On the Convergence of Martingales and the "Littlewood-Paley" Operators

Motohiro YAMASAKI*
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Let \( f = \{ f_n, \Omega_n; n \geq 1 \} \) be a martingale on a probability space \((\Omega, \mathcal{F}, P)\), where \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \) are sub-\(\sigma\)-fields of \( \mathcal{F} \). Denote \( d_n = f_n - f_{n-1} \) (\( n \geq 2 \)), \( d_1 = f_1 \), and \( s_n = \frac{1}{n} (f_1 + \cdots + f_n) \). We define the so-called "Littlewood-Paley" operators \( \lambda(f) \) and \( \pi(f) \) by

\[
\lambda(f) = \sum_{n=1}^{\infty} \frac{(f_n - s_n)^2}{n}
\]  

(1)

and

\[
\pi(f) = \sum_{k=1}^{\infty} (f_{nk} - s_{nk})^2
\]  

(2)

where \( n_1 < n_2 < \cdots \) are positive integers such that \( 1 < q_1 \leq \frac{n_k+1}{n_k} \leq q_2 \) \((k = 1, 2, \ldots)\) with constants \( q_1 \) and \( q_2 \). Tsuchikura ([2] Th. 2.4) showed that if \( \sup_n |d_n| < \infty \) and one of the expectations \( E[\lambda(f)^{\frac{1}{2}}] \) and \( E[\pi(f)^{\frac{1}{2}}] \) is finite, then \( \lim_{n \to \infty} f_n \) exists almost surely. In this note we show that the above theorem of Tsuchikura is valid without the assumption \( E[\sup_n |d_n|] < \infty \).

Burkholder and Gundy [1] introduced more wide class of operators on martingales. Let \( a_{jk} \) be a \( \mathcal{F}_{k-1} \)-measurable function, and for all \( k \geq 1 \),

\[
C_1 \leq \sum_{j=1}^{\infty} a_{jk}^2 \leq C_2
\]

where \( C_1 \) and \( C_2 \) are positive constants. Then an operator \( M \):

\[
M(f) = \left( \sum_{j=1}^{\infty} \left( \lim_{n \to \infty} \sup_{k \leq n} \left| \sum_{k=1}^{n} a_{jk} d_k \right|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]  

(3)

is called to be of matrix type. Let \( f^n = \{ f^n_1, f^n_2, \cdots \} \) be a martingale \( f \) stopped at \( n \), that is \( f^n_k = f_k (k \leq n), f_n (k > n) \). It was shown ([1] Th. 6. 1) that if \( \sup_n E[M(f^n)] < \infty \), then \( \lim_{n \to \infty} f_n \) exists almost surely, where \( M \) is of matrix type.

* Lecturer, Institute of Mathematics.
Theorem. Let $f=\{f_n, \mathfrak{F}_n; n \ge 1\}$ be a martingale. If $E[\lambda(f)^{\frac{1}{2}}] < \infty$ or $E[\tau(f)^{\frac{1}{2}}] < \infty$, then $\lim f_n$ exists almost surely.

Proof. Note that
\[
\lambda(f) = \sum_{n=1}^{\infty} \frac{(f_n - s_n)^2}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k-1}{n} d_k)^2
\]
\[= \sum_{n=2}^{\infty} \left( \sum_{k=2}^{n} \frac{k-1}{n^2} d_k)^2.\]

We put
\[a_{jk} = \begin{cases} 1 & \text{if } j = k = 1, \\ j^{\frac{3}{2}}(k - 1) & \text{if } 2 \le k \le j, \\ 0 & \text{otherwise} \end{cases} \]
in (3), and denote the resulting operator of matrix type by $A(f)$:
\[A(f) = \left[ d_1^2 + \sum_{j=2}^{\infty} \left( \limsup_{n \to \infty} \sum_{k=2}^{n} j^{-\frac{3}{2}} (k - 1) d_k \right)^2 \right]^{\frac{1}{2}}\]
\[= \left[ d_1^2 + \sum_{j=2}^{\infty} \left( \sum_{k=2}^{j} j^{-\frac{3}{2}} (k - 1) d_k \right)^2 \right]^{\frac{1}{2}}\]

Now,
\[A(f) = \left\{ (d_{n_1})^2 + \sum_{j=2}^{\infty} \left( \sum_{k=2}^{j} j^{-\frac{3}{2}} (k - 1) d_{n_1} \right)^2 \right\}^{\frac{1}{2}}\]
\[= \left\{ d_1^2 + \sum_{j=2}^{\infty} \left( \sum_{k=2}^{j} j^{-\frac{3}{2}} (k - 1) d_k \right)^2 \right\}^{\frac{1}{2}}\]
\[\le \left\{ d_1^2 + \sum_{j=2}^{\infty} \left( \sum_{k=2}^{j} j^{-\frac{3}{2}} (k - 1) d_k \right)^2 + \frac{1}{n^2} \left( \sum_{k=2}^{n} (k - 1) d_k \right)^2 \right\}^{\frac{1}{2}}\]
\[\le \left\{ d_1^2 + \lambda(f) + (f_n - s_n)^2 \right\}^{\frac{1}{2}}\]
\[\le \left\{ d_1^2 + \lambda(f) + |f_n - s_n| \right\}^{\frac{1}{2}}\]

But by Lem. 2.3 [2], $\sup_n E[|f_n - s_n|] \le c \cdot E[\lambda(f)^{\frac{1}{2}}]$ where $c$ is a positive constant. So,
\[\sup_n E[A(f^n)] \le E[|d_1|] + E[\lambda(f)^{\frac{1}{2}}] + \sup_n E[|f_n - s_n|] \le E[|f_1|] + (c + 1) E[\lambda(f)^{\frac{1}{2}}].\]

Therefore, if $E[\lambda(f)^{\frac{1}{2}}] < \infty$, then $\sup E[A(f^n)] < \infty$, and by the theorem of Burkholder and Gundy we can conclude that $\lim f_n$ exists almost surely.
Now we turn to $\pi(f)$. Because
\[
\pi(f) = \sum_{k=1}^{\infty} \left( f_{n_k} - s_{n_k} \right)^2 = \sum_{j=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n_j} k - 1 \right) d_k^2,
\]
we put in (3)
\[
a_{jk} = \begin{cases} 
1 & \text{if } j = k = 1, \\
n_{j-1}(k-1) & \text{if } 2 \leq k \leq n_j, \\
0 & \text{otherwise},
\end{cases}
\]
and denote the matrix type operator from these $a_{jk}$'s by $II(f)$:
\[
II(f) = \left[ d_1^2 + \sum_{j=2}^{\infty} \left( \lim_{n \to \infty} \sup_{1 \leq k \leq n_j} \left( \frac{n_j}{n} \sum_{k=1}^{n_j} (k - 1) d_k \right) \right)^2 \right]^\frac{1}{2}.
\]
Then,
\[
II(f^n) = \left\{ d_1^2 + \sum_{j=2}^{\infty} \left( \sum_{k=1}^{n_j} n_j^{-1}(k - 1) d_k \right)^2 \right\}^\frac{1}{2}
\]
\[
= \left[ d_1^2 + \sum_{j=2}^{\infty} \left( \sum_{k=1}^{n_j} n_j^{-1}(k - 1) d_k \right)^2 + \sum_{j: n_j > n} \left( \sum_{k=1}^{n_j} n_j^{-1}(k - 1) d_k \right)^2 \right]^\frac{1}{2}
\]
\[
\leq \left[ d_1^2 + \pi(f) + \sum_{j: n_j > n} \left( \frac{n_j}{n} \left( f_n - s_n \right)^2 \right) \right]^\frac{1}{2}
\]
\[
\leq |d_1| + \pi(f)^{\frac{1}{2}} + \left( \frac{q_1^2}{q_1^2 - 1} \right)^{\frac{1}{2}} |f_n - s_n|.
\]
Note that $\{n(f_n - s_n); n \geq 1\}$ is a martingale, so, by the submartingale inequality,
\[
E\left[ |f_n - s_n| \right] = \frac{1}{n} E\left[ |f_n - s_n| \right] \leq \frac{1}{n} E\left[ n |f_n - s_n| \right] \leq E\left[ n \pi(f) \right] \leq q_2 E\left[ \pi(f) \right]^{\frac{1}{2}}
\]
where $n_{k-1} \leq n \leq n_k$. Therefore
\[
E\left[ II(f^n) \right] \leq E\left[ |d_1| \right] + \left( 1 + \frac{q_1 q_2}{q_1^2 - 1} \right) E\left[ \pi(f) \right]^{\frac{1}{2}}.
\]
So, if $E[\pi(f)]^{\frac{1}{2}} < \infty$, then $\sup_n E[II(f^n)] < \infty$ and $\lim_{n \to \infty} f_n$ exists almost surely. q.e.d.

We give an example of a martingale $f$, such that $E[\lambda(f)]^{\frac{1}{2}} < \infty$ and
$E[\pi(f)^{1/2}] < \infty$, but $E[\sup_{n} d_n] = \infty$. Let $\Omega = (0, 1]$, $\mathcal{B}$ be a family of all Borel sets on $\Omega$, $P$ be a Lebesgue measure on $\mathcal{B}$, $\mathcal{F}_n$ be a $\sigma$-field generated by the sets $(0, 1/n], (1/n, 1/(n-1)], (1/(n-1), 1/(n-2)], \ldots, (1/2, 1]$. We define $d_n's$ by $d_1 \equiv 0$, $d_n = -\frac{2^n}{n-1} I_{(0, 1/n]} + \frac{n-1}{n} I_{1/(n-1), 1/(n-2)]} (n \geq 2)$. This gives a desired example.

References