The Solution of the Generalized Boussinesq's Problem for Elastic Foundation. Part II

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Synopsis. The purpose of the present work is to show the way of getting the stress distribution in the three-dimensional semi-infinite elastic solid, when any distributions of one kind of normal pressure and of two kinds of shearing forces are applied on the bounding plane.

CONTENTS

§ 1. INTRODUCTION

First, the general solution of this boundary-value problem is recapitulated,[1] which is expressed in the form of Fourier integral.

Secondly, as important applications of the general solution, two kinds of examples are given. One is to the normal pressure of quadratic distribution, and the other to the shearing force of linear distribution; both extending over a rectangular area. All the results are still in the form of Fourier integral.

Thirdly, a tentative rule for its numerical integration is therefore given, which is derived from interpolation formula in two dimensions[2]. By means of this rule a simplest case was treated with a hand-operated calculating machine.

§ 2. THE BOUNDARY-VALUE PROBLEM

Boundary conditions with which we are to be concerned are thus:

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$$\left.\begin{aligned} &\text{i)}\quad \left(\widehat{zz}\right)_{z=0} = F_1(x,\ y),\\ &\text{ii)}\quad \left(\widehat{yz}\right)_{z=0} = F_2(x,\ y),\\ &\text{iii)}\quad \left(\widehat{zx}\right)_{z=0} = F_3(x,\ y), \end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

$F_i\ (x,\ y)$ being external forces distributed on the bounding plane $z=0$ within some prescribed area.

Coordinates to be referred to are illustrated in Fig. 1.

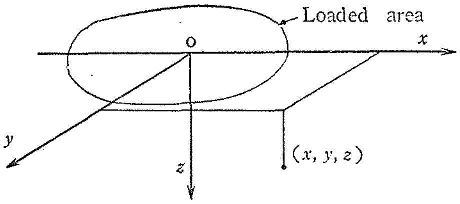


Fig. 1. Coordinates

§ 3. THE GENERAL SOLUTION

The general solution of the above boundary-value problem is given in the following[3]:

$$\left.\begin{aligned}
\widehat{zz} = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[&(1+\gamma z)\,F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\\
&+\beta z\,F_2(\xi,\ \eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\\
&+\alpha z\,F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta,\\
\widehat{yz} = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[&\beta z\,F_1(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\\
&+\left(1-\frac{\beta^2}{\gamma}z\right)F_2(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\\
&+\frac{\alpha\beta}{\gamma}z\,F_3(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta,\\
\widehat{zx} = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[&\alpha z\,F_1(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\\
&+\frac{\alpha\beta}{\gamma}z\,F_2(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)\\
&+\left(1-\frac{\alpha^2}{\gamma}z\right)F_3(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta,\\
\widehat{xx} = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[&\frac{\alpha^2}{\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\\
&+\frac{\beta}{\gamma^2}\left(2\sigma\frac{\beta^2}{\gamma}-\alpha^2 z\right)F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\\
&+\frac{2\alpha}{\gamma}\left(1+\sigma\frac{\beta^2}{\gamma^2}-\frac{\alpha^2}{2\gamma}z\right)F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta,
\end{aligned}\right\}(2)$$

$$\left.\begin{aligned}
\widehat{yy} &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[\frac{\beta^2}{\gamma^2}\Big(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\Big)\, F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta) \\
&\quad +\frac{2\beta}{\gamma}\Big(1+\sigma\frac{\alpha^2}{\gamma^2}-\frac{\beta^2}{2\gamma}z\Big)\, F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta) \\
&\quad +\frac{\alpha}{\gamma^2}\Big(2\sigma\frac{\alpha^2}{\gamma}-\beta^2 z\Big)\, F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]\, e^{-\gamma z}\, d\xi\, d\eta, \\
\widehat{xy} &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[-\frac{\alpha\beta}{\gamma^2}(1-2\sigma-\gamma z)\, F_1(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta) \\
&\quad +\frac{\alpha}{\gamma}\Big(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\Big)\, F_2(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta) \\
&\quad +\frac{\beta}{\gamma}\Big(1-2\sigma\frac{\alpha^2}{\gamma^2}-\frac{\alpha^2}{\gamma}z\Big)\, F_3(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\Big]\, e^{-\gamma z}\, d\xi\, d\eta;
\end{aligned}\right\}$$

$$\left.\begin{aligned}
u &= \frac{1}{2\pi^2\mu}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[\frac{\alpha}{\gamma^2}(1-2\sigma-\gamma z)\, F_1(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta) \\
&\quad -\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\, F_2(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta) \\
&\quad -\frac{1}{\gamma}\Big\{2-\frac{\alpha^2}{\gamma^2}(2\sigma+\gamma z)\Big\}\, F_3(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\Big]\, e^{-\gamma z}\, d\xi\, d\eta, \\
v &= \frac{1}{2\pi^2\mu}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[\frac{\beta}{\gamma^2}(1-2\sigma-\gamma z)\, F_1(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta) \\
&\quad -\frac{1}{\gamma}\Big\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\Big\}\, F_2(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta) \\
&\quad -\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\, F_3(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)\Big]\, e^{-\gamma z}\, d\xi\, d\eta, \\
w &= \frac{1}{2\pi^2\mu}\int_0^\infty\int_0^\infty d\alpha\, d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[-\frac{1}{\gamma}\,\{2(1-\sigma)+\gamma z\}\, F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta) \\
&\quad -\frac{\beta}{\gamma^2}(1-2\sigma+\gamma z)\, F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta) \\
&\quad -\frac{\alpha}{\gamma^2}(1-2\sigma+\gamma z)\, F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]\, e^{-\gamma z}\, d\xi\, d\eta.
\end{aligned}\right\} \tag{3}$$

Here $\widehat{zz}$, $\widehat{yz}, \cdots \widehat{xy}$ denote six stress-components referred to rectangular coordinates (Fig. 1), and u, v, w are displacement-components in directions of axes of x, y, z respectively. μ is the modulus of rigidity for the solid, and σ Poisson's ratio, so that $\mu = E/2(1+\sigma)$; α, β, γ being parameters provided

$$\alpha^2+\beta^2=\gamma^2.$$

It has been verified by substitution that equations (2) and (3) satisfy the three stress-equations

$$\frac{\partial\widehat{xx}}{\partial x}+\frac{\partial\widehat{xy}}{\partial y}+\frac{\partial\widehat{zx}}{\partial z}=0, \quad \cdots\cdots,$$

and the Hooke's law

$$\frac{\partial u}{\partial x}=\frac{1}{E}\{\widehat{xx}-\sigma(\widehat{yy}+\widehat{zz})\}, \quad \cdots\cdots \qquad \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{2(1+\sigma)}{E}\widehat{yz}, \quad \cdots\cdots,$$

and also that, when $z=0$, $\widehat{zz}$, $\widehat{zy}$, $\widehat{zx}$ in (2) reduce to

$$\left(\widehat{zz}\right)_{z=0}=\frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\,d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta,$$

$$\left(\widehat{yz}\right)_{z=0}=\frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\,d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_2(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta,$$

$$\left(\widehat{zx}\right)_{z=0}=\frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\,d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_3(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta,$$

the Fourier representations of the boundary conditions (1). This proves the validity of our solutions (2) and (3).

§ 4. APPLICATION. I. ···NORMAL PRESSURE

Our attention is here confined to the case of normal pressure distribution of the quadratic form

$$\left(\widehat{zz}\right)_{z=0}=F_1(x,y)=A+Bx+Cy+Dx^2+Exy+Fy^2, \quad\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

which is valid for a rectangular area whose sides are $2a$ and $2b$, and vanishes outside this rectangular area. The two kinds of shearing forces $F_2(x, y)$ and $F_3(x, y)$ are supposed to be zero throughout (Fig. 2).

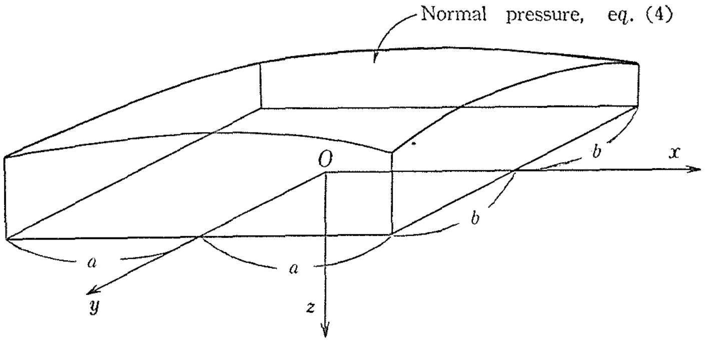


Fig. 2. Normal pressure distribution

In this case the double integral in (2) and (3) must be written

$$\int_{-\infty}^\infty\int_{-\infty}^\infty d\xi\,d\eta=\int_{-b}^b\int_{-a}^a d\xi\,d\eta.$$

Then by substituting (4) into the first equation of (2), we have

$$\widehat{zz}=\frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\,d\beta\int_{-b}^b\int_{-a}^a(1+\gamma z)(A+B\xi+C\eta+D\xi^2+E\xi\eta+F\eta^2)$$
$$\times\cos\alpha(x-\xi)\cos\beta(y-\eta)\,e^{-\gamma z}\,d\xi\,d\eta$$
$$=\frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\,d\beta(1+\gamma z)\,e^{-\gamma z}$$
$$\times\int_{-b}^b\int_{-a}^a(A+B\xi+C\eta+D\xi^2+E\xi\eta+F\eta^2)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta.$$

Having performed the integrations with respect to ξ and η, the first stress-component $\widehat{zz}$ in (2) is written

$$\widehat{zz} = A\,\widehat{zz}_0 + B\,\widehat{zz}_x + C\,\widehat{zz}_y + D\,\widehat{zz}_{x^2} + E\,\widehat{zz}_{xy} + F\,\widehat{zz}_{y^2}\,, \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

in which

$$\left.\begin{aligned}
\widehat{zz}_0 &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty d\alpha\,d\beta(1+\gamma z)\,e^{-\gamma z}\int_{-b}^{b}\int_{-a}^{a}\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta\\
&= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty(1+\gamma z)\,e^{-\gamma z}\frac{1}{\alpha\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{zz}_x &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty(1+\gamma z)\,e^{-\gamma z}\frac{1}{\alpha\beta}\Big(-a\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\qquad\qquad + \frac{1}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zz}_y &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty(1+\gamma z)\,e^{-\gamma z}\frac{1}{\alpha\beta}\Big(-b\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b\\
&\qquad\qquad + \frac{1}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zz}_{x^2} &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty(1+\gamma z)\,e^{-\gamma z}\frac{1}{\alpha\beta}\Big(a^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad + \frac{2a}{\alpha}\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b - \frac{2}{\alpha^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zz}_{xy} &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty(1+\gamma z)\,e^{-\gamma z}\frac{1}{\alpha\beta}\Big(ab\sin\alpha x\sin\beta y\cos\alpha a\cos\beta b\\
&\qquad - \frac{b}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b - \frac{a}{\beta}\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad + \frac{1}{\alpha\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zz}_{y^2} &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty(1+\gamma z)\,e^{-\gamma z}\frac{1}{\alpha\beta}\Big(b^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad + \frac{2b}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b - \frac{2}{\beta^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,
\end{aligned}\right\}(6)$$

where, as before,

$$\alpha^2 + \beta^2 = \gamma^2.$$

The second stress-component $\widehat{yz}$ in (2) is written

$$\widehat{yz} = A\,\widehat{yz}_0 + B\,\widehat{yz}_x + C\,\widehat{yz}_y + D\,\widehat{yz}_{x^2} + E\,\widehat{yz}_{xy} + F\,\widehat{yz}_{y^2}\,, \quad\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

in which

$$\begin{aligned}
\widehat{yz}_0 &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{yz}_x &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(-a\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad\qquad + \frac{1}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yz}_y &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(b\mathrm{c\ os}\,\alpha x\cos\beta y\sin\alpha a\cos\beta b
\end{aligned}$$

$$\left.\begin{aligned}
&\qquad\qquad -\frac{1}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yz}_{x^2} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(a^2\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\\
&\quad+\frac{2a}{\alpha}\cos\alpha x\sin\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yz}_{xy} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(-ab\sin\alpha x\cos\beta y\cos\alpha a\cos\beta b\\
&\quad+\frac{b}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b+\frac{a}{\beta}\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b\\
&\qquad\qquad-\frac{1}{\alpha\beta}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yz}_{y^2} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(b^2\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\\
&\quad+\frac{2b}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
&\qquad\qquad\alpha^2+\beta^2=\gamma^2.
\end{aligned}\right\}\quad(8)$$

The third stress-component $\widehat{zx}$ in (2) is written

$$\widehat{zx}=A\,\widehat{zx}_0+B\,\widehat{zx}_x+C\,\widehat{zx}_y+D\,\widehat{zx}_{x^2}+E\,\widehat{zx}_{xy}+F\,\widehat{zx}_{y^2},\quad\cdots\cdots\cdots(9)$$

in which

$$\left.\begin{aligned}
\widehat{zx}_0 &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\beta}e^{-\gamma z}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{zx}_x &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\beta}e^{-\gamma z}\Big(a\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\qquad\qquad-\frac{1}{\alpha}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zx}_y &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\beta}e^{-\gamma z}\Big(-b\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b\\
&\qquad\qquad+\frac{1}{\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zx}_{x^2} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\beta}e^{-\gamma z}\Big(a^2\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\quad+\frac{2a}{\alpha}\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zx}_{xy} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\beta}e^{-\gamma z}\Big(-ab\cos\alpha x\sin\beta y\cos\alpha a\cos\beta b\\
&\qquad+\frac{b}{\alpha}\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b+\frac{a}{\beta}\cos\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad-\frac{1}{\alpha\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{zx}_{y^2} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{z}{\beta}e^{-\gamma z}\Big(b^2\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b
\end{aligned}\right\}\quad(10)$$

$$
\left.\begin{aligned}
&+\frac{2b}{\beta}\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
&\qquad\qquad\alpha^2+\beta^2=\gamma^2.
\end{aligned}\right\}
$$

The fourth stress-component $\widehat{xx}$ in (2) is written

$$
\widehat{xx}=A\,\widehat{xx}_0+B\,\widehat{xx}_x+C\,\widehat{xx}_y+D\,\widehat{xx}_{x^2}+E\,\widehat{xx}_{xy}+F\,\widehat{xx}_{y^2}, \qquad \cdots\cdots(11)
$$

in which

$$
\left.\begin{aligned}
\widehat{xx}_0&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\alpha}{\beta\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)e^{-\gamma z}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{xx}_x&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\alpha}{\beta\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)e^{-\gamma z}\Big(-a\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\qquad\qquad+\frac{1}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xx}_y&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\alpha}{\beta\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)e^{-\gamma z}\Big(-b\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b\\
&\qquad\qquad+\frac{1}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xx}_{x^2}&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\alpha}{\beta\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)e^{-\gamma z}\Big(a^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad+\frac{2a}{\alpha}\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xx}_{xy}&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\alpha}{\beta\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)e^{-\gamma z}\Big(ab\sin\alpha x\sin\beta y\cos\alpha a\cos\beta b\\
&\qquad-\frac{b}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b-\frac{a}{\beta}\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad+\frac{1}{\alpha\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xx}_{x^2}&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\alpha}{\beta\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)e^{-\gamma z}\Big(b^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad+\frac{2b}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
&\qquad\qquad\alpha^2+\beta^2=\gamma^2.
\end{aligned}\right\}(12)
$$

The fifth stress-component $\widehat{yy}$ in (2) is written

$$
\widehat{yy}=A\,\widehat{yy}_0+B\,\widehat{yy}_x+C\,\widehat{yy}_y+D\,\widehat{yy}_{x^2}+E\,\widehat{yy}_{xy}+F\,\widehat{yy}_{y^2}, \qquad \cdots\cdots(13)
$$

in which

$$
\begin{aligned}
\widehat{yy}_0&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\beta}{\alpha\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)e^{-\gamma z}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{yy}_x&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\beta}{\alpha\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)e^{-\gamma z}\Big(-a\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\qquad\qquad\qquad+\frac{1}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yy}_y&=\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\beta}{\alpha\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)e^{-\gamma z}\Big(-b\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b
\end{aligned}
$$

$$\begin{aligned}
&\qquad +\frac{1}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yy}_{x^2} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\beta}{\alpha\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)e^{-\gamma z}\Big(a^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad +\frac{2a}{\alpha}\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yy}_{xy} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\beta}{\alpha\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)e^{-\gamma z}\Big(ab\sin\alpha x\sin\beta y\cos\alpha a\cos\beta b\\
&\qquad -\frac{b}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b-\frac{a}{\beta}\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad +\frac{1}{\alpha\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{yy}_{y^2} &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{\beta}{\alpha\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)e^{-\gamma z}\Big(b^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad +\frac{2b}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\,\mathrm{con}\,\beta b-\frac{2}{\beta^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
&\qquad \alpha^2+\beta^2=\gamma^2.
\end{aligned}\tag{14}$$

The sixth stress-component $\widehat{xy}$ in (2) is written

$$\widehat{xy} = A\,\widehat{xy}_0 + B\,\widehat{xy}_x + C\,\widehat{xy}_y + D\,\widehat{xy}_{x^2} + E\,\widehat{xy}_{xy} + F\,\widehat{xy}_{y^2}\,, \tag{15}$$

in which

$$\begin{aligned}
\widehat{xy}_0 &= -\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{1}{\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{xy}_x &= -\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{1}{\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(a\cos\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad -\frac{1}{\alpha}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xy}_y &= -\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{1}{\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(b\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b\\
&\qquad -\frac{1}{\beta}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xy}_{x^2} &= -\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{1}{\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(a^2\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\\
&\qquad +\frac{2a}{\alpha}\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xy}_{xy} &= -\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{1}{\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(ab\cos\alpha x\cos\beta y\cos\alpha a\cos\beta b\\
&\qquad -\frac{b}{\alpha}\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b-\frac{a}{\beta}\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\qquad +\frac{1}{\alpha\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
\widehat{xy}_{y^2} &= -\frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty\frac{1}{\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(b^2\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b
\end{aligned}\tag{16}$$

$$\left.\begin{aligned}
&+\frac{2b}{\beta}\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
&\qquad\qquad\alpha^2+\beta^2=\gamma^2.
\end{aligned}\right|$$

The first displacement-component u in (3) is written

$$u=A\,u_0+B\,u_x+C\,u_y+D\,u_{x^2}+E\,u_{xy}+F\,u_{y^2}\,,\quad\cdots\cdots\cdots\cdots(17)$$

in which

$$\left.\begin{aligned}
u_0&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\beta\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
u_x&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\beta\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(a\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\qquad-\frac{1}{\alpha}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
u_y&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\beta\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(-b\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b\\
&\qquad+\frac{1}{\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
u_{x^2}&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\beta\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(a^2\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad+\frac{2a}{\alpha}\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
u_{xy}&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\beta\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(-ab\cos\alpha x\sin\beta y\cos\alpha a\cos\beta b\\
&\qquad+\frac{b}{\alpha}\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b+\frac{a}{\beta}\cos\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad-\frac{1}{\alpha\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
u_{y^2}&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\beta\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(b^2\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\qquad+\frac{2b}{\beta}\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
&\qquad\qquad\alpha^2+\beta^2=\gamma^2.
\end{aligned}\right\}(18)$$

The second displacement-component v in (3) is written

$$v=A\,v_0+B\,v_x+C\,v_y+D\,v_{x^2}+E\,v_{xy}+F\,v_{y^2}\,,\quad\cdots\cdots\cdots\cdots(19)$$

in which

$$\left.\begin{aligned}
v_0&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\alpha\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
v_x&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\alpha\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Big(-a\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\qquad+\frac{1}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\
v_y&=\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty\frac{1}{\alpha\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}(b\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b
\end{aligned}\right|$$

$$\begin{aligned}
&\qquad -\frac{1}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
v_{x^2} &= \frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Bigl(a^2\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\\
&\quad +\frac{2a}{\alpha}\cos\alpha x\sin\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
v_{xy} &= \frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Bigl(-ab\sin\alpha x\cos\beta y\cos\alpha a\cos\beta b\\
&\quad +\frac{b}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b+\frac{a}{\beta}\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\quad -\frac{1}{\alpha\beta}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
v_{y^2} &= \frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\gamma^2}(1-2\sigma-\gamma z)\,e^{-\gamma z}\Bigl(b^2\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\\
&\quad +\frac{2b}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha d\beta,\\
&\qquad \alpha^2+\beta^2=\gamma^2.
\end{aligned}\tag{20}$$

The third displacement–component w in (3) is written

$$w=A\,w_0+B\,w_x+C\,w_y+D\,w_{x^2}+E\,w_{xy}+F\,w_{y^2}\,,\quad\cdots\cdots\cdots\cdots\tag{21}$$

in which

$$\begin{aligned}
w_0 &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta\gamma}\{2(1-\sigma)+\gamma z\}\,e^{-\gamma z}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
w_x &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta\gamma}\{2(1-\sigma)+\gamma z\}\,e^{-\gamma z}\Bigl(-a\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\\
&\quad +\frac{1}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
w_y &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta\gamma}\{2(1-\sigma)+\gamma z\}\,e^{-\gamma z}\Bigl(-b\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b\\
&\quad +\frac{1}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
w_{x^2} &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta\gamma}\{2(1-\sigma)+\gamma z\}\,e^{-\gamma z}\Bigl(a^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\\
&\quad +\frac{2a}{\alpha}\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b-\frac{2}{\alpha^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
w_{xy} &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta\gamma}\{2(1-\sigma)+\gamma z)\,e^{-\gamma z}\Bigl(ab\sin\alpha x\sin\beta y\cos\alpha a\cos\beta b\\
&\quad -\frac{b}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b-\frac{a}{\beta}\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\
&\quad +\frac{1}{\alpha\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Bigr)\,d\alpha\,d\beta,\\
w_{y^2} &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta\gamma}\{2(1-\sigma)+\gamma z\}\,e^{-\gamma z}\Bigl(b^2\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b
\end{aligned}\tag{22}$$

$$\left.\begin{aligned} &+\frac{2b}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b-\frac{2}{\beta^2}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\ &\qquad\qquad \alpha^2+\beta^2=\gamma^2. \end{aligned}\right\}$$

§ 5. APPLICATION. II. …SHEARING FORCE

This article is devoted to the case of shearing force, whose distribution is of the linear form

$$\left(\widehat{yz}\right)_{z=0}=F_2(x,y)=A'+B'x+C'y,\quad\cdots\cdots(23)$$

which is valid within the rectangular area $2a\times 2\,b$, and vanishes outside this area. The two remaining external forces $F_1(x,y)$ and $F_3(x,y)$ vanish throughout on the bounding plane.

The first stress-component $\widehat{zz}$ in (2) is written

$$\widehat{zz}=A'\widehat{zz}_0+B'\widehat{zz}_x+C'\widehat{zz}_y,\quad\cdots\cdots(24)$$

in which

$$\left.\begin{aligned} \widehat{zz}_0&=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\ \widehat{zz}_x&=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(-a\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b\\ &\qquad+\frac{1}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\ \widehat{zz}_y&=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{z}{\alpha}e^{-\gamma z}\Big(b\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b\\ &\qquad-\frac{1}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta. \end{aligned}\right\}\cdots(25)$$

The second stress-component $\widehat{yz}$ in (2) is written

$$\widehat{yz}=A'\widehat{yz}_0+B'\widehat{yz}_x+C'\widehat{yz}_y,\quad\cdots\cdots(26)$$

in which

$$\left.\begin{aligned} \widehat{yz}_0&=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{e^{-\gamma z}}{\alpha\beta}\left(1-\frac{\beta^2}{\gamma}z\right)\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\ \widehat{yz}_x&=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{e^{-\gamma z}}{\alpha\beta}\left(1-\frac{\beta^2}{\gamma}z\right)\Big(-a\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\\ &\qquad+\frac{1}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta,\\ \widehat{yz}_y&=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{e^{-\gamma z}}{\alpha\beta}\left(1-\frac{\beta^2}{\gamma}z\right)\Big(-b\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b\\ &\qquad+\frac{1}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta. \end{aligned}\right\}\cdots(27)$$

The third stress-component $\widehat{zx}$ in (2) is written

$$\widehat{zx}=A'\widehat{zx}_0+B'\widehat{zx}_x+C'\widehat{zx}_y,\quad\cdots\cdots(28)$$

in which

$$\left.\begin{aligned}
\overset{\frown}{zx}_0 &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{z}{\gamma} e^{-\gamma z} \sin \alpha x \sin \beta y \sin \alpha a \sin \beta b \; d\alpha\, d\beta, \\
\overset{\frown}{zx}_x &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{z}{\gamma} e^{-\gamma z} \Big(a \cos \alpha x \sin \beta y \cos \alpha a \sin \beta b \\
&\qquad - \frac{1}{\alpha} \cos \alpha x \sin \beta y \sin \alpha a \sin \beta b\Big)\, d\alpha\, d\beta, \\
\overset{\frown}{zz}_y &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{z}{\gamma} e^{-\gamma z} \Big(b \sin \alpha x \cos \beta y \sin \alpha a \cos \beta b \\
&\qquad - \frac{1}{\beta} \sin \alpha x \cos \beta y \sin \alpha a \sin \beta b\Big)\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{29}$$

The fourth stress-component $\overset{\frown}{xx}$ in (2) is written

$$\overset{\frown}{xx} = A'\overset{\frown}{xx}_0 + B'\overset{\frown}{xx}_x + C'\overset{\frown}{xx}_y\,, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{30}$$

in which

$$\left.\begin{aligned}
\overset{\frown}{xx}_0 &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma^2} \Big(2\sigma\frac{\beta^2}{\gamma} - \alpha^2 z\Big) \cos \alpha x \sin \beta y \sin \alpha a \sin\beta b \; d\alpha\, d\beta, \\
\overset{\frown}{xx}_x &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma^2} \Big(2\sigma\frac{\beta^2}{\gamma} - \alpha^2 z\Big) \Big(-a \sin \alpha x \sin \beta y \cos \alpha a \sin \beta b \\
&\qquad + \frac{1}{\alpha} \sin \alpha x \sin \beta y \sin \alpha a \sin \beta b\Big)\, d\alpha\, d\beta, \\
\overset{\frown}{xx}_y &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma^2} \Big(2\sigma\frac{\beta^2}{\gamma} - \alpha^2 z\Big) \Big(b \cos \alpha x \cos \beta y \sin \alpha a \cos \beta b \\
&\qquad - \frac{1}{\beta} \cos \alpha x \cos \beta y \sin \alpha a \sin \beta b\Big)\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{31}$$

The fifth stress-component $\overset{\frown}{yy}$ in (2) is written

$$\overset{\frown}{yy} = A'\overset{\frown}{yy}_0 + B'\overset{\frown}{yy}_x + C'\overset{\frown}{yy}_y\,, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{32}$$

in which

$$\left.\begin{aligned}
\overset{\frown}{yy}_0 &= \frac{8}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma} \Big(1 + \sigma\frac{\alpha^2}{\gamma^2} - \frac{\beta^2}{2\gamma} z\Big) \cos \alpha x \sin \beta y \sin \alpha a \sin \beta b \; d\alpha\, d\beta, \\
\overset{\frown}{yy}_x &= \frac{8}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma} \Big(1 + \sigma\frac{\alpha^2}{\gamma^2} - \frac{\beta^2}{2\gamma} z\Big) \Big(-a \sin \alpha x \sin \beta y \cos \alpha a \sin \beta b \\
&\qquad + \frac{1}{\alpha} \sin \alpha x \sin \beta y \sin \alpha a \sin \beta b\Big)\, d\alpha\, d\beta, \\
\overset{\frown}{yy}_y &= \frac{8}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma} \Big(1 + \sigma\frac{\alpha^2}{\gamma^2} - \frac{\beta^2}{2\gamma} z\Big) \Big(b \cos \alpha x \cos \beta y \sin \alpha a \cos \beta b \\
&\qquad - \frac{1}{\beta} \cos \alpha x \cos \beta y \sin \alpha a \sin \beta b\Big)\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{33}$$

The sixth stress-component $\overset{\frown}{xy}$ in (2) is written

$$\overset{\frown}{xy} = A'\overset{\frown}{xy}_0 + B'\overset{\frown}{xy}_x + C'\overset{\frown}{xy}_y\,, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{34}$$

in which

$$\left.\begin{aligned}
\widehat{xy}_0 &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\beta\gamma}\left(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\right)\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
\widehat{xy}_x &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\beta\gamma}\left(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\right)\left(a\cos\alpha x\cos\beta y\cos\alpha a\sin\beta b\right.\\
&\qquad\left.-\frac{1}{\alpha}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\right)d\alpha\,d\beta,\\
\widehat{xy}_y &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\beta\gamma}\left(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\right)\left(-b\sin\alpha x\sin\beta y\sin\alpha a\cos\beta b\right.\\
&\qquad\left.+\frac{1}{\beta}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\right)d\alpha\,d\beta.
\end{aligned}\right\}\text{(35)}$$

The first displacement-component u in (3) is written

$$u = A'u_0 + B'u_x + C'u_y\,, \quad\cdots\cdots\cdots\cdots\text{(36)}$$

in which

$$\left.\begin{aligned}
u_0 &= -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\gamma^3}(2\sigma+\gamma z)\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
u_x &= -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\gamma^3}(2\sigma+\gamma z)\left(a\cos\alpha x\sin\beta y\cos\alpha a\sin\beta b\right.\\
&\qquad\left.-\frac{1}{\alpha}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\right)d\alpha\,d\beta,\\
u_y &= -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\gamma^3}(2\sigma+\gamma z)\left(b\sin\alpha x\cos\beta y\sin\alpha a\cos\beta b\right.\\
&\qquad\left.-\frac{1}{\beta}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\right)d\alpha\,d\beta.
\end{aligned}\right\}\text{(37)}$$

The second displacement-component v in (3) is written

$$v = A'v_0 + B'v_x + C'v_y\,, \quad\cdots\cdots\cdots\cdots\text{(38)}$$

in which

$$\left.\begin{aligned}
v_0 &= -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\alpha\beta\gamma}\left\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\right\}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,\\
v_x &= -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\alpha\beta\gamma}\left\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\right\}\left(-a\sin\alpha x\cos\beta y\cos\alpha a\sin\beta b\right.\\
&\qquad\left.+\frac{1}{\alpha}\sin\alpha x\cos\beta y\sin\alpha a\sin\beta b\right)d\alpha\,d\beta,\\
v_y &= -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\alpha\beta\gamma}\left\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\right\}\left(-b\cos\alpha x\sin\beta y\sin\alpha a\cos\beta b\right.\\
&\qquad\left.+\frac{1}{\beta}\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\right)d\alpha\,d\beta.
\end{aligned}\right\}\text{(39)}$$

The third displacement-component w in (3) is written

$$w = A'w_0 + B'w_x + C'w_y\,, \quad\cdots\cdots\cdots\cdots\text{(40)}$$

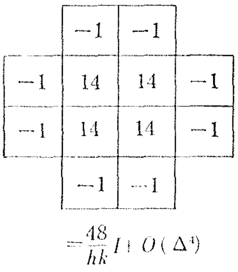
in which

$$w_0 = -\frac{2}{\pi^2\mu}\int_0^\infty\!\!\int_0^\infty \frac{e^{-\gamma z}}{\alpha\gamma^2}(1-2\sigma+\gamma z)\cos\alpha x\sin\beta y\sin\alpha a\sin\beta b\,d\alpha\,d\beta,$$

$$\left.\begin{aligned} w_x &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{e^{-\gamma z}}{\alpha\gamma^2}(1-2\sigma+\gamma z)\Big(-a\sin\alpha x\sin\beta y\cos\alpha a\sin\beta b \\ &\qquad +\frac{1}{\alpha}\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta, \\ w_y &= -\frac{2}{\pi^2\mu}\int_0^\infty\int_0^\infty\frac{e^{-\gamma z}}{\alpha\gamma^2}(1-2\sigma+\gamma z)\Big(b\cos\alpha x\cos\beta y\sin\alpha a\cos\beta b \\ &\qquad -\frac{1}{\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\Big)\,d\alpha\,d\beta. \end{aligned}\right\}(41)$$

§ 6. NUMERICAL EVALUATION

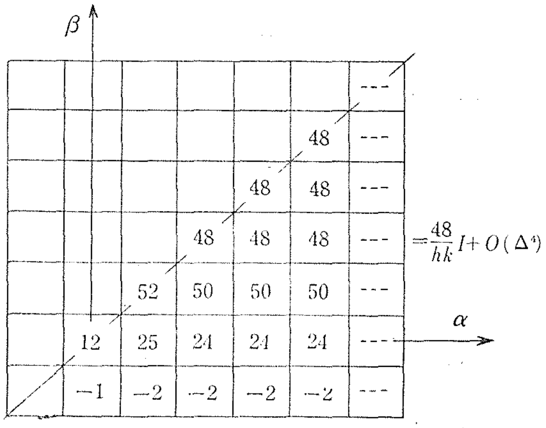
For the evaluation of the above integrals, I adopted a method of numerical integration which was derived from interpolation formulas in two dimensions[4]. The rule for the present adopted here is given in Fig. 3. This will be obtained from the interpolation formula of modified Bessel type, by curtailling 4th and higher differences. Repeated application of Fig. 3 gives rise to Fig. 4. Since the integrals above are of rapid convergence, it is not so laborious to secure first two or three significant figures of numerical results.



	−1	−1	
−1	14	14	−1
−1	14	14	−1
	−1	−1	

$=\frac{48}{hk}I+O(\Delta^4)$

Fig. 3. Unit rule for integration (Domain $= h\times k$)



					48	---
				48	48	---
			48	48	48	---
		52	50	50	50	---
	12	25	24	24	24	---
	−1	−2	−2	−2	−2	---

$=\frac{48}{hk}I+O(\Delta^4)$

Fig. 4. Aggregate of unit rules (Domain of integration $=\infty\times\infty$)

For instance, let us find the value of $\widehat{zz}$ at point $x=0$, $y=0$, $z=1$, when a uniform pressure, its intensity being unity, is extended over a square domain $a=1, b=1$ (Fig. 5). The integrand in the $\widehat{zz}$, from the first equation of (6), reduce to

$$\frac{1+\gamma}{\alpha\beta}\sin\alpha\sin\beta\, e^{-\gamma}, \qquad = f(\alpha,\ \beta;\ 0,\ 0,\ 1) \quad \text{say.}$$

Lattice-point values of $f(\alpha,\ \beta;\ 0,\ 0,\ 1)$ for $h=k=\pi/4$ is given in Fig. 6. In this figure, those in the upper half are not written, since they can be written down from the symmetry $f(\alpha,\ \beta)=f(\beta,\ \alpha)$.

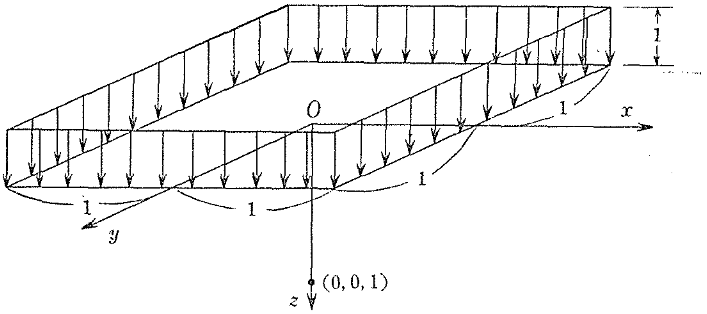
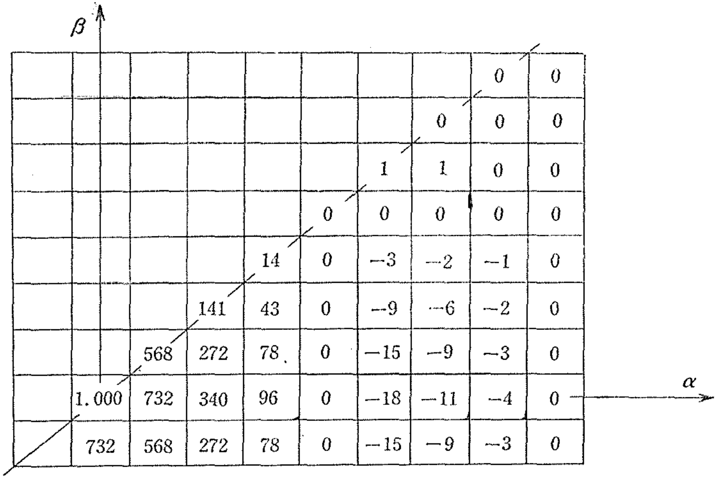


Fig. 5. Uniform pressure over square domain

By multiplying corresponding values in Figs. 4 and 6, and adding the results, we get

$$\widehat{zz} = \frac{4}{\pi^2}\frac{hk}{48}\sum \kappa\, f(\alpha,\beta) = \frac{134.26}{192} = 0.699,$$

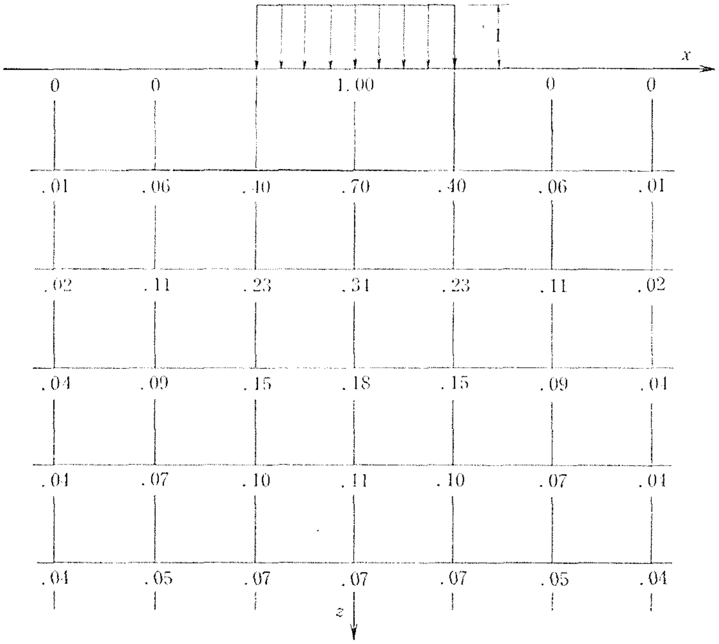
κ representing weights in Fig. 4. This value has been confirmed by a finer net in which $h = k = \pi/8$.



								0	0
							0	0	0
						1	1	0	0
					0	0	0	0	0
				14	0	−3	−2	−1	0
			141	43	0	−9	−6	−2	0
		568	272	78	0	−15	−9	−3	0
	1.000	732	340	96	0	−18	−11	−4	0
	732	568	272	78	0	−15	−9	−3	0

Fig. 6. Lattice-point values of $f(\alpha,\beta) = \frac{1+\gamma}{\alpha\beta} \sin\alpha \sin\beta\, e^{-\gamma}$
(Pitch $h = k = \pi/4$)

In this way the stress distribution of $\widehat{zz}$ is obtained as is given in Fig. 7. This numerical result is in accordance with the Love's by means of the Boussinesq's potential method[5], the numerical computation of which was due to J. Kimura[6].



0	0		1.00		0	0
.01	.06	.40	.70	.40	.06	.01
.02	.11	.23	.31	.23	.11	.02
.04	.09	.15	.18	.15	.09	.04
.04	.07	.10	.11	.10	.07	.04
.04	.05	.07	.07	.07	.05	.04

Fig. 7. Distribution of $\widehat{zz}$ in the plane $y=0$
$(a=b=1)$

In conclusion it is noted that further numerical computations are now being planned by the aid of a digital computer.

References

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4) Loc. cit. 2).
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