Integrability theorems for Walsh Fourier series

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(Received July 4, 1966)

SYNOPSIS

Integrability theorems for trigonometric series have been researched by B. Sz-Nagy (2), P. Heywood (4), R. P. Boas (3), M. M. Robertson (6) and so on.

The purpose of this note is to obtain certain analogues for Walsh Fourier series of M. M. Robertson (6) concerning the integrability of trigonometric series.

INTRODUCTION

First, we begin with some notations and definitions :

The Rademacher functions are defined by

$$\phi_0(x) = 1 \quad (0 \leqq x < \frac{1}{2}), \qquad \phi_0(x) = -1 \, (\frac{1}{2} \leqq x < 1)$$

$$\phi_0(x) = \phi_0(x+1), \qquad \phi_n(x) = \phi_0(2^n x) \quad (n = 1, \ 2, \cdots\cdots).$$

The Walsh functions are then given by

$$\psi_0(x) \equiv 1, \qquad \psi_n(x) = \phi_{n1}(x)\,\phi_{n2}(x)\cdots\cdots\phi_{nr}(x)$$

for $n = 2^{n_1} + 2^{n_2} + \cdots\cdots + 2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1} < n_i$. Walsh proves that $\{\psi_n(x)\}$ form a complete orthonormal system.

Every periodic function $f(x)$ which is integrable in the sense of Lebesgue on $(0, 1)$ will associate with it a Walsh Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} a_k \psi_k(x)$$

where the coefficients are given by

$$a_k = \int_0^1 f(x)\,\psi_k(x)\,dx, \quad (k = 0, \ 1, \ 2, \cdots\cdots).$$

We write

$$J_k(x) = \int_0^x \psi_k(t) dt.$$

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For basic properties of Walsh functions, the reader is referred to N. J. Fine (1). Finally, C denotes a positive absolute constant not always the same. Though our method of proof is essentially the same as that used by M. M. Robertson (6), we shall prove these theorems for the convenience of the reader.

I

Theorem 1. Suppose that $0 \leqq \eta(x) \in L(0,1)$, $f(x) \in L(0,1)$ and

$$a_k = \int_0^1 f(x)\psi_k(x)\,dx, \quad \text{for } k=1,\ 2,\cdots\cdots.$$

If $\eta(x) f(x) \in L(0,1)$ and if also there is a positive number $\delta \leqq 1$ such that, for all t in $0 < t \leqq \delta$,

$$t^{-1}\int_0^t \eta(x)dx \leqq c\eta(t), \quad \int_t^\delta x^{-1}\eta(x)dx \leqq c\eta(t), \tag{1.1}$$

then the series

$$\sum_{k=1}^{\infty} a_k \int_0^{1/k} \eta(x)dx$$

is convergent.

Proof

For every positive integer N greater than $\{\min(1,\delta)\}^{-2}$, we have

$$\begin{aligned}\sum_{k=1}^{N} a_k \int_0^{1/k} \eta(x)dx &= \sum_{k=1}^{N} \int_0^1 f(t)\,\psi_k(t)dt \int_0^{1/k} \eta(x)dx \\ &= \int_0^1 f(t)\left\{\sum_{k=1}^{N} \psi_k(t)\int_0^{1/k}\eta(x)\,dx\right\}\mathrm{dt}.\end{aligned} \tag{1.2}$$

We write $\varepsilon = N^{-1}$, $\zeta = \varepsilon^{\frac{1}{2}} = N^{-\frac{1}{2}}$ and note that $0 < \varepsilon < \zeta < \min(1,\delta)$. Since

$$\left\{\int_0^{1/k} \eta(x)\,dx\right\}$$

is a positive decreasing null sequence, the series

$$\sum_{k=1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x)dx$$

is convergent for all real valued of t in $0 < t \leqq 1$. By (1.2), we obtain

$$\begin{aligned}\sum_{k=1}^{N} a_k \int_0^{1/k} \eta(x)dx &= \int_0^\varepsilon f(t)\left\{\sum_{k=1}^{N}\psi_k(t)\int_0^{1/k}\eta(x)dx\right\}\mathrm{dt} \\ &+ \int_\varepsilon^1 f(t)\left\{\sum_{k=1}^{\infty}\psi_k(t)\int_0^{1/k}\eta(x)dx\right\}\mathrm{dt} \\ &- \int_\varepsilon^\zeta f(t)\left\{\sum_{k=N+1}^{\infty}\psi_k(t)\int_0^{1/k}\eta(x)dx\right\}\mathrm{dt}\end{aligned} \tag{1.3}$$

$$- \int_{\zeta}^{\delta} f(t) \Big\{ \sum_{k=N+1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x) dx \Big\} \mathrm{dt}$$

$$- \int_{\delta}^{1} f(t) \Big\{ \sum_{k=N+1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x) dx \Big\} \mathrm{dt}$$

$$= I_1 + I_2 - I_3 - I_4 - I_5, \text{ say.}$$

We shall complete the proof of the theorem by showing that, as N tends to infinity, I_2 tends to a finite limit and I_1, I_3, I_4 and I_5 tend to zero.

We write $T = [t^{-1}]$, $\varDelta = [\delta^{-1}]$. By Abel's lemma and (1. 1), we have, for all t in $0 < t \leqq \delta$,

$$\Big| \sum_{k=\varDelta+1}^{N} \psi_k(t) \int_0^{1/k} \eta(x) dx \Big| \leqq \sum_{k=\varDelta+1}^{T} \int_0^{1/k} \eta(x) dx + \int_0^t \eta(x) dx \max_{T+1 \leqq r \leqq N} \Big| \sum_{k=T+1}^{r} \psi_k(t) \Big|$$

$$\leqq T \int_0^{1/T+1} \eta(x) dx + \sum_{k=\varDelta+1}^{T} k \int_{1/k+1}^{1/k} \eta(x) dx + \frac{c}{t} \int_0^t \eta(x) dx \tag{1. 4}$$

$$\leqq ct^{-1} \int_0^t \eta(x) dx + c \int_{1/T+1}^{\delta} x^{-1} \eta(x) dx \leqq c\eta(t).$$

It follows from (1. 3) that

$$|I_1| = \Big| \int_0^{\varepsilon} f(t) \Big\{ \sum_{k=1}^{N} \psi_k(t) \int_0^{1/k} \eta(x) dx \Big\} dt$$

$$\leqq \int_0^{\varepsilon} |f(t)| \varDelta \int_0^1 \eta(x) dx + c\eta(t) \Big\} dt \tag{1. 5}$$

$$\leqq c \int_0^{\varepsilon} |f(t)| dt + c \int_0^{\varepsilon} \eta(t) |f(t)| dt.$$

Since $f(x)$, $\eta(x) f(x) \epsilon L(0, 1)$, $I_1 \to 0$ as $\varepsilon \downarrow 0$, i. e. as $N \to \infty$.
By (1. 3) and (1. 1), we have

$$|I_3| = \Big| \int_{\varepsilon}^{\zeta} f(t) \Big\{ \sum_{k=N+1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x) dx \Big\} dt \Big|$$

$$\leqq c \int_{\varepsilon}^{\zeta} |f(t)| \frac{1}{t} dt \int_0^{\varepsilon} \eta(x) dx$$

$$\leqq c \int_{\varepsilon}^{\zeta} |f(t)| \Big\{ \frac{1}{t} \int_0^t \eta(x) dx \Big\} dt$$

$$\leqq c \int_{\varepsilon}^{\zeta} \eta(t) |f(t)| dt,$$

and thus $I_3 \to 0$ as $N \to \infty$. By (1. 3) and (1. 1), we have

$$|I_5| = \left| \int_\delta^1 f(t) \left\{ \sum_{k=N+1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x)dx \right\} dt \right|$$

$$\leqq c \int_\delta^1 |f(t)| t^{-1}\, dt \int_0^\varepsilon \eta(x)\, dx$$

$$\leqq c \int_\delta^1 |f(t)|\, dt \int_0^\varepsilon \eta(x)\, dx$$

$$\leqq c \int_0^\varepsilon \eta(x)\, dx,$$

and thus $I_5 \to 0$ as $N \to \infty$. From (1.1), we have

$$\int_0^\zeta \eta(t)\, dt \geqq \int_\varepsilon^\zeta \eta(t)dt \geqq c \int_\varepsilon^\zeta \left(t^{-1} \int_0^t \eta(x)\, dx\right) dt \geqq c \log(\zeta/\varepsilon) \int_0^\varepsilon \eta(x)\, dx. \tag{1.6}$$

Hence it follows by (1.3) that

$$|I_4| = \left| \int_\zeta^\delta f(t) \left\{ \sum_{k=N+1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x)\, dx \right\} dt \right|$$

$$\leqq c \int_\zeta^\delta |f(t)| t^{-1}\, dt \int_0^\varepsilon \eta(x)\, dx$$

$$\leqq c \left\{ \log(\zeta/\varepsilon) \right\}^{-1} \int_\zeta^\delta |f(t)| t^{-1}\, dt \int_0^\zeta \eta(x)\, dx$$

$$\leqq c \left\{ \log(\zeta/\varepsilon) \right\}^{-1} \int_\zeta^\delta |f(t)| \left\{ t^{-1} \int_0^t \eta(x)\, dx \right\} dt$$

$$\leqq c \left\{ \log(\zeta/\varepsilon) \right\}^{-1} \int_0^\delta \eta(t) |f(t)|\, dt$$

$$\leqq c \left\{ \log(\zeta/\varepsilon) \right\}^{-1},$$

and so $I_4 \to 0$ as $N \to \infty$.
From (1.3), we have

$$I_2 = \int_\varepsilon^1 f(t) \left\{ \sum_{k=1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x)\, dx \right\} dt.$$

For all t in $\delta \leqq t \leqq 1$, we have

$$\left| \sum_{k=1}^{\infty} \psi_k(t) \int_0^{1/k} \eta(x)dx \right| \leqq c\, \frac{1}{t} \int_0^1 \eta(x)\, dx$$

$$\leqq c\, \frac{1}{\delta} \int_0^1 \eta(x)\, dx \leqq c,$$

and, for all t in $0 < t \leqq \delta$, we have

(1. 7) $$\left|\sum_{k=1}^{\infty}\psi_k(t)\int_0^{1/k}\eta(x)\,dx\right| \leqq c\,\eta(t)+c$$

by (1. 4). Therefore

$$\int_0^1 f(t)\left\{\sum_{k=1}^{\infty}\psi_k(t)\int_0^{1/k}\eta(x)\,dx\right\}dt$$

exists as a Lebesque integral and so I_2 tends to this finite limit as $N\to\infty$, q. e. d.

II

Theorem 2. Suppose that $\eta(x)$ is a positive monotone decreasing function for $0<x\leqq 1$, $f(x)\in L(0,1)$ and

$$a_k=\int_0^1 f(x)\,\psi_k(x)\,dx, \quad \text{for } k=1,2,\cdots.$$

If the series

$$\sum_{k=1}^{\infty}|a_k|\int_0^{1/k}\eta(x)\,dx$$

is convergent and if also there is a positive number $\delta\leqq 1$ such that, for all t in $0<t\leqq\delta$,

(2. 1) $$t^{-1}\int_0^t \eta(x)\,dx \leqq c\,\eta(t),$$

then the integral

$$\int_{\to 0}^1 \eta(x)\,f(x)\,dx$$

exists as a Cauchy limit.

For the proof of this theorem, we require the following lemma.

Lemma. If we write $k=2^n+k'(0\leqq k'<2^n)$, then we have

$$J_k(x)=2^{-(n+2)}\left\{\psi_{k'}(x)-\sum_{r=1}^{\infty}2^{-r}\psi_{2^{n+r}+2^n+k'}(x)\right\}.$$

This lemma is due to N. J. Fine (1). From the above lemma, we obtain

(2. 2) $$|J_k(x)|\leqq 2^{-(n+2)}\sum_{r=0}^{\infty}2^{-r}=2^{-(n+1)}<1/k.$$

Proof of Theorem 2.

Let ε be any positive number less than min $(1,\delta)$. We write $N=[\varepsilon^{-1}]$, $M=[\varepsilon^{-\frac{1}{2}}]$ and

$$\varphi(x) = f(x) - \int_0^1 f(t)\,dt - \sum_{k=1}^{N} a_k\phi_k(x).$$

Then we have

$$\begin{aligned}\int_\varepsilon^1 \eta(x) f(x)\,dx &= \int_0^1 \eta(x) \sum_{k=1}^{N} a_k\phi_k(x)\,dx - \int_0^\varepsilon \eta(x) \sum_{k=1}^{M} a_k\phi_k(x)\,dx \\ &\quad - \int_0^\varepsilon \eta(x) \sum_{k=M+1}^{N} a_k\phi_k(x)\,dx + \int_\varepsilon^1 \eta(x)\,dx \int_0^1 f(t)\,dt \\ &\quad + \int_\varepsilon^1 \eta(x)\varphi(x)\,dx \\ &= J_1 - J_2 - J_3 + J_4 + J_5, \text{ say.}\end{aligned} \tag{2.3}$$

We shall complete the proof by showing that J_1 and J_4 tend to finite limits and J_2, J_3 and J_5 tend to zero as ε decreases to zero.
Since $\eta(x) \in L(0,1)$, clearly

$$J_4 \to \int_0^1 \eta(x)\,dx \int_0^1 f(t)dt.$$

Next, we have

$$\begin{aligned}\left|\int_0^1 \eta(x)\,\phi_k(x)\,dx\right| &= \left|\int_0^{1/k} \eta(x)\,\phi_k(x)\,dx + \int_{1/k}^1 \eta(x)\,\phi_k(x)\,dx\right| \\ &\leqq \int_0^{1/k} \eta(x)dx + \left|\int_{1/k}^1 \eta(x)\,\phi_k(x)\,dx\right|.\end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}\int_{1/k}^1 \eta(x)\phi_k(x)\,dx &= \eta(1/k) \int_{1/k}^{\xi} \phi_k(x)dx \quad (1/k \leqq \xi \leqq 1) \\ &= \eta(1/k)\ \Big[J_k(x)\Big]_{1/k}^{\xi} = \eta(1/k)\Big[J_k(\xi) - J_k(1/k)\Big]\end{aligned}$$

by the mean value theorem. Hence we have by (2.4) and (2.2)

$$\begin{aligned}\left|\int_0^1 \eta(x)\,\phi_k(x)\,dx\right| &\leqq \int_0^{1/k} \eta(x)dx + \eta(1/k)\ \Big\{|J_k(\xi)| + |J_k(1/k)|\Big\} \\ &\leqq \int_0^{1/k} \eta(x)dx + 2\frac{1}{k}\eta(1/k) \\ &\leqq \int_0^{1/k} \eta(x)dx + 2\int_0^{1/k} \eta(x)dx \leqq c\int_0^{1/k} \eta(x)dx.\end{aligned}$$

Since

$$\sum_{k=1}^{\infty} |a_k| \int_0^{1/k} \eta(x)dx$$

is convergent, it follows that

$$\sum_{k=1}^{\infty} a_k \int_0^1 \eta(x)\, \psi_k(x)\, dx$$

is absolutely convergent. Therefore, by (2. 3)

$$J_1 = \sum_{k=1}^{N} a_k \int_0^1 \eta(x)\, \psi_k(x)\, dx \longrightarrow \sum_{k=1}^{\infty} a_k \int_0^1 \eta(x)\, \psi_k(x)\, dx$$

as $N \rightarrow \infty$, i. e. as $\varepsilon \downarrow 0$.
It follows from (2. 1) and (1. 6) that, if $\zeta = \varepsilon^{\frac{1}{2}}$,

$$\int_0^{\zeta} \eta(x)dx \geqq c\, log\, (\zeta/\varepsilon) \int_0^{\varepsilon} \eta(x)dx.$$

By (2. 3), we have

$$\begin{aligned} |J_2| = \left| \int_0^{\varepsilon} \eta(x) \sum_{k=1}^{M} a_k\, \psi_k(x)\, dx \right| &\leqq \sum_{k=1}^{M} |a_k| \int_0^{\varepsilon} \eta(x)\, dx \\ &\leqq c \left\{ log\, (\zeta/\varepsilon) \right\}^{-1} \sum_{k=1}^{M} |a_k| \int_0^{\zeta} \eta(x)\, dx \\ &\leqq c \left\{ log\, (\zeta/\varepsilon) \right\}^{-1} \sum_{k=1}^{\infty} |a_k| \int_0^{1/k} \eta(x)dx, \end{aligned}$$

and thus $J_2 \rightarrow 0$ as $\varepsilon \downarrow 0$. From (2, 3), we have

$$\begin{aligned} |J_3| = \left| \int_0^{\varepsilon} \eta(x) \sum_{k=M+1}^{N} a_k \psi_k(x) dx \right| &\leqq \sum_{k=M+1}^{N} |a_k| \int_0^{\varepsilon} \eta(x)\, dx \\ &\leqq \sum_{k=M+1}^{N} |a_k| \int_0^{1/k} \eta(x)\, dx \end{aligned}$$

and so $J_3 \rightarrow 0$ as $\varepsilon \downarrow 0$.
Since $f(x) \in L(0, 1)$, it is clear that $\varphi(x) \in L(0, 1)$. Then if

$$\Phi(x) = \int_0^x \varphi(t) dt,$$

we obtain

$$\int_{\varepsilon}^1 \eta(x)\varphi(x)dx = \eta(1)\Phi(1) - \eta(\varepsilon)\Phi(\varepsilon) - \int_{\varepsilon}^1 \eta'(x)\Phi(x)dx. \tag{2. 5}$$

On the other hand,

$$\sum_{k=N+1}^{\infty} a_k \psi_k(x)$$

is the Walsh Fourier series of $\varphi(x)$, it may be integrated term by term whether it converges or not, and then we obtain

$$\Phi(x) = \sum_{k=N+1}^{\infty} a_k J_k(x)$$

for $0 \leqq x \leqq 1$ (see (1)). Therefore

$$\left| \eta(\varepsilon)\, \Phi(\varepsilon) \right| \leqq \eta(\varepsilon) \sum_{k=N+1}^{\infty} |a_k|\, |J_k(x)| \leqq \eta(\varepsilon) \sum_{k=N+1}^{\infty} \frac{1}{k} |a_k|$$

$$\leqq \sum_{k=N+1}^{\infty} |a_k| \frac{1}{k}\, \eta({}^1/_k)$$

$$\leqq \sum_{k=N+1}^{\infty} |a_k| \int_0^{1/k} \eta(x)\, dx$$

and

$$\left| \eta(1) \Phi(1) \right| \leqq \eta(1) \sum_{k=N+1}^{\infty} |a_k|\, |J_k(1)| \leqq \sum_{k=N+1}^{\infty} |a_k| \frac{1}{k}\, \eta({}^1/_k)$$

$$\leqq \sum_{k=N+1}^{\infty} |a_k| \int_0^{1/k} \eta(x) dx$$

and so, both $\eta(\varepsilon)\, \Phi(\varepsilon)$ and $\eta(1)\, \Phi(1)$ tend to zero as $\varepsilon \downarrow 0$. Next, we obtain

$$\left| \int_\varepsilon^1 \eta'(x)\, \Phi(x)\, dx \right| \leqq \left| \int_\varepsilon^1 \eta'(x)\, dx \right| \left| \max_{0 \leqq x \leqq 1} |\Phi(x)| \right|$$

$$\leqq \eta(\varepsilon) \sum_{k=N+1}^{\infty} |a_k|\, |J_k(x) \leqq \eta(\varepsilon) \sum_{k=N+1}^{\infty} \frac{1}{k} |a_k|$$

$$\leqq \sum_{k=N+1}^{\infty} |a_k| \frac{1}{k}\, \eta({}^1/_k) \leqq \sum_{k=N+1}^{\infty} |a_k| \int_0^{1/k} \eta(x) dx,$$

which tends to zero as $\varepsilon \downarrow 0$. Thus, by (2.3) and (2.5), $I_5 \to 0$ as $\varepsilon \downarrow 0$, q.e.d.

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