ON SEVERAL TYPES OF LINGUISTIC ISOMORPHISM

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In the present paper, we shall investigate the diverse types of linguistic isomorphism. For the purpose, before preceding, we shall explain the notations and terminology which are used in the Book [2] by S. Marcus.

Let \( \Gamma \) be a given finite set — vocabulary. Elements of \( \Gamma \) will be called words. We denote by \( T \) the free semi-group generated by \( \Gamma \). By definition, the elements of \( T \) will be finite strings (briefly, strings). Now let \( \Phi \) be a subset of \( T \). Then we shall call the strings which belong to \( \Phi \) marked strings. If \( P \) is a given partition of \( \Gamma \), each set of \( P \) will be called a P-cell, and we denote by \( P(a) \) a P-cell containing the word \( a \). Moreover we notice that for two distinct words \( a, b \) we have either \( P(a) = P(b) \) or \( P(a) \cap P(b) = \emptyset \) (where \( \emptyset \) is the empty set). Now consider a triple \( \{\Gamma, P, \Phi\} \), and we shall call such a triple a language with paradigmatic structure (briefly, language). Let \( x \in \Gamma \), and \( y \in \Gamma \). We shall say that \( x \) dominates \( y \), and we shall write \( x \rightarrow y \), if for each pair of strings \( p \) and \( q \) such that the string \( pxq \) is marked, then the string \( pyq \) is also marked. Thus for any \( x \in \Gamma \), \( y \in \Gamma \), the set \( S(x) = \{ y ; x \rightarrow y \text{ and } y \rightarrow x \} \) determines a partition of \( \Gamma \) into disjoint sets and such a partition is called a family \( S \). Furthermore, the unit partition of \( T \) is, by definition that partition for which \( E(x) = \{ x \} \) when \( x \in \Gamma \).

The above partitions \( S \) and \( E \) are the most useful in the study of linguistics. The notion of marked strings and family has been introduced by O. Koulaudgina in his paper [1]. The notions of domination and of family have been studied in detail by S. Marcus in his papers (for example, [3]). A finite sequence \( P(x_1), P(x_2), \ldots, P(x_n) \) of the cells of a partition \( P \) of \( \Gamma \), is called a P-structure, and we shall say that this P-structure is marked if there exists a marked string \( y_1, y_2, \ldots, y_n \) such that \( y_1 \in P(x_1), y_2 \in P(x_2), \ldots, y_n \in P(x_n). \) Let \( P(x) \) and \( P(y) \) be two cells of \( P \). Then we shall say that \( P(x) \) and \( P(y) \) are P-equivalent and we shall write \( P(x) \leftrightarrow P(y), \) if for each pair of P-structures \( P_1 \) and \( P_2 \), the P-structure \( P_1 P(x) P_2 \) and \( P_1 P(y) P_2 \) are either both marked or both unmarked. Let us consider a language \( \{\Gamma, P, \Phi\} \). Put, for each \( x \in \Gamma \),

\[
P'(x) = \bigcup_{P(x) \leftrightarrow P(y)} P(y)
\]
Then the set $P'(x)$ determines a partition of $\Gamma$ into disjoint sets. The partition $P'$ is called the **derivative** of the partition $P$.

Now let us consider two languages $L_1 = \{\Gamma_1, P, \Phi_1\}$ and $L_2 = \{\Gamma_2, P, \Phi_2\}$.

In the Book [2] by S. Marcus already referred to, various types of isomorphism of $L_1$ and $L_2$ are introduced as follows:

1. **P $\Phi$-isomorphism**: there exists a 1 : 1 mapping $f$ of $\Gamma_1$ onto $\Gamma_2$, such that $P_2(f(x)) = f(P_1(x))$ for each $x \in \Gamma_1$ and such that the string $f(x_1) f(x_2) \ldots f(x_n)$ is $\Phi_2$ if and only if the string $x_1 x_2 \ldots x_n \in \Phi_1$ $(x_i \in \Gamma_1, 1 \leq i \leq n)$.

2. **P $\Phi$-isomorphism**: there exists a 1 : 1 mapping $g$ of $\Gamma_1$ onto $\Gamma_2$, such that $P_2'(g(x)) = g(P_1(x))$ and $S_2(g(x)) = g(S_1(x))$ for each $x \in \Gamma_1$, where $S_1$ and $S_2$ are the partitions into families in $L_1$ and $L_2$, respectively.

3. **PP $\Phi$-isomorphism**: there exists a 1 : 1 mapping $h$ of $\Gamma_1$ onto $\Gamma_2$, such that $P_2(h(x)) = h(P_1(x))$, $P_2'(h(x)) = h(P_1'(x))$ and $S_2(h(x)) = h(S_1(x))$ for any $x \in \Gamma_1$.

Now we shall also define the new types of isomorphism, as follows.

1. **P $\Phi$-isomorphism**: there exists a 1 : 1 mapping $r$ of $\Gamma_1$ onto $\Gamma_2$, such that $P_2'(r(x)) = r(P_1'(x))$ for each $x \in \Gamma_1$ and such that the string $r(x_1) r(x_2) \ldots r(x_n)$ is $\Phi_2$ if and only if the string $x_1 x_2 \ldots x_n \in \Phi_1$ $(x_i \in \Gamma_1, 1 \leq i \leq n)$.

2. **PP $\Phi$-isomorphism**: there exists a 1 : 1 mapping $v$ of $\Gamma_1$ onto $\Gamma_2$, such that $P_2(v(x)) = v(P_1(x))$, $P_2'(v(x)) = v(P_1'(x))$ for any $x \in \Gamma_1$, and such that the string $v(x_1) v(x_2) \ldots v(x_n) \in \Phi_2$ if and only if the string $x_1 x_2 \ldots x_n \in \Phi_1$ $(x_i \in \Gamma_1, 1 \leq i \leq n)$.

Thus we shall have some results on the above described types of isomorphism of $L_1$ and $L_2$. In the first place, we have the following proposition.

**Proposition 1.** If two languages $L_1$ and $L_2$ are $P'$ $\Phi$-isomorphic, they are also $P$ $\Phi$-isomorphic.

**Proof.** Let $y \in S_1(x)$. By definition, for any pair of strings $p$ and $q$ we have either $pq \in \Phi_1$, $pq \notin \Phi_1$ or $pq \notin \Phi_1$, $pq \notin \Phi_1$. Hence, by hypothesis, there exists a 1 : 1 mapping $r$ of $\Gamma_1$ onto $\Gamma_2$ such that $P_2'(r(x)) = r(P_1'(x))$ and such that the string $r(p) r(q) \in \Phi_2$. The latter means $y \in S_2(r(x))$. Thus all of the required conditions of $P'$ $\Phi$-isomorphic languages $L_1$ and $L_2$ are fulfilled.

**Proposition 2.** There exist two $P$ $\Phi$-isomorphic languages $L_1$ and $L_2$ which are not $P'$ $\Phi$-isomorphic.

**Proof.** Let $\Gamma_1 = \{a, b, c\} = \Gamma_2$, $P_1 = E = P_2$, $\Phi_1 = \{a b, a c\}$, and $\Phi_2 = \{a a b, a a c\}$. Then we have $S_1(a) = P_1'(a) = \{a\} = S_2(a) = P_2'(a)$, $S_1(b) = P_2'(b) = \{b, c\} = S_2'(b) = P_2(b)$. By taking as $\varphi$ the identical mapping of $\Gamma_1$, it follows that for any $x \in \Gamma_1$,

$$\varphi(P_1'(x)) = P_2'(\varphi(x)) \text{ and } \varphi(S_1(x)) = S_2(\varphi(x)).$$

Hence $L_1$ and $L_2$ are $P'$ $\Phi$-isomorphic. On the other hand, $L_1$ and $L_2$ are not $P'$ $\Phi$-isomorphic, since the length of each string of $L_1$ is equal to 2, whereas the
length of each string of $L_2$ is equal to 3.

Now we shall have the following propositions.

Proposition 3. There exist two $P'$ $\Phi$-isomorphic languages $L_1$ and $L_2$ which are not $P$ $\Phi$-isomorphic.

Proof. Let $\Gamma_1 = \{a, b, c, d\} = \Gamma_2$, $P_1 = E$, $\Phi_1 = \{aa, ab, ba, cc, cd, dc\} = \Phi_2$, $P_2'(a) = \{a, b\}$, $P_2'(c) = \{c, d\}$. Then we have $P_1'(a) = \{a, b\}$, $P_1'(c) = \{c, d\}$, $P_2'(a) = \{a, b\}$, $P_2'(c) = \{c, d\}$. By taking as $\varphi$ the identical mapping of $\Gamma_1$, it follows that for each $x \in \Gamma_1$ $P_2'\varphi(x) = \varphi(P_1'(x))$ and the string $x_1 x_2 \in \Phi_1$ if and only if $\varphi(x_1) \varphi(x_2) \in \Phi_2$ for any $x_1, x_2$.

Hence $L_1$ and $L_2$ are $P'$ $\Phi$-isomorphic. On the other hand, $L_1$ and $L_2$ are not $P$ $\Phi$-isomorphic, since $P_1$ is the unit partition $E$, whereas $P_2 \neq P_1$.

Proposition 4. There exist two $P$ $\Phi$-isomorphic languages $L_1$ and $L_2$ which are not $P'$ $\Phi$-isomorphic.

Proof. Let $\Gamma_1 = \{a, b, c, d\} = \Gamma_2$, $P_1 = P_2$, $\Phi_2 = \{a b, a c, c d, d c\}$ and $\Phi_2 = \{a b, a d, d c, c d\}$. Then we have $P_2'(a) = \{a\}$, $P_2'(b) = \{b, c\}$, $P_2'(d) = \{d\}$, $P_2'(a) = \{a, d\}$, $P_2'(b) = \{b\}$, $P_2'(c) = \{c\}$. Now define a 1 : 1 mapping $\Psi$ of $\Gamma_1$ onto $\Gamma_2$ as follows: $\Psi(a) = b$, $\Psi(b) = a$, $\Psi(c) = d$, $\Psi(d) = c$. Thus using the mapping $\Psi$, we see easily that $L_1$ and $L_2$ are $P$ $\Phi$-isomorphic, but these languages are not $P'$ $\Phi$-isomorphic, since $\Psi(P_1'(x)) = P_2'(\Psi(x))$ for any $x \in \Gamma_1$.

By V. A. Uspenskii a language is said to be adequate if we have $S(x) \subseteq P'(x)$ for any $x \in \Gamma$ (see, [5]). Then we obtain the following proposition.

Proposition 5. If $L_1$ and $L_2$ are $P'$ $\Phi$-isomorphic and $L_1$ is adequate, then $L_2$ is also adequate.

Proof. If $L_1$ and $L_2$ are $P'$ $\Phi$-isomorphic, in view of proposition 1, these languages are $P'$ $S$-isomorphic. Moreover, since $L_1$ is adequate, by Proposition 57 of [2], $L$ is also adequate.

By S. Marcus (see [4]), a language is said to be completely adequate, if for any two words $x$ and $y$ such that $x$ dominates $y$ we have $y \in P'(x)$.

Proposition 6. If $L_1$ and $L_2$ are $P$ $\Phi$-isomorphic and $L_1$ is completely adequate, $L_2$ is also completely adequate.

Proof. Since $L_1$ is completely adequate, for any pair of the strings $p$ and $q$ we have $y \in P_1'(x)$, $pxq \in \Phi_1$ and $pyq \in \Phi_1$. By hypotese, there exists a 1 : 1 mapping $r$ of $\Gamma_1$ onto $\Gamma_2$ such that $P_2'(r(x)) = r(P_1'(x))$ and such that $r(p)$ $r(x)$ $r(q) \in \Phi_2$ if and only if $pxq \in \Phi_1$. Hence we have $r(y) \in r(P_1'(x)) = P_2'(r(x))$, $r(p)$ $r(x)$ $r(q) \in \Phi_2$, $r(p)$ $r(y)$ $r(q) \in \Phi_2$. That is $L_2$ is completely adequate.

Now we shall have the following statements.

Proposition 7. If $L_1$ and $L_2$ are $PP'$ $\Phi$-isomorphic, these languages are also $PP'$ $S$-isomorphic.

Proof. This proof follows immediately from both the proof of proposition 1
and the definitions of PP' \( \Phi \)-isomorphism and PP' \( S \)-isomorphism.

Proposition 8. There exist two PP' \( S \)-isomorphic languages which are not PP' \( \Phi \)-isomorphic.

Proof. Let \( \Gamma_1 = \{ a, b, c, d \} = \Gamma_2 \), \( P_1(a) = \{ a \} = P_2(a) \), \( P_1(b) = \{ b \} = P_2(b) \), \( P_1(c) = \{ c, d \} = P_2(c) \), \( \Phi_1 = \{ a, c, b, a, d, b, d \} \) and \( \Phi_2 = \{ a, c, d, b, c, a, b, d, a, b \} \). Then we have \( P_1'(a) = \{ a, b \} = S_1(a) \), \( P_1'(c) = \{ c, d \} = S_1(c) \), \( P_2'(a) = \{ a, b \} = S_2(a) \) and \( P_2'(c) = \{ c, d \} = S_2(c) \). Taking for \( h \) the identical mapping of \( \Gamma_1 \), it is easy to see that \( L_1 \) and \( L_2 \) are PP' \( S \)-isomorphic. However, these languages are not PP' \( \Phi \)-isomorphic, since the length of each string of \( L_1 \) is equal to 2 whereas the length of each string of \( L_2 \) is equal to 3.

Proposition 9. If \( L_1 \) and \( L_2 \) are PP' \( \Phi \)-isomorphic, these languages are also PP \( \Phi \)-isomorphic.

Proof. This proof follows immediately from the definitions of PP' \( \Phi \)-isomorphism and PP \( \Phi \)-isomorphism.

Proposition 10. There exist two PP \( \Phi \)-isomorphic languages \( L_1 \) and \( L_2 \) which are not PP' \( \Phi \)-isomorphic.

Proof. Let \( \Gamma_1 = \{ a, b, c \} = \Gamma_2 = \{ x, y, z \} \), \( P_1 = E = P_2 \), \( \Phi_1 = \{ a, b, a, c, c \} \) and \( \Phi_2 = \{ x, y, x, z, z \} \). Then we have \( P_1'(a) = \{ a \} \), \( P_1'(b) = \{ b, c \} \), \( P_1'(x) = \{ x, z \} \), and \( P_2'(b) = \{ y \} \). Now define a \( 1:1 \) mapping \( \phi \) of \( \Gamma_1 \) onto \( \Gamma_2 \) as follows: \( \phi(a) = y, \phi(b) = x, \phi(c) = z \). Thus using the mapping \( \phi \), it is easy to see that \( L_1 \) and \( L_2 \) are PP \( \Phi \)-isomorphic, but these languages are not PP' \( \Phi \)-isomorphic, since \( \phi(P_1'(x)) \neq P_2'(\phi(x)) \) for any \( x \in \Gamma_1 \).

Proposition 11. If \( L_1 \) and \( L_2 \) are PP' \( \Phi \)-isomorphic. theses languages are also PP' \( \Phi \)-isomorphic.

Proof. This proof follows immediately from the definitions of PP' \( \Phi \)-isomorphism and PP' \( \Phi \)-isomorphism.

Proposition 12. There exist two PP' \( \Phi \)-isomorphic languages \( L_1 \) and \( L_2 \) which are not PP' \( \Phi \)-isomorphic.

Proof. Let \( \Gamma_1 = \{ a, b, c \} = \Gamma_2 \), \( P_1 = E \), \( P_2(a) = \{ a \} \), \( P_2(b) = \{ b, c \} \) and \( \Phi_1 = \{ a, b, a, c, a, a \} = \Phi_2 \). Then we have \( P_1'(a) = \{ a \} \), \( P_1'(b) = \{ b, c \} \), \( P_1'(a) = \{ a \} \) and \( P_2'(b) = \{ b, c \} \). Now by taking as \( \phi \) the identical mapping of \( \Gamma_1 \), it follows that \( L_1 \) and \( L_2 \) are PP \( \Phi \)-isomorphic. However, these languages are not PP' \( \Phi \)-isomorphic, since \( P_1 \) is the unit partition \( E \), whereas \( P_2 \neq P_1 \).

References


