Decreasing Rearrangements of Non-Negative \((c_0)\) Sequences and Some Extensions of Hardy–Littlewood–Pólya’s Theorems

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Decreasing rearrangements of non-negative \((c_0)\) sequences and three preorder relations which are extensions of Hardy–Littlewood–Pólya’s one are defined. Some generalizations of Hardy–Littlewood–Pólya’s inequalities for rearrangements and convex functions are given.

1 Introduction

In recent years a number of inequalities have appeared which involve rearrangements of vectors in \(\mathbb{R}^n\) or sequences in \((l^1)\) and of measurable functions on a finite measure space or non-negative \(L^1\) functions on an infinite measure space \([1; 6]\). These inequalities are not only interesting themselves, but also have many applications in probability theory, information theory, mathematical economics, and so on \([8]\). But many times we are forced to consider sequences which belong to \((c_0)\).

In this paper we define decreasing rearrangements of non-negative \((c_0)\) sequences and we introduce three preorder relations in the positive cone \((c_0)^+\), of, \((c_0)\), two of which are new and one is equivalent to that of Markus \([7, \text{p. 103}]\). Consequently, some generalizations of well-known results of Hardy–Littlewood–Pólya \([5, \text{Theorem 108, p. 89}]\) and Pólya \([9]\) are given. Moreover, two results of Chong \([3, \text{Theorem 2.7, p. 158}; 4, \text{Theorem 3.9, p. 434}]\) are generalized.

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2 Notations and Preliminaries

Let \(\mathbb{R}^n\) denote the set of all \(n\)-tuples of real numbers. For any \(n\)-tuple \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), we denote by

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the \( n \)-tuple whose components are those of \( x \) arranged in decreasing order of magnitude. If \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) and \( b = (b_1, \ldots, b_n) \in \mathbb{R}^n \), then \( a \prec b \) means that
\[
\sum_{i=1}^{k} a_i^* \leq \sum_{i=1}^{k} b_i^*
\]
for \( 1 \leq k \leq n \), and we write \( a < b \) if, in addition to \( a \prec b \), there is equality in (2.1) for \( k = n \). These two preorder relations in \( \mathbb{R}^n \) were originally defined by Hardy–Littlewood–Pólya [5], and the following theorems give characterizations of \(< \) and \( \prec \) [5, Theorem 108, p. 89; 9].

Suppose \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) are \( n \)-tuples in \( \mathbb{R}^n \), then the following hold.

\((H_1)\) \( a < b \) is equivalent to
\[
\sum_{i=1}^{n} \phi(a_i) \leq \sum_{i=1}^{n} \phi(b_i)
\]
for all convex functions \( \phi: \left[ b_n^*, b_1^* \right] \to \mathbb{R} \).

\((H_2)\) \( a \prec b \) is equivalent to
\[
\sum_{i=1}^{n} \phi(a_i) \leq \sum_{i=1}^{n} \phi(b_i)
\]
for all non-decreasing convex functions \( \phi: \left[ b_n^*, b_1^* \right] \to \mathbb{R} \), and this is equivalent to
\[
\sum_{i=1}^{n} (a_i - u)^+ \leq \sum_{i=1}^{n} (b_i - u)^+
\]
for all real numbers \( u \), where \( x^* = \max\{x, 0\} \) for any \( x \in \mathbb{R} \).

If \( f(x) \) is non-increasing and right continuous on \([0, \infty)\),
\[
f^*(x) = \sup \{ \lambda : f(\lambda) > x \} \quad (x \geq 0)
\]
is called the right continuous inverse of \( f \) on \([0, \infty)\), and the following are well known [1, p. 24].

\((R_1)\) \( f^*(x) \) is right continuous and decreasing.

\((R_2)\) \( f^*(x) \geq \lambda \) is equivalent to \( f(\lambda) > x \).

\((R_3)\) \( d_{f^*}(\lambda) = \mu\{ x \in \mathbb{R} : f^*(x) > \lambda \} = f(\lambda) \),

where \( \mu \) is the Lebesgue measure on \([0, \infty)\).

Throughout this paper, we write \( N \) in place of the set of all positive integers, and \( \mathbb{Z}_+ \) denotes the set of all non-negative integers. Also \( R_+ \) stands for the
positive cone of $R$, and $\bar{R}$, for the set of all non-negative extended real numbers, while $(P)_+$ denotes the positive cone of $(P) (1 \leq p)$. Moreover, if $a = (a_1, a_2, \ldots) \in (c_0)_+$, we write $a_i = a(i)$ for any integer $i \in N$, and $S_+$ stands for a set \{a: a \in (c_0)_+, there exists an $m \in N$ so that $i > m$ implies $a(i) = 0$\}. $d_\phi(\lambda) = \text{Card} \{i: a(i) > \lambda\}$ is called the distribution of $a$. Then $(c_0)_+$ is characterized by a set such that \{a: a = (a_1, a_2, \ldots) \geq 0, \ d_\phi(\lambda) < \infty \text{ for any } \lambda > 0\}.

In the sequel, we use the term "convex" in a narrow meaning: a convex function is a function $\phi$ such that $R_1, R_2 \leq 0$ and $z_1 + z_2 = 1$ imply $\phi(z_1x + z_2y) \leq z_1\phi(x) + z_2\phi(y)$ for any $x$ and $y$ in the domain of $\phi$.

3 Decreasing Rearrangements of Non-Negative $(c_0)_+$ Sequences and Some Extensions of H-L-P's Theorems

Our results are based on the next existence theorem for rearrangements of sequences in $(c_0)_+$.

**Theorem 1.** If $a$ belongs to $(c_0)_+$, then we can rearrange all the components $a_i > 0$ of $a$ in a non-increasing order of magnitude so that $a_1^* \geq a_2^* \geq \cdots$ holds.

**Proof.** If $a \in S_+$, then the statement in our theorem is evident, therefore we may assume $a \notin S_+$ and $a \in (c_0)_+$. Then there exists a component $a_j > 0$ of $a$. Put $A_1 = N$, and then $d_\phi(\frac{a_j}{2}) = \text{Card} \{i: a(i) > \frac{a_j}{2}\}$ is finite, which insures the existence of $i_1 \in N$ so that $a_{i_1} = \max \{a_i: i \in A_1\}$. Define $A_n$ and $a_{i_n}$ ($n = 2, 3, \ldots$) by induction as follows:

$$a_{i_n} = \max \{a_i: i \in A_n\}, \ A_{n+1} = A_n - \{i_n\}. \quad (3.1)$$

If we set $A_+ = \{i: a_i > 0\}$, we can define a single valued mapping

$$\phi: N \rightarrow A_+, \ \phi(i) = i_j \quad (3.2)$$

by means of (3.1). It is easy to see that $\phi$ is one-to-one; for any $a_k \in A_+$ there exists one coordinate $i_j$ such that $a_k = a_{i_j}$, since $d_\phi(\frac{a_k}{2}) = \text{Card} \{i: a_i > \frac{a_k}{2}\}$ is finite. That is, $\phi: N \rightarrow A_+$ is a one-to-one and onto mapping, and if we put

$$a^*(j) = a(\phi(j)), \quad (3.3)$$

$a_i^*$ ($i \in N$) is the desired one.

**Definition 1.** If $a \in (c_0)_+$ and $a \notin S_+$, then we define $a^* = (a(\phi(1)), a(\phi(2)), \ldots)$, where $\phi$ is the mapping defined by (3.2). If $a \in S_+$, assume $\text{Card} \{i: a_i > 0\} = m$, and denote by $a_i^*$ ($i = 1, \ldots, m$) the positive components of $a$ rearranged in non-increasing order of magnitude. In this case, we define $a^* = (a_1^*, \ldots, a_m^*, 0, \ldots, 0)$.

* This easy but important fact is suggested by Mr. Yukio Takeuchi.
we call $a^*$ the decreasing rearrangement of $a \in (c_0)^+$. It is easy to see that $d_a(\lambda) = d_{a^*}(\lambda)$ for any $\lambda \in R$. Therefore $a^* \in (c_0)^+$ if $a \in (c_0)^+$.

**Definition 2.** If $a, b \in (c_0)^+$, we write

$$a \sim b \text{ if and only if } d_a(\lambda) = d_b(\lambda)$$

for any $\lambda \in R$, and we say that $a$ and $b$ are *equidistributed* if $a \sim b$.

It is easy to see that $\sim$ is a preorder relation in $(c_0)^+$, and that $a \sim a^*$.

**Proposition 1.** If $a, b \in (c_0)^+$, then

$$a \sim b \text{ if and only if } a^* = b^*.$$  

(3.4)

**Proof.** Both $a \sim a^*$ and $b \sim b^*$ with $a \sim b$ imply $a^* \sim b^*$; hence $a^* = b^*$ is clear. The proof of the converse implication is clear from Definition 1.

In the sequel, we regard $\kappa_0$ as $\infty$, an element of $\overline{R}_+$, and we consider $d_a(\cdot)$ as a mapping from $R$ to $\overline{R}_+$.

**Theorem 2.** A mapping $f(\cdot)$ from $R$ to $\overline{R}_+$ is a *distribution* $d_a(\cdot)$ for some $a \in (c_0)^+$, if and only if, $f(\cdot)$ satisfies the following three conditions $D_1, D_2,$ and $D_3$.

(D1) $f(\lambda) \in Z_+$ for any $\lambda > 0$ and $f(\lambda) = \infty$ for any $\lambda < 0$.

(D2) There exists a $\lambda_0 \in R_+$ such that $f(\lambda) = 0$ for any $\lambda > \lambda_0$.

(D3) $f(\lambda)$ is a non-increasing and right continuous function on $R$.

**Proof.** If $f(\cdot) = d_a(\cdot)$ for some $a \in (c_0)^+$, then we have an alternative expression $f(\cdot) = d_{a^*}(\cdot)$; hence $D_1$ and $D_2$ are clear. $D_3$ is a consequence of the continuity of a measure $\text{Card} \{\cdot\}$.

To prove the converse implication, consider the right continuous inverse $f^*$ of $f$. Then, as mentioned already, (2.6), (2.7), and (2.8) hold. Moreover, if $f^*(0) = \infty$, then $f^*(0) > K$ for any $K > 0$, which is equivalent to $f(K) > 0$ for any $K > 0$ by (2.7), contradictory to $D_2$; hence $f^*(0) < \infty$. Now, define $\tilde{a}(s) = f^*(s-1)$ for any $s \in N$. Then,

$$\tilde{a}(s) \geq 0 \text{ is non-increasing for any } s \in N, \text{ and } \tilde{a}(1) = f^*(0) < \infty.$$  

(3.6)

It is clear that $d_{\tilde{a}}(\lambda) = f(\lambda) = \infty$ holds for any $\lambda < 0$. Assume $0 \leq \lambda < \tilde{a}(1) = f^*(0)$, then

$$d_{\tilde{a}}(\lambda) = \max \{s: s \in N, \tilde{a}(s) > \lambda\}$$

$$= \max \{s: s \in N, f^*(s-1) > \lambda\}$$

$$= \max \{s: s \in N, f(\lambda) > s-1\}$$

$$= f(\lambda).$$
Next, assume $\lambda \geq a(1) = f^*(0)$, then (3.6) implies $d_\lambda(\lambda) = 0$, while $f(\lambda) \leq 0$ follows from (2.7); hence $f(\lambda) = 0$. Thus we have proved that $d_\lambda(\lambda) = f(\lambda)$ for any $\lambda \in R$, and $d_\lambda(\lambda) = f(\lambda) < \infty$ for any $\lambda > 0$. Therefore $\bar{a}$ belongs to $(c_0)_+$, and $f(\cdot) = d_\bar{a}(\cdot)$: the proof is completed.

**Corollary 1.** If $a$ belongs to $(c_0)_+$, then

$$a^*(s) > \lambda \text{ if and only if } d_a(\lambda) > s - 1 \tag{3.7}$$

for any $s \in N$.

**Proof.** Suppose $a \in (c_0)_+$, and put $f = d_a = d_\bar{a}$. Then $f = d_\bar{a}$, where $\bar{a}$ is the element in $(c_0)_+$ defined in the proof of Theorem 2. Hence,

$$d_a(\lambda) > s - 1 \iff f(\lambda) > s - 1 \iff f^*(s - 1) > \lambda \iff a^*(s) > \lambda$$

is clear from (2.7).

**Corollary 2.** If $a \in (c_0)_+$, then

$$a^*(s) = \sup \{\lambda: d_a(\lambda) > s - 1\} = \inf \{\lambda: d_a(\lambda) \leq s - 1\}$$

necessarily holds for any $s \in N$.

**Proof.** Both $a^*(s) = \sup \{\lambda: a^*(s) > \lambda\} = \sup \{\lambda: d_a(\lambda) > s - 1\}$ and $a^*(s) = \inf \{\lambda: a^*(s) \leq \lambda\} = \inf \{\lambda: d_a(\lambda) \leq s - 1\}$ are immediate consequences of (3.7).

By virtue of Corollary 1 and Corollary 2, we can easily obtain the next convergence theorem for rearrangement.

**Theorem 3.** If $a_n$ and $a \in (c_0)_+$, then

$$a_n \uparrow a \text{ implies both } d_{a_n} \uparrow d_a \text{ and } a_n^* \uparrow a^*. \tag{3.8}$$

**Proof.** It is easy to see that $a_n \uparrow a$ implies $d_{a_n}(\lambda) \leq d_{a_{n+1}}(\lambda) \leq d_a(\lambda)$ for any $\lambda \in R$. Then $a_n^* \leq a_{n+1}^* \leq a^*$ is immediate by Corollary 2, and $d_{a_n} \uparrow d_a$ is a mere consequence of the continuity of a measure. Hence $\lim_{n \to \infty} a_n^*(s) \leq a^*(s) (s \in N)$ is immediate. To obtain the opposite side inequality, assume $a^*(s) > \lambda$. Then we have $d_a(\lambda) = \lim d_{a_n}(\lambda) > s - 1$ by (3.7), which implies the existence of an integer $m$ so that $d_{a_n}(\lambda) > s - 1$ holds for any $n > m$. Hence $a_n^*(s) > \lambda$ for any $n > m$ and $\lim_{n \to \infty} a_n^*(s) \geq \lambda$ hold. That is $\lim_{n \to \infty} a_n^* \geq a^*$. Thus we have completed the proof.

Now we shall extend the preorders of Hardy–Littlewood–Pólya in $R^n$ to the sequences belonging to $(c_0)_+$.

**Definition 3.** If $a, b \in (c_0)_+$, then we write

$$a \ll b \text{ if and only if } \sum_{i=1}^k a_i^* \leq \sum_{i=1}^k b_i^* \tag{3.9}$$

for any $k \in N$, and
Here we write $\sum_{i=1}^{\infty} b_i = \infty$, whenever $\sum_{i=1}^{\infty} b_i$ is divergent. We say $a$ is weakly (strongly) majorized by $b$ if $a \ll b$ ($a \gg b$).

It should be noted that (3.9) is a generalization of the preorder of Markus [7, p. 103]. It is clear that $a \sim b$ is equivalent to $a \ll b$ and $b \ll a$, and that $a \ll b$ ($a < b$) is equivalent to $a^* \ll b^*$ ($a^* < b^*$).

**Proposition 2.** If $0 \leq a_n \uparrow a \in (c_0)_+$ and $0 \leq b_n \uparrow b \in (c_0)_+$ with $a_n \ll b_n$ ($a_n < b_n$) for any $n \in N$, then $a \ll b$ ($a < b$) necessarily holds.

**Proof.** $a_n \ll b_n$ ($a_n < b_n$) is equivalent to $a^*_n \ll b^*_n$ ($a^*_n < b^*_n$). Hence $a^* \ll b^*$ ($a^* < b^*$) is readily seen.

**Lemma 1.** If $a \ll b$ ($a < b$), then there exist two sequences $\{a_n\} \subset S_+$ and $\{b_n\} \subset S_+$ such that $a_n < b_n$ and $a_n \uparrow a$, $b_n \downarrow b$ hold.

**Proof.** If $a$, $b \in S_+$, then our theorem is clear. In the other case, firstly we shall prove that there are two sequences $\{a_n^*\} \subset S_+$ and $\{b_n^*\} \subset S_+$ such that $a_n^* \uparrow a^*$, $b_n^* \uparrow b^*$, and $a_n^* < b_n^*$ ($n \in N$) hold. If $b \neq 0$ and $b \in S_+$, then there exists a unique $k \in N$ so that $b_k^* > 0$, and $b_{k+1}^* = 0$ hold. For this $k$, choose any $j \in N$ so that $a_i^* + \cdots + a_j^* > b_i^* + \cdots + b_{j-k+1}^*$ holds, and put $j_0 = j$, $a_n^* = (a_1^*, \ldots, a_{j_0}^*, \ldots, a_{j_0+k}^* \ldots, 0, 0, \ldots)$ and $b_n^* = (b_1^*, \ldots, b_{k-1}^*, \sum_{i=1}^{j_0+k} a_i^* - \sum_{i=1}^{k} b_i^*, 0, 0, \ldots)$. On the other hand, if $b \in S_+$, then there exists an unique $k_n \in N$ so that $a_1^* + \cdots + a_{k_1}^* \leq b_1^* + \cdots + b_{k_1}^*$ and $a_1^* + \cdots + a_{k_{n+1}}^* > b_1^* + \cdots + b_{k_{n+1}}^*$ hold. For any $n \in N$, and set $b_n^* = (b_1^*, \ldots, b_{k_n}^*, 0, 0, \ldots)$ and $a_n^* = (a_1^*, \ldots, a_{k_n}^* = \sum_{i=1}^{k_n} b_i^* - \sum_{i=1}^{k_n} a_i^*, 0, 0, \ldots)$. Then $\{a_n^*\}$ and $\{b_n^*\}$ satisfy our requirements. Secondly, according to Definition 1, if $a \in S_+$, then there exists a one–to–one mapping $\phi : N \rightrightarrows A_+$ which satisfies (3.2), and we define $\tilde{a}_n(i) = a_n(\phi(i)) = a_n^*(i)$ for any $i \in A_+$, and $\tilde{a}_n(i) = 0$ for any $i \in A_+$, where $n$ is any positive integer. On the other hand, if $a$ belong to $S_+$, then there exists a permutation $\Pi$ over $N$ such that $\phi(\Pi(j)) = a^*(j)$ holds. For this case, set $\tilde{a}_n(i) = \tilde{a}_n(\Pi(j)) = a_n^*(j)$. If we define $\tilde{b}_n$ similarly as above, $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ satisfy the whole requirements in our theorem. Finally, if $a \ll b$, then a proof of our theorem is obtained similarly as above.

**Lemma 2.** If $\phi : R_+ \rightrightarrows R$ is convex with $\phi(0) = 0$, then $\sum_{i=1}^{\infty} \phi(a_i)$ is defined for any $a \in (c_0)_+$, and the next holds:

$$0 \leq a_n \uparrow a \in (c_0)_+ \text{ implies } \lim_{n \to \infty} \sum_{i=1}^{\infty} \phi(a_n(i)) = \sum_{i=1}^{\infty} \phi(a(i)).$$

(3.11)
**Proof.** If $\phi: R_+ \rightarrow R$ is convex with $\phi(0) = 0$, then the next four cases occur:

1. $\phi(t)$ is non-decreasing on $R_+$, and hence continuous at $t = 0$,
2. $\phi(t)$ is non-increasing on $R_+$,
3. there exist $t_1, t_2 > 0$ so that $\phi(t_1) \phi(t_2) < 0$,

and

4. $\phi(t)$ is non-decreasing on $(0, \infty)$ and non-continuous at $t = 0$.

We recall that

$$a \in (c_0), \text{ is equivalent to } d_a(\lambda) = \text{Card } \{i: a(i) > \lambda\} < \infty \quad (3.12)$$

for any $\lambda > 0$; hence $\sum_{i=1}^{\infty} \phi(a(i))$ is defined for all convex functions $\phi$ with $\phi(0) = 0$, which may be $+\infty$ or $-\infty$. Besides, $\phi(\cdot)$ is necessarily continuous at any $t > 0$, therefore it is easy to see that

$$0 \leq a_n \uparrow a \in (c_0)_+ \text{ implies } \lim_{n \rightarrow \infty} \phi(a_n(i)) = \phi(a(i)) \quad (3.13)$$

for any $i \in N$. In the case $C_1$ or $C_2$, (3.11) follows from Levi's Monotone Convergence Theorem with (3.13), and in the case $C_3$, there exists an $\alpha > 0$ such that $\phi(t)$ is non-increasing on $[0, \alpha]$, and non-decreasing on $[\alpha, \infty)$, Set $A_1 = \{i: a(i) \leq \alpha\}$ and $A_2 = \{i: a(i) > \alpha\}$. Then $A_2$ is a finite set of indices; hence follows

$$\lim_{n \rightarrow \infty} \sum_{i \in A_1} \phi(a_n(i)) = \sum_{i \in A_2} \phi(a(i)). \quad (3.14)$$

On the other hand, if $i \in A_1$, then

$$\phi(a(i)) \leq \phi(a_{n+1}(i)) \leq \phi(a_n(i)) \leq 0 \quad (n \in N)$$

holds, and we have

$$\lim_{n \rightarrow \infty} \sum_{i \in A_1} \phi(a_n(i)) = \sum_{i \in A_1} \phi(a(i)), \quad (3.15)$$

again by Levi's theorem. Consequently, (3.11) follows from (3.14) and (3.15).

Finally, in the case $C_4$, there exists an $a_0 > 0$ so that $\phi(a_0) = 0$, set $B_1 = \{i : 0 < a(i) \leq a_0\}$ and $B_2 = \{i : a(i) > a_0\}$, where $B_2$ is also a finite set of indices. If we note that $\phi(a_n(i)) \leq \phi(a_{n+1}(i)) \leq \phi(a(i)) \leq 0$ holds for any $i \in B_1$, it is easy to see that

$$\sum_{i \in B_1} \phi(a_n(i)) = -\infty \text{ follows from } \sum_{i \in B_1} \phi(a(i)) = -\infty.$$

Moreover,
\[ \phi(t) \leq -\frac{\phi(0_+)}{\alpha_0} t + \phi(0_+) \leq 0 \]  

(3.16)

holds for any \( t \in (0, \alpha_0] \), so we can claim that \( B_i \) is again a finite set of indices, provided \( \sum \phi(a(i)) \neq -\infty \). The rest of the proof is easy.

**Theorem 4.** Suppose \( a, b \in (c_0)_+ \), then,

\[
\begin{align*}
(1) \quad & a \ll b \text{ is equivalent to } \sum_{i=1}^{\infty} \phi(a) \leq \sum_{i=1}^{\infty} \phi(b) \\
(2) \quad & a \ll b \text{ is equivalent to } \sum_{i=1}^{\infty} (a_i - u)^+ \leq \sum_{i=1}^{\infty} (b_i - u)^+ \\
(3) \quad & a \ll b \text{ is equivalent to } \sum_{i=1}^{\infty} \phi(a_i) \leq \sum_{i=1}^{\infty} \phi(b_i)
\end{align*}
\]

(3.17, 3.18, 3.19)

for all non-decreasing convex functions \( \phi : R_+ \rightarrow R \) with \( \phi(0) = 0 \). In particular,

\[
\begin{align*}
(2) \quad & a \ll b \text{ is equivalent to } \sum_{i=1}^{\infty} (a_i - u)^+ \leq \sum_{i=1}^{\infty} (b_i - u)^+ \\
& \text{for all positive real numbers } u.
\end{align*}
\]

(3.18)

**Proof.** According to Lemma 1, if \( a, b \in (c_0)_+ \) satisfy \( a \ll b \), then there exist two sequences \( \{a_n\} \) and \( \{b_n\} \subset S_+ \) which satisfy

\[
a_n \uparrow a, \ b_n \uparrow b, \text{ and } a_n \ll b_n.
\]

Then \( \sum_{i=1}^{\infty} \phi(a_n(i)) \leq \sum_{i=1}^{\infty} \phi(b_n(i)) \) follows from (2.3), where \( \phi \) is any non-decreasing convex function on \( R_+ \), and the necessary conditions in (3.17), and in (3.18) follow from Lemma 2. Now we recall that

\[
(x - u - v)^+ = ((x - u)^+ - v)^+
\]

holds for any \( u, v > 0 \). If \( \sum_{i=1}^{\infty} (a_i - u)^+ \leq \sum_{i=1}^{\infty} (b_i - u)^+ \) is valid for any \( u > 0 \), then

\[
\sum_{i=1}^{\infty} ((a_i - u)^+ - v)^+ \leq \sum_{i=1}^{\infty} ((b_i - u)^+ - v))^+
\]

(3.20)

is so, for any \( u, v > 0 \). Since \( (a-u)^+ = ((a_1-u)^+, \ldots) \) and \( (b-u)^+ = ((b_1-u)^+, \ldots) \) belong to \( S_+ \), \( (a-u)^+ \ll (b-u)^+ \) follows from (2.4) and
(3.20), and Proposition 2 implies $a \ll b$. Thus (3.17) and (3.18) are obtained. The sufficient condition in (3.19) is easily obtained if we put $\phi(t) = -t$, and the converse implication is also obtained similarly as above.

**Corollary 3.** If $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and convex, with $\phi(0) = 0$, then

$$a \ll b \implies (\phi(a_1), \phi(a_2), \ldots) \ll (\phi(b_1), \phi(b_2), \ldots).$$

**Proof.** If we put $\phi(t) = (t - u)^* \phi(t)$, then $\phi$ is again a non-decreasing convex function with $\phi(0) = 0$, and (3.17) and (3.18) imply

$$(\phi(a_1), \phi(a_2), \ldots) \ll (\phi(b_1), \phi(b_2), \ldots)^*.$$ 

**Example 1.**

(1) If $a \ll b$, then $||a||_p \leq ||b||^*_p$$

necessarily holds for any $p \geq 1$, where $||\cdot||_p$ denotes the $(l^p)$ norm, whether the right side is finite or infinite.

(2) If $a < b$, then $||b||_q \leq ||a||_q$

necessarily holds for any $0 < q \leq 1$, where $||\cdot||_q$ denotes the formal $(l^q)$ norm, whether the right side is finite or infinite.

**Example 2.** Suppose $a < b$, then $h(b) \leq h(a)$ necessarily holds, where $h(a) = - \sum_{i=1}^{\infty} a_i \log a_i$ denotes an entropy of $a \in (c_0)_+$, provided $0 \cdot \log 0 = 0$.

**References**


* This argument is borrowed from Chong [2, P. 1330].

** If $a \in l^1$, then our example is easily obtained from [10, Examples (1), p. 19].
Papers in Pure and Applied Mathematics 10 (1967), 83-144.


