theorem.

Theorem C. *If $f(x)$ and $g(x)$ are continuous even function, of period 2π, with Fourier coefficients f_n and g_n, if $f(x)$ is a contraction of $g(x)$, and if $|g_n|\leq\gamma_n$ where*

$$\sum_{n=1}^{\infty} n^{-1/2}\{\sum_{k=n}^{\infty}\gamma_k^2\}^{1/2}<\infty \tag{1}$$

and

$$\sum_{n=1}^{\infty} n^{-3/2}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}<\infty, \tag{2}$$

then $\sum|f_n|<\infty$.

Sunouchi [10] proved that the convergence of two series (1) and (2) is equivalent. Kinukawa [3] extended this theorem and gave its dual theorems.

In this section, we generalize Theorem C in the following form by using the absolute summability in place of the absolute convergence.

Theorem 3. *Let*

$$f(x)\sim\sum f_n e^{inx}$$

and

$$g(x)\sim\sum g_n e^{inx}.$$

Suppose that

$$\int_0^{2\pi}|f(x+t)-f(x)|^2dx\leq\int_0^{2\pi}|g(x+t)-g(x)|^2dx$$

for any t, and suppose that there exists a positive sequence $\{\gamma_n\}$ such that $|g_n|\leq\gamma_n$ and

$$\sum_{n=2}^{\infty} n^{-1/2}(\log\ n)^{-1}\{\sum_{k=n}^{\infty}\gamma_k^2\}^{1/2}<\infty \tag{3}$$

or

$$\sum_{n=2}^{\infty} n^{-3/2}(\log\ n)^{-1}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2}<\infty, \tag{4}$$

then the series $\sum f_n e^{inx}$ is summable $|N, 1/n+1|$ almost everywhere.

Theorem 4. *Let*

$$f(x) \sim \sum f_n e^{inx}$$

and

$$g(x) \sim \sum g_n e^{inx}.$$

Suppose that

$$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leq \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$

for any t, *and suppose that there exists a positive sequence* $\{\gamma_n\}$ *such that* $|g_n| \leqq \gamma_n$ *and*

$$\sum_{n=1}^{\infty} n^{-1/2-\alpha}\{\sum_{k=n}^{\infty} \gamma_k^2\}^{1/2} < \infty \tag{5}$$

or

$$\sum_{n=1}^{\infty} n^{-3/2-\alpha}\{\sum_{k=1}^{n} k^2\gamma_k^2\}^{1/2} < \infty, \tag{6}$$

then the series $\sum f_n e^{inx}$ *is summable* $|C, \alpha|$ *almost everywhere, where* $0 \leqq \alpha < 1/2$.

Theorem 5. *Let*

$$f(x) \sim \sum f_n e^{inx}$$

and

$$g(x) \sim \sum g_n e^{inx}.$$

Suppose that

$$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leq \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$

for any t, *and suppose that there exists a positive sequence* $\{\gamma_n\}$ *such that* $|g_n| \leqq \gamma_n$ *and*

$$\sum_{n=1}^{\infty} n^{-1}\{\sum_{k=n}^{\infty} \gamma_k^2\}^{1/2} < \infty, \tag{7}$$

then the series $\sum f_n e^{inx}$ *is summable* $|C, 1/2|$ *almost everywhere.*

Theorem 6. *Let*

$$f(x) \sim \sum f_n e^{inx}$$

and

$$g(x) \sim \sum g_n e^{inx}.$$

Suppose that

$$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leq \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$

for any t, *and suppose that there exists a positive sequence* $\{\gamma_n\}$ *such that* $|g_n| \leqq \gamma_n$ *and*

$$\sum_{n=2}^{\infty} n^{-1}(\log n)^{-1/2}\{\sum_{k=n}^{\infty} \gamma_k^2\}^{1/2} < \infty, \tag{8}$$

then the series $\sum f_n e^{inx}$ *is summable* $|C, \alpha|$ *almost everywhere,* where $1/2 < \alpha \leqq 1$.

For the proofs of these theorems, we require the structure theorem due to Leindler [4].

Theorem D. *Let* $0 < \beta \leq 2$. *Let* $\lambda(x)(x \geq 1)$ *be a positive monotone function such that*

$$\sum_{k=n}^{\infty} k^{-\beta}\lambda(k)^{-1} \leq An^{-\beta+1}\lambda(n)^{-1}.$$

Then the conditions

$$\int_0^1 t^{-2}\lambda(1/t)^{-1}\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2 dx\}^{\beta/2} dt < \infty$$

and

$$\sum_{n=1}^{\infty} \lambda(n)^{-1}\{E_n^{(2)}(f)\}^{\beta} < \infty$$

are mutually equivalent.

Here we prove only Theorem 3, because the other theorems can be shown similarly.

Proof of Theorem 3. By the hypotheses of Theorem 3 and Theorem D, we obtain

$$\int_0^1 t^{-3/2}(\log 1/t)^{-1}\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2 dx\}^{1/2} dt$$

$$\leq \int_0^1 t^{-3/2}(\log 1/t)^{-1}\{\int_0^{2\pi}[g(x+t)-g(x-t)]^2 dx\}^{1/2} dt < \infty.$$

Thus, by Theorem D, the series $\sum_{n=2}^{\infty} n^{-1/2}(\log n)^{-1}E_n^{(2)}(f)$ converges. Therefore we see from Theorem 1(ii) that the series $\sum f_n e^{inx}$ is summable $|N, 1/n+1|$ almost everywhere.

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