Stress Analysis of a Gravitating Simply-Supported Beam

Bennosuke TANIMOTO*, Dr. Eng.

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Synopsis. A gravitating long beam supported simply at both ends is analyzed by the polynomial method. The results obtained indicate that the replacement of the body force or the gravitation by the surface traction in the elementary theory of bending is sufficiently legitimate, except for the deflection in case of short span.

A beam in constructional engineering has always its own weight, and the usual elementary theory of bending assumes *this body force to be replaced by surface traction.* This article will reveal the legitimacy of this replacement, except for the deflection. As for the deflection, the usual elementary theory of bending gives a considerably low estimation for the beam of short span, as is the case in the beam subjected to a uniformly distributed load. In fact, the present calculation was made by being stimulated by the known procedure of the beam of uniformly distributed load [1),2),3)].

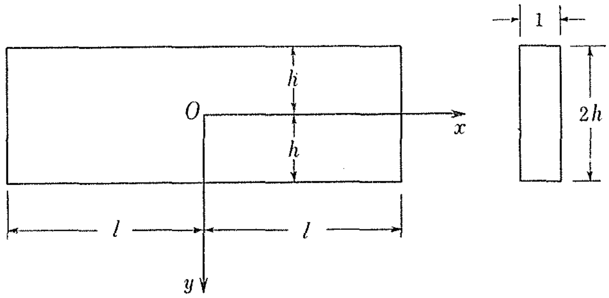

Fig. 1

Various cases of the beam of simple support and of cantilever type seem to have been treated, but all external forces to be applied to the beam are restricted to the surface traction only, and no attention, so far as I am aware, has been directed to the case of body force.

The beam is here supposed to be supported simply at both ends; its span being $2l$, its height $2h$, and its width, in the direction perpendicular to the

* Professor of Civil Engineering, Faculty of Engineering, Shinshu University, Nagano, Japan.

plane of this sheet, unity, as shown in Fig. 1. The origin of coordiates is situated at the centroid of the beam and the axis of x is directed rightwards, and the axis of y downwards.

Boundary conditions with which we shall be concerned are taken for convenience to be as follows:

1) The top and bottom surfaces of the beam are free from traction, which are expressed

$$(\widehat{xy})_{y=\pm h}=0, \qquad (\widehat{yy})_{y=\pm h}=0. \quad \cdots\cdots(1)$$

2) Both terminal surfaces $x=\pm l$ of the beam satisfy the conditions

$$\int_{-h}^{h}(\widehat{xy})_{x=\pm l}dy=\mp 2lh\rho, \qquad \int_{-h}^{h}(\widehat{xx})_{x=\pm l}dy=0, \qquad \int_{-h}^{h}(\widehat{xx})_{x=\pm l}\, ydy=0, \cdots(2)$$

ρ being the density of the beam. The second and the third equations in (2) imply that there are no resultant force and no resultant couple on the terminal planes $x=\pm l$.

The Airy's stress–function suitable for the present boundary–value problem will be found to be

$$\chi(x,y)=a\left(x^2y^3-\frac{1}{5}y^5\right)+cx^2y+dy^3,$$

where a, c, and d are constants to be determined.

Stresses can then be calculated by the operations

$$\widehat{xx}=\frac{\partial^2\chi}{\partial y^2}-\rho y, \qquad \widehat{yy}=\frac{\partial^2\chi}{\partial x^2}-\rho y, \qquad \widehat{xy}=-\frac{\partial^2\chi}{\partial x\partial y}.$$

Here the gravitation is directed downwards or to the positive direction of the axis of y (Fig. 1), and the unit of stress is measured by the gravitational one which is commonly adopted in practical engineering. We then have by substitution

$$\left.\begin{aligned}\widehat{xx}&=2a(3x^2y-2y^3)+(6d-\rho)y,\\ \widehat{yy}&=2ay^3+(2c-\rho)y,\\ \widehat{xy}&=-6axy^2-2cx.\end{aligned}\right\} \quad \cdots\cdots(3)$$

The boundary conditions (1) require the equations

$$-3ah^2-c=0, \qquad ah^3+ch=\frac{\rho h}{2},$$

from which we find

$$a=-\frac{\rho}{4h^2}, \qquad c=\frac{3}{4}\rho.$$

It can be seen that the first and the second equations in (2) are satisfied by these values. Then the remaining step is to determine d by the third equation in (2). This gives

$$d=\frac{\rho}{4}\left(\frac{l^2}{h^2}+\frac{4}{15}\right).$$

In virtue of these values, equations (3) become

$$\widehat{xx} = -\frac{3\rho}{2h^2}\left(x^2y - \frac{2}{3}y^3\right) + \frac{3}{2}\left(\frac{l^2}{h^2} - \frac{2}{5}\right)\rho y,$$

$$\widehat{yy} = -\frac{\rho}{2h^2}y^3 + \frac{\rho y}{2},$$

$$\widehat{xy} = -\frac{3\rho}{2h^2}(h^2 - y^2)x.$$

Noting that $2h^3/3$ is equal to the moment of inertia I of the rectangular cross-sectional area of unit width (Fig. 1), the above equations may take the forms

$$\left.\begin{aligned} \widehat{xx} &= \frac{\rho h}{I}(l^2 - x^2)y + \frac{2\rho h}{I}\left(\frac{y^2}{3} - \frac{h^2}{5}\right)y, \\ \widehat{yy} &= \frac{\rho h}{3I}(h^2 - y^2)y, \\ \widehat{xy} &= -\frac{\rho h}{I}(h^2 - y^2)x, \end{aligned}\right\} \quad \cdots\cdots(4)$$

which is the require stress distribution in the gravitating beam.

When the beam with no body force is subjected to a uniformly distributed load, say q, per unit of length of the beam, on its top surface, the stress distribution in the beam is [4]

$$\left.\begin{aligned} \widehat{xx} &= \frac{q}{2I}(l^2 - x^2)y + \frac{q}{I}\left(\frac{y^2}{3} - \frac{h^2}{5}\right)y, \\ \widehat{yy} &= -\frac{q}{2I}\left(\frac{1}{3}y^3 - h^2y + \frac{2}{3}h^3\right), \\ \widehat{xy} &= -\frac{q}{2I}(h^2 - y^2)x. \end{aligned}\right\} \quad \cdots\cdots(5)$$

It can be seen that $\widehat{xx}$ and $\widehat{xy}$ in (4) and (5) above are entirely of the same form. Since $2\rho h$ is equal to the weight of the slice of the beam of unit length, we may regard the solution (4) as if the weight $2\rho h$ were a uniformly distributed load applied on the top surface of the beam.

The first term in the right-hand side of $\widehat{xx}$ in equations (4) represents the stress given by the usual elementary theory of bending, and its second term gives the necessary correction. This corrrection does not depend on x, and is small in comparison with the maximum bending stress, when the span of the beam is large in comparison with its depth. For such beams the elementary theory of bending gives a sufficiently accurate value for the stress $\widehat{xx}$. This equation may be written in the form

$$\widehat{xx} = \frac{\rho h}{I}(l^2 - x^2)y\,(1 + \varepsilon_1), \quad \cdots\cdots(6)$$

where

$$\varepsilon_1 = \frac{2}{15}\left(\frac{h}{l}\right)^2 \frac{5\left(\frac{y}{h}\right)^2 - 3}{1 - \left(\frac{x}{l}\right)^2}, \quad \cdots\cdots(7)$$

which amounts to only 3.0% even when $l/h = 3.0$. It should be noticed, however, that the elementary theory of bending affords a little low estimation for the bending stress.

The shearing stress $\widehat{xy}$ in (4) is of parabolic distribution for any vertical cross section and increases with x, which is the same as in the elementary theory. The normal stress $\widehat{yy}$ is constant along the length of the beam, since it does not depend on x, and its distribution is also of parabolic one.

We shall find displacements corresponding to the stress system (4). We now have

$$\frac{\partial u}{\partial x} = \frac{1}{E}(\widehat{xx} - \sigma\widehat{yy}), \quad \frac{\partial v}{\partial y} = \frac{1}{E}(\widehat{yy} - \sigma\widehat{xx}), \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1+\sigma)}{E}\widehat{xy}. \quad \cdots(8)$$

Substituting (4) into the first two equations and integrating with respect to x and y respectively, we first have

$$\left.\begin{aligned} u &= \frac{\rho h}{EI}\left[xy\left(l^2 - \frac{x^2}{3}\right) + 2xy\left(\frac{y^2}{3} - \frac{h^2}{5}\right) - \frac{\sigma}{3}xy(h^2 - y^2)\right] + f_1(y), \\ v &= \frac{\rho h}{EI}\left[\frac{1}{6}\left(h^2 - \frac{y^2}{2}\right)y^2 - \sigma\left\{(l^2 - x^2)\frac{y^2}{2} + \left(\frac{y^2}{6} - \frac{h^2}{5}\right)y^2\right\}\right] + f_2(x), \end{aligned}\right\} \cdots\cdots(9)$$

where $f_1(y)$ and $f_2(x)$ are arbitrary functions. These functions will be determined in the following.

On the middle plane of the beam, the horizontal displacement u must vanish by symmetry, from which

$$(u)_{x=0} = 0 = f_1(y).$$

Substituting (9) into the third equation of (8), we have

$$\frac{\rho h}{EI}\left[x\left(l^2 - \frac{x^2}{3}\right) + \left(\frac{8}{5} + \frac{5}{3}\sigma\right)xh^2\right] + \frac{df_2(x)}{dx} = 0,$$

or, on integrating,

$$f_2(x) = -\frac{\rho h}{EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} + \left(\frac{4}{5} + \frac{5\sigma}{6}\right)x^2h^2\right] + \delta,$$

in which δ denotes

$$\delta = (v)_{\substack{x=0 \\ y=0}};$$

and this will be evaluated soon (equation (11)).

Thus the displacements become

$$\left.\begin{aligned} u &= \frac{\rho h}{EI}\left[xy\left(l^2 - \frac{x^2}{3}\right) + 2xy\left(\frac{y^2}{3} - \frac{h^2}{5}\right) - \frac{\sigma}{3}xy(h^2 - y^2)\right], \\ v &= \frac{\rho h}{EI}\left[\frac{1}{6}\left(h^2 - \frac{y^2}{2}\right)y^2 - \sigma\left\{(l^2 - x^2)\frac{y^2}{2} + \left(\frac{y^2}{6} - \frac{h^2}{5}\right)y^2\right\}\right] \\ &\quad - \frac{\rho h}{EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} + \left(\frac{4}{5} + \frac{5\sigma}{6}\right)x^2h^2\right] + \delta. \end{aligned}\right\} \cdots\cdots(10)$$

If in the first equation we put $y = 0$, we then have $u = 0$. This indicates that the neutral surface of the beam is perfectly coincides with the center line, whereas in the case of surface traction (equations (5)) there occurs a slight discrepancy between them. From the expression for v in (10) we find the equation for the deflection curve, *viz.*

$$(v)_{y=0} = -\frac{\rho h}{EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} + \left(\frac{4}{5} + \frac{5\sigma}{6}\right)x^2h^2\right] + \delta. \quad \cdots\cdots(11)$$

Assuming that the deflection is zero at the ends $x = \pm l$ of the center line, we obtain

$$\delta = \frac{5\rho hl^4}{12EI}\left[1 + \frac{12}{5}\left(\frac{4}{5} + \frac{5\sigma}{6}\right)\frac{h^2}{l^2}\right], \quad \cdots\cdots(12)$$

or

$$\delta = \frac{5\rho hl^4}{12EI}[1 + \varepsilon_2],$$

where

$$\varepsilon_2 = \frac{12}{5}\left(\frac{4}{5} + \frac{5\sigma}{6}\right)\frac{h^2}{l^2}. \quad \cdots\cdots(13)$$

In equation (12), the factor before the brackets is the deflection which is derived by the elementary theory, assuming as if the body force were a surface traction. The second term in the brackets represents the correction, which is usually called the *effect of shearing stress.*

The corresponding correction for the case of a uniformly distributed load is

$$\varepsilon_2' = \frac{12}{5}\left(\frac{4}{5} + \frac{\sigma}{2}\right)\frac{h^2}{l^2}, \quad \cdots\cdots(14)$$

and this holds also for any case of continuously varying intensity of load, which has been given by Th. v. Kármán. ε_2 of (13) is obviously a little greater than ε_2' of (14).

By differentiating equation (11) for the deflection curve twice with respect to x, we find the equation for the curvature:

$$\left(\frac{d^2v}{dx^2}\right)_{y=0} = -\frac{2\rho h}{EI}\left[\frac{l^2-x^2}{2} + \left(\frac{4}{5} + \frac{5\sigma}{6}\right)h^2\right]. \quad \cdots\cdots(15)$$

It is seen here that the curvature is not exactly proportional to $2\rho h(l^2 - x^2)/2$ which is the bending moment derived by the elementary analysis. The additional term in the brackets represents the necessary correction to the usual elementary formula. In this connection equation (15) may be written in the form

$$\left(\frac{d^2v}{dx^2}\right)_{y=0} = -\frac{2\rho h}{EI}\frac{l^2-x^2}{2}(1 + \varepsilon_3),$$

where

$$\varepsilon_3 = \frac{2\left(\frac{4}{5} + \frac{5\sigma}{6}\right)}{1 - \left(\frac{x}{l}\right)^2}\left(\frac{h}{l}\right)^2.$$

The maximum bending moment occurs at $x = 0$, in which case

$$(\varepsilon_3)_{x=0} = 2\left(\frac{4}{5} + \frac{5\sigma}{6}\right)\left(\frac{h}{l}\right)^2, \quad \cdots\cdots (16)$$

whose numerical values are given in Table 1.

Table 1. Correction for bending moment

l/h	$(\varepsilon_3)_{x=0}$ by eq. (16)	
	$\sigma = 1/3$	$\sigma = 1/4$
3.0	0.240	0.224
4.0	0.135	0.126
5.0	0.086	0.081
6.0	0.060	0.056
7.0	0.044	0.041

In conclusion it can be stated that, for a gravitating beam of short span, there is an insignificant effect on the stress distribution calculated by the elementary theory of bending, while there is a certain amount of effect on the deflection or the bending moment, as given in Table 1.

References

1) S. Timoshenko: "Theory of Elasticity," 1934, pp. 38-41.
2) A. E. H. Love: "Theory of Elasticity," 4th ed. (1934), pp. 138-140.
3) W. J. Ibbetson: "Theory of Elasticity," 1887, pp. 351-358.
4) Loc. cit. 1), p. 39.