On the Absolute Nörlund Summability of Orthogonal Series

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We investigate the absolute Nörlund summability with index k of orthogonal series, and give a generalization of various known results, e. g., U'yanov [9], Wang [11], Tsuchikura [7] and the author [3] and so on. Further we show that some sufficient conditions for the summability of orthogonal series are the best possible ones.

1. Let \( \Sigma a_n \) be a given infinite series with \( s_n \) as its n-th partial sum. If \( \{p_n\} \) is a sequence of constants, and \( P_n = p_0 + p_1 + \cdots + p_n \) \((n = 0, 1, \cdots)\), then the Nörlund mean \( t_n \) of \( \Sigma a_n \) is defined by

\[
(1.1) \quad t_n = \frac{1}{P_n} \sum_{j=0}^{n} p_{n-j} \frac{a_j}{P_{n-j}} (P_n \neq 0).
\]

For a constant \( k, 1 \leq k \leq 2 \), if the series

\[
(1.2) \quad \sum_{n=1}^{\infty} |P_n| |p_n|^{k-1} |t_n - t_{n-1}|^k
\]

converges, then the series \( \Sigma a_n \) is said to be summable \( |N, p_n|_k \). For the definition of this summability, the reader is referred to Umar and Khan [10]. The case \( k=1 \) is reduced to the absolute Nörlund summability \( |N, p_n|_1 \), and further if

\[
p_n = \Gamma(n+\alpha)/\{\Gamma(\alpha)\Gamma(n+1)\},
\]

we have the absolute Cesàro summability \( |C, \alpha| \).

Let \( \{\phi_n(x)\} \) be an orthonormal system defined in the interval \( (a, b) \). For a function \( f(x) \in L^2(a, b) \) such that \( f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \) we denote by \( E^{(2)}_n(f) \) the best approximation to \( f \) in the metric of \( L^2 \) by means of polynomials of \( \phi_0, \cdots, \phi_{n-1} \). It is well known that

\[
E^{(2)}_n(f) = \left( \sum_{j=0}^{\infty} |a_j|^2 \right)^{1/2}.
\]

We write

\[
W^{(k)}_j = \frac{1}{\sum_{j=0}^{\infty} n^{2k/\alpha} p_n^{2k/\alpha} p_{n-j}^{2k/\alpha}} \left( \frac{P_n}{p_n} \frac{P_{n-j}}{p_{n-j}} \right)^{\alpha/2}.
\]

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In the following, we use the notations:

\[ L_0(t) = 1, \quad L_1(t) = \log t, \quad L_p(t) = L_1(L_{p-1}(t)) = \log \cdots \log t, \quad (p \text{ times}) \]
\[ L_p^{(\varepsilon)}(t) = L_1(t) \cdots L_{p-1}(t)(L_p(t))^{1+\varepsilon} \quad (\varepsilon \geq 0, p = 1, 2, \ldots), \]

where, if the right hand sides are not determined as positive numbers, we replace them by 1's.

\[ \Delta \lambda_n = \lambda_n - \lambda_{n-1} \quad \text{for any sequence} \{ \lambda_n \}. \quad A \text{ is a positive constant not necessarily the same at each occurrence.} \]

2. For the trigonometric series, Singh [5] proved the following theorem, which is an extension of theorems due to Pati [4], Ul'yanov [9] and Wang [11].

Theorem A. Let \{\Omega_n\} and \{\lambda_n\} be two positive sequences such that \{\Omega_n \lambda_n^2\} is a monotonic increasing sequence and that

\[ (2.1) \quad \sum_{n=1}^{\infty} \lambda_n^{\gamma-1} \Omega_n^{-\beta} \]

is convergent. If the series

\[ (2.2) \quad \sum_{n=0}^{\infty} |a_n|^2 \Omega_n \]

converges, then the trigonometric series \( \sum_{n=0}^{\infty} \lambda_n a_n \cos (nx + \alpha_n) \) is summable \( |C, \alpha| (\alpha > 1/2) \) almost everywhere.

One of the authors [3] established the following theorem.

Theorem B. Let \( \{\Omega_n\} \) be a positive sequence such that \( \{\Omega_n/n\} \) is a non-increasing sequence and the series \( \Sigma n^{-1} \Omega_n^{-1} \) converges. Let \( \{p_n\} \) be non-negative and non-increasing. If the series \( \Sigma |a_n|^2 \Omega_n W_n \) converges, then the orthogonal series \( \Sigma a_n \phi_n(x) \) is summable \( |N, p_n| \) almost everywhere, where \( W_n = W_n^{(k)} \) is defined by (1.3).

In this paper, we shall first generalize these two theorems.

Theorem 1. Let \( 1 \leq k \leq 2 \) and \{\lambda_n\} be a positive sequence. If \{p_n\} is a positive sequence and the series

\[ (2.3) \quad \sum_{n=1}^{\infty} \frac{\lambda_n}{P_n P_n^{k-1}} \left\{ \sum_{j=1}^{n} \frac{P_n}{P_{n-j}} \left( \frac{P_n}{P_{n-j}} \right)^2 \lambda_j^2 |a_j|^2 \right\}^{k/2} \]

converges, then the orthogonal series

\[ (2.4) \quad \Sigma \lambda_n a_n \phi_n(x) \]

is summable \( |N, p_n|^k \) almost everywhere.

This theorem is also a generalization of theorems due to Tsuchikura [7, 8].
Proof of Theorem 1. Let $t_n(x)$ be the $n$-th Nörlund mean of the series (2.4). Then, as Banerji shown,

$$A_t(x) = t_n(x) - t_{n-1}(x)$$

Using the Hölder inequality and the orthogonality,

$$\int_a^b |A_t(x)|^k dx \leq A\left( \int_a^b |A_t(x)|^2 dx \right)^{k/2}$$

and then,

$$\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{k-1} \int_a^b |A_t(x)|^k dx \leq A \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{k-1} \left( \frac{p_n}{p_n} \right)^{k} \left( \sum_{j=1}^{n} P_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 j^2 a_j^2 \right)^{k/2}$$

which is convergent by the assumption and from the Beppo-Lévi lemma we complete the proof.

3. Now we shall show that Theorem 1 includes Theorems A and B.

Lemma 1. Let $w(x)$ be a positive and non-decreasing function of $x$ over the interval $[N, \infty]$. Then the two series $\sum_{n=N}^{\infty} n^{-1} w(n^{-1})$ and $\sum_{n=N}^{\infty} n^{-1} w(n^{-1} n^{-1})$ converge or diverge simultaneously.

This lemma is due to Ul'yanov [9].

For $k=1$ and $p_n = \Gamma(n+1)/\Gamma(n+1)$ we have $n^{-1} w(n^{-1}) \leq A \sum_{n=1}^{\infty} \frac{1}{n^{a+1}} \left( \sum_{j=1}^{n} (n-j+1)^{2a-2} j^2 |a_j|^2 \right)^{1/2}$

say. Under the assumption of Theorem A we have $\lambda_j^2 / \Omega_j \leq \lambda_i^2 / \Omega_i$ and by (2.2) we get

$$A \sum_{n=1}^{\infty} \frac{1}{n^{a+1}} \left( \sum_{j=1}^{n} (n-j+1)^{2a-2} j^2 |a_j|^2 \right)^{1/2}$$
Similarly we have

\[
T \leq A \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \left( \sum_{j=1}^{n^{1/2}} (n-j+1)^{2\alpha-2} j^2 \Omega_j^{-1} |a_j|^2 \Omega_j^{1/2} \right)
\]

By the Schwarz inequality and Lemma 1, we get

\[
T \leq A \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \left( \sum_{j=1}^{n^{1/2}} (n-j+1)^{2\alpha-2} j^2 \Omega_j^{-1} |a_j|^2 \Omega_j^{1/2} \right)
\]

Hence from the assumption of Theorem A we can apply Theorem 1 and we see that Theorem 1 contains Theorem A.

Theorem B is also deduced from Theorem 1 setting \( \lambda_n=1 \) and \( k=1 \).

4. Applying Theorem 1 we shall show some generalization of known theorems.

The following Lemma will be proved by easy calculations.

**Lemma 2** For \( p_n=\Gamma(n+\alpha)/\{\Gamma(\alpha)\Gamma(n+1)\} \) \((\alpha>0)\) the sum \( W_j^{(k)} \) is, as \( j \to \infty \), (i) \( O(1) \) if \( 12\alpha>1/2 \), (ii) \( O(L_2(j)) \) if \( \alpha=1/2 \) and (iii) \( O(j^{-2\alpha}) \) if \( 0<\alpha<1/2 \). (iv) For \( p_n=L_3(n+2)^{1/2} [L_2(n+2)]^{1/2} \) \((r>-1)\),

\[
W_j^{(k)} = O(jL_2(j)^{-2r-2/k} L_2^{-1/2}(j)^{2-2/k}),
\]

and (v) for \( p_n=(n+2)^{-1} L_2^{1/2} (n+2)^{-1} \),

\[
W_j^{(k)} = O(jL_2^{1/2}(j)^{-2/k} L_2^{-1/2}(j)^{2-2/k})
\]
Theorem 2. Let $1 \leq k \leq 2$ and $\{\Omega_n\}$ be a positive sequence such that $\{\Omega_n/n\}$ is non-increasing and the series $\sum n^{-1} \Omega_n^{-1}$ converges. If $\{p_n\}$ is a positive non-increasing sequence and the series $\sum |a_n|^2 W_n^{(k)} \Omega_n^{j/k-1}$ converges, then the orthogonal series $\sum a_n \phi_n(x)$ is summable $|N, p_n|_k$ almost everywhere.

Proof. To apply Theorem 1, we shall make an estimation of the sum (2.3) with $\lambda_j=1$ ($j=1, 2, \cdots$). By the Hölder inequality,

$$S \leq \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-j}} \left( \sum_{j=1}^{\infty} p_n^{\alpha_j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 |a_j|^2 \right)^{k/2}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n \Omega_n} \left( \sum_{n=1}^{\infty} \frac{\Omega_n^{j/k-1} p_n^{\alpha_j} \Omega_n^{j/k-1} p_n^{\alpha_j}}{P_n^{j/k} P_{n-j}^{j/k}} \sum_{j=1}^{\infty} p_n^{\alpha_j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 |a_j|^2 \right)^{k/2}$$

$$\leq A \sum_{n=1}^{\infty} |a_j|^2 \frac{1}{n} \left( \sum_{n=1}^{\infty} \frac{\Omega_n^{j/k-1} p_n^{\alpha_j} \Omega_n^{j/k-1} p_n^{\alpha_j}}{P_n^{j/k} P_{n-j}^{j/k}} \sum_{j=1}^{\infty} p_n^{\alpha_j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 |a_j|^2 \right)^{k/2},$$

since $\Omega_n^{j/k-1}/j=(\Omega_n/j)^{j/k-1}$ is non-increasing, and then

$$S \leq A \left( \sum_{j=1}^{\infty} |a_j|^2 W_j^{(k)} \Omega_j^{j/k-1} \right)^{k/2}.$$

which is finite by the assumption, and we complete the proof.

For each sequence $\{p_n\}$ treated in Lemma 2, the above Theorem 2 implies the following result.

Corollary 1. Let $1 \leq k \leq 2$ and $p$ be a positive integer.

(i) If the series

$$(4.1) \quad \sum |a_n|^2 L_p^{(\epsilon)}(n)^{\alpha/k-1}$$

converges for some $\epsilon > 0$, then the series $\sum a_n \phi_n(x)$ is summable $|C, \alpha|_k$ almost everywhere for any $1 \geq \alpha > 1/2$.

(ii) If the series

$$(4.2) \quad \sum |a_n|^2 L_n(n)L_p^{(\epsilon)}(n)^{\alpha/k-1}$$

converges for some $\epsilon > 0$, then $\sum a_n \phi_n(x)$ is summable $|C, 1/2|_k$ almost everywhere.

(iii) If the series

$$(4.3) \quad \sum |a_n|^2 n^{-1-\alpha} L_p^{(\epsilon)}(n)^{\alpha/k-1}$$
converges for some $\epsilon > 0$, then $\Sigma a_n \phi_n(x)$ is summable $|C, \alpha|_k$ almost everywhere for $0 < \alpha < 1/2$.

Further let $q$ be a non-negative integer and $s$ a positive integer.

(iv) Let $r > -1$ be a real number. If the series

\[ \Sigma |a_n|^3 n L_{q+1} (n)^{-2r+2/k} L_{s-1}^{(o)}(n) L_{s+q}^{(e)}(n)^{2/k-1} \]

converges for some $\epsilon > 0$, then $\Sigma a_n \phi_n(x)$ is summable $|N, p_n|_k$ almost everywhere for $p_n = L_s(n+2)^r \{ (n+2) L_{s-1}^{(o)}(n+2) \}^{-1}$.

(v) If the series

\[ \Sigma |a_n|^3 n L_{s+1} (n)^{-2/k} L_{s-1}^{(o)}(n) L_{s+q+1}^{(e)}(n)^{2/k-1} \]

converges for some $\epsilon > 0$, then $\Sigma a_n \phi_n(x)$ is summable $|N, p_n|_k$ almost everywhere for $p_n = 1/\{ (n+2) L_{s-1}^{(o)}(n+2) \}$.

For $k=1$, the cases $b=1$ and $b=2$ in this Corollary are the results obtained by Wang [11] and Ul'yanov [9] respectively; and the case $s=q=k=1$ and $r=0$ in (iv) is due to Okuyama [3].

Now, if $\{ \Omega_n \}$ is a positive non-decreasing sequence with $\Omega_0 = 0$, then by using the best approximation, we see that

\[ \Sigma |a_n|^3 \Omega_n \]

therefore, estimating the corresponding $\Delta \Omega_j$ for the series (4. 1)~(4. 5), we have easily the following:

Corollary 2. In the Corollary 1 the series (4. 1)~(4. 5) can be replaced by the following series (4. 7)~(4. 11) respectively:

\[ \Sigma n^{-1} L_1 (n)^{-1} L_{s-1}^{(o)}(n) L_{s+q}^{(e)}(n)^{2/k-1} \{ E_n^{(2)}(f) \}^2, \]

\[ \Sigma n^{-1} L_{s-1}^{(o)}(n) L_{s+q}^{(e)}(n)^{2/k-1} \{ E_n^{(2)}(f) \}^2, \]

\[ \Sigma n^{-2} L_{s}^{(e)}(n)^{2/k-1} \{ E_n^{(2)}(f) \}^2, \]

\[ \Sigma L_s(n)^{-2r+2/k} L_{s+1}^{(o)}(n) L_{s+q+1}^{(e)}(n)^{2/k-1} \{ E_n^{(2)}(f) \}^2, \]

\[ \Sigma L_{s+1} (n)^{-2/k} L_{s}^{(o)}(n) L_{s+q+1}^{(e)}(n)^{2/k-1} \{ E_n^{(2)}(f) \}^2. \]
For the trigonometric orthogonal system we shall make a remark. Let \( f(x) \in L^2(0, 2\pi) \), and denote by \( \Omega(\delta, f) \) one of the following integral moduli:

\[
\omega^{(2)}(\delta, f) = \sup_{0 \leq t \leq \delta} \left\{ \int_0^{2\pi} \left[ f(x + t) - f(x - t) \right]^2 dx \right\}^{1/2},
\]

\[
w^{(2)}(\delta, f) = \sup_{0 \leq t \leq \delta} \left\{ \int_0^{2\pi} \left[ f(x + 2t) + f(x - 2t) - 2f(x) \right]^2 dx \right\}^{1/2},
\]

\[
W^{(2)}(\delta, f) = \left\{ \frac{1}{\delta} \int_0^{2\pi} \left[ f(x + t) - f(x - t) \right]^2 dx \right\}^{1/2},
\]

\[
W^{(2)}(\delta, f) = \left\{ \frac{1}{\delta} \int_0^{2\pi} \left[ f(x + 2t) + f(x - 2t) - 2f(x) \right]^2 dx \right\}^{1/2}.
\]

Let \( \{\lambda_n\} \) be a positive monotone sequence such that

\[
\sum_{j=n}^{\infty} j^{-s} \lambda_j^{-1} \leq A n^{-s} \lambda_n^{-1}.
\]

Then, Leindler [2] proved that the conditions \( \Sigma \lambda_n^{-1} \Omega(1/n, f)^2 < \infty \) and \( \Sigma \lambda_n^{-1} [E_n^{(2)}(f)]^2 < \infty \) are equivalent. So that we get easily the following result.

**Corollary 3.** For the case of trigonometric series, sufficient conditions for the conclusions (i)-(v) in Corollary 1 are, for some \( \varepsilon > 0 \),

(i) \( \Omega(\delta, f) = O(L_s(1/\delta)^{1/2} L_p^{(2)}(1/\delta)^{-1/2}) \),

(ii) \( \Omega(\delta, f) = O(L_p^{(2)}(1/\delta)^{-1/2}) \),

(iii) \( \Omega(\delta, f) = O(\delta^{1/2 - s} L_p^{(2)}(1/\delta)^{-1/2}) \),

(iv) \( \Omega(\delta, f) = O(\delta^{1/2} L_s(1/\delta)^{r + 1/2} L_s^{(2)}(1/\delta)^{1/2 - 1} L_s^{(4)}(1/\delta)^{-1/2}) \),

(v) \( \Omega(\delta, f) = O(\delta^{1/2} L_s(1/\delta)^{1/2} L_s^{(2)}(1/\delta)^{1/2 - 1} L_s^{(4)}(1/\delta)^{-1/2}) \)

respectively.

The case \( k=1 \) and \( p=2 \) in the results (i)-(iii) are due to Ul'yanov [9], and the case \( s=p=k=1 \) and \( r=0 \) in (iv) is due to Okuyama [3].

**5.** Let \( \{r_n(t)\} \) be the Rademacher system. For the series

\[
\sum \lambda_n a_n r_n(t)
\]

instead of general orthogonal series (2. 4), we shall establish an inverse of Theorem 1.

**Theorem 3.** Let \( k \geq 1 \) and let \( \{p_n\} \) be a positive sequence such that for any fixed integer \( j_0 > 0 \),

\( p_n-i (P_n/p_n-P_{n-i}/p_{n-i}) = O(1) \) for \( n \geq j_0 \geq j \geq 1 \). Suppose that the
set of points \( t \) for which the series (5. 1) is summable \( |N, p_n|^{k} \) is of positive measure, then the series (2. 3) converges.

Proof. We may suppose that the \( |N, p_n|^{k} \) sum of the series (5. 1) is uniformly bounded for all \( t \in E \subset (0, 1) \) where \( m(E) > 0 \). Then,

\[
\int \sum_{n=1}^{\infty} \frac{p_n}{p_n p_n^{k}} \sum_{j=1}^{n} p_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^k \lambda_j a_j r_j(t)^k \ dt < \infty.
\]

Let \( N \) be a positive integer and replace \( a_1, a_2, \ldots, a_{N-1} \) in the series (5. 1) by zeros. This replacement has no influence on the summability, since

\[
\sum_{n=1}^{\infty} \frac{p_n}{p_n p_n^{k}} \sum_{j=1}^{n-1} p_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^k \lambda_j a_j r_j(t)^k \ dt < \infty
\]

which is finite if \( \Sigma p_n < \infty \), and by Pringsheim's theorem \( \Sigma p_n P_n^{-1} p_n^{-k} < \infty \) for any \( k > 0 \), if \( \Sigma p_n = \infty \). Therefore we may suppose that

\[
\int \sum_{n=1}^{\infty} \frac{p_n}{p_n p_n^{k}} \sum_{j=1}^{N-1} p_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^k \lambda_j a_j r_j(t)^k \ dt < \infty
\]

(5. 2)

where \( N = N(E) \) is determined by the well known Khinchin inequality:

\[
\int \sum_{j=1}^{N} p_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 \lambda_j a_j^2 \ dt < \infty
\]

(5. 3)

From (5. 2) and (5. 3) we can conclude the convergence of the series (2. 3), since repeating the similar argument as above, the integer \( N \) may be replaced by 1.

6. We shall show that the positive number \( \epsilon \) in \( L_{p}^{(c)} (t) \) is indispensable in Corollaries 1, 2 and 3 for the case of trigonometric series.

Lemma 3. Let \( 1 \leq k \leq 2 \) and \( \{p_n\} \) be the same sequence as in Theorem 3. Put \( A_j(x) = \rho_j \cos (jx + \theta) \). If the series

\[
\sum_{n=1}^{\infty} \frac{p_n}{p_n p_n^{k}} \left( \sum_{j=1}^{n} p_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 A_j^2 (x) \right)^{k/2}
\]

(6. 1)

converges for every \( x \) in a set of positive measure, then the series

\[
\sum_{n=1}^{\infty} \frac{p_n}{p_n p_n^{k}} \left( \sum_{j=1}^{n} p_{n-j} \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 \rho_j^2 \right)^{k/2}
\]

(6. 2)
On the Absolute Nörlund Summability of Orthogonal Series

converges. Conversely, the convergence of (6. 2) implies that of (6. 1) for every \(x\).

Proof. We may suppose that the sum (6. 1) is uniformly bounded by a constant \(A\) in a set \(E, m(E) > 0\) and denote, for the simplicity, \(\alpha_n = \frac{p_n}{p_n^{k-1}} \beta_n, \beta_n, j = \frac{p_n-j}{p_n} \beta_n-j\). Then we have

\[
I = \sum_{n=1}^{\infty} \alpha_n \left\{ \sum_{j=1}^{n} \beta_n,j \beta_j^2 \cos^2 (jx + \theta_j) \right\} \geq A m(E).
\]

Using the Minkowski inequality, we get

\[
I \geq \sum_{n=1}^{\infty} \alpha_n \left\{ \sum_{j=1}^{n} \beta_n,j \rho_j \cos (jx + \theta_j) \right\}^2 \geq A m(E).
\]

By the Riemann–Lebesgue theorem, we have

\[
\int_E \cos (jx + \theta_j) dx \geq \int_E \cos (jx + \theta_j) dx = \frac{1}{2} m(E) + \frac{1}{2} \int_E \cos 2(jx + \theta_j) dx.
\]

for sufficiently large \(j\), say \(j \geq N\). Therefore, by (6. 4) and (6. 5)

\[
I \geq A \sum_{n=1}^{\infty} \alpha_n \left( \sum_{j=N}^{n} \beta_n,j \rho_j^2 \right)^{k/2} \geq A \sum_{n=1}^{\infty} \alpha_n \left( \sum_{j=N}^{n} \beta_n,j \rho_j^2 \right)^{k/2}.
\]

By the same reason as in Theorem 3, we replace \(N\) by 1 and we conclude the convergence of (6. 2). The converse is obvious.

Theorem 4. Let \(2 \geq k \geq 1\) and let \(|\beta_n|\) be the same as in Theorem 3. If the series (6. 2) converges, then almost all series of

\[
\Sigma^\pm (a_n \cos nx + b_n \sin nx),
\]

where \(A_n(x) = \rho_n \cos (nx + \theta_n) = a_n \cos nx + b_n \sin nx\), are summable \(|N, p_n|_k\) for
almost every $x$, and if (6. 2) diverges, then almost all series of (6. 6) are non-summable $|N, p_n|_k$ for almost every $x$.

Proof. Considering the series $\sum r_n(t) A_n(x)$, the first part is an easy consequence of Theorem 1 putting $\lambda_j a_j = A_j(x)$. The latter part is also a consequence of Theorem 3 and Lemma 3 following the well known Paley–Zygmund argument.

Corollary 4. Let $1 \leq k < 2$. In the assumptions of Corollaries 1, 2 or 3 the positive number $\varepsilon$ in $L_p^{(s)}$ or $L_{p+k}^{(s)}$ is indispensable.

Proof. We treat the case (iv) of Corollary 1, because the other cases can be shown similarly. It is sufficient to show the existence of a Rademacher–trigonometric series $\sum a_n r_n(t) \cos nx$ which is non-summable $|N, p_n|_k$ for almost every $(t, x)$ in $(0, 1) \times (0, 2\pi)$ and the series (4. 4) is convergent for $\varepsilon = 0$. For this purpose we put

$$a_n = n^{-1} L_s (n)^{r+1/k} L_{s+1}^{(0)} (n)^{1/k-1} L_{s+q+1}^{(0)} (n)^{-1/k},$$

then as we see easily the series (4. 4) with $\varepsilon = 0$ is

$$\sum n^{-1} L_{s+q}^{(0)} (n)^{-1} L_{s+q+1}^{(0)} (n)^{-1/k}$$

which is convergent for $1 \leq k < 2$. On the other hand, since $p_n = n^{-1} L_{s+1}^{(0)} (n)^{-1} L_s (n)^r$, we see $P_n \sim L_s (n)^{r+1}$ and $P_n/p_n \sim n L_{s+q}^{(0)}(n)$. Hence it is easy to see that the series (6. 2) is not smaller than

$$A \sum_{n=1}^\infty \frac{p_n}{P_n} \left( \sum_{j=\lfloor n/2 \rfloor}^{n-1} p_n^{-j} \left( \frac{P_n}{p_n} \right)^2 a_j^{r+1/k} \right) \geq A \sum_{n=1}^\infty n^{-1} L_{s+q+1}^{(0)} (n)^{-1}$$

which is divergent.

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On the Absolute Nörlund Summability of Orthogonal Series

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