A Note on the Dimension Subgroups

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§ 1. Introduction

The $n$th dimension subgroup $D_n(G)$ of a group $G$ is the subgroup of $G$ consisting of elements $x$ such that $x^{-1} \in I_G^{n+1}$, where $I_G$ is the augmentation ideal of the group ring $\mathbb{Z}G$.

On the other hand, we write $G_n$ as the $n$th term in the lower central series of $G$.

The dimension conjecture is referred to:

$D_n(G) = G_n$ for all groups $G$ and all integers $n \geq 0$.

The case $n=0$ is trivial, and the case $n=1$ is proved from the fact $G/G_1 = I/\mathcal{P}$, etc. (1).

This conjecture was introduced by Magnus.

He established when $G$ is any free group. (5).

Attempts by Corn and Losey to establish this conjecture were unsuccessful except in a special case (6).

Higmann proved that to establish the dimension conjecture it is enough to give an affirmative answer when $G$ is any finite $p$-group (1), and from the process of his proof, the following important theorem was induced (1, 6):

**Theorem 1.** If $G$ is a group such that $D_n(G) \not\subset G_n$, then there is a subquotient group $L$ of $G$ with the properties:

(a) $L$ is a finite $p$-group of class $\leq n$.

(b) $D_n(L) \not= \{1\}$.

In this paper the author tries to give another proof of this theorem, which is successful in a very special case.

**Remarks**

(1) $D_n(G) \supset G_n$ is an easy exercise.

(2) Passi proved the following results using Theorem 1. and various cohomological methods (6):

(a) $D_2(G) = G_2$ for all groups $G$.

(b) If $G$ is a group such that none of its subquotients is a 2-group of class 3, then $D_3(G) = G_3$.

(c) If $G$ is a group such that all its subquotients which are 2-groups of class 3 have order 64, then $D_3(G) = G_3$.

(3) Corn and Losey proved $D_n(G) = G_n$, for all $n \geq 0$, when $G$ satisfies:

$x \in G_n$, $x \notin G_{n+1}$, $x^k \in G_{n+1} \Rightarrow x^k = 1$. 

§ 2. Dimension subgroups of a quotient group

Let $G$ be a group, $T=\mathbb{Q}/\mathbb{Z}$ the additive group of rationals mod 1, and $\varphi$ a mapping $G\to T$ of the underlying set.

The mapping $\varphi$ can be extended by linearity to a homomorphism of the additive group of the group ring $\mathbb{Z}G$ to $T$.

$\varphi : G\to T$ is called a polynomial map of degree $\leq n$ if $\varphi$ vanishes on $I^{n+1}$.

**Lemma 1.** If $G$ is a group, then $D_n(G) = \{x|x \in G, \text{ and } \varphi(x) = \varphi(1)\}$ for all polynomial maps $\varphi : G \to T$ of degree $\leq n$ (6).

**Lemma 2.** If $G$ is a group, and $N$ a normal subgroup of $G$, then $I_{G/N}^{n+1} = (I_{G}^{n+1}+(ZG)I_N)/(ZG)I_N \subseteq I_{G/N}^{n+1}/(ZG)I_N \cap I_{G/N}^{n+1}$.

If $N$ is contained in $D_n(G)$, then $I_{G/N}^{n+1} \cong I_{G}^{n+1}/(ZG)I_N$.

**Proof.** Let $\pi : \mathbb{Z}G \to (\mathbb{Z}/N)G$ be the ring epimorphism induced from the natural group epimorphism $\pi : G\to G/N$, then $Ker \pi$ is obviously $(ZG)I_N$, the ideal of $\mathbb{Z}G$ generated by the augmentation ideal of $\mathbb{Z}N$. $\pi$ maps $I_{G}^{n+1}$ onto $I_{G/N}^{n+1}$, so the upper isomorphisms are established.

Now let $N$ be contained in $D_n(G)$, then $\{g(h-1)\}$, where $g \in G$ and $h \in N$, generates $(ZG)I_N$ over $\mathbb{Z}$. Observing $g(h-1) \in I_{G}^{n+1}$, we can see $(ZG)I_N \subseteq I_{G/N}^{n+1}$, which leads to the lower isomorphism. Therefore the lemma is proved.

**Proposition 1.** If $N$ is a normal subgroup of a group $G$, then $D_n(G/N) = D_n(G)N/N$.

Especially if $N$ is contained in $D_n(G)$, then $D_n(G/N) = D_n(G)N/N$.

**Proof.** Let $gN \in D_n(G)N/N$, $\varphi : G/N \to T$ be any polynomial map of degree $\leq n$, then we can assume that $g$ belongs to $D_n(G)$. Now $\varphi = \varphi \pi$, where $\pi$ is the natural epimorphism of $G$ onto $G/N$, is a polynomial map $G\to T$ of degree $\leq n$ because $\varphi(I_{G}^{n+1}) \subseteq \varphi(I_{G/N}^{n+1}) = 0$. So from $g \in D_n(G)$ and **Lemma 1**, it follows $\varphi(gN) = \varphi(g) = \varphi(1) = \varphi(N)$. Therefore $gN \in D_n(G/N)$.

Conversely let $gN \in D_n(G/N)$. Then $gN-N \in I_{G/N}^{n+1}$, and observing the upper isomorphisms of **Lemma 2**, we can see $g-1+(ZG)I_N \in (I_{G}^{n+1}+(ZG)I_N)/(ZG)I_N$, which leads to $gh-1 \in I_{G}^{n+1}$, where $h$ is a suitable element in $N$. Therefore $gh \in D_n(G)$, i.e. $gN \in D_n(G)N/N$.

Now the first equality is established.

The second equality is naturally induced from the first.

**Corollary 1.** If $G$ is a group such that $D_n(G) \neq G_n$, then there is a quotient group $K$ of $G$ with the properties:

(a) $K$ is a nilpotent group of class $\leq n$.

(b) $D_n(K) = \{1\}$

**Proof.** Put $K = G/G_n$, then clearly $K$ is a nilpotent group of class $\leq n$, and by the proposition,
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\[ D_n(K) = D_n(G)/G_n \cong \{ 1 \} \]

This completes the proof.

**Corollary 2.** In the hypothesis of the proposition, there is a one-to-one correspondence between the set of all polynomial maps \( G \to T \) of degree \( \leq n \) and the set of all polynomial maps \( G/N \to T \) of degree \( \leq n \).

Especially there is a one-to-one correspondence between the set of all polynomial maps \( G \to T \) of degree \( \leq n \) and the set of all polynomial maps \( G/D_n(G) \to T \) of degree \( \leq n \).

**Proof.** For a polynomial map \( \phi : G/N \to T \) of degree \( \leq n \), we correspond \( \phi_\pi : G \to G/N \to T \) which, as was seen in the proof of the proposition, is a polynomial map of degree \( \leq n \).

Conversely let \( \psi : G \to T \) be a polynomial map of degree \( \leq n \).

Then observing \( h-1 \in I_G^{n+1} \), we see:

\[ \phi(gh) - \phi(g) = \phi(gh-g) = \phi(g(h-1)) \in \phi(I_G^{n+1}) = 0. \]

So the image of \( gN \) by \( \phi \) is uniquely determined and the map \( \overline{\phi} : G/N \to T \) is induced such that \( \overline{\phi} = \phi_\pi \).

The fact that \( \overline{\phi} \) is a polynomial map of degree \( \leq n \) is easily seen by

\[ \overline{\phi}(I_G^{n+1}) = \overline{\phi}(I_G^{n+1}) = \phi(I_G^{n+1}) = 0. \]

The fact that those correspondences are inverse to each other is trivial.

This completes the proof.

§ 3. A proof of Theorem 1 in the case \( G/G_n \) is finite

**Lemma 3.** If \( G, H, \ldots, K \) are groups, then

\[
(G \times H \times \ldots \times K)_n = G_n \times H_n \times \ldots \times K_n,
\]

\[
D_n(G \times H \times \ldots \times K) = D_n(G) \times D_n(H) \times \ldots \times D_n(K).
\]

**Proof.** The upper equality is suitable as an exercise in Group Theory, and we omit its proof.

Let \( f : Z(G \times H) \to ZG \) be the ring epimorphism induced from the projection \( G \times H \to G \). Then easily \( f(I_G^{n+1}) = I_G \).

Let \( \alpha \in D_n(G \times H) \). We write \( \alpha = (g, h) \), \( g \in G, h \in H \).

From the fact \( \alpha - 1 \in I_G^{n+1} \times H \), it follows \( g - 1 = f(\alpha - 1) \in I_G^{n+1} \), so \( g \in D_n(G) \). Similarly \( h \in D_n(H) \).

Now we have \( D_n(G \times H) \subset D_n(G) \times D_n(H) \), and the converse is trivial.

Consequently it follows:

\[
D_n(G \times H \times \ldots \times K) = D_n(G) \times D_n(H \times \ldots \times K) = \ldots = D_n(G) \times D_n(H) \times \ldots \times D_n(K).
\]

This completes the proof.

**Corollary.** If there is a finite nilpotent group \( G \) of class \( \leq n \) such that \( D_n(G) \cong \{ 1 \} \), then there is a finite \( p \)-group \( L \) of class \( \leq n \) such that \( D_n(L) \cong \{ 1 \} \).
PROOF. As any finite nilpotent group is the direct product of its Sylow subgroups, we find a suitable Sylow subgroup of $G$ as $L$ in the corollary.

This completes the proof.

Combining COROLLARY 1 to PROPOSITION 1 and COROLLARY to LEMMA 3, THEOREM 1 is verified in the case $G/G_m$ is finite.

References


Supplementary Note

The author lectured on the contents of this paper in the 18th, 3, 1974 at the society for the study of Cohomology Theory of Finite Groups in The Institute of Mathematical Analysis, Kyoto.