On the integrability of functions defined by Walsh series

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Synopsis

It is known that the function defined by Walsh series with monotone coef ficients is very delicate in a neighborhood of the origin. The purpose of this note is to prove Integrability theorems for Walsh Fourier series.

I. Introduction

Let the Rademacher functions be defined by

$$\phi_o(x) = 1\ (0 \leqq x < 1/2),\ \ \phi_o(x) = -1\ (1/2 \leqq x < 1),$$

$$\phi_o(x+1) = \phi_o(x),\ \ \phi_n(x) = \phi_o(2^n . x)\ (n = 1, 2, \cdots).$$

Then the Walsh functions are given by

$$\psi_o(x) \equiv 1,\ \ \psi_n(x) = \phi_{n(1)}(x)\phi_{n(2)}\cdots\phi_{n(r)}(x)$$

for $n = 2^{n(1)} + 2^{n(2)} + \cdots + 2^{n(r)}$, where the integers $n(i)$ are uniquely determined by $n(i+1) < n(i)$. As is well known, $\{\psi_n(x)\}$ form a complete orthonormal system. Every periodic function $f(x)$ which is integrable on (0, 1) can be expa-nded into a Walsh Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} a_n\psi_n(x) \tag{1}$$

where the coefficients are given by

$$a_n = \int_0^1 f(x)\psi_n(x)dx\ \ (n = 0, 1, 2, \cdots\cdots). \tag{2}$$

If $f(x)$ has (1) as its WFS, we shall set

$$s_n(x) \equiv s_n(x; f) \equiv \sum_{k=0}^{n-1} a_k\psi_k(x)\ \ (n = 1, 2, \cdots\cdots).$$

The "Dirichlet kernel" of WFS is defined by

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(3) $$D_n(x) \equiv \psi_0(x) + \psi_1(x) + \cdots\cdots + \psi_{n-1}(x).$$

The size of $D(x)$ is given by

(4) $$|D_n(x)| < 2/x.$$

We write

(5) $$J_n(x) = \int_0^x \psi_n(t)dt \ (n = 0, 1, 2, \cdots\cdots).$$

For basic properties of Walsh functions, the readers are referred to N. J. Fine (3). A denotes a positive absolute constant that is not always the same.

II. Preliminary theorems

To prove main theorems, we shall state the necessary theorems;

Theorem 1. (S. Yano (5)). If $c_n \geqq c_{n+1} \to 0$ and the series $\sum_{n=0}^{\infty} c_n\psi_n(x)$ converges, save for $x = 0$, to an integrable function $f(x)$, then the series $\sum_{n=0}^{\infty} c_n\psi_n(x)$ is the Fourier series of $f(x)$.

Theorem 2. (N. J Fine (3)). Let

$$f(x) \sim \sum_{k=0}^{\infty} a_k\psi_k(x).$$

Then

$$\int_0^x f(t)dt = \sum_{k=0}^{\infty} a_k J_k(x).$$

Theorem 3. (G. Sunouchi (4)). If $f(x) \geqq 0$, $s > 0$ and $\int_0^x f(t)dt = F(x)$, then

$$\int_0^\infty \left(\frac{F(x)}{x}\right) x^{-s}dx = \frac{1}{s}\int_0^\infty f(x)x^{-s}dx.$$

Theorem 4. Let $f(x)$ be a non-negative function defined for $x \geqq 0$, and let be $p > 1$, $s < p-1$. Then if $f(x)$ is integrable over $(0, \infty)$ so is $\{x^{-1}F(x)\}^p x^s$, where $F(x) = \int_0^x f(t)dt$. Moreover,

$$\int_0^\infty \left\{\frac{F(x)}{x}\right\}^p x^s dx \leqq \left(\frac{p}{p-s-1}\right)^p \int_0^\infty f^p(x)x^s dx.$$

This is due to A. Zygmund (6).

Lemma 1. (N. J. Fine (3)). If $J_k(x) = \int_0^x \psi_k(t)dt$, then

$$J_k(x) = 2^{-(n+2)}\left\{\psi_{k'}(x) - \sum_{r=1}^{\infty} 2^{-r}\psi_{2^{n+r}+k}(x)\right\}$$

where $k = 2^n + k'$, $0 \leqq k' < 2^n$.

From the lemma above, we obtain

(6) $$|J_k(x)| < {}^1/_k.$$

Lemma 2. Suppose that $a_1 \geqq a_2 \geqq \cdots \to 0$ and

$$f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x) \in L(0, 1).$$

Then we have

$$a_{2^l} \leqq 2F\left(\frac{1}{2^l}\right), \quad l = 1, 2, \cdots\cdots,$$

where $F(x) = \int_0^x f(t)dt$

Proof

It follows from Theorem 4 that

$$F(x) = \int_0^x f(t)dt = \sum_{k=1}^{\infty} a_k J_k(x).$$

Thus

$$F\left(\frac{1}{2^l}\right) = \sum_{k=1}^{\infty} a_k J_k\left(\frac{1}{2^l}\right) = \sum_{k=1}^{2^l-1} a_k J_k\left(\frac{1}{2^l}\right) + \sum_{k=2^l}^{\infty} a_k J_k\left(\frac{1}{2^l}\right).$$

By the properties of Walsh functions, we get

$$J_k\left(\frac{1}{2^l}\right) = \frac{1}{2^l} \text{ for } 0 \leqq k \leqq 2^l - 1$$

$$J_k\left(\frac{1}{2^l}\right) = 0 \text{ for } 2^l \leqq k < \infty.$$

Hence we have

$$F\left(\frac{1}{2^l}\right) = \sum_{k=1}^{2^l-1} a_k J_k\left(\frac{1}{2^l}\right) = \sum_{k=1}^{2^l-1} \frac{1}{2^l} a_k \geqq \frac{a_{2^l}}{2^l} \sum_{k=1}^{2^l-1} 1$$

$$= \frac{a_{2^l}}{2^l}(2^l - 1) \geqq \frac{1}{2} a_{2^l}.$$

Lemma 3. If $f(x)$ is positive and decreases in $(0, 1)$, and a_n are Walsh Fourier coefficients of $f(x)$, then we have

(i) $$|a_n| \leqq 3F\left(\frac{1}{n}\right) \ (n = 1, 2, \cdots\cdots)$$

(ii) $$f\left(\frac{1}{2^n}\right) \leqq s_{2^n} \ (n = 0, 1, \cdots\cdots)$$

where $F(x) = \int_0^x f(t)dt$ and $s_n = \sum_{k=0}^{n-1} |a_k|$.

Proof

Since

$$a_n = \int_0^1 f(t)\phi_n(x)dx = \int_0^{\frac{1}{n}} f(x)\phi_n(x)dx + \int_{\frac{1}{n}}^1 f(x)\phi_n(x)dx,$$

we get

(7) $$\begin{aligned} |a_n| &\leqq \int_0^{\frac{1}{n}} f(x)dx + \left|\int_{\frac{1}{n}}^1 f(x)\phi_n(x)dx\right| \\ &\leqq F\left(\frac{1}{n}\right) + \left|\int_{\frac{1}{n}}^1 f(x)\phi_n(x)dx\right|. \end{aligned}$$

By the second mean theorem, we have

$$\int_{\frac{1}{n}}^1 f(x)\phi_n(x)dx = f\left(\frac{1}{n}\right)\int_{\frac{1}{n}}^{\xi} \phi_n(x)dx = f\left(\frac{1}{n}\right)\left((J_n(\xi) - J_n\left(\frac{1}{n}\right)\right)\left(\frac{1}{n} \leqq \xi \leqq 1\right).$$

Hence we obtain by lemma 1

(8) $$\left|\int_{\frac{1}{n}}^1 f(x)\phi_n(x)dx\right| \leqq f\left(\frac{1}{n}\right)\left((|J_n(\xi)| + |J_n\left(\frac{1}{n}\right)|\right) \leqq \frac{2}{n} f\left(\frac{1}{n}\right).$$

On the other hand,

(9) $$F\left(\frac{1}{n}\right) = \int_0^{\frac{1}{n}} f(x)dx \geqq \frac{1}{n} f\left(\frac{1}{n}\right).$$

By (8) and (9), we have

(10) $$\left|\int_{\frac{1}{n}}^1 f(x)\phi_n(x)dx\right| \leqq 2F\left(\frac{1}{n}\right).$$

Therefore, it follows from (7) and (10) that

$$|a_n| \leqq 3F\left(\frac{1}{n}\right).$$

If we set $s_{2^n} = \sum_{k=0}^{2^n-1} |a_k|$, then we obtain

$$\sum_{k=0}^{2^n-1} a_k = \sum_{k=0}^{2^n-1} \int_0^1 f(x)\psi_k(x)\,dx = \int_0^1 f(x)\left(\sum_{k=0}^{2^n-1} \psi_k(x)\right)dx$$

$$= \int_0^1 f(x) D_{2^n}(x)dx = 2^n \int_0^{\frac{1}{2^n}} f(x)dx$$

$$\geqq 2^n f\left(\frac{1}{2^n}\right)\int_0^{\frac{1}{2^n}} dx = f\left(\frac{1}{2^n}\right).$$

Hence

$$f\left(\frac{1}{2^n}\right) \leqq s_{2^n}.$$

III. Main theorems

Theorem A. Suppose that $a_1 \geqq a_2 \geqq \cdots\cdots \to 0$ and

$$f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x).$$

Then for : (i) $p \geqq 1$, (ii) $1-p < \gamma < 1$; $x^{-\gamma}\{f(x)\}^p \in L^p$ (0, 1) if and only if the series

$$\sum_{n=1}^{\infty} n^{\gamma+p-2} a_n{}^p$$

converges.

This theorem is the Walsh-analogue of results due to Y. M. Chen ((1), (2)) and G. Sunouchi (4).

Proof of Theorem A.

First we shall prove the case $p = 1$ in Theorem A. If $x^{-\gamma} f(x) \in L(0, 1)$, then $f(x) \in L(0, 1)$.

We now set

$$F(x) = \int_0^x f(t)dt, \quad H(x) = \int_0^x |f(t)|\,dt.$$

Then we have by lemma 2

$$\sum_{n=1}^{\infty} n^{\gamma-1} a_n = \sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} k^{\gamma-1} a_k \leqq \sum_{n=0}^{\infty} a_{2^n} \sum_{k=2^n}^{2^{n+1}} k^{\gamma-1}$$

$$(11) \qquad \leqq A\sum_{n=0}^{\infty} 2^{n\gamma} a_{2^n} \leqq A\sum_{n=1}^{\infty} 2^{n\gamma} F\left(\frac{1}{2^n}\right) + A\int_0^1 |f(x)|\, dx$$

$$\leqq A\sum_{n=1}^{\infty} 2^{n\gamma} H\left(\frac{1}{2^n}\right) + A\int_0^1 | f(x)|\, dx.$$

On the other hand, we get

$$(12) \qquad \int_0^1 \left\{\frac{H(x)}{x}\right\} x^{-\gamma} dx = \sum_{n=1}^{\infty} \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \left\{\frac{H(x)}{x}\right\} x^{-\gamma} dx$$

and

$$(13) \qquad \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \left\{\frac{H(x)}{x}\right\} x^{-\gamma} dx \geqq H\left(\frac{1}{2^n}\right) \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} x^{-\gamma-1} dx \geqq A 2^{n\gamma} H\left(\frac{1}{2^n}\right).$$

By (11), (12) and (13), it follows that

$$\sum_{n=1}^{\infty} n^{\gamma-1} a_n \leqq A\sum_{n=1}^{\infty} 2^{n\gamma} H\left(\frac{1}{2^n}\right) + A\int_0^1 |f(x)|\, dx$$

$$\leqq A\sum_{n=1}^{\infty} \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \left\{\frac{H(x)}{x}\right\} x^{-\gamma} dx + A\int_0^1 |f(x)|\, dx$$

$$\leqq A\int_0^1 \left\{\frac{H(x)}{x}\right\} x^{-\gamma} dx + A\int_0^1 |f(x)|\, dx < \infty,$$

and so the necessity of the condition is established.
To prove that the condition is sufficicient, we observe that

$$|f(x)| \leqq \sum_{k=1}^{n} a_k + \left| \sum_{k=n+1}^{\infty} a_k \phi_k(x) \right| \leqq s_n^* + \frac{A\, a_n}{x}, \quad (s_n^* = \sum_{k=1}^{n} a_k).$$

Then we get

$$|f(x)| \leqq A s_n^*, \quad \text{for} \frac{1}{n+1} \leqq x < \frac{1}{n}.$$

Hence we have by Theorem 3

$$\int_0^1 x^{-\gamma} |f(x)|\, dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} x^{-\gamma} |f(x)|\, dx \leqq A\sum_{n=1}^{\infty} s_n^* \int_{\frac{1}{n+1}}^{\frac{1}{n}} x^{-\gamma} dx$$

$$\leqq A\sum_{n=1}^{\infty} s_n^*((n+1)^{\gamma-1} - n^{\gamma-1}) \leqq A\sum_{n=1}^{\infty} n^{\gamma-2}s_n^*$$

$$= A\sum_{n=1}^{\infty} n^{\gamma-1}\left(\frac{s_n^*}{n}\right) = A\sum_{n=1}^{\infty} n^{\gamma-1}a_n < \infty.$$

If we use Theorem 4 in place of Theorem 3, the remainning case is proved by the similar method to the proof of the case $p=1$ in Theorem A.

The following theorem is the dual for Theorem A.

Theorem B. If $f(x)$ is positive and decreases in $(0, 1)$, and a_n are Walsh-Fourier coefficients of $f(x)$, then for : (i) $p \geqq 1$, (ii) $1-p < \gamma < 1$; the series $\sum_{n=1}^{\infty} n^{\gamma+p-2}|a_n|^p$ converges if and only if $x^{-\gamma}f(x)^p \in L(0, 1)$.

Though our method of proof is essentially the same that is used in Theorem A, we shall prove this theorem for the convenience of the reader.

Proof of Theorem B.

We shall prove the sufficient condition. If we put $F(x) = \int_0^x f(t)dt$, then we get by Theorem 3

$$\int_0^1 \left\{\frac{F(x)}{x}\right\} x^{-\gamma}dx = \frac{1}{\gamma}\int_0^1 x^{-\gamma}f(x)dx.$$

Therefore, we have by lemma 3

$$\infty > \int_0^1 F(x)x^{-1-\gamma}dx = \sum_{n=2}^{\infty}\int_{\frac{1}{n}}^{\frac{1}{n-1}} F(x)x^{-1-\gamma}dx$$

$$\geqq \sum_{n=2}^{\infty} F\left(\frac{1}{n}\right)\int_{\frac{1}{n}}^{\frac{1}{n-1}} x^{-1-\gamma}dx$$

$$\geqq A\sum_{n=2}^{\infty} F\left(\frac{1}{n}\right)(n^{\gamma} - (n-1)^{\gamma})$$

$$\geqq A\sum_{n=2}^{\infty} F\left(\frac{1}{n}\right)n^{\gamma-1} \geqq A\sum_{n=1}^{\infty} n^{\gamma-1}|a_n|.$$

Next we shall prove the necessary condition. If we set

$$a(x) = |a_n| \text{ for } n-1 \leqq x < n (n = 1, 2, \cdots)$$

$$s(x) = \int_0^x a(t)dt,$$

then the finiteness of $\sum_{n=1}^{\infty} n^{\gamma-1}|a_n|$ implies the finiteness of $\int_0^{\infty} x^{\gamma-1}a(x)dx$.

By using Theorem 3, we obtain

$$(14)\qquad \int_0^\infty \left\{\frac{s(x)}{x}\right\} x^{\gamma-1}dx = \frac{1}{1-\gamma}\int_0^\infty a(x)x^{\gamma-1}dx < \infty.$$

Hence

$$\int_0^\infty \left\{\frac{s(x)}{x}\right\} x^{\gamma-1}dx = \int_0^\infty s(x)x^{\gamma-2}dx \geqq \int_1^\infty s(x)x^{\gamma-2}dx$$

$$(15)\qquad = \sum_{n=0}^\infty \int_{2^n}^{2^{n+1}} s(x)x^{\gamma-2}dx \geqq \sum_{n=0}^\infty s(2^n)\int_{2^n}^{2^{n+1}} x^{\gamma-2}dx$$

$$\geqq A\sum_{n=0}^\infty \frac{2^{n\gamma}}{2^n}s(2^n).$$

On the other hand, since $f(x)$ is monotone and $s_{2^n} \leqq As(2^n)$, we have

$$\int_0^1 x^{-\gamma}f(x)dx = \sum_{n=1}^\infty \int_{\frac{1}{2}}^{\frac{1}{2^{n-1}}} {}^{n}x^{-\gamma}f(x)dx \leqq \sum_{n=1}^\infty f\left(\frac{1}{2^n}\right)\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} x^{-\gamma}dx$$

$$\leqq A\sum_{n=1}^\infty \frac{2^{n\gamma}}{2^n} f\left(\frac{1}{2^n}\right) \leqq A\sum_{n=1}^\infty \frac{2^{n\gamma}}{2^n} s_{2^n}$$

$$\leqq A\sum_{n=1}^\infty \frac{2^{n\gamma}}{2^n} s(2^n).$$

By (14), (15) and (16), it follows that

$$\int_0^1 x^{-\gamma}f(x)\,dx \leqq A\int_0^\infty \left\{\frac{s(x)}{x}\right\} x^{\gamma-1}dx \leqq A\int_0^\infty a(x)x^{\gamma-1}dx < \infty.$$

Hence the case $p=1$ of theorem is proved. The remainning case is also done by the same method.

REFERENCES

(1) Y. M. Chen, On the integrability of functions defined by trigonometrical series, Math. Zeit., 66 (1956), 9–12.

(2) Y. M. Chen, Some Asymptotic Properties of Fourier Constants and Integrability Theorem, Math. Zeit., 68 (1957), 227–244.

(3) N. J. Fine, On the Walsh functions, Trans. Amer. Math. Soc., 65 (1949), 372–414.

(4) G. Sunouchi, Integrability of Trigonometric Series, Jour. Math., 1 (1953), 99–103.

(5) S. Yano, On Walsh-Fourier series, Tôhoku Math. J., 3 (1951), 223–242.

(6) A. Zygmund, Trigonometric series, I, Cambridge (1959).