Elementary Properties of Infinitely Divisible Probability Distributions for $\tau_T$-Semigroups

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§ 1. Introduction.

Let $T$ be a binary operation on $[0, 1]$ and $\Delta^+$ be a class of probability distributions. For any $F, G$ in $\Delta^+$ and for any real $x$, Moynihan [2] defined 

$$\tau_T(F, G) = \sup \{ T(F(u), G(v)) ; u + v = x \}$$

(1. 1)

and obtained a commutative semigroup $(\Delta^+, \tau_T)$, called the $\tau_T$-semigroup. Moynihan [4], [5] defined the conjugate transform $C_T F$ and obtained the similar properties of characteristic functions in Lukacs [1].

As pointed out by Schweizer and Sklar in [6], several open problems on the arithmetic for $\tau_T$-semigroups are left. One of the interesting problems is a characterization of the class of infinitely divisible elements in $(\Delta^+, \tau_T)$. In this paper we study two elementary properties for this class (Theorem 3. 1 and Theorem 3. 2). These are well-known for the semigroup $(\Delta^+, *)$, the semigroup of probability distributions under the convolution $*$. But $(\Delta^+, \tau_T)$ is not isomorphic to $(\Delta^+, *)$ (Moynihan [3] Theorem 1. 5). Furthermore, $\tau_T$ is not derivable from any function on random variables ([6] p13 Theorem 7. 6. 5).

Therefore our results are not trivial ones.

§ 2. $\tau_T$-Semigroups.

Definition 2. 1. A t-norm is any two-place function $T : [0, 1] \times [0, 1]$ satisfying

(a) $T(a, 1) = a$
(b) $T(c, d) \geq T(a, b)$ for $c \geq a, d \geq b$
(c) $T(a, b) = T(b, a)$
(d) $T(T(a, b), c) = T(a, T(b, c))$.

A t-norm $T$ is strict if $T$ is continuous on $[0, 1] \times [0, 1]$ and strictly increasing in each place on $[0, 1] \times [0, 1]$.

A strict t-norm has the following representation ([6] p68 Theorem 5. 5. 5. and Theorem 5. 5. 7).

Theorem 2. 1. (1) A t-norm $T$ is strict if and only if it admits the representation

$$T(x, y) = k^{-1}(k(x) k(y)) \quad \text{for all } x, y \text{ in } [0, 1]$$

(2. 1)

where $k$ is a continuous and strictly increasing function from $[0, 1]$ onto $[0, 1]$, so that
\( k(0) = 0 \) and \( k(1) = 1. \)

(We say that \( k(\cdot) \) is the multiplicative generator of \( T \).)

(2) If \( k_1 \) and \( k_2 \) are multiplicative generators of a strict t-norm \( T \) then there is an \( r > 0 \) such that

\[
k_2(x) = (k_1(x))^r \quad \text{for all } x \in [0, 1]
\]

Examples.

1. \( T(x, y) = xy \) is a strict t-norm with the multiplicative generator \( k(x) = x \).

2. \( 7rb(\cdot, y) = (\text{Max}(x^p + y^p - 1, 0))^{1/p} \) for \( p \neq 0 \) and \( 7rb(\cdot) = x \) for \( p = 0 \).

\( 7rb \) is a strict t-norm if and only if \( p \leq 0 \), and has the multiplicative generator

\[
k_r(x) = \begin{cases} \exp\{(x^p - 1)/p\} & p < 0 \\ x & p = 0 \end{cases}
\]

Definition 2.2. \( \Delta^+ \) = \( \{F : \mathbb{R} \rightarrow [0, 1] ; F \text{ is left-continuous nondecreasing and } F(0) = 0\} \).

For \( F, G \) in \( \Delta^+ \), \( x \) in \( \mathbb{R} \) and for a strict t-norm \( T \),

\( \tau_T(F, G)(x) \) is defined by (1.1). Then we have ([2])

Theorem 2.2. (1) \( (\Delta^+, \tau_T) \) is a commutative semigroup, that is

(a) \( \tau_T(F, G) \) is in \( \Delta^+ \) for any \( F, G \) in \( \Delta^+ \).

(b) \( \tau_T(F, e_0) = F, \) where \( e_0(x) = 1 \) for \( x > 0 \) and \( = 0 \) for \( x \leq 0 \).

(c) \( \tau_T(F, G) = \tau_T(G, F) \).

(d) \( \tau_T(\tau_T(F, G), H) = \tau_T(F, \tau_T(G, H)) \).

(2) If we introduce the modified Lévy metric \( L \) in \( \Delta^+ \), then \( (\Delta^+, \tau_T, L) \) is a topological semigroup, where

\[
L(F, G) = \inf \{ \delta ; F(x) \leq G(x + \delta) + \delta, \\
G(x) \leq F(x + \delta) + \delta, 0 < x < 1/\delta \}.
\]

Following [2], we say that \( (\Delta^+, \tau_T) \) is the \( \tau_T \)-semigroup. In [4], [5] Moynihan introduced the conjugate transform in \( (\Delta^+, \tau_T) \), which plays the similar role as the characteristic function in usual probability theory.

Definition 2.3. Let \( T \) be a strict t-norm and \( k(\cdot) \) be the multiplicative generator of \( T \).

For any \( F \) in \( \Delta^+ \) and for \( z \geq 0 \) the conjugate transform of \( T \) is defined by

\[
C_T,F(z) = \sup \{ e^{-xz} kF(x) ; x \geq 0 \}
\]

where \( kF(x) = k(F(x)) \) for \( x > 0 \) and \( = 0 \) for \( x \leq 0 \).

Definition 2.4. For any strict t-norm \( T \)

(i) Let \( A_T = \{ \phi : [0, \infty] \rightarrow [0, 1] ; \phi \text{ is positive, non-increasing, log-convex and continuous} \} \cup \{ \theta_0 \}, \) where \( \theta_0(z) = 0 \) for all \( z \geq 0 \).

(ii) For any \( \phi \) in \( A_T \)
\[ C_T^* \cdot \phi(x) = k^{-1}(\inf \{e^{xz} \phi(z) ; z \geq 0\}). \] (2.6)

(iii) For \( F \) in \( \Delta^+ \), \( b_T = \sup \{x ; F(x) = 0\} \).

The following fundamental properties of conjugate transforms are in Moynihan [4], [5].

**Theorem 2.3.** (1) \( C_T(\tau_T(F, G))(z) = C_T F(z) \cdot C_T G(z) \).

(2) \( A_T = \{C_T F ; F \in \Delta^+\} \).

(3) \( C_T : \Delta^+_T \rightarrow A_T \) is a bijection with inverse \( C_T^* \), where \( \Delta^+_T = \{F \in \Delta^+ ; kF \text{ is log-concave}\} \).

(4) \( C_T^*(C_T F \cdot C_T G) = \tau_T(F, G) \), where \( F_T \) and \( G_T \) are the log-concave envelope of \( F \) and \( G \) respectively.

(5) Let \( \{G_n\} \) be a sequence in \( \Delta^+ \), which converges weakly to some \( G \) in \( \Delta^+ \) then \( C_T G_n(z) \) converges to \( C_T G(z) \) for any \( z > 0 \).

(6) Let \( \{\phi_n\} \) be a sequence in \( A_T \setminus \{\theta_n\} \) and \( \lim_{n \to \infty} \phi_n(z) = \phi(z) \) for any \( z \geq 0 \) and if \( \phi(z) \) is continuous at 0 then there exists a \( F \) in \( \Delta^+ \) such that \( \phi(z) = C_T F(z) \) for any \( z \geq 0 \).

(6) is an analogy of the Lévy’s continuity theorem for characteristic functions.

**Definition 2.5.** \( F \) in \( \Delta^+ \) is **infinitely divisible under** \( \tau_T \) if for any positive integer \( n \) there exists a \( G \) in \( \Delta^+ \) such that 
\[ G^n = F, \] where \( G^n \) is recursively defined by \( G^1 = G \), \( G^n = \tau_T(G^{n-1}, G) \) for \( n \geq 2 \).

Let \( B_T = \{F \in \Delta^+ ; kF \text{ is log-concave on } (b_T, \infty)\} \), then following result is in Moynihan [5] Theorem 4.2.

**Theorem 2.4.** Every \( F \) in \( B_T \setminus \{\epsilon_m\} \) is infinitely divisible under \( \tau_T \).

§ 3. Main results.

**Theorem 3.1.** Let \( T \) be a strict t-norm. Then the \( \tau_T \)-product of a finite number of infinitely divisible probability distributions in \( \Delta^+ \) is infinitely divisible under \( \tau_T \).

**Proof** It is sufficient to prove the theorem for the case of two factors. Suppose that \( F \), \( G \) in \( \Delta^+ \) are infinitely divisible under \( \tau_T \). Then there exist for any positive integer \( n \) two probability distributions \( F_n \) and \( G_n \) in \( \Delta^+ \) such that \( F = (F_n)^n, \ G = (G_n)^n \), where \( (F_n)^n \) and \( (G_n)^n \) are defined in Definition 2.5.

Set \( H = \tau_T(F, G) \) and \( H_n = \tau_T(F_n, G_n) \). Then by Theorem 2.3 (1) we have
\[ C_T H(z) = C_T F(z) \cdot C_T G(z) \]
\[ = (C_T F_n)^n(z)(C_T G_n)^n(z) \]
\[ = (C_T H_n)^n(z) = (C_T H_n)^n(z) \]

Operating \( C_T^* \) on both sides, we have by Theorem 2.3 (3), (4)
\[ H = ((H_n)^n). \]
Since \((H_n)_\tau\) is the log-cocave envelope of \(H_n\) and so, it is in \(\Delta^+\), \(H\) is infinitely divisible under \(\tau_\tau\). (Q. E. D.)

**Theorem 3.2.** A distribution function in \(\Delta^+\), which is the weak limit of a sequence of infinitely divisible distribution functions in \(\Delta^+\) is infinitely divisible under \(\tau_\tau\).

**Proof.** Let \(\{F_k\}\) be a sequence of infinitely divisible distribution functions in \(\Delta^+\) and suppose that this sequence converges weakly to a probability distribution function \(F\) in \(\Delta^+\), By Theorem 2.3 (5) we have

\[
(Cr F)(z) = \lim_{k \to \infty} (Cr F_k)(z) \quad \text{for any } z > 0. \tag{3.1}
\]

Since \(\{F_k\}\) is infinitely divisible, there exists for any positive integer \(n\) a sequence \(\{F_{k,n}\}\) in \(\Delta^+\) such that \(F_k = (F_{k,n})^n\).

Then, by Theorem 2.3 (1) we have

\[
Cr F_k(z) = (Cr F_{k,n})^n(z) \tag{3.2}
\]

It follows from (3.1) and (3.2) that

\[
\lim_{k \to \infty} (Cr F_{k,n})(z) = \lim_{k \to \infty} (Cr F_k)^{1/n}(z)
\]

\[
= \lim_{k \to \infty} \exp\left\{\frac{\log Cr F_k(z)}{n}\right\}(z)
\]

\[
= \exp\left\{\frac{\log Cr F(z)}{n}\right\}(z)
\]

If we define \((Cr F)(0)\) by \(\lim_{z \downarrow 0} (Cr F)(z)\), then \(\{(Cr F)(z)\}^{1/n}\) is continuous at \(z = 0\) and by Theorem 2.3 (6) there exists a \(G\) in \(\Delta^+ \setminus \{e_m\}\) such that

\[
(Cr G)(z) = \{(Cr F)(z)\}^{1/n}.
\]

Therefore we have

\[
(Cr F)(z) = (Cr G)(z)^n = (Cr G^n)(z) \tag{3.3}
\]

Operating \(C_{r^*}\) on both sides, we have by Theorem 2.3 (3), (4) that \(F = (G_r)^n\) and since \(G_r\) is in \(\Delta^+\), \(F\) is infinitely divisible under \(\tau_\tau\). (Q. E. D.)

**Corollary of Theorem 3.2.** \(F \in \Delta^+\) is infinitely divisible under \(\tau_\tau\) if and only if \((Cr F)^r \in A_\tau\) for any positive \(r\), where

\[
(Cr F)^r(\tau) = \exp\{r \log Cr F(\tau)\}.
\]

**Proof.** Since the only part is trivial, we prove the if part. The general case is proved by Theorem 3.2, it is sufficient to prove in the case of \(r\) is a positive rational number.

If \(r = n/m\) then \(F^n\) is infinitely divisible by Theorem 3.1. Then there exists a \(G_m\) in \(\Delta^+\) such that \(F^n = (G_m)^m\).

Therefore we have

\[
F^r = F^{nm} = G_m \in \Delta^+ \text{ and by Theorem 2.3 (3), } (Cr F)^r \in A_\tau. \tag{Q. E. D.}
\]
References


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