On the Ricci tensor and the generalized
Tanaka-Webster connection of real hypersurfaces
in a complex space form

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Abstract. We prove that the Ricci tensor $\hat{S}$ with respect to the generalized Tanaka-Webster connection of a real hypersurface with the almost contact structure $(\eta,\phi,\xi, g)$ in a complex space form of complex dimension $n \geq 3$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector field $X$ and $Y$, $\lambda$ being a function, if and only if the real hypersurface is locally congruent to some type $(A)$ hypersurface.

1. Introduction

Tanaka-Webster connection is a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifold which associated with the almost contact structure ([12], [14]). Tanno [13] gave the generalized Tanaka-Webster connection (g-Tanaka-Webster connection) for contact metric manifolds, which coincides with Tanaka-Webster connection if the associated CR-structure is integrable. For a real hypersurface in a Kählerian manifold with an almost contact metric structure $(\eta, \phi, \xi, g)$, in [3] and [4], Cho defined the g-Tanaka-Webster connection...
connection $\hat{\nabla}^{(k)}$ for a non-zero real number $k$. Then we can see that $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. Moreover, if the shape operator $A$ of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

For real hypersurfaces in a complex space form $M^n(c)$ of constant holomorphic sectional curvature $4c \neq 0$, one of the major problem is to determine real hypersurfaces satisfying certain geometrical assumptions. Cho [5] determined flat Hopf hypersurfaces in a non-flat complex space form with respect to the g-Tanaka-Webster connection. Besides, he classified Hopf hypersurfaces in a non-flat complex space form which admits a pseudo-Einstein $CR$-structure for the g-Tanaka-Webster connection.

The purpose of this paper is to study real hypersurfaces in a complex space form whose Ricci tensor $\hat{S}$ with respect to the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields $X$ and $Y$.

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2. Preliminaries

Let $M^n(c)$ denote the complex space from of complex dimension $n$ (real dimension $2n$) of constant holomorphic sectional curvature $4c$. For the sake of simplicity, if $c > 0$, we only use $c = +1$ and call it the complex projective space $\mathbb{C}P^n$, and if $c < 0$, we just consider $c = -1$, so that we call it the complex hyperbolic space $\mathbb{C}H^n$. We denote by $J$ the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ will be denoted by $G$.

Let $M$ be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. We denote by $g$ the Riemannian metric induced on $M$ from $G$. We take the unit normal vector field $V$ of $M$ in $M^n(c)$. For any vector field $X$ tangent to $M$, we define $\phi$, $\eta$ and $\xi$ by

$$JX = \phi X + \eta(X)V, \quad JV = -\xi,$$
where $\phi X$ is the tangential part of $JX$, $\phi$ is a tensor field of type $(1,1)$, $\eta$ is a 1-form, and $\xi$ is the unit vector field on $M$. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. Let $H_0$ denote the holomorphic distribution on $M$ defined by $H_0(x) = \{X \in T_x(M) | \eta(X) = 0\}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by $\nabla$ the one in $M$ determined by the induced metric. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)V,$$

$$\tilde{\nabla}_X V = -AX$$

for any vector fields $X$ and $Y$ tangent to $M$. We call $A$ the shape operator of $M$.

From the Gauss and Weingarten formulas, we have

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by $R$ the Riemannian curvature tensor field of $M$. Then the equation of Gauss is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

and the equation of Codazzi by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

If $A\xi = \lambda\xi$, $\lambda$ being a function, then $M$ is called a Hopf hypersurface. There are many results for real hypersurfaces in complex space forms under the assumption that they are Hopf hypersurfaces. By the Codazzi equation, we have the following result (c.f. [8]).
Proposition A. Let $M$ be a Hopf hypersurface in $M^n(c)$, $n \geq 2$. If $X \perp \xi$ and $AX = \beta X$, then $\alpha = g(A\xi, \xi)$ is constant and

$$(2\beta - \alpha)A\phi X = (\beta\alpha + 2c)\phi X.$$ 

We use the following results for the proof of the main theorem.

Theorem B ([7]). Let $M$ be a Hopf hypersurface in $\mathbb{C}P^n$. Then $M$ has constant principal curvatures if and only if $M$ is locally congruent to one of the following:

(A) a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$,
(B) a tube over a totally geodesic $\mathbb{C}P^l$ ($1 \leq l \leq n - 2$), where $0 < r < \pi/2$,

(C) a tube of radius $r$ over a complex quadric $Q^{n-1}$ and $\mathbb{R}P^n$, where $0 < r < \pi/4$.

(D) a tube of radius $r$ over $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$, where $0 < r < \pi/4$ and $n$ ($\geq 5$) is odd,
(E) a tube of radius $r$ over a totally real hyperbolic space $\mathbb{R}H^n$.

Theorem C ([1]). Let $M$ be a Hopf hypersurface in $\mathbb{C}H^n$. Then $M$ has constant principal curvatures if and only if $M$ is locally congruent to one of the following:

(A) a horosphere,
(B) a tube over a complex hyperbolic hyperplane $\mathbb{C}H^k$ ($k = 0, n - 1$),
(C) a tube over a totally geodesic $\mathbb{C}H^l$ ($1 \leq l \leq n - 2$),
(D) a tube over a totally real hyperbolic space $\mathbb{R}H^n$. 

Next we introduce the notion of Tanaka-Webster connection and its generalization. Tanaka [12] defined the canonical affine connection on a non-degenerate, pseudo-Hermitian CR manifold. As a generalization of Tanaka-Webster connection, Tanno [13] defined the g-Tanaka-Webster connection for contact metric manifolds by

\[ \nabla_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y, \]

where \( (\eta, \phi, \xi, g) \) is a contact metric structure. Using the naturally extended affine connection of Tanno’s g-Tanaka-Webster connection, the g-Tanaka-Webster connection \( \hat{\nabla}^{(k)} \) for real hypersurfaces in Kähler manifold is given by,

\[ \hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \]

for a non-zero real number \( k \) (see Cho [3], [4]). Then we see that

\[ \hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0. \]

In particular, if the shape operator of a real hypersurface satisfies \( \phi A + A\phi = 2k\phi \), then the g-Tanaka-Webster connection coincides with the Tanaka-Webster connection. Next we define the g-Tanaka-Webster curvature tensor \( \hat{R} \) with respect to \( \hat{\nabla}^{(k)} \) by

\[ \hat{R}(X, Y)Z = \hat{\nabla}_X (\hat{\nabla}_Y Z) - \hat{\nabla}_Y (\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]} Z \]

for all vector fields \( X, Y, Z \) in \( M \). We denote by \( \hat{S} \) the g-Tanaka Webster Ricci tensor, which is defined by

\[ \hat{S}(Y, Z) = \text{trace of } \{ X \mapsto \hat{R}(X, Y)Z \} . \]

3. The Ricci tensor of real hypersurfaces in a complex space form

To prove the theorem, we prepare the following lemma.
Lemma 3.1. Let $M$ be a real hypersurface in a complex space form $M^n(c)$, $n \geq 3$, $c \neq 0$. If there exists an orthonormal frame \( \{e_1, \cdots, e_{2n-2}, \xi\} \) on a sufficiently small neighborhood $N$ of $x \in M$ such that the shape operator $A$ can be represented as

\[
A = \begin{pmatrix}
    a_1 & 0 & \cdots & 0 & h_1 \\
    \vdots & \ddots & \ddots & \vdots & \vdots \\
    0 & \cdots & a_{2n-2} & 0 & \alpha \\
    h_1 & 0 & \cdots & 0 & \alpha
\end{pmatrix},
\]

then we have

\[
(a_1 - a_j)g(\nabla_{e_i}e_1, e_j) + (a_j - a_i)g(\nabla_{e_i}e_i, e_j) + a_i h_1 g(\phi e_i, e_j) = 0, \quad (3.1)
\]

\[
(a_j - a_1)g(\nabla_{e_i}e_1, e_1) - (a_i - a_1)g(\nabla_{e_i}e_i, e_1) + h_1 (a_i + a_j)g(\phi e_i, e_j) = 0, \quad (3.2)
\]

\[
\{2c - 2a_i a_j + \alpha (a_i + a_j)\}g(\phi e_i, e_j) - h_1 g(\nabla_{e_i}e_j, e_1) + h_1 g(\nabla_{e_i}e_i, e_1) = 0, \quad (3.3)
\]

\[
(a_1 - a_i)g(\nabla_{e_i}e_1, e_i) - (e_1 a_i) = 0, \quad (3.4)
\]

\[
h_1 (2a_i + a_1)g(\phi e_i, e_1) + (a_1 - a_i)g(\nabla_{e_i}e_i, e_1) + (e_1 a_i) = 0, \quad (3.5)
\]

\[
(c + a_i \alpha - a_1 a_i - h_1^2)g(\phi e_i, e_1) - (a_1 - a_i)g(\nabla_{\xi}e_1, e_1)
+ h_1 g(\nabla_{e_i}e_1, e_i) = 0, \quad (3.6)
\]

for any $i, j \geq 2$, $i \neq j$.

**Proof.** By the equation of Codazzi, we have

\[
g((\nabla_{e_i} A)e_1 - (\nabla_{e_1} A)e_i, e_j) = 0,
\]

where $i, j = 2, \cdots, 2n - 2$. On the other hand, we have

\[
g((\nabla_{e_i} A)e_1 - (\nabla_{e_1} A)e_i, e_j)
= g(\nabla_{e_i} (A e_1) - A \nabla_{e_i} e_1 - \nabla_{e_1} (A e_i) + A \nabla_{e_1} e_i, e_j)
= (a_1 - a_j)g(\nabla_{e_i}e_1, e_j) + (a_j - a_i)g(\nabla_{e_i}e_i, e_j) + a_i h_1 g(\phi e_i, e_j).
\]

Thus we obtain (3.1). By the similar computation, we have our results. \(\square\)
Theorem 3.2. Let $M$ be a real hypersurface in a complex space form $M^n(c)$, $n \geq 3$, $c \neq 0$. We suppose that the Ricci tensor $\hat{S}$ of the generalized Tanaka-Webster connection $\nabla^{(k)}$ satisfies $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields $X$ and $Y$, $\lambda$ being a function.

1. If $c > 0$ and $k^2 \neq 4c$, then $M$ is a Hopf hypersurface.
2. If $c < 0$, then $M$ is a Hopf hypersurface.

Proof. By the definition of the $g$-Tanaka-Webster connection, we have (see [5])

\[ \hat{R}(X, Y)Z = R(X, Y)Z + g(\phi((\nabla_X A)Y - (\nabla_Y A)X), Z)\xi \\
+ 2g(\phi AX, Z)\phi AY - 2g(\phi AX, Z)\phi AY \\
+ g((\nabla_X \phi)AY - (\nabla_Y \phi)AX, Z)\xi \\
- \eta(Z)\left(\phi((\nabla_X A)Y - (\nabla_Y A)X) + (\nabla_X \phi)AY - (\nabla_Y \phi)AX\right) \\
- k\left(g((\phi A + A\phi)X, Y)\phi Z + \eta(Y)(\nabla_X \phi)Z - \eta(X)(\nabla_Y \phi)Z\right) \\
+ g(\phi AX, F_Y Z)\xi - \eta(F_Y Z)\phi AX - k\eta(X)\phi F_Y Z \\
- g(\phi AY, F_X Z)\xi + \eta(F_X Z)\phi AY + k\eta(Y)\phi F_X Z, \]

where $F$ is given by

\[ F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \]

By the definition of $g$-Tanaka-Webster Ricci tensor, equation of Gauss and Codazzi, direct calculation shows that

\[ \hat{S}(Y, Z) = 2ncg(Y, Z) + (\text{tr} A - \eta(A\xi) + k)g(AY, Z) \\
- g(A^2 Y, Z) - g(\phi A\phi AY, Z) - k\eta(\phi A\phi Y, Z) + \eta(AY)g(A\xi, Z) \\
+ \eta(Z)(-2nc\eta(Y) - \eta(AY)\text{tr} A + \eta(A^2 Y) - k\eta(AY)). \]

Now we use the following lemma of Ryan [10].

Lemma D. Let $A$ be a symmetric tensor field of type (1,1) on a
connected Riemannian manifold $M^n$. Then there exists $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that for each point $x$, $\{\lambda_i(x)\} (i = 1, \cdots, n)$ are the eigenvalues of $A_x$.

For the shape operator $A$ of a real hypersurface $M$, we consider the symmetric tensor field $\phi A \phi$ of type (1,1). By the above lemma, we can take an orthonormal frame $\{v_1, ..., v_{2n-2}, \xi\}$ in a neighborhood of a point $x$ such that $\phi A \phi \xi = 0$, $\phi A \phi v_1 = -a_1 v_1$, $\cdots$, $\phi A \phi v_{2n-2} = -a_{2n-2} v_{2n-2}$. Then we have

$$g(A\phi v_i, \phi v_j) = -g(\phi A \phi v_i, v_j) = 0 \quad (i \neq j),$$
$$g(A\phi v_i, \phi v_i) = -g(\phi A \phi v_i, v_i) = a_i.$$

We take an orthonormal frame $\{e_1 = \phi v_1, ..., e_{2n-2} = \phi v_{2n-2}, \xi\}$ in a neighborhood $\mathcal{N}$ of a point $x$. Then, in the neighborhood, $A$ is of the form

$$A = \begin{pmatrix} a_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{2n-2} & h_{2n-2} \\ h_1 & \cdots & h_{2n-2} & \alpha \end{pmatrix},$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \cdots, 2n-2$, and $\alpha = g(A\xi, \xi)$.

The condition $\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)$ for any vector fields $X$ and $Y$ is equivalent to $\hat{S}(X, Y) = \lambda g(X, Y)$ for any vector field $X$ and any vector field $Y$ orthogonal to $\xi$. By the direct computation using the previous equation, we have

$$\hat{S}(\xi, \xi) = 0, \hat{S}(e_i, \xi) = 0,$$
$$\hat{S}(\xi, e_i) = (\text{tr} A - \alpha + k - a_i)h_i - g(\phi A \phi A \xi, e_i) = 0, \quad (3.8)$$
$$\hat{S}(e_i, e_i)$$
$$= 2nc + (\text{tr} A)a_i - a_i^2 - \alpha a_i + ka_i + (a_i + k)g(A\phi e_i, \phi e_i) = \lambda, \quad (3.9)$$
$$\hat{S}(e_i, e_j) = (a_i + k)g(A\phi e_i, \phi e_j) = 0 \quad (i \neq j). \quad (3.10)$$

In the following, we suppose that $M$ is not a Hopf hypersurface. Then there is a point $x$ and hence an open neighborhood $\mathcal{N}$ of $x$ where $A\xi \neq \alpha \xi$ on $\mathcal{N}$. Then $h_i \neq 0$ for some $i$.  

If $a_i = -k$ for all $i$ at some $x \in \mathcal{N}$, then (3.9) and $\text{tr}A = -(2n - 2)k + \alpha$ imply that
\[
2nc + (2n - 4)k^2 = \lambda.
\]
By (3.8),
\[
(\text{tr}A - \alpha + 2k)h_i + g(\phi A\xi, A\phi e_i) = 0.
\]
Since $g(\phi A\xi, A\phi e_i) = -kh_i$, $\text{tr}A - \alpha = -(2n - 2)k$, we have
\[
(2n - 3)kh_i = 0.
\]
for all $i$. Thus we have $k = 0$. This contradicts to our assumption. Therefore, $a_i \neq -k$ for some $i$. From (3.10), if $a_i \neq -k$, then $g(A\phi e_i, \phi e_j) = 0$ for all $j \neq i$. Thus we set
\[
A\phi e_i = \bar{a}_i \phi e_i + \bar{h}_i \xi,
\]
where we have put $\bar{a}_i = g(A\phi e_i, \phi e_i)$ and $\bar{h}_i = g(A\phi e_i, \xi)$. We also have
\[
\hat{S}(\phi e_i, \phi e_i) = 2nc + (\text{tr}A)\bar{a}_i - \bar{a}_i^2 - \alpha \bar{a}_i + k\bar{a}_i + (\bar{a}_i + k)a_i = \lambda. \tag{3.11}
\]
Using (3.9) and (3.11), we obtain
\[
(a_i - \bar{a}_i)(\text{tr}A - \alpha - a_i - \bar{a}_i) = 0.
\]
When $a_i = \bar{a}_i$, (3.9) implies
\[
2nc - \lambda = a_i(\alpha - 2k - \text{tr}A).
\]
Otherwise, if $a_i \neq \bar{a}_i$, then $\text{tr}A - \alpha = a_i + \bar{a}_i$. Using (3.9), we obtain
\[
2a_i^2 - 2(\text{tr}A - \alpha)a_i - k(\text{tr}A - \alpha) - 2nc + \lambda = 0,
\]
from which
\[
(a_i - a_j)(\text{tr}A - \alpha - a_i - a_j) = 0
\]
for \( a_j \) that satisfies \( a_j \neq k \) and \( a_j \neq \bar{a}_j \). If \( a_i \neq a_j \), then \( \text{tr}A - \alpha = a_i + a_j = a_i + \bar{a}_i \). Hence we have \( a_j = \bar{a}_i \). We put \( b = a_i \) and \( \bar{b} = \bar{a}_i \). They satisfy

\[
\begin{align*}
b + \tilde{b} &= \text{tr}A - \alpha, \\
\tilde{b}b &= -\frac{k}{2}(\text{tr}A - \alpha) - nc + \frac{\lambda}{2}.
\end{align*}
\]

We remark that \( b \neq -k \) or \( \bar{b} \neq -k \).

From these, in \( \mathcal{N} \), we have

\[
A = \begin{pmatrix}
b & & & \\
& \ddots & & \\
& b & \ddots & \\
& \bar{b} & \ddots & \ddots \\
& & d & \ddots & \\
& & & \ddots & \ddots \\
& h_1 & \cdots & & \ddots \\
& & & & \ddots \\
& & & h_{2n-2} & \alpha
\end{pmatrix},
\]

where

\[
\begin{align*}
d &= g(A\epsilon_y, \epsilon_s) = g(A\phi\epsilon_s, \phi\epsilon_s) \neq -k, \\
2nc - \lambda &= d(\alpha - 2k - \text{tr}A).
\end{align*}
\]

In the following, we use integers \( y, z, \cdots \) for \( A\epsilon_y = b\epsilon_y + h_y\xi \), \( s \cdots \) for \( A\epsilon_s = d\epsilon_s + h_s\xi \) and \( v \cdots \) for \( A\epsilon_v = -k\epsilon_v \). We denote by \( H_1(x) \), \( H_2(x) \), \( H_3(x) \) and \( H_4(x) \) the subspaces of a tangential space at \( x \) spanned by \( \{\epsilon_y\}, \{\phi\epsilon_y\}, \{\epsilon_s\} \) and \( \{\epsilon_v\} \), respectively.
We suppose that \( \dim H_3(x) \neq 0 \) and \( \dim H_4(x) \neq 0 \) at some \( x \in N \). Taking \( e_s \in H_3(x) \) and \( e_v \in H_4(x) \), (3.9) implies

\[
\hat{S}(e_v, e_v) = 2nc - k(\tr A) - 2k^2 + \alpha k = \lambda.
\]

From this and (3.14), we have

\[
(d + k)(\alpha - 2k - \tr A) = 0.
\]

Since \( d \neq -k \), then we have \( \tr A - \alpha = -2k \) and \( 2nc - \lambda = 0 \).

Moreover, if \( \dim H_1(x) = \dim H_2(x) \neq 0 \), taking \( e_y \in H_1(x) \), (3.12), (3.13) and (3.14) imply \( a_y = b = -k \) and \( \bar{a}_y = \bar{b} = -k \). This case cannot be occurred. Hence we have \( \dim H_1(x) = \dim H_2(x) = 0 \).

Then, by \( \phi e_s \in H_3(x) \) and \( \phi e_v \in H_4(x) \), we have \( a_i = \bar{a}_i \) for any \( i \in \{1, \ldots, 2n - 2\} \). Thus, by (3.8) and \( \tr A - \alpha = -2k \),

\[
(-k - a_i)h_i - g(\phi A\phi A\xi, e_i) = -kh_i = 0
\]

for all \( i \). This implies \( k = 0 \). This contradicts to our assumption.

So, we see that \( \dim H_3(x) = 0 \) or \( \dim H_4(x) = 0 \) at any point \( x \in N \), that is,

\[
A = \begin{pmatrix}
\begin{array}{cccc}
  b & \cdots & b \\
  \bar{b} & \cdots & \bar{b} \\
  f & \cdots & f \\
  h_1 & \cdots & h_{2n-2} & \alpha \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
h_1 \\
\vdots \\
h_{2n-2} \\
\alpha
\end{pmatrix}
\]

When \( \dim H_4 = 0 \), \( f \) denotes \( a_s = d \). We remark that \( f = d \) satisfies (3.14). Otherwise, when \( \dim H_3 = 0 \), \( f \) denotes \( a_v = -k \). In this case, we see that \( \bar{a}_v = -k \) by the definition of \( b \) and \( \bar{b} \). Thus, using (3.9), \( f = -k \) also satisfies

\[
2nc - \lambda = -k(\alpha - 2k - \tr A).
\]
Hence, \( f = \bar{f} \) and \( f \) satisfies
\[
2nc - \lambda = f(\alpha - 2k - \text{tr}A)
\] (3.15)
in both cases.

In the following, we use integers \( s \cdots \) for \( Ae_s = fe_s + h_s\xi \) and redefine \( H_3(x) \) as the subspaces of a tangential space at \( x \) spanned by \( \{e_s\} \).

By a direct computation using (3.8),
\[
\begin{align*}
(\text{tr}A - \alpha + k + b - \bar{b})h_y &= 0, \quad (3.16) \\
(\text{tr}A - \alpha + k - b + \bar{b})\bar{h}_y &= 0, \quad (3.17) \\
(\text{tr}A - \alpha + k)h_s &= 0. \quad (3.18)
\end{align*}
\]

**Lemma 3.3.** We have \( h_s = 0 \) for all \( e_s \in H_3 \).

**Proof.** If there exists \( e_s \in H_3 \) that satisfies \( h_s \neq 0 \) at some \( x \), and hence on some neighborhood \( \mathcal{N}' \subset \mathcal{N} \), then
\[
\text{tr}A - \alpha + k = 0.
\]

From (3.16) and (3.17), we have
\[
(-b + \bar{b})h_y = 0, \quad (b - \bar{b})\bar{h}_y = 0.
\]
Since \( b \neq \bar{b} \), we have \( h_y = 0 \) and \( \bar{h}_y = 0 \) for all \( y \). The direct computation shows that
\[
|tE - A| = (t - b)^p(t - \bar{b})^p(t - f)^{q-1}\{(t - f)(t - \alpha) - \sum_{s=1}^q h_s^2\},
\]
where \( p \) and \( q \) are the multiplicities of \( b \) and \( f \), respectively. We remark that \( 2p + q = 2n - 2 \).

Suppose \( Ae'_e = fe'_e \) is satisfied by \( e'_e = X + \beta\xi \), where \( X \in H_3 \).

Since \( AX = fX + h\xi \) for some \( h \), we obtain
\[
Ae'_e = fX + h\xi + \beta(\sum h_se_s + \alpha\xi).
\]
On the other hand, we have

\[ Ae' = f(X + \beta \xi) = fx + f\beta\xi. \]

From these equations, we obtain

\[ \beta \sum h_x e_x + (h + \alpha \beta - f \beta) \xi = 0. \]

Since \( h_x \neq 0 \) for some \( e_x \), we have \( \beta = 0 \), that is, \( g(e', \xi) = 0 \). Thus, in \( N' \), we can represent the shape operator \( A \) by a following matrix with respect to a local orthonormal frame \( \{ e_1, \cdots, e_p, \phi e_1, \cdots, \phi e_p, e_{2p+1}, \cdots, e_{2n-2}, \xi \} \):

\[
A = \begin{pmatrix}
 b \\
 \ddots \\
 b \\
 \bar{b} \\
 \ddots \\
 \bar{f} \\
 \ddots \\
 f \\
 0 & \cdots & 0 & h_{2n-2} & \alpha
\end{pmatrix}
\]

From (3.15) and (3.18) we obtain

\[ 2nc - \lambda = -fk, \quad \text{tr}A - \alpha = -k. \]

We now suppose that there is a point \( x \) in \( N' \) where \( p \neq 0 \). Then (3.12) implies

\[ -(p - 1)k + qf = 0. \]

By (3.13), we also have

\[ \bar{b}\bar{b} = \frac{1}{2}(k^2 + fk). \]

Using \( b + \bar{b} = \text{tr}A - \alpha = -k \), we see

\[ (b + \frac{k}{2})^2 + \frac{1}{4}(k + 2f)k = 0. \]
Since \((p - 1)k = qf\), we see \(fk \geq 0\). This implies that \(k + 2f = 0\) and hence \((2p - 2 + q)k = 0\). Thus we have \(k = 0\). This contradicts to our assumption.

Let us suppose that \(p = 0\) on \(\mathcal{N}'\) of \(x\). Then \(\text{tr}A - \alpha = (2n - 2)f = -k\) shows that \(f\) is non-zero constant on \(\mathcal{N}'\) of \(x\). By (3.5), we see that \(h_{2n-2}f = 0\). This is also a contradiction. This proves our lemma.

If there exist \(e_y \in H_1\) and \(\phi e_z \in H_2\) that satisfy \(h_y \neq 0\) and \(\bar{h}_z \neq 0\), (3.16) and (3.17) implies \(b = \bar{b}\). This case cannot be occurred. So it is sufficient to consider the case that \(\bar{h}_y = 0\) for any \(\phi e_y \in H_2\). Using (3.12) and (3.16), we have

\[
b = \text{tr}A - \alpha + \frac{k}{2}, \quad \bar{b} = -\frac{k}{2}.
\]

By the similar calculation as Lemma 3.3, in \(\mathcal{N}\), we can represent the shape operator \(A\) by a following matrix with respect to an orthonormal frame \(\{e_1, \cdots, e_p, \phi e_1, \cdots, \phi e_p, e_{2p+1}, \cdots, e_{2n-2}, \xi\}\):

\[
A = \begin{pmatrix}
b & \cdots & \cdots & \cdots & h_1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]

Then we have

\[
\text{tr}A = p(b + \bar{b}) + qf + \alpha.
\]

Using (3.12),

\[
(p - 1)(b + \bar{b}) + qf = 0.
\]

First, we suppose that \(\text{tr}A - \alpha = b + \bar{b} \neq 0\) at a point \(x\) and hence an open neighborhood \(\mathcal{N}'' \subset \mathcal{N}\) of \(x\). Then (3.20) implies that \(q \neq 0\).
on $\mathcal{N}''$. Because, if $q = 0$ at some point $x \in \mathcal{N}''$, then $p - 1 = 0$ and hence $n = 2$. This contradicts to $n \geq 3$. From (3.13) and (3.19), we have
\[
-\frac{k^2}{4} = -nc + \frac{\lambda}{2},
\]
(3.21)
from which we see that $-nc + (\lambda/2) \neq 0$ and $\lambda$ is constant on $\mathcal{N}''$. Thus, by (3.15) and (3.20), we obtain $f \neq 0$ and $p \neq 1$. So we have $p \geq 2$. Using (3.15) and (3.19),
\[
2nc - \lambda = f(\alpha - 2k - \text{tr}A) = f\left(-b - \frac{3}{2}k\right),
\]
(3.22)
From (3.19), (3.20), (3.22) and $2p + q = 2n - 2$, we obtain
\[
b^2 + kb - \frac{3}{4}k^2 - \frac{(2nc - \lambda)(2n - 2p - 2)}{p - 1} = 0.
\]
Since $b$ is continuous and $p$ is positive integer, we see that $b$ is constant. So (3.22) implies that $f$ is also constant on $\mathcal{N}''$.

We put $AU = bU + h_1\xi$ and $AZ = fZ$. By the equation of Codazzi, computing $g((\nabla_Z A)U - (\nabla_U A)Z, \phi Z)$, we have
\[
(b - f)g(\nabla_Z U, \phi Z) + fh_1 = 0
\]
on $\mathcal{N}''$. Similarly, computing $g((\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z, Z)$,
\[
(\bar{b} - f)g(\nabla_Z \phi U, Z) = 0.
\]
If $\bar{b} = f$, then (3.21) and (3.22) imply that $b = \bar{b} = -k/2$. This case cannot be occured. So we have $g(\nabla_Z \phi U, Z) = 0$. On the other hand, we obtain
\[
g(\nabla_Z U, \phi Z) = -g(U, (\nabla_Z \phi)Z) - g(U, \phi \nabla_Z Z) \\
g = g(\phi U, \nabla_Z Z) = g(\nabla_Z \phi U, Z) = 0.
\]
From these we have $fh_1 = 0$. This contradicts to $f \neq 0$.

Finally, we consider the case $\text{tr}A - \alpha = b + \bar{b} = 0$ on $\mathcal{N}''$. Then (3.20) implies that $qf = 0$. If $f = 0$, then (3.15) gives $2nc - \lambda = 0$ and hence, by (3.13), we see
\[
\bar{b}b = -\frac{k^2}{4} = 0,
\]
which contradicts to $k \neq 0$. So we have $q = 0$ on $\mathcal{N}''$.

From (3.13), (3.19) and (3.20),

$$b = -\bar{b} = \frac{k}{2}, \quad \bar{b}b = -nc + \frac{\lambda}{2}.$$ 

We can choose an orthonormal frame \( \{e_1, e_2, \cdots, e_{n-1}, e_{n-2}, \xi\} \) on \( M \) which satisfies \( Ae_1 = be_1 + h_1\xi, Ae_y = be_y \) for \( y = 2, \cdots, n-1 \) and \( A\phi e_y = \bar{b}\phi e_y \) for \( y = 1, \cdots, n-1 \). Then, in \( \mathcal{N}'' \), the shape operator \( A \) is represented by the following

\[
A = \begin{pmatrix}
    b & -\bar{b} & 0 & 0 \\
    \vdots & b & \bar{b} & \vdots \\
    h_1 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & \alpha 
\end{pmatrix}
\]

Using Lemma 3.1, we have

**Lemma 3.4.** Let $\phi e_y \in H_2$ be perpendicular to $\phi e_1$. Then,

\[
\nabla_{e_1} e_1 = \frac{h_1}{2} \phi e_1, \quad (3.23) \\
\nabla_{\phi e_y} e_1 = \frac{2c + 2nc - \lambda}{h_1} e_y. \quad (3.24)
\]

**Proof.** Using (3.5), we have \( g(\nabla_{e_1} \phi e_y, e_1) = -g(\nabla_{e_1} e_1, \phi e_y) = 0 \). On the other hand, putting $e_i = \phi e_1$ in (3.5),

\[
h_1(2\bar{b} + b)g(\phi^2 e_1, e_1) + (b - \bar{b})g(\nabla_{e_1} \phi e_1, e_1) = 0,
\]

from which we obtain

\[
g(\nabla_{e_1} e_1, \phi e_1) = \frac{h_1}{2}.
\]

By (3.6), we see that \( g(\nabla_{e_1} e_1, e_y) = 0 \) for any $e_y \in H_1$. Since \( g(\nabla_{e_1} e_1, \xi) = -g(e_1, \phi Ae_1) = 0 \), we have (3.23).
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Next, putting $e_i = \phi e_y$ and $e_j = \phi e_z$ in (3.1), we have $g(\nabla_{\phi e_y} e_1, \phi e_z) = 0$ for any $\phi e_y, \phi e_z \in H_2$, $y \neq z$. Moreover, we have $g(\nabla_{\phi e_y} e_1, \phi e_y) = 0$ by (3.4). On the other hand, using (3.2), we see that

$$g(\nabla_{e_z \phi e_y} e_1, \phi e_y) = 0$$

for any $e_z \in H_1$. Thus, putting $e_i = e_z$ and $e_j = \phi e_y$ in (3.3), direct calculation shows that

$$g(\nabla_{\phi e_y} e_1, e_z) = 2c + 2nc - \lambda h_1 g(\phi e_z, \phi e_y).$$

Since $g(\nabla_{\phi e_y} e_1, \xi) = 0$ and $g(\nabla_{\phi e_y} e_1, e_1) = 0$, we have (3.24).

Using this lemma, we compute the sectional curvature spanned by $e_1$ and $\phi e_y \perp \phi e_1$. From (3.23), we have

$$g(\nabla_{\phi e_y} \nabla_{e_1} e_1, \phi e_y) = -\frac{h_1}{2} g(e_1, \nabla_{\phi e_y} \phi e_y).$$

Since $g(\phi e_1, \phi e_y) = 0$, we have

$$g(\phi e_1, \nabla_{\phi e_y} \phi e_y) = -g(\nabla_{\phi e_y} \phi e_1, \phi e_y) = -g(\phi \nabla_{\phi e_y} e_1, \phi e_y)$$

$$= -g(\nabla_{\phi e_y} e_1, e_y) = \frac{-2c - 2nc + \lambda}{h_1}.$$  

Thus we obtain

$$g(\nabla_{\phi e_y} \nabla_{e_1} e_1, \phi e_y) = c + nc - \lambda/2.$$  

On the other hand, by (3.24),

$$g(\nabla_{e_1} \nabla_{\phi e_y} e_1, \phi e_y) = \nabla_{e_1} g(\nabla_{\phi e_y} e_1, \phi e_y) - g(\nabla_{\phi e_y} e_1, \nabla_{e_1} \phi e_y)$$

$$= \frac{-2c - 2nc + \lambda}{h_1} g(e_y, \nabla_{e_1} \phi e_y).$$

Putting $e_i = \phi e_y$ and $e_j = e_y$ in (3.1), we have $g(\nabla_{e_1} \phi e_y, e_y) = -h_1/2$. From these equations, we obtain

$$g(\nabla_{e_1} \nabla_{\phi e_y} e_1, \phi e_y) = c + nc - \lambda/2.$$
Next, we see that
\[
g(\nabla_{[\phi e_y, e_1]} e_1, \phi e_y) = g(\nabla_{\xi} e_1, \phi e_y) + \sum_{z \geq 2} g(\nabla_{e_z} e_1, \phi e_y)g(e_z, [\phi e_y, e_1]) + \sum_{z \geq 1} g(\nabla_{[\phi e_y, e_1]} e_1, \phi e_y)g(\phi e_z, [\phi e_y, e_1]) = 0.
\]

Here we note that we have \(g(\nabla_{\phi e_z} \phi e_y, e_1) = 0\) for \(z \neq y\) from (3.1) and \(g(\nabla_{\phi e_y} \phi e_y, e_1) = 0\) from (3.4).

From these equations, we see that
\[
g(R(\phi e_y, e_1)e_1, \phi e_y) = g(\nabla_{\phi e_y} \nabla_{e_1} e_1, \phi e_y) - g(\nabla_{e_1} \nabla_{\phi e_y} e_1, \phi e_y) - g(\nabla_{[\phi e_y, e_1]} e_1, \phi e_y) = 0.
\]

On the other hand, the equation of Gauss implies that
\[
g(R(\phi e_y, e_1)e_1, \phi e_y) = c + b\bar{b} = c - nc + \frac{\lambda}{2}.
\]
So we have \(nc - \lambda/2 = c\). Since \(b\bar{b} = -c\) and \(b = -\bar{b} = k/2\), we see that \(c > 0\), \(b^2 = c\) and \(k^2 = 4c\). This contradicts to our assumption \(k^2 \neq 4c\).

From these considerations we see that \(M\) has no point \(x\) where \(A\xi \neq \alpha\xi\), and hence \(M\) is a Hopf hypersurface. This proves our theorem.

Using Theorem 3.2 and Theorem B-C, we have our main result.

**Theorem 3.5.** Let \(M\) be a real hypersurface in a complex space form \(M^n(c)\), \(n \geq 3\), \(c \neq 0\). We suppose that the Ricci tensor \(\hat{S}\) of the generalized Tanaka-Webster connection \(\hat{\nabla}^{(k)}\) satisfies \(\hat{S}(X, \phi Y) = \lambda g(X, \phi Y)\) for any vector fields \(X\) and \(Y\), \(\lambda\) being a function.

(1) If \(M\) is a real hypersurface in \(\mathbb{C}P^n\) and \(k^2 \neq 4\), then \(M\) is locally congruent to one of the following:

(a) a geodesic hypersphere with \(k^2 \geq (2n - 2)(2n - \lambda)\),

(b) a tube over a totally geodesic \(\mathbb{C}P^l\) \((1 \leq l \leq n - 2)\) with \(\lambda = 2n\).
(2) If $M$ is a real hypersurface in $\mathbb{C}H^n$, then $M$ is locally congruent to one of the following:

(a) a geodesic hypersphere with $k^2 \geq (-2n - 2)(2n - \lambda)$,
(b) a tube over a complex hyperbolic hyperplane with $k^2 \geq (-2n - 2)(2n - \lambda)$,
(c) a horosphere with $\lambda = 2k - 2$,
(d) a tube over a totally geodesic $\mathbb{C}H^l$ ($1 \leq l \leq n - 2$) with $\lambda = -2n$.

**Proof.** From Theorem 3.2, $M$ is a Hopf hypersurface of $M^n(c)$. Then Proposition A shows

$$(2\beta - \alpha)A\phi X = (\beta\alpha + 2c)\phi X,$$

where $AX = \beta X$, $g(X, \xi) = 0$ and $\alpha = g(A\xi, \xi)$. We notice that $\alpha$ is constant. If $2\beta - \alpha = 0$, then $\beta\alpha + 2c = 0$, and hence $\alpha^2 + 4c = 0$. Thus we have $c < 0$ and $M$ has two distinct constant principal curvatures $\alpha$ and $b$ with multiplicities $1$ and $2n - 2$ respectively. Moreover $b$ is constant and $M$ is a horosphere of principal curvatures $2$ and $1$ with multiplicities $1$ and $2n - 2$, respectively (see Berndt [1]). By (3.9) and $c = -1$, we have $\lambda = 2k - 2$.

In the following, we assume that $2\beta - \alpha \neq 0$. Then

$$A\phi X = \frac{\beta\alpha + 2c}{2\beta - \alpha}\phi X.$$

We put $\bar{\beta} = (\beta\alpha + 2c)/(2\beta - \alpha)$. Then, by the assumption on $\hat{S}$, we obtain

$$\lambda = 2nc + (\text{tr}A - \alpha + k)\beta - \beta^2 + \bar{\beta}^2 + k\bar{\beta},$$

$$\lambda = 2nc + (\text{tr}A - \alpha + k)\beta - \bar{\beta}^2 + \bar{\beta}^2 + k\bar{\beta}.$$  \hspace{1cm} (3.26)

These imply

$$0 = (\beta - \bar{\beta})(\text{tr}A - \alpha - \beta - \bar{\beta}).$$

Suppose $\beta \neq \bar{\beta}$. Then $\text{tr}A - \alpha - \beta - \bar{\beta} = 0$. Substituting $\bar{\beta} = \text{tr}A - \alpha - \beta$ into the equation above, we obtain

$$2\beta^2 - 2(\text{tr}A - \alpha)\beta - k(\text{tr}A - \alpha) - 2nc + \lambda = 0.$$  \hspace{1cm} (3.27)
Therefore, β satisfies the quadratic equation

\[ 2t^2 - 2(\text{tr} A - \alpha)t - k(\text{tr} A - \alpha) - 2nc + \lambda = 0. \]

From this we see that at most two distinct β satisfies the above equation. But \( \overline{\beta} \) also satisfies the above quadratic equation, and \( M \) has two principal curvatures \( b \) and \( \overline{b} \) with multiplicities \( p \) and \( q \), \( 0 \leq p \leq n-1 \), that satisfies \( b \neq \overline{b} \).

We next suppose that \( \beta = \overline{\beta} \). Then \( \beta^2 - \alpha \beta - c = 0 \). Therefore, \( M \) has at most two non-zero distinct constant principal curvatures \( d \) and \( f \) such that \( d = d, \ f = f \) with multiplicities \( q \) and \( r \), respectively, where \( 2p + q + r = 2n - 2 \). On the other hand, from (3.26), we have

\[
2nc - \lambda + (\text{tr} A - \alpha + 2k)d = 0, \\
2nc - \lambda + (\text{tr} A - \alpha + 2k)f = 0.
\]

(3.28)

If \( M \) has 5 distinct principal curvatures \( b \neq \overline{b}, \ d, \ f \) and \( \alpha \), then the above equations show that \( \text{tr} A - \alpha + 2k = 0 \) and \( 2nc - \lambda = 0 \) since \( d \neq f \). Moreover, from (3.27), we have \( 2b^2 + 4kb + 2k^2 = 2(b+k)^2 = 0 \) and \( (b+k)^2 = 0 \). Hence we obtain \( b = \overline{b} = -k \). This contradicts to the assumption \( b \neq \overline{b} \).

We now suppose that \( M \) has 4 distinct principal curvatures \( b \neq \overline{b}, \ d, \ \alpha \). Then we have

\[ \text{tr} A - \alpha = b + \overline{b} = p(b + \overline{b}) + qd. \]

From this and \( 2p + q = 2n - 2 \),

\[ (p - 1)(b + \overline{b}) + (2n - 2p - 2)d = 0. \]

We notice that \( b \) and \( \overline{b} \) is continuous. Since \( p \) is positive integer and \( d \) is non-zero constant, we see that \( p \neq 1 \) and \( b + \overline{b} \) is constant. Moreover, \( \text{tr} A - \alpha \) is constant. So (3.28) shows that \( \lambda \) is constant. Hence, from (3.27), \( b \) and \( \overline{b} \) are also constant. But there is no Hopf hypersurface with constant four principal curvatures.

If \( M \) has two constant principal curvatures \( d \) and \( \alpha \), then \( \text{tr} A - \alpha = (2n - 2)d \). From (3.26),

\[ (2n - 2)d^2 + 2kd + 2nc - \lambda = 0. \]
This gives a root when
\[ k^2 - (2n - 2)(2nc - \lambda) \geq 0. \]

Next, if \( M \) has three distinct principal curvatures \( b, \bar{b} \) and \( \alpha \), then
\[ \text{tr} A - \alpha = b + \bar{b} = (n - 1)(b + \bar{b}). \]
Hence we have \( b + \bar{b} = \text{tr} A - \alpha = 0 \). On the other hand, \( b \) and \( \bar{b} \) satisfy
\[ b + \bar{b} = \frac{2b^2 + 2c}{2b - \alpha} = 0. \]
Thus we have \( c < 0 \). But the condition \( c < 0 \) implies that the principal curvatures \( b \) and \( \bar{b} \) are positive. This contradicts to \( b + \bar{b} = 0 \).

Finally we consider the case that \( M \) has three constant principal curvatures \( d, f, \alpha \), where \( d = \bar{d}, f = \bar{f} \). Since \( d \neq f \), we have
\[ \text{tr} A - \alpha = -2k, \quad 2nc - \lambda = 0. \]
From these considerations and Thereoms B, C we have our assertion.

\[ \square \]

References


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