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On the Absolute Nörlund Summability of Orthogonal Series

Yasuo OKUYAMA*

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We investigate the absolute Nörlund summability with index k of orthogonal series, and give a generalization of various known results, e. g., U'yanov [9], Wang [11], Tsuchikura [7] and the author [3] and so on. Further we show that some sufficient conditions for the summability of orthogonal series are the best possible ones.

1. Let Σa_n be a given infinite series with s_n as its n-th partial sum. If $\{p_n\}$ is a sequence of constants, and $P_n=p_0+p_1+\cdots+p_n$ $(n=0,1,\cdots)$, then the Nörlund mean t_n of Σa_n is defined by

$$(1.\ 1)\qquad t_n=\frac{1}{P_n}\sum_{j=0}^{n}p_{n-j}\,s_j=\frac{1}{P_n}\sum_{j=0}^{n}P_{n-j}\,a_j\ (P_n\neq 0).$$

For a constant k, $1\leqq k\leqq 2$, if the series

$$(1.\ 2)\qquad \sum_{n=1}^{\infty}|P_n/p_n|^{k-1}|t_n-t_{n-1}|^k$$

converges, then the series Σa_n is said to be summable $|N, p_n|_k$. For the definition of this summability, the reader is referred to Umar and Khan [10]. The case $k=1$ is reduced to the absolute Nörlund summability $|N, p_n|$, and further if $p_n=\Gamma(n+\alpha)/\{\Gamma(\alpha)\Gamma(n+1)\}$, we have the absolute Cesàro summability $|C, \alpha|$.

Let $\{\phi_n(x)\}$ be an orthonormal system defined in the interval (a, b). For a function $f(x)\in L^2(\mathrm{a, b})$ such that $f(x)\sim\sum_{n=0}^{\infty}a_n\phi_n(\mathrm{x})$ we denote by $E_n^{(2)}(f)$ the best approximation to f in the metric of L^2 by means of polynomials of $\phi_0,\cdots,\phi_{n-1}$. It is well known that $E_n^{(2)}(f)=(\sum_{j=n}^{\infty}|a_j|^2)^{1/2}$. We write

$$(1.\ 3)\qquad W_j^{(k)}=\frac{1}{j}\sum_{n=j}^{\infty}\frac{n^{2/k}\,p_n^{2/k}p_{n-j}^2}{P_n^{2+2/k}}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2.$$

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* Associate Professor, Institute of Mathematics

In the following, we use the notations :

$$L_0(t)=1,\ \ L_1(t)=\log\ t,\ \ L_p(t)=L_1(L_{p-1}(t))=\log\cdots\log\ t,\ \ (p\ \text{times})$$
$$L_p^{(\varepsilon)}(t)=L_1(t)\cdots L_{p-1}(t)(L_p(t))^{1+\varepsilon}\ \ (\varepsilon\geqq 0, p=1,\ 2,\cdots),$$

where, if the right hand sides are not determined as positive numbers, we replace them by 1's.

$\Delta\lambda_n=\lambda_n-\lambda_{n-1}$ for any sequence $\{\lambda_n\}$. A is a positive constant not necessarily the same at each occurrence.

2. For the trigonometric series, Singh [5] proved the following theorem, which is an extension of theorems due to Pati [4], Ul'yanov [9] and Wang [11].

Theorem A. *Let* $\{\Omega_n\}$ *and* $\{\lambda_n\}$ *be two positive sequences such that* $\{\Omega_n\lambda_n^{-2}\}$ *is a monotonic increasing sequence and that*

$$\sum_{n=1}^{\infty}\lambda_n^2 n^{-1}\Omega_n^{-1} \tag{2. 1}$$

is convergent. If the series

$$\sum_{n=0}^{\infty}|a_n|^2\Omega_n \tag{2. 2}$$

converges, then the trigonometric series $\sum_{n=0}^{\infty}\lambda_n a_n\cos\,(nx+\alpha_n)$ *is summable* $|C,\ \alpha|$ $(\alpha>1/2)$ *almost everywhere.*

One of the authors [3] established the following theorem.

Theorem B. *Let* $\{\Omega_n\}$ *be a positive sequence such that* $\{\Omega_n/n\}$ *is a non-increasing sequence and the series* $\Sigma n^{-1}\Omega_n^{-1}$ *converges. Let* $\{p_n\}$ *be non-negative and non-increasing. If the series* $\Sigma|a_n|^2\Omega_n W_n$ *converges, then the orthogonal series* $\Sigma a_n\,\phi_n(x)$ *is summable* $|N,\ p_n|$ *almost everywhere, where* $W_n=W_n^{(1)}$ *is defined by* (1. 3).

In this paper, we shall first generalize these two theorems.

Theorem 1. *Let* $1\leqq k\leqq 2$ *and* $\{\lambda_n\}$ *be a positive sequence. If* $\{p_n\}$ *is a positive sequence and the series*

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\{\sum_{j=1}^{n}p_{n-j}^2(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}})^2\ \lambda_j^2\,|a_j|^2\ \}^{k/2} \tag{2. 3}$$

converges, then the orthogonal series

$$\Sigma\lambda_n a_n\phi_n(x) \tag{2. 4}$$

is summable $|N,\ p_n|_k$ *almost everywhere.*

This theorem is also a generalization of theorems due to Tsuchikura [7, 8] and Banerji [1].

Proof of Theorem 1. Let $t_n(x)$ be the n-th Nörlund mean of the series (2. 4). Then, as Banerji shown,

$$\Delta t_n(x)=t_n(x)-t_{n-1}(x)$$
$$=\frac{p_n}{P_nP_{n-1}}\sum_{j=1}^{n}p_{n-j}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)\lambda_j a_j\phi_j(x).$$

Using the Hölder inequality and the orthogonality,

$$\int_a^b|\Delta t_n(x)|^k dx\leqq A\{\int_a^b|\Delta t_n(x)|^2dx\}^{k/2}$$
$$=A\left(\frac{p_n}{P_nP_{n-1}}\right)^k\{\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2\lambda_j^2a_j^2\}^{k/2},$$

and then,

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\int_a^b|\Delta t_n(x)|^k dx$$
$$\leqq A\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\{\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2\lambda_j^2a_j^2\}^{k/2}$$

which is convergent by the assumption and from the Beppo-Lévi lemma we complete the proof.

3. Now we shall show that Theorem 1 includes Theorems A and B.

Lemma 1. *Let $w(x)$ be a positive and non-decreasing function of x over the interval $[N,\infty]$. Then the two series $\sum_{n=N}^{\infty}n^{-1}w(n)^{-1}$ and $\sum_{n=N^2}^{\infty}n^{-1}w(n^{1/2})^{-1}$ converge or diverge simultaneously.*

This lemma is due to Ul'yanov [9].

For $k=1$ and $p_n=\Gamma(n+\alpha)/\{\Gamma(\alpha)\Gamma(n+1)\}\cong n^{\alpha-1}/\Gamma(\alpha)$ the sum (2. 3) is not greater than

$$A\sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}\{\sum_{j=1}^{n}(n-j+1)^{2\alpha-2}j^2\lambda_j^2|a_j|^2\}^{1/2}$$
$$\leqq A\sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}\{\sum_{j=1}^{[n^{1/2}]}\cdots\}^{1/2}+A\sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}\{\sum_{j=[n^{1/2}]+1}^{\infty}\cdots\}^{1/2}=S+T,$$

say. Under the assumption of Theorem A we have $\lambda_j^2/\Omega_j\leqq\lambda_1^2/\Omega_1$ and by (2. 2) we get

$$S \leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} n^{\alpha-1}\ n^{1/2} \{ \sum_{j=1}^{[n^{1/2}]} \lambda_j^2 \Omega_j^{-1} |a_j|^2 \Omega_j \}^{1/2}$$

$$\leqq A \sum_{n=1}^{\infty} n^{-3/2} \{ \sum_{j=1}^{[n^{1/2}]} |a_j|^2 \Omega_j \}^{1/2} < \infty.$$

Similarly we have

$$T \leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \{ \sum_{j=[n^{1/2}]}^{n} (n-j+1)^{2\alpha-2} j^2 \lambda_j^2 \Omega_j^{-1} |a_j|^2 \Omega_j \}^{1/2}$$

$$\leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} (\lambda_{[n^{1/2}]} \Omega_{[n^{1/2}]}^{-1/2}) \{ \sum_{j=[n^{1/2}]}^{n} (n-j+1)^{2\alpha-2} j^2 |a_j|^2 \Omega_j \}^{1/2}.$$

By the Schwarz inequality and Lemma 1, we get

$$T \leqq A \{ \sum_{n=1}^{\infty} \lambda_{[n^{1/2}]}^2 n^{-1} \Omega_{[n^{1/2}]}^{-1} \}^{1/2} \{ \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{j=1}^{n} (n-j+1)^{2\alpha-2} j^2 |a_j|^2 \Omega_j \}^{1/2}$$

$$\leqq A \{ \sum_{j=1}^{\infty} j^2 |a_j|^2 \Omega_j \sum_{n=j}^{\infty} (n-j+1)^{2\alpha-2} n^{-2\alpha-1} \}^{1/2}$$

$$= A \{ \sum_{j=1}^{\infty} j^2 |a_j|^2 \Omega_j O(j^{-2}) \}^{1/2}$$

$$\leqq A \{ \sum_{j=1}^{\infty} |a_j|^2 \Omega_j \}^{1/2} < \infty.$$

Hence from the assumption of Theorem A we can apply Theorem 1 and we see that Theorem 1 contains Theorem A.

Theorem B is also deduced from Theorem 1 setting $\lambda_n = 1$ and $k=1$.

4. Applying Theorem 1 we shall show some generalization of known theorems. The following Lemma will be proved by easy calculations.

Lemma 2 *For* $p_n = \Gamma(n+\alpha)/\{\Gamma(\alpha)\Gamma(n+1)\}$ $(\alpha>0)$ *the sum* $W_j^{(k)}$ *is, as* $j \longrightarrow \infty$, *(i)* $O(1)$ *if* $1 \geqq \alpha > 1/2$, *(ii)* $O(L_1(j))$ *if* $\alpha = 1/2$ *and (iii)* $O(j^{1-2\alpha})$ *if* $0<\alpha<1/2$. *(iv) For* $p_n = L_s(n+2)^r / \{(n+2) L_{s-1}^{(0)}(n+2)\}$ $(r>-1)$,

$$W_j^{(k)} = O(j L_s(j)^{-2r-2/k} L_{s-1}^{(0)}(j)^{2-2/k}),$$

and (v) for $p_n = (n+2)^{-1} L_s^{(0)}(n+2)^{-1}$,

$$W_j^{(k)} = O(j L_{s+1}(j)^{-2/k} L_s^{(0)}(j)^{2-2/k})$$

Theorem 2. *Let* $1\leqq k\leqq 2$ *and* $\{\Omega_n\}$ *be a positive sequence such that* $\{\Omega_n/n\}$ *is non-increasing and the series* $\Sigma n^{-1}\Omega_n^{-1}$ *converges. If* $\{p_n\}$ *is a positive non-increasing sequence and the series* $\Sigma|a_n|^2W_n^{(k)}\Omega_n^{2/k-1}$ *converges, then the orthogonal series* $\Sigma a_n\phi_n(x)$ *is summable* $|N,\ p_n|_k$ *almost everywhere.*

Proof. To apply Theorem 1, we shall make an estimation of the sum (2. 3) with $\lambda_j=1$ $(j=1,\ 2,\cdots)$. By the Hölder inequality,

$$S=\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\{\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2|a_j|^2\}^{k/2}$$
$$\leqq\left[\sum_{n=1}^{\infty}\frac{1}{n\Omega_n}\right]^{1-k/2}\left[\sum_{n=1}^{\infty}\frac{n^{2/k-1}\Omega_n^{2/k-1}p_n^{2/k}}{P_{n-1}^{2+2/k}}\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2|a_j|^2\right]^{k/2}$$
$$\leqq A[\sum_{j=1}^{\infty}|a_j|^2\sum_{n=j}^{\infty}\frac{n^{2/k-1}\Omega_n^{2/k-1}p_n^{2/k}p_{n-j}^2}{P_{n-1}^{2+2/k}}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2]^{k/2}$$
$$\leqq A[\sum_{j=1}^{\infty}|a_j|^2\frac{\Omega_j^{2/k-1}}{j}\sum_{n=j}^{\infty}\frac{n^{2/k}p_n^{2/k}p_{n-j}^2}{P_{n-1}^{2+2/k}}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2]^{k/2},$$

since $\Omega_j^{2/k-1}/j=(\Omega_j/j)^{2/k-1}\ j^{2/k-2}$ is non-increasing, and then

$$S\leqq A[\sum_{j=1}^{\infty}|a_j|^2W_j^{(k)}\Omega_j^{2/k-1}]^{k/2}$$

which is finite by the assumption, and we complete the proof.

For each sequence $\{p_n\}$ treated in Lemma 2, the above Theorem 2 implies the following result.

Corollary 1. *Let* $1\leqq k\leqq 2$ *and* p *be a positive integer.*

(i) *If the series*

$$\Sigma|a_n|^2L_p^{(\varepsilon)}(n)^{2/k-1} \tag{4. 1}$$

converges for some $\varepsilon>0$, *then the series* $\Sigma a_n\phi_n(x)$ *is summable* $|C,\ \alpha|_k$ *almost everywhere for any* $1\geqq\alpha>1/2$.

(ii) *If the series*

$$\Sigma|a_n|^2 L_1(n)L_p^{(\varepsilon)}(n)^{2/k-1} \tag{4. 2}$$

converges for some $\varepsilon>0$, *then* $\Sigma a_n\phi_n(x)$ *is summable* $|C,\ 1/2|_k$ *almost everywhere.*

(iii) *If the series*

$$\Sigma|a_n|^2n^{1-2\alpha}L_p^{(\varepsilon)}(n)^{2/k-1} \tag{4. 3}$$

converges for some $\varepsilon>0$, *then* $\Sigma a_n\phi_n(x)$ *is summable* $|C, \alpha|_k$ *almost everywhere for* $0<\alpha<1/2$.

Further let q *be a non-negative integer and* s *a positive integer.*

(iv) *Let* $r>-1$ *be a real number. If the series*

$$\Sigma|a_n|^2 nL_s(n)^{-2r-2/k}L_{s-1}^{(0)}(n)^{2-2/k}L_{s+q}^{(\varepsilon)}(n)^{2/k-1} \tag{4. 4}$$

converges for some $\varepsilon>0$, *then* $\Sigma a_n\phi_n(x)$ *is summable* $|N, p_n|_k$ *almost everywhere for* $p_n=L_s(n+2)^r\{(n+2)L_{s-1}^{(0)}(n+2)\}^{-1}$.

(v) *If the series*

$$\Sigma|a_n|^2 nL_{s+1}(n)^{-2/k}L_s^{(0)}(n)^{2-2/k}L_{s+q+1}^{(\varepsilon)}(n)^{2/k-1} \tag{4. 5}$$

converges for some $\varepsilon>0$, *then* $\Sigma a_n\phi_n(x)$ *is summable* $|N, p_n|_k$ *almost everywhere for* $p_n=1/\{(n+2)L_s^{(0)}(n+2)\}$.

For $k=1$, the cases $p=1$ and $p=2$ in this Corollary are the results obtained by Wang [11] and Ul'yanov [9] respectively; and the case $s=q=k=1$ and $r=0$ in (iv) is due to Okuyama [3].

Now, if $\{\Omega_n\}$ is a positive non-decreasing sequence with $\Omega_0=0$, then by using the best approximation, we see that

$$\Sigma|a_n|^2\Omega_n=\sum_{j=1}^{\infty}\Delta\Omega_j\sum_{n=j}^{\infty}|a_n|^2=\sum_{j=1}^{\infty}\{E_j^{(2)}(f)\}^2\,\Delta\Omega_j, \tag{4. 6}$$

therefore, estimating the corresponding $\Delta\Omega_j$ for the series (4. 1)~(4. 5), we have easily the following:

Corollary 2. *In the Corollary* 1 *the series* (4. 1)~(4. 5) *can be replaced by the following series* (4. 7)~(4. 11) *respectively*:

$$\Sigma n^{-1}L_1(n)^{-1}L_p^{(\varepsilon)}(n)^{2/k-1}\{E_n^{(2)}(f)\}^2, \tag{4. 7}$$

$$\Sigma n^{-1}L_p^{(\varepsilon)}(n)^{2/k-1}\{E_n^{(2)}(f)\}^2, \tag{4. 8}$$

$$\Sigma n^{-2\alpha}L_p^{(\varepsilon)}(n)^{2/k-1}\{E_n^{(2)}(f)\}^2, \tag{4. 9}$$

$$\Sigma L_s(n)^{-2r-2/k}L_{s-1}^{(0)}(n)^{2/k-1}L_{s+q}^{(\varepsilon)}(n)^{2/k-1}\{E_n^{(2)}(f)\}^2, \tag{4. 10}$$

$$\Sigma L_{s+1}(n)^{-2/k}L_s^{(0)}(n)^{2-2/k}L_{s+q+1}^{(\varepsilon)}(n)^{2/k-1}\{E_n^{(2)}(f)\}^2. \tag{4. 11}$$

For the trigonometric orthogonal system we shall make a remark. Let $f(x)\in L^2(0, 2\pi)$, and denote by $\Omega(\delta, f)$ one of the following integral moduli :

$$w^{(2)}(\delta, f)=\sup_{0\leqq t\leqq \delta}\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\}^{1/2},$$

$$w_2^{(2)}(\delta, f)=\sup_{0\leqq t\leqq \delta}\{\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\}^{1/2},$$

$$W^{(2)}(\delta, f)=\{\frac{1}{\delta}\int_0^{\delta}dt\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\}^{1/2},$$

$$W_2^{(2)}(\delta, f)=\{\frac{1}{\delta}\int_0^{\delta}dt\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\}^{1/2}.$$

Let $\{\lambda_n\}$ be a positive monotone sequence such that

$$\sum_{j=n}^{\infty}j^{-2}\lambda_j^{-1}\leqq An^{-1}\lambda_n^{-1}.$$

Then, Leindler [2] proved that the conditions $\Sigma\lambda_n^{-1}\Omega(1/n, f)^2<\infty$ and $\Sigma\lambda_n^{-1}\{E_n^{(2)}(f)\}^2<\infty$ are equivalent. So that we get easily the following result.

Corollary 3. *For the case of trigonometric series, sufficient conditions for the conclusions* (i)~(v) *in Corollary* 1 *are, for some* $\varepsilon>0$,

(i) $$\Omega(\delta, f)=O(L_1(1/\delta)^{1/2}L_p^{(\varepsilon)}(1/\delta)^{-1/k}),$$

(ii) $$\Omega(\delta, f)=O(L_p^{(\varepsilon)}(1/\delta)^{-1/k}),$$

(iii) $$\Omega(\delta, f)=O(\delta^{1/2-\alpha}L_p^{(\varepsilon)}(1/\delta)^{-1/k}),$$

(iv) $$\Omega(\delta, f)=O(\delta^{1/2}L_s(1/\delta)^{r+1/k}L_{s-1}^{(0)}(1/\delta)^{1/k-1}L_{s+q}^{(\varepsilon)}(1/\delta)^{-1/k}),$$

(v) $$\Omega(\delta, f)=O(\delta^{1/2}L_{s+1}(1/\delta)^{1/k}L_s^{(0)}(1/\delta)^{1/k-1}L_{s+q+1}^{(\varepsilon)}(1/\delta)^{-1/k})$$

respectively.

The case $k=1$ and $p=2$ in the results (i)~(iii) are due to Ul'yanov [9], and the case $s=p=k=1$ and $r=0$ in (iv) is due to Okuyama [3].

5. Let $\{r_n(t)\}$ be the Rademacher system. For the series

$$\Sigma\lambda_n a_n r_n(t) \tag{5. 1}$$

instead of general orthogonal series (2. 4), we shall establish an inverse of Theorem 1.

Theorem 3. *Let* $k\geqq 1$ *and let* $\{p_n\}$ *be a positive sequence such that for any fixed integer* $j_0>0$, $p_{n-j}(P_n/p_n-P_{n-j}/p_{n-j})=O(1)$ *for* $n\geqq j_0\geqq j\geqq 1$. *Suppose that the* *set of points t for which the series* (5. 1) *is summable* $|N, p_n|_k$ *is of positive measure, then the series* (2. 3) *converges.*

Proof. We may suppose that the $|N, p_n|_k$ sum of the series (5. 1) is uniformly bounded for all $t \in E \subset (0, 1)$ where $m(E)>0$. Then,

$$\int_E \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \Big| \sum_{j=1}^{n} p_{n-j} \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big) \lambda_j a_j r_j(t) \Big|^k dt < \infty.$$

Let N be a positive integer and replace $a_1, a_2, \cdots, a_{N-1}$ in the series (5. 1) by zeros. This replacement has no influence on the summability, since

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \Big| \sum_{j=1}^{N-1} p_{n-j} \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big) \lambda_j a_j r_j(t) \Big|^k \\ &\leqq A \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \{ \sum_{j=1}^{N-1} \lambda_j |a_j| \}^k \end{aligned}$$

which is finite if $\Sigma p_n < \infty$, and by Pringsheim's theorem $\Sigma p_n\ P_n^{-1} P_{n-1}^{-k} < \infty$ for any $k>0$, if $\Sigma p_n = \infty$. Therefore we may suppose that

$$\int_E \sum_{n=N}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \Big| \sum_{j=N}^{n} p_{n-j} \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big) \lambda_j a_j r_j(t) \Big|^k dt < \infty \tag{5. 2}$$

where $N=N(E)$ is determined by the well known Khinchin inequality :

$$\begin{aligned} &\int_E \Big| \sum_{j=N}^{n} p_{n-j} \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big) \lambda_j a_j r_j(t) \Big|^k dt \\ &\geqq A \{ \sum_{j=N}^{n} p_{n-j}^2 \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2 \lambda_j^2 |a_j|^2 \}^{k/2}. \end{aligned} \tag{5. 3}$$

From (5. 2) and (5. 3) we can conclude the convergence of the series (2. 3), since repeating the similar argument as above, the integer N may be replaced by 1.

6. We shall show that the positive number ε in $L_p^{(\varepsilon)}(t)$ is indispensable in Corollaries 1, 2 and 3 for the case of trigonometric series.

Lemma 3. *Let* $1 \leqq k \leqq 2$ *and* $\{p_n\}$ *be the same sequence as in Theorem* 3. *Put* $A_j(x) = \rho_j \cos (jx + \theta_j)$. *If the series*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \{ \sum_{j=1}^{n} p_{n-j}^2 \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2 A_j^2(x) \}^{k/2} \tag{6. 1}$$

converges for every x in a set of positive measure, then the series

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \{ \sum_{j=1}^{n} p_{n-j}^2 \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2 \rho_j^2 \}^{k/2} \tag{6. 2}$$

converges. Conversely, the convergence of (6. 2) *implies that of* (6. 1) *for every* x.

Proof. We may suppose that the sum (6. 1) is uniformly bounded by a constant A in a set E, $m(E)>0$ and denote, for the simplicity, $\alpha_n=p_nP_n^{-1}P_{n-1}^{-k}$, $\beta_{n,j}=p_{n-j}|\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}|$. Then we have

$$I=\sum_{n=1}^{\infty}\alpha_n\int_E\{\sum_{j=1}^{n}\beta_{n,j}^2\rho_j^2\cos^2(jx+\theta_j)\}^{k/2}dx\leqq Am(E). \tag{6. 3}$$

Using the Minkowski inequality, we get

$$\begin{aligned} I&\geqq\sum_{n=1}^{\infty}\alpha_n\{\sum_{j=1}^{n}(\int_E\beta_{n,j}\ \rho_j|\cos(jx+\theta_j)\ |dx)^2\}^{k/2} \\ &=\sum_{n=1}^{\infty}\alpha_n\{\sum_{j=1}^{n}\beta_{n,j}^2\rho_j^2(\int_E|\cos(jx+\theta_j)|dx)^2\}^{k/2}. \end{aligned} \tag{6. 4}$$

By the Riemann-Lebesgue theorem, we have

$$\begin{aligned} &\int_E|\cos(jx+\theta_j)|dx\geqq\int_E\cos^2(jx+\theta_j)dx \\ &=\frac{1}{2}\int_E(1+\cos 2(jx+\theta_j))dx=\frac{1}{2}m(E)+\frac{1}{2}\int_E\cos 2(jx+\theta_j)dx \\ &\geqq\frac{1}{4}m(E), \end{aligned} \tag{6. 5}$$

for sufficiently large j, say $j\geqq N$.
Therefore, by (6. 4) and (6. 5)

$$\begin{aligned} I&\geqq\sum_{n=1}^{\infty}\alpha_n\{\sum_{j=N}^{n}\beta_{n,j}^2\rho_j^2\left(\frac{1}{4}m(E)\right)^2\}^{k/2} \\ &\geqq A\sum_{n=1}^{\infty}\alpha_n\{\sum_{j=N}^{n}\beta_{n,j}^2\rho_j^2\}^{k/2}. \end{aligned}$$

By the same reason as in Theorem 3, we replace N by 1 and we conclude the convergence of (6. 2). The converse is obvious.

Theorem 4. *Let* $2\geqq k\geqq 1$ *and let* $\{p_n\}$ *be the same as in Theorem* 3. *If the series* (6. 2) *converges, then almost all series of*

$$\Sigma\pm(a_n\cos nx+b_n\sin nx), \tag{6. 6}$$

where $A_n(x)=\rho_n\cos(nx+\theta_n)=a_n\cos nx+b_n\sin nx$, *are summable* $|N, p_n|_k$ *for* *almost every* x, *and if* (6. 2) *diverges, then almost all series of* (6. 6) *are non-summable* $|N, p_n|_k$ *for almost every* x.

Proof. Considering the series $\Sigma r_n(t)\ A_n(x)$, the first part is an easy consequence of Theorem 1 putting $\lambda_j a_j = A_j(x)$. The latter part is also a consequence of Theorem 3 and Lemma 3 following the well known Paley–Zygmund argument.

Corollary 4. *Let* $1 \leqq k < 2$. *In the assumptions of Corollaries* 1, 2 *or* 3 *the positive number* ε *in* $L_p^{(\varepsilon)}$ *or* $L_{s+q}^{(\varepsilon)}$ *is indispensable.*

Proof. We treat the case (iv) of Corollary 1, because the other cases can be shown similarly. It is sufficient to show the existence of a Rademacher–trigonometric series $\Sigma a_n r_n(t)$ cos nx which is non–summable $|N, p_n|_k$ for almost every (t, x) in $(0, 1)\times(0, 2\pi)$ and the series (4. 4) is convergent for $\varepsilon = 0$. For this purpose we put

$$a_n = n^{-1}\ L_s\ (n)^{r+1/k}\ L_{s-1}^{(0)}\ (n)^{1/k-1}\ L_{s+q+1}^{(0)}\ (n)^{-1/k},$$

then as we see easily the series (4. 4) with $\varepsilon = 0$ is

$$\Sigma n^{-1}\ L_{s+q}^{(0)}\ (n)^{-1}\ L_{s+q+1}\ (n)^{-2/k}$$

which is convergent for $1 \leqq k < 2$. On the other hand, since $p_n = n^{-1} L_{s-1}^{(0)}(n)^{-1}\ L_s(n)^r$, we see $P_n \sim L_s\ (n)^{r+1}$ and $P_n/p_n \sim n L_s^{(0)}(n)$. Hence it is easy to see that the series (6. 2) is not smaller than

$$A\sum_{n=1}^{\infty}\frac{p_n}{P_n^{k+1}}\{\sum_{j=[n/2]}^{n} p_{n-j}^2\left(\frac{P_n}{p_n}\right)^2 a_j^2\}^{k/2}$$

$$\geqq A\sum_{n} n^{-1}\ L_{s+q+1}^{(0)}(n)^{-1}$$

which is divergent.

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