On the Norlund Summability of Laguerre Series

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1. Definitions and Notations

Let \( f(t) \) be a Lebesgue-measurable function such that the integral

\[
\int_0^\infty e^{-x^\alpha} f(x) L_n^{(\alpha)}(x) \, dx, \quad \alpha > -1
\]

exists, where \( L_n^{(\alpha)}(x) \) denotes the \( n \)th Laguerre polynomial of order \( \alpha \).

The Laguerre series corresponding to this function \( f(x) \) is

\[
f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)
\]

in which

\[
a_n = \frac{1}{\Gamma(\alpha + 1)} A_n^\alpha \int_0^\infty e^{-y^\alpha} f(y) L_n^{(\alpha)}(y) \, dy
\]

and

\[
A_n^\alpha = \left( \frac{n+\alpha}{n} \right) \sim n^\alpha.
\]

Let \( \sum a_n \) be a given infinite series and \( \{s_n\} \) the sequence of its partial sums. A sequence \( \{s_n\} \) is said to be summable by harmonic means,\(^1\) if

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_{n-k}}{k+1}
\]

exists.

Let \( \{p_n\} \) be a sequence of real constants such that \( p_0 > 0, \ p_n \geq 0 \) and let us write

\[
P_n = p_0 + p_1 \cdots + p_n, \quad P_{-1} = p_{-1} = 0.
\]

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The sequence-to-sequence transformations:

\[ \tau_n = \frac{1}{N_n} \sum_{k=0}^{n} p_k s_{n-k} (P_n \neq 0) \]  

(4)

defines the sequence of Nörlund means of the sequence \( \{s_n\} \) generated by the sequence of coefficients \( \{p_n\} \).

The series \( \sum a_n \) is said to be summable \((N, p_n)\) to the sum \( s \) if \( \lim_{n \to \infty} \tau_n \) exists and is equal to \( s \), and further is said to be regular\(^1\) if it sums every convergent series to its ordinary sum.

In the special case in which \( p_n = \frac{1}{n+1} \), the Nörlund mean reduces to the harmonic mean stated above.

2. Introduction

Recently B. S. Pandey has proved the following theorem.

Theorem A.\(^2\) For \(-\frac{1}{2} > \alpha \geq -\frac{5}{6}\), the series \( \sum_n a_n L_n^{(\alpha)} (x) \) is summable to sum \( s \) by harmonic means at the point \( x = 0 \), provided,

\[ \int_0^t |\varphi(y)| \, dy = o(t^{\alpha+1}), \text{ as } t \to +0, \]

\[ \int_0^\infty e^{y/2} y^{-\alpha/2-3/4} |\varphi(y)| \, dy = o(n^{-\alpha/2-1/4}) \]

and

\[ \int_n^\infty e^{y/2} y^{-1/3} |\varphi(y)| \, dy = o(1), \]

where

\[ \varphi(y) = \frac{1}{\Gamma(\alpha + 1)} e^{-y} \{ f(y) - s \} y^\alpha. \]

In this note we shall prove a theorem concerning Nörlund summability which includes, as a particular case, theorem A stated above.

3. The main theorem

We establish the following theorem which includes, as a particular case, the theorem due to B. S. Pandey.\(^2\)

Theorem. we write

\[ \varphi(y) = \frac{1}{\Gamma(\alpha + 1)} e^{-y} \{ f(y) - s \} y^\alpha, \]
and let $e^{y/2} y^{-1/3} \varphi(y)$ be Lebesgue integrable over $(1, \infty)$.

If

$$\int_0^t |\varphi(y)| \, dy = o(t^{s+1})$$

as $t \to +0$, and

$$\int_{\omega}^{n^2/2} e^{y/2} y^{-3/4} \varphi(y) \, dy = o\left(n^{-\alpha/2-1/4}\right)$$

as $n \to \infty$, then for $-\frac{1}{2} > \alpha \geq -\frac{5}{6}$ Laguerre series

$$\sum_{n=0}^\infty a_n L_n^{(a)}(x)$$

is summable to the sum $s$ by regular Nörlund means $(N, p_n)$ at the point $x = 0$.

4. Preliminary lemmas

Lemma 1. \(\text{Let } \alpha \text{ be arbitrary and real, } c \text{ and } \omega \text{ be fixed positive constants. Then for } n \to \infty\)

$$L_n^{(a)}(x) = \begin{cases} \frac{x^{-\alpha/2-1/4}}{\omega^{\alpha/2-1/4}} O(\omega^{\alpha/2-1/4}) & \text{if } \frac{c}{n} \leq x \leq \omega, \\ O(\omega^n) & \text{if } 0 \leq x \leq \frac{c}{n}. \end{cases}$$

Lemma 2. \(\text{Let } \alpha \text{ be arbitrary and real, } \omega > 0, 0 < \eta < 4. \text{ We have for } n \to \infty\)

$$\max e^{-x/2} x^{\alpha/2+1/4} \left| L_n^{(a)}(x) \right| \sim \begin{cases} n^{\alpha/2-1/4} & \text{if } \omega \leq x \leq (4-\eta) \omega, \\ n^{\alpha/2-1/11} & \text{if } x \geq \omega. \end{cases}$$

5. Proof of the Theorem.

Let $s_n$ denote the $n$th partial sum of the series $\sum_{n=0}^\infty a_n L_n^{(a)}(x)$ at the point $x = 0$, and $\tau_n$ denote the Nörlund means of the sequence $\{s_n\}$.

In order to prove the theorem, it is sufficient to demonstrated that

$$\tau_n - s = o(1), \text{ as } n \to \infty.$$

Since the integral in the left side of (6) increases with the increase of $n$, we have obviously $\alpha < -\frac{1}{2}$.

Now we have
\[ s_n = \sum_{k=0}^{n-1} \frac{1}{\Gamma(\alpha+1)} \frac{1}{A_k} L_k^{(\alpha)}(0) \int_0^{\infty} e^{-y} y^s f(y) L_k^{(\alpha)}(y) \, dy \]

\[ = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^s f(y) L_n^{(\alpha+1)}(y) \, dy. \]

Hence by the definition (4)

\[ \tau_n - s = \frac{1}{P_n} \sum_{k=0}^{n-1} p_k (s_{n-k}-s) \]

\[ = \frac{1}{P_n} \sum_{k=0}^{n-1} p_k (s_{n-k}-s) + \frac{b_n}{P_n} (s_0-s). \]

But, by (1)

\[ s_0 = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^s f(y) L_0^{(\alpha+1)}(y) \, dy \]

\[ = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^s f(y) \, dy \]

\[ = O(1), \text{ for } \alpha > -1, \]

and hence, by the regularity for Nörlund means, we have for \( \alpha > -1, \)

\[ \frac{b_n}{P_n} (s_0-s) = o(1), \text{ as } n \to \infty. \]

Therefore we have

\[ \tau_n - s = \frac{1}{P_n} \sum_{k=0}^{n-1} \frac{b_k}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-y} y^s \{ f(y) - s \} L_n^{(\alpha+1)}(y) \, dy + o(1) \]

\[ = \frac{1}{P_n} \sum_{k=0}^{n-1} \frac{b_k}{\Gamma(\alpha+1)} \int_0^{\infty} \varphi(y) L_n^{(\alpha+1)}(y) \, dy + o(1). \]

We now divide the integral into four parts such that

\[ \int_0^{\infty} \varphi(y) L_n^{(\alpha+1)}(y) \, dy \]

\[ = \left\{ \int_0^{c/(n-k)} + \int_{c/(n-k)}^{n-k} + \int_{n-k}^{\infty} \right\} \varphi(y) L_n^{(\alpha+1)}(y) \, dy \]

\[ = A_1 + B_1 + C_1 + D_1, \text{ say.} \]
Furthermore we set

$$
\tau_n - s = \frac{1}{P} \sum_{n,k=0}^{w-1} p_k (A_1 + B_1 + C_1 + D_1) + o(1)
$$

$$
= A + B + C + D + o(1), \text{ say.}
$$

In the estimation of $A$, we use Lemma 1 and our hypothesis (5), then

$$
|A| \leq \frac{1}{P} \sum_{n,k=0}^{w-1} p_k \int_0^c \varphi(y) |L_{n-k}^{(a+1)}(y)| \, dy
$$

$$
= \frac{1}{P} \sum_{n,k=0}^{w-1} p_k O(n-k)^{a+1} o(n-k)^{-(a+1)}
$$

$$
= o(1), \text{ as } n \to \infty.
$$

Similarly, by Lemma 1

$$
|B| \leq \frac{1}{P} \sum_{n,k=0}^{w-1} p_k \int_0^c \varphi(y) |L_{n-k}^{(a+1)}(y)| \, dy
$$

$$
= \frac{1}{P} \sum_{n,k=0}^{w-1} p_k O(n-k)^{a+1/2} \int_0^c \varphi(y) |y^{-a/2-5/4}| \, dy
$$

$$
= \frac{1}{P} \sum_{n,k=0}^{w-1} p_k O(n-k)^{a+1/4} \int_0^c \varphi(y) |y^{-a/2-3/4}| \, dy. \quad (7)
$$

Next, by integration by parts and hypothesis (5) we get

$$
\int_0^c \varphi(y) |y^{-a/2-3/4}| \, dy = \left\{ \Phi(y) y^{-a/2-3/4} \right\}_{c/(n-k)}^{e} + \left( \frac{\alpha}{2} + \frac{3}{4} \right) \int_0^c \Phi(y) y^{-a/2-7/4} \, dy
$$

$$
= K + o\left( \frac{C}{n-k} \right)^{a+1/4} + \int_0^c \Phi(y) y^{-a/2-3/4} \, dy
$$

$$
= K + o(n-k)^{-a/2-1/4}, \text{ as } n \to \infty, \ \alpha < -\frac{1}{2}, \quad (8)
$$

where

$$
\Phi(t) = \int_0^t |\varphi(y)| \, dy.
$$

Hence we have for $\alpha < -\frac{1}{2}$
\[ |B| \leq \frac{1}{P} \sum_{n-k=0}^{n-1} p_k O(n-k)^{\alpha/2+1/4} \{K+o(n-k)^{-\alpha/2-1/4}\} \]

\[ = o(1), \text{ as } n \to \infty, \]

by (7) and (8).

In the estimation of \( C \), we use Lemma 2 and hypothesis (6), then

\[ |C| \leq \frac{1}{P} \sum_{n-k=0}^{n-1} p_k \left| \frac{\varphi(y)}{L_{n-k}^{(\alpha+1)}} \right| dy \]

\[ \leq \frac{K}{P} \sum_{n-k=0}^{n-1} \left| \frac{\varphi(y)}{e^{\gamma y - (\alpha+1)/2 - 1/4} (n-k)^{(\alpha+1)/2 - 1/4}} \right| dy \]

\[ = \frac{K}{P} \sum_{n-k=0}^{n-1} \left| \frac{\varphi(y)}{e^{\gamma y - (\alpha+2)/2 - 3/4} (n-k)^{(\alpha+2)/4}} \right| dy \]

\[ = o(1), \text{ as } n \to \infty. \]

Lastly we shall estimate \( D \).

By hypothesis on \( \varphi(t) \) we get

\[ \int_{n}^{\infty} e^{\gamma y - 1/2} \varphi(y) dy = o(1), \text{ as } n \to \infty. \]

Hence we use Lemma 2, then for \( -\frac{1}{2} > \alpha \geq -\frac{5}{6} \),

\[ |D| \leq \frac{1}{P} \sum_{n-k=0}^{n-1} p_k \int_{n-k}^{\infty} \varphi(y) \left| L_{n-k}^{(\alpha+1)} \right| dy \]

\[ = \frac{1}{P} \sum_{n-k=0}^{n-1} p_k \int_{n-k}^{\infty} \varphi(y) e^{\gamma y - (\alpha+1)/2 - 1/4} (n-k)^{(\alpha+1)/2 - 1/4} dy \]

\[ = \frac{1}{P} \sum_{n-k=0}^{n-1} p_k \int_{n-k}^{\infty} \varphi(y) e^{\gamma y - (\alpha+2)/2 - 3/4} (n-k)^{(\alpha+2)/4} dy \]

\[ = \frac{1}{P} \sum_{n-k=0}^{n-1} p_k (n-k)^{\alpha/2 + 5/12} \int_{n-k}^{\infty} \varphi(y) e^{\gamma y - 1/3} dy \]

\[ \leq \frac{1}{P} \sum_{n-k=0}^{n-1} p_k \int_{n-k}^{\infty} \varphi(y) e^{\gamma y - 1/2} dy \]
Collecting above estimations we have
\[ \tau_n - s = o(1), \text{ as } n \rightarrow \infty. \]

This completes the proof of our theorem.

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References