Doctoral Dissertation (Shinshu University)

# Frobenius extensions and Auslander-Gorenstein rings

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# Abstract


The aim of this paper is to study relationships between Frobenius extensions and Auslander-Gorenstein rings and to give constructions of Auslander-Gorenstein rings using Frobenius extensions.

Let $G$ be a non-trivial finite multiplicative group with the unit element $e$ and $A = \oplus_{x \in G} A_x$ a $G$-graded ring. We construct a Frobenius extension $\Lambda$ of $A$ and study when the ring extension $A$ of $A_e$ can be a Frobenius extension. Also, formulating the ring structure of $\Lambda$, we introduce the notion of $G$-bigraded rings and show that every $G$-bigraded ring is isomorphic to the $G$-bigraded ring $\Lambda$ constructed above.

The case that $G$ is a cyclic group. Starting from an arbitrary ring $R$ we provide a systematic construction of $\mathbb{Z}/n\mathbb{Z}$-graded rings $A$ which are Frobenius extensions of $R$, and show that under mild assumptions $A$ is an Auslander-Gorenstein local ring if and only if so is $R$.

Moreover, formulating the construction of Clifford algebras we introduce the notion of Clifford extensions and show that Clifford extensions are Frobenius extensions. Consequently, Clifford extensions of Auslander-Gorenstein rings are Auslander-Gorenstein rings.

# Contents

# Chapter1

# Introduction

## 1.1 Background and Motivation

Auslander-Gorenstein rings (see Definition 2.1.2) appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type $A$ in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [5], [6] and [21], respectively). Nevertheless, ring theoritical studies of Auslander-Gorenstein rings are seen little. In particular, the construction of Auslander-Gorenstein rings are very little. Recently, M. Hoshino and H. Koga has shown in [13, Section 3] that a left and right noetherian ring $B$ is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over an Auslander-Gorenstein ring . A special ring extension $A/R$ of a noetherian ring $R$ which is called "Frobenius extension" is a typical example which admits an Auslander-Gorenstein resolution. Commutative Gorenstein local rings have been studied extensively (see e.g. [16]). It is needed to study Auslander-Gorenstein local rings.

In this paper, starting from an arbitrary Auslander-Gorenstein ring (respectively, Auslander-Gorenstein local ring) we will provide a method of construction of Auslander-Gorenstein rings (respectively, Auslander-Gorenstein local rings) using Frobenius extensions. We will introduce the notion of Clifford extensions and show that Clifford extensions are Frobenius extensions. Since Frobenius extensions of Auslander-Gorenstein rings are Auslander-Gorenstein ([10, Proposition 1.9]), it follows that Clifford extensions of Auslander-Gorenstein rings are Auslander-Gorenstein.

Now we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [17, 18] which we modify as follows (cf. [1, Section 1]). We use the notation $A/R$ to denote that a ring $A$ contains a ring $R$ as a subring. We say that $A/R$ is a Frobenius extension if the following conditions are satisfied:

(F1) $A$ is finitely generated as a left $R$-module;

(F2) $A$ is finitely generated projective as a right $R$-module;

(F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ as right $A$-modules.

Note that for any $f \in \mathrm{Hom}_R(A, R)$ and any $a \in A$, define $(fa)b = f(ab)$ for any $b \in A$, $\mathrm{Hom}_R(A, R)$ is a right $A$-module. Note also that $\phi$ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$ (cf. Proposition 2.1.4). A Frobenius extension $A/R$ is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as $R$-$A$-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of $R$. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let $A/R$ be a Frobenius extension. Then $A$ is an Auslander-Gorenstein ring if so is $R$, and the converse holds true if $A$ is projective as a left $R$-module, and if $A/R$ is split, i.e., the inclusion $R \to A$ is a split monomorphism of $R$-$R$-bimodules. It should be noted that $A$ is projective as a left $R$-module if $A/R$ is of second kind.

## 1.2 Results

In Chapter 2, we first recall the notion of Frobeniu extensions of rings and that of Auslander-Gorenstein rings.

In Chapter 3, let $G$ be a non-trivial finite multiplicative group with the unit element $e$ and $A = \oplus_{x \in G} A_x$ a $G$-graded ring. In this paper, assuming $A_e$ is a local ring, we study when a ring extension $A$ of $A_e$ can be a Frobenius extension, the notion of which we recall below. To state our main theorem we have to construct a Frobenius extension $\Lambda/A$ of first kind. Namely, we will define an appropriate multiplication on a free right $A$-module $\Lambda$ with a basis $\{v_x\}_{x \in G}$ so that $\Lambda/A$ is a Frobenius extension of first kind. Denote by $\{\gamma_x\}_{x \in G}$ the dual basis of $\{v_x\}_{x \in G}$ for the free left $A$-module $\mathrm{Hom}_A(\Lambda, A)$ and set $\gamma = \Sigma_{x \in G} \gamma_x$. Assume $A_e$ is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and $A$ is reflexive as a right $A_e$-module. Our main theorem states that the following are equivalent (Theorem 3.2.3):

(1) $A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right $A$-modules;

(2) There exist a unique $s \in G$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that $\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$ for all $x \in G$;

(3) There exist a unique $s \in G$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in G$.

Assume $A/A_e$ is a Frobenius extension. We show that it is of second kind (Corollary 3.2.5), and that $A$ is an Auslander-Gorenstein ring if and only if so is $\Lambda$ (Theorem 3.2.6). As we saw above, the ring $\Lambda$ plays an essential role in our argument. Formulating the ring structure of $\Lambda$, we introduce the notion of group-bigraded rings as follows. A ring $\Lambda$ together with a group homomorphism $\eta : G^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$ is said to be a $G$-bigraded ring, denoted by $(\Lambda, \eta)$, if

$1 = \sum_{x \in G} v_x$ with the $v_x$ orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in G$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in G$. We conclude that every $G$-bigraded ring is isomorphic to the $G$-bigraded ring $\Lambda$ constructed above (Proposition 3.3.3).

In Chapter 4, we fix a set of integers $G = \{0, 1, \dots, n-1\}$ with $n \geq 2$ and a cyclic permutation

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

of $G$. Note that the law of composition $G \times G \to G, (i, j) \mapsto \pi^j(i)$ makes $G$ a cyclic group with 0 the unit element. Note also that if $A = F[X]$ is a polynomial ring in one variable $X$ over a ring $F$ and $R = F[X^n]$ is a subring of $A$ then $A$ can be considered as a $G$-graded ring over $R$. In this paper, we will formulate this example and, starting from an arbitrary ring $R$, provide a systematic way to construct $G$-graded rings $A$ so that the ring extensions $A/R$ are split Frobenius extensions of second kind. Namely, we will define an appropriate multiplication on a free right $R$-module $A$ with a basis $\{e_i\}_{i \in G}$ using the following two data: a certain pair $(q, \chi)$ of an integer $q$ and a mapping $\chi : G \to \mathbb{Z}$; a certain triple $(\sigma, c, t)$ of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R$. Our main results in this Chapter state that if either $t \in \mathrm{rad}(R)$, or $c \in \mathrm{rad}(R)$ and $n\chi(i) > iq$ for all $i \neq 0$, then $A$ is an Auslander-Gorenstein local ring if and only if so is $R$ (Theorems 4.1.6 and 4.1.7). Also, we will provide a way to obtain every pair $(q, \chi)$ mentioned above.

In Chapter 5, we will provide a systematic way to construct split Frobenius extensions of second kind. We fix a set of integers $G = \{0, 1, \dots, n-1\}$ with $n \geq 2$ and a ring $R$ together with a pair $(\sigma, c)$ of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the condition: (*) $\sigma^n = \mathrm{id}_R$ and $c \in R^\sigma \cap \mathrm{Z}(R)$. Let $\Lambda$ be a free right $R$-module with a basis $\{v_i\}_{i \in G}$ and, setting $v_{i+kn} = v_i c^k$ for $i \in G$ and $k \in \mathbb{Z}_+$, define a multipleication on $\Lambda$ subject to the following axioms: (L1) $v_i v_j = v_{i+j}$ for all $i, j \in G$; (L2) $av_i = v_i\sigma^i(a)$ for all $a \in R$ and $i \in G$. We will show that $\Lambda/R$ is a split Frobenius extension of second kind, that $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda/\mathrm{rad}(\Lambda)$ canonically if $c \in \mathrm{rad}(R)$, and that for any $\varepsilon \in R^\sigma \cap \mathrm{Z}(R)$ with $\varepsilon^n = 1$ there exists $\tilde{\sigma} \in \mathrm{Aut}(\Lambda)$ such that for any $c' \in R^\sigma \cap \mathrm{Z}(R)$ the pair $(\tilde{\sigma}, c')$ satisfies the condition (*) (Proposition 5.1.2). In case $n = 2$, we use the notation $Cl_1(R; \sigma, c)$ to denote the ring $\Lambda$ constructed above. Moreover, we restrict ourselves to the case where $n = 2$. We fix a set of integers $G = \{0, 1\}$ and a ring $R$ together with a sequence of elements $c_1, c_2, \dots$ in $\mathrm{Z}(R)$. We will construct a sequence $\{(\Lambda_k, \sigma_k)\}_{k \geq 0}$ of pairs $(\Lambda_k, \sigma_k)$ of rings $\Lambda_k$ and certain $\sigma_k \in \mathrm{Aut}(\Lambda_k)$ inductively. Namely, setting $\Lambda_0 = R$ and $\sigma_0 = \mathrm{id}_R$, for $k \geq 1$ we set $\Lambda_k = Cl_1(\Lambda_{k-1}; \sigma_{k-1}, c_k)$ with which a certain $\sigma_k \in \mathrm{Aut}(\Lambda_k)$ is associated (see the proof of Theorem 5.2.1(1)). For any $k \geq 1$ we will show that the ring extension $\Lambda_k/R$ is a split Frobenius extension of first kind, and that $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_k/\mathrm{rad}(\Lambda_k)$ canonically if $c_i \in \mathrm{rad}(R)$ for all $1 \leq i \leq k$ (Theorem 5.2.1). We use the notation $Cl_k(R; c_1, \dots, c_k)$ to denote the ring $\Lambda_k$ constructed above for $k \geq 1$ and we call those rings Clifford extensions of $R$. It should be noted that if $R$ is an Auslander-Gorenstein local ring and if $c_i \in \mathrm{rad}(R)$ for all $i \geq 1$ then every $Cl_k(R; c_1, \dots, c_k)$ is an Auslander-Gorenstein local ring.

In Appendix, we describe some examples.

# Chapter2

# Preliminaries

## 2.1 Definitions and Basic Properties

For a ring $R$ we denote by $\mathrm{rad}(R)$ the Jacobson radical of $R$, by $R^\times$ the set of units in $R$, by $\mathrm{Z}(R)$ the center of $R$, by $\mathrm{Aut}(R)$ the group of ring automorphisms of $R$, for $\sigma \in \mathrm{Aut}(R)$ by $R^\sigma$ the subring of $R$ consisting of all $x \in R$ with $\sigma(x) = x$, and for $n \geq 2$ by $\mathrm{M}_n(R)$ the ring of $n \times n$ full matrices over $R$. Usually, the identity element of a ring is simply denoted by 1. Sometimes, the notation $1_R$ is used to stress that it is the identity element of the ring $R$. We denote by Mod-$R$ the category of right $R$-modules. Left $R$-modules are considered as right $R^{\mathrm{op}}$-modules, where $R^{\mathrm{op}}$ denotes the opposite ring of $R$. In particular, we denote by inj dim $R$ (resp., inj dim $R^{\mathrm{op}}$) the injective dimension of $R$ as a right (resp., left) $R$-module. Sometimes, we use the notation $X_R$ (resp., ${}_RX$) to stress that the module $X$ considered is a right (resp., left) $R$-module.

We start by recalling the notion of Auslander-Gorenstein rings.

**Proposition 2.1.1** (Auslander)**.** *Let $R$ be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

*(1) In a minimal injective resolution $I^\bullet$ of $R$ in* Mod-$R$*,* flat dim $I^i \leq i$ *for all $0 \leq i \leq n$.*

*(2) In a minimal injective resolution $J^\bullet$ of $R$ in* Mod-$R^{\mathrm{op}}$*,* flat dim $J^i \leq i$ *for all $0 \leq i \leq n$.*

*(3) For any $1 \leq i \leq n+1$, any $M \in$ mod-$R$ and any submodule $X$ of $\mathrm{Ext}^i_R(M,R) \in$ mod-$R^{\mathrm{op}}$ we have $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ for all $0 \leq j < i$.*

*(4) For any $1 \leq i \leq n+1$, any $X \in$ mod-$R^{\mathrm{op}}$ and any submodule $M$ of $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$ mod-$R$ we have $\mathrm{Ext}^j_R(M,R) = 0$ for all $0 \leq j < i$.*

*Proof.* See e.g. [7, Theorem 3.7]. □

**Definition 2.1.2** ([6])**.** A right and left noetherian ring $R$ is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 2.1.1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring $R$ we have inj dim $R =$ inj dim $R^{\mathrm{op}}$ whenever inj dim $R < \infty$ and inj dim $R^{\mathrm{op}} < \infty$ (see [22, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [17, 18], which we modify as follows (cf. [1, Section 1]).

**Definition 2.1.3.** A ring $A$ is said to be an extension of a ring $R$ if $A$ contains $R$ as a subring, and the notation $A/R$ is used to denote that $A$ is an extension ring of $R$. A ring extension $A/R$ is said to be Frobenius if the following conditions are satisfied:

(F1) $A$ is finitely generated as a left $R$-module;
(F2) $A$ is finitely generated projective as a right $R$-module;
(F3) $A \cong \mathrm{Hom}_R(A, R)$ as right $A$-modules.

In case $R$ is a right and left noetherian ring, for any Frobenius extension $A/R$ the isomorphism $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-$A$ yields an Auslander-Gorenstein resolution of $A$ over $R$ in the sense of [13, Definition 3.5].

The next proposition is well-known and easily verified.

**Proposition 2.1.4.** *Let $A/R$ be a ring extension and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in* Mod-$A$*. Then the following hold.*

*(1) There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*

*(2) If $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ is another isomorphism in* Mod-$A$*, then there exists $u \in A^{\times}$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*

*(3) $\phi$ is an isomorphism of $R$-$A$-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

*Proof.* For the benefit of the reader we include a proof.

(1) Since $\mathrm{Hom}_R(A, R) \in$ Mod-$R^{\mathrm{op}}$, $x\phi(1) \in \mathrm{Hom}_R(A, R)$ for all $x \in R$. Then, since $\mathrm{Hom}_R(A, R)$ is a free right $A$-module of rank 1 with a basis $\{\phi(1)\}$, for all $x \in R$ there exists a unique element $\theta(x) \in A$ such that $x\phi(1) = \phi(1)\theta(x)$. Next, we show that $\theta : R \to A$ is a ring homomorphism. We have $\phi(1)\theta(1_R) = 1_R\phi(1) = \phi(1) = \phi(1)1_A$ and hence $\theta(1_R) = 1_A$. Also, for all $x, y \in R$ we have $\phi(1)\theta(xy) = xy\phi(1) = x\phi(1)\theta(y) = \phi(1)\theta(x)\theta(y)$ and hence $\theta(xy) = \theta(x)\theta(y)$.

(2) Set $u = \phi^{-1}(\phi'(1))$. Then $u \in A^{\times}$ and we have $\phi(1)u = \phi(u) = \phi'(1)$. Also, for all $x \in R$ we have $\phi'(1)\theta'(x) = x\phi'(1) = x\phi(1)u = \phi(1)\theta(x)u = \phi(1)uu^{-1}\theta(x)u = \phi'(1)u^{-1}\theta(x)u$ and hence $\theta'(x) = u^{-1}\theta(x)u$.

(3) "Only if" part. We have $\phi(1)\theta(x) = x\phi(1) = \phi(x) = \phi(1)x$ and hence $\theta(x) = x$.

"If" part. For any $x \in R$ and $a \in A$ we have $\phi(xa) = \phi(1)xa = \phi(1)\theta(x)a = x\phi(1)a = x\phi(a)$. □

**Definition 2.1.5** (cf. [17, 18])**.** A Frobenius extension $A/R$ is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as $R$-$A$-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

**Proposition 2.1.6.** *If $A/R$ is a Frobenius extension of second kind, then $A$ is projective as a left $R$-module.*

*Proof.* Let $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ be an isomorphism in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Then $\theta$ induces an equivalence $U_\theta : \text{Mod-}R^{\mathrm{op}} \xrightarrow{\sim} \text{Mod-}R^{\mathrm{op}}$ such that for any $M \in \text{Mod-}R^{\mathrm{op}}$ we have $U_\theta M = M$ as an additive group and the left $R$-module structure of $U_\theta M$ is given by the law of composition $R \times M \to M, (x, m) \mapsto \theta(x)m$. Since $\phi$ yields an isomorphism of $R$-$A$-bimodules $U_\theta A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$, and since $\mathrm{Hom}_R(A, R)$ is projective as a left $R$-module, it follows that $U_\theta A$ and hence $A$ are projective as left $R$-modules. □

**Proposition 2.1.7.** *For any Frobenius extensions $\Lambda/A$, $A/R$ the following hold.*

*(1) $\Lambda/R$ is a Frobenius extension.*

*(2) Assume $\Lambda/A$ is of first kind. If $A/R$ is of second (resp., first) kind, then so is $\Lambda/R$.*

*Proof.* (1) Obviously, (F1) and (F2) are satisfied. Also, we have

$$\begin{aligned}\Lambda &\cong \mathrm{Hom}_A(\Lambda, A)\\ &\cong \mathrm{Hom}_A(\Lambda, \mathrm{Hom}_R(A, R))\\ &\cong \mathrm{Hom}_R(\Lambda \otimes_A A, R)\\ &\cong \mathrm{Hom}_R(\Lambda, R)\end{aligned}$$

in Mod-$\Lambda$.

(2) Let $\psi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A)$ be an isomorphism of $A$-$\Lambda$-bimodules and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Setting $\gamma = \psi(1)$ and $\alpha = \phi(1)$, as in (1), we have an isomorphism in Mod-$\Lambda$

$$\xi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

For any $x \in R$, we have

$$\begin{aligned}x\xi(1)(\mu) &= x\alpha(\gamma(\mu))\\ &= \alpha(\theta(x)\gamma(\mu))\\ &= \alpha(\gamma(\theta(x)\mu))\\ &= \xi(1)(\theta(x)\mu)\end{aligned}$$

for all $\mu \in \Lambda$ and $x\xi(1) = \xi(1)\theta(x)$. □

**Definition 2.1.8** ([1])**.** A ring extension $A/R$ is said to be split if the inclusion $R \to A$ is a split monomorphism of $R$-$R$-bimodules.

**Proposition 2.1.9** (cf. [1])**.** *For any Frobenius extension $A/R$ the following hold.*

*(1) If $R$ is an Auslander-Gorenstein ring, then so is $A$ with* $\text{inj dim } A \le \text{inj dim } R$.

*(2) Assume $A$ is projective as a left $R$-module and $A/R$ is split. If $A$ is an Auslander-Gorenstein ring, then so is $R$ with* $\text{inj dim } R = \text{inj dim } A$.

*Proof.* (1) See [13, Theorem 3.6].

(2) It follows by [1, Proposition 1.7] that $R$ is a right and left noetherian ring with $\text{inj dim } R = \text{inj dim } R^{\text{op}} = \text{inj dim } A$. Let $A \to E^\bullet$ be a minimal injective resolution in Mod-$A$. For any $i \ge 0$, $\text{Hom}_R(-, E^i) \cong \text{Hom}_A(- \otimes_R A, E^i)$ as functors on Mod-$R$ and $E^i_R$ is injective, and $E^i \otimes_R - \cong E^i \otimes_A A \otimes_R -$ as functors on Mod-$R^{\text{op}}$ and $\text{flat dim } E^i_R \le \text{flat dim } E^i_A \le i$. Now, since $R_R$ appears in $A_R$ as a direct summand, it follows that $R$ satisfies the Auslander condition. □

# Chapter3

# Group-graded and group-bigraded rings

## 3.1 Graded rings

In this chapter, $G$ stands for a non-trivial finite multiplicative group with the unit element $e$.

**Definition 3.1.1.** Let $A$ be a ring. A $G$-grading for $A$ is a family $\{A_x\}_{x\in G}$ such that $A = \oplus_{x\in G} A_x$ and $A_x A_y \subset A_{xy}$ for all $x, y \in G$.

Throughout this and the next sections, we fix a ring $A$ together with a family $\{\delta_x\}_{x\in G}$ in $\mathrm{End}_{\mathbb{Z}}(A)$ satisfying the following conditions:
(D1) $\delta_x\delta_y = 0$ unless $x = y$ and $\sum_{x\in G} \delta_x = \mathrm{id}_A$;
(D2) $\delta_x(a)\delta_y(b) = \delta_{xy}(\delta_x(a)b)$ for all $a, b \in A$ and $x, y \in G$.
Namely, setting $A_x = \mathrm{Im}\ \delta_x$ for $x \in G$, $A = \oplus_{x\in G} A_x$ is a $G$-graded ring. In particular, $A/A_e$ is a split ring extension.

To prove our main theorem (Theorem 3.2.3), we use an extension ring $\Lambda$ of $A$ such that $\Lambda/A$ is a Frobenius extension of first kind. We construct such a ring $\Lambda$. Let $\Lambda$ be a free right $A$-module with a basis $\{v_x\}_{x\in G}$ and define a multiplication on $\Lambda$ subject to the following axioms:
(M1) $v_xv_y = 0$ unless $x = y$ and $v_xv_x = v_x$ for all $x \in G$;
(M2) $av_x = \sum_{y\in G} v_y\delta_{yx^{-1}}(a)$ for all $a \in A$ and $x \in G$.
We denote by $\{\gamma_x\}_{x\in G}$ the dual basis of $\{v_x\}_{x\in G}$ for the free left $A$-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{x\in G} v_x\gamma_x(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{x,y\in G} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)$$

for all $\lambda, \mu \in \Lambda$. Also, setting $\gamma = \sum_{x\in G} \gamma_x$, we define a mapping

$$\phi : \Lambda \to \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda.$$

**Proposition 3.1.2.** *The following hold.*

(1) *$\Lambda$ is an associative ring with $1 = \sum_{x \in G} v_x$ and contains $A$ as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{x \in G} v_x a$.*

(2) *$\phi$ is an isomorphism of $A$-$\Lambda$-bimodules, i.e., $\Lambda/A$ is a Frobenius extension of first kind.*

*Proof.* (1) Let $\lambda \in \Lambda$. Obviously, $\sum_{x \in G} v_x \cdot \lambda = \lambda$. Also, by (D1) we have

$$\begin{aligned} \lambda \cdot \sum_{y \in G} v_y &= \sum_{x,y \in G} v_x \delta_{xy^{-1}}(\gamma_x(\lambda)) \\ &= \sum_{x \in G} v_x \gamma_x(\lambda) \\ &= \lambda. \end{aligned}$$

Next, for any $\lambda, \mu, \nu \in \Lambda$ by (D2) we have

$$\begin{aligned} (\lambda\mu)\nu &= \sum_{x,y,z \in G} v_x \delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu))\gamma_z(\nu) \\ &= \sum_{x,y,z \in G} v_x \delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu))\gamma_z(\nu) \\ &= \lambda(\mu\nu). \end{aligned}$$

The remaining assertions are obvious.

(2) Let $\lambda \in \operatorname{Ker} \phi$. For any $y \in G$ we have $0 = \gamma(\lambda v_y) = \sum_{x \in G} \delta_{xy^{-1}}(\gamma_x(\lambda))$ and $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $x \in G$. Thus for any $x \in G$ we have $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $y \in G$ and by (D1) $\gamma_x(\lambda) = 0$, so that $\lambda = 0$. Next, for any $f = \sum_{x \in G} a_x \gamma_x \in \operatorname{Hom}_A(\Lambda, A)$, setting $\lambda = \sum_{x,z \in G} v_x \delta_{xz^{-1}}(a_z)$, by (D1) we have

$$\begin{aligned} (\gamma\lambda)(v_y) &= \gamma(\lambda v_y) \\ &= \sum_{x \in G} \delta_{xy^{-1}}(\gamma_x(\lambda)) \\ &= \sum_{x,z \in G} \delta_{xy^{-1}}(\delta_{xz^{-1}}(a_z)) \\ &= a_y \\ &= f(v_y) \end{aligned}$$

for all $y \in G$ and $f = \gamma\lambda$. Finally, for any $a \in A$ by (D1) we have

$$\begin{aligned} (\gamma a)(\lambda) &= \gamma(a\lambda) \\ &= \sum_{x,y \in G} \delta_{yx^{-1}}(a)\gamma_x(\lambda) \\ &= a\gamma(\lambda) \end{aligned}$$

for all $\lambda \in \Lambda$ and $\gamma a = a\gamma$. $\square$

*Remark* 3.1.3. Denote by $|G|$ the order of $G$. If $|G| \cdot 1_A \in A^\times$, then $\Lambda/A$ is a split ring extension.

**Lemma 3.1.4.** *The following hold.*

*(1)* $v_x \lambda v_y = v_x \delta_{xy^{-1}}(\gamma_x(\lambda))$ *for all* $\lambda \in \Lambda$ *and* $x, y \in G$.

*(2)* $v_x \Lambda v_y = v_x A_{xy^{-1}}$ *for all* $x, y \in G$.

*(3)* $v_x a \cdot v_y b = v_x ab$ *for all* $x, y, z \in G$ *and* $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.

*Proof.* Immediate by the definition. □

Setting $\Lambda_{x,y} = v_x \Lambda v_y$ for $x, y \in G$, we have $\Lambda = \oplus_{x,y \in G} \Lambda_{x,y}$ with $\Lambda_{x,y}\Lambda_{z,w} = 0$ unless $y = z$ and $\Lambda_{x,y}\Lambda_{y,z} \subseteq \Lambda_{x,z}$ for all $x, y, z \in G$. Also, setting $\lambda_{x,y} = \delta_{xy^{-1}}(\gamma_x(\lambda)) \in A_{xy^{-1}}$ for $\lambda \in \Lambda$ and $x, y \in G$, we have a group homomorphism

$$\eta : G^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$$

such that $\eta_x(\lambda)_{y,z} = \lambda_{yx^{-1},zx^{-1}}$ for all $\lambda \in \Lambda$ and $x, y, z \in G$. We denote by $\Lambda^G$ the subring of $\Lambda$ consisting of all $\lambda$ such that $\eta_x(\lambda) = \lambda$ for all $x \in G$.

**Proposition 3.1.5.** *The following hold.*

*(1)* $\eta_y(v_x) = v_{xy}$ *for all* $x, y \in G$.

*(2)* $\Lambda^G = A$.

*(3)* $(\lambda\mu)_{x,z} = \sum_{y \in G} \lambda_{x,y}\mu_{y,z}$ *for all* $\lambda, \mu \in \Lambda$ *and* $x, z \in G$.

*Proof.* (1) Since $\eta_y(v_x)_{z,w} = \delta_{zw^{-1}}(\gamma_{zy^{-1}}(v_x))$ for all $z, w \in G$, we have

$$\eta_y(v_x)_{z,w} = \begin{cases} 1 & \text{if } z = w \text{ and } x = zy^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $a \in A$, since $\eta_x(a)_{y,z} = a_{yx^{-1},zx^{-1}} = \delta_{(yx^{-1})(zx^{-1})^{-1}}(a) = \delta_{yz^{-1}}(a) = a_{y,z}$ for all $x, y, z \in G$, we have $a \in \Lambda^G$. Conversely, for any $\lambda \in \Lambda^G$ we have $\delta_{y^{-1}}(\gamma_x(\lambda)) = \lambda_{x,yx} = \eta_{x^{-1}}(\lambda)_{e,y} = \lambda_{e,y} = \delta_{y^{-1}}(\gamma_e(\lambda))$ for all $x, y \in G$, so that $\gamma_x(\lambda) = \gamma_e(\lambda)$ for all $x \in G$.

(3) For any $\lambda, \mu \in \Lambda$ and $x, z \in G$ by (D2) we have

$$\begin{aligned} (\lambda\mu)_{x,z} &= \sum_{y \in G} \delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)) \\ &= \sum_{y \in G} \delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu)) \\ &= \sum_{y \in G} \lambda_{x,y}\mu_{y,z}. \end{aligned}$$

□

*Remark* 3.1.6. We have $\eta_y(v_x a_x)v_y b_y = v_{xy} a_x b_y$ for all $a_x \in A_x$ and $b_y \in A_y$.

**Proposition 3.1.7.** *The following hold.*

*(1)* $\mathrm{End}_\Lambda(v_x\Lambda) \cong A_e$ *as rings for all* $x \in G$.

*(2)* $v_x\Lambda \ncong v_y\Lambda$ *in* Mod-$\Lambda$ *for all* $x, y \in G$ *with* $A_{xy^{-1}}A_{yx^{-1}} \subseteq \mathrm{rad}(A_e)$.

*Proof.* (1) We have $\mathrm{End}_\Lambda(v_x\Lambda) \cong v_x\Lambda v_x \cong A_e$ as rings.

(2) For any $f : v_x\Lambda \to v_y\Lambda$ and $g : v_y\Lambda \to v_x\Lambda$ in Mod-$\Lambda$, since $f(v_x) = v_y a$ with $a \in A_{yx^{-1}}$ and $g(v_y) = v_x b$ with $b \in A_{xy^{-1}}$, we have $g(f(v_x)) = v_x ba$ with $ba \in \mathrm{rad}(A_e)$. □

The proposition above asserts that if $A_e$ is local and $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ then $\Lambda$ is semiperfect and basic. We refer to [3] for semiperfect rings.

## 3.2 Auslander-Gorenstein rings

In this section, we will ask when $A/A_e$ is a Frobenius extension.

**Lemma 3.2.1.** *For any* $x \in G$ *the following hold.*

*(1)* $av_x = v_x a$ *for all* $a \in A_e$ *and* $\Lambda v_x$ *is a* $\Lambda$-$A_e$*-bimodule.*

*(2)* $\Lambda v_x = \sum_{y \in G} v_y A_{yx^{-1}}$.

*(3)* $A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y\in G} v_y \delta_{yx^{-1}}(a)$ *as* $A$-$A_e$*-bimodules.*

*(4) If* $\Lambda v_x$ *is reflexive as a right* $A_e$*-module, then* $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong A_e$ *as rings.*

*Proof.* (1) and (2) Immediate by the definition.

(3) By (2) we have a bijection $f_x : A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y\in G} v_y \delta_{yx^{-1}}(a)$. Since every $\delta_{yx^{-1}}$ is a homomorphism in Mod-$A_e$, so is $f_x$. Finally, for any $a, b \in A$ we have

$$\begin{aligned} a \cdot (\sum_{y\in G} v_y \delta_{yx^{-1}}(b)) &= \sum_{y,z\in G} v_z \delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b) \\ &= \sum_{z\in G} v_z (\sum_{y\in G} \delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b)) \\ &= \sum_{z\in G} v_z \delta_{zx^{-1}}(\sum_{y\in G} \delta_{zy^{-1}}(a)b) \\ &= \sum_{z\in G} v_z \delta_{zx^{-1}}(ab) \end{aligned}$$

and $f_x$ is a homomorphism in Mod-$A^{\mathrm{op}}$.

(4) Since the canonical homomorphism

$$\Lambda v_x \to \mathrm{Hom}_{A_e^{\mathrm{op}}}(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e), A_e), \lambda \mapsto (f \mapsto f(\lambda))$$

is an isomorphism, $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(\Lambda v_x)^{\mathrm{op}} \cong v_x\Lambda v_x \cong A_e$ as rings. □

It follows by Lemma 3.2.1(1) that $\delta_e\gamma_e : \Lambda \to A_e$ is a homomorphism of $A_e$-$A_e$-bimodules and $\Lambda/A_e$ is a split ring extension.

**Lemma 3.2.2.** *For any $x, y \in G$ and $a, b \in A$ we have*

$$v_x a \cdot (\sum_{z\in G} v_z \delta_{zy^{-1}}(b)) = v_x(\sum_{z\in G} \delta_{xz^{-1}}(a)\delta_{zy^{-1}}(b))$$

*Proof.* Immediate by the definition. □

**Theorem 3.2.3** ([10] Theorem 3.3)**.** *Assume $A_e$ is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and $A$ is reflexive as a right $A_e$-module. Then the following are equivalent.*

*(1) $A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right $A$-modules.*

*(2) There exist a unique $s \in G$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that*

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

*for all $x \in G$.*

*(3) There exist a unique $s \in G$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that*

$$\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$$

*for all $x \in G$.*

*Proof.* (1) ⇒ (2). Let $A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), 1 \mapsto \alpha$ in Mod-$A$. Then, since by Proposition 3.1.2(2) $\Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda$ in Mod-$\Lambda$, by adjointness we have an isomorphism in Mod-$\Lambda$

$$\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

By Proposition 3.1.7(1) $\Lambda = \oplus_{x\in G} v_x\Lambda$ with the $\mathrm{End}_\Lambda(v_x\Lambda)$ local. Also, by (1) and (4) of Lemma 3.2.1

$$\mathrm{Hom}_{A_e}(\Lambda, A_e) \cong \oplus_{x\in G}\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)$$

with the $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e))$ local. Now, according to Proposition 3.1.7(2), it follows by the Krull-Schmidt theorem that there exists a unique $s \in G$ such that

$$\phi_{s,e} : v_s\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_e, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

Thus, setting $\alpha_s = \alpha|_{A_s}$, by Lemmas 3.2.1(2) and 3.2.2 we have

$$\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab))).$$

It then follows again by Lemmas 3.2.1(2) and 3.2.2 that

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

for all $x \in G$.

(2) $\Rightarrow$ (3). Since $A = \oplus_{x \in G} A_{sx} = \oplus_{x \in G} A_{x^{-1}}$, and since $A_{sx} A_{x^{-1}} \subseteq A_s$ for all $x \in G$, $\psi$ induces $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in G$.

(3) $\Rightarrow$ (1). Setting $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for each $x \in G$, the $\psi_x$ yields $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$. □

*Remark* 3.2.4. In the theorem above, $\alpha_s$ is an isomorphism and $A_e \xrightarrow{\sim} \mathrm{End}_{A_e}(A_s)$ canonically.

*Proof.* For any $b \in A_e$, setting $f : A_e \to A_e, 1 \mapsto b$, we have $f = \psi_e(a)$ and hence $b = \alpha_s(a)$ for some $a \in A_s$. Also, Ker $\alpha_s$ = Ker $\psi_s = 0$. Then, since the composite $A_e \to \mathrm{End}_{A_e}(A_s) \to \mathrm{Hom}_{A_e}(A_s, A_e)$ is an isomorphism, the last assertion follows. □

**Corollary 3.2.5.** *Assume $A_e$ is local and $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$. If $A/A_e$ is a Frobenius extension, then it is of second kind.*

*Proof.* Set $t = \alpha_s^{-1}(1) \in A_s$. Then for any $u \in A_s$ there exists $f \in \mathrm{End}_{A_e}(A_s)$ such that $u = f(t)$ and hence $u = at$ for some $a \in A_e$. Thus $A_e t = A_s$ and there exists $\theta \in \mathrm{Aut}(A_e)$ such that $\theta(a)t = ta$ for all $a \in A_e$. Then $(\alpha_s \theta(a))(t) = \alpha_s(\theta(a)t) = \alpha_s(ta) = \alpha_s(t)a = a = (a\alpha_s)(t)$ and $\alpha_s \theta(a) = a\alpha_s$ for all $a \in A_e$. Now, setting $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$, we have $(a\psi(1))(b) = a\alpha_s(\delta_s(b)) = (a\alpha_s)(\delta_s(b)) = (\alpha_s\theta(a))(\delta_s(b)) = \alpha_s(\theta(a)\delta_s(b)) = \alpha_s(\delta_s(\theta(a)b)) = (\psi(1)\theta(a))(b)$ for all $a, b \in A$, so that $a\psi(1) = \psi(1)\theta(a)$ for all $a \in A$. □

**Theorem 3.2.6** ([10] Theorem 3.6)**.** *Assume $A_e$ is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$, and $A/A_e$ is a Frobenius extension. Then $A$ is an Auslander-Gorenstein ring if and only if so is $\Lambda$.*

*Proof.* The "only if" part follows by Propositions 2.1.9(1) and 3.1.2(2). Assume $\Lambda$ is an Auslander-Gorenstein ring. By Proposition 3.1.2(2) $\Lambda/A$ is a Frobenius extension of first kind, and by Corollary 3.2.5 $A/A_e$ is a Frobenius extension of second kind. Thus by Proposition 2.1.7 $\Lambda/A_e$ is a Frobenius extension of second kind. Also, by Lemma 3.2.1(1) $\Lambda/A_e$ is split. Hence by Propositions 2.1.6 and 2.1.9(2) $A_e$ is an Auslander-Gorenstein ring and by Proposition 2.1.9(1) so is $A$. □

*Remark* 3.2.7. Assume $A_e$ is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and $A/A_e$ is a Frobenius extension. Let $s \in G$ be as in Theorem 3.2.3. Then the following hold.

(1) $s \neq e$ unless $A = A_e$.

(2) Let $H$ be a subgroup of $G$ containing $s$ and $A_H = \oplus_{x \in H} A_x$. Then $A_H/A_e$ is a Frobenius extension and, unless $s = e$, the mapping cone of the multiplication map

$$\bigoplus_{x \in H} \Lambda v_x \otimes_{A_e} v_x \Lambda \to \Lambda$$

is a tilting complex for right $\Lambda$-modules (see [20] for tilting complexes).

*Proof.* (1) Suppose to the contrary that $s = e$. Let $x \in G$ with $x \neq e$ and $A_x \neq 0$. Then by Remark 3.2.4 there exists $u \in A_e{}^{\times}$ such that $A_x \xrightarrow{\sim} \mathrm{Hom}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto uab)$. Note that $uab \in \mathrm{rad}(A_e)$ for all $a \in A_x$ and $b \in A_{x^{-1}}$. On the other hand, since $A_{x^{-1}}$ is nonzero projective, and since $A_e$ is local, there exists an epimorphism $f : A_{x^{-1}} \to A_e$ in Mod-$A_e$, a contradiction.

(2) Since $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in H$, the $\psi_x$ yields $\psi_H : A_H \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_H, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$. The first assertion follows by Theorem 3.2.3.

Next, let $v_H = \sum_{x \in H} v_x$. Then by Lemma 3.2.1(1) $av_H = v_H a$ for all $a \in A_e$. Since $\Lambda/A_e$ is a Frobenius extension, $\Lambda v_H$ is finitely generated projective as a right $A_e$-module and by Theorem 3.2.3 $v_H\Lambda \cong \mathrm{Hom}_{A_e}(\Lambda v_H, A_e)$ as right $\Lambda$-modules. Note that $v_x\Lambda v_x \neq 0$ and $v_{sx}\Lambda v_x \neq 0$ for all $x \in H$. Thus the last assertion follows by the same argument as in [1, Example 4.3]. □

We will see in the final section that the element $s \in G$ in Theorem 3.2.3 does not necessarily depend on the structure of the group $G$ (Example 3.4.3).

## 3.3 Bigraded rings

Formulating the ring structure of $\Lambda$ constructed in Section 2, we make the following.

**Definition 3.3.1.** A ring $\Lambda$ together with a group homomorphism

$$\eta : G^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$$

is said to be a $G$-bigraded ring, denoted by $(\Lambda, \eta)$, if $1 = \sum_{x \in G} v_x$ with the $v_x$ orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in G$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in G$.

Throughout this section, we fix a $G$-bigraded ring $(\Lambda, \eta)$. Set $A_x = v_x\Lambda v_e$ for $x \in G$ and $A = \oplus_{x \in G} A_x$. Note that $\eta_y(A_x) = v_{xy}\Lambda v_y$ for all $x, y \in G$. For any $a_x \in A_x$ and $b_y \in A_y$ we define the multiplication $a_x \cdot b_y$ in $A$ as the multiplication $\eta_y(a_x)b_y$ in $\Lambda$ (cf. Remark 3.1.6).

**Proposition 3.3.2.** *The following hold.*

*(1) $A$ is an associative ring with $1 = v_e$.*

*(2) $A$ is a $G$-graded ring.*

*Proof.* (1) For any $a_x \in A_x$, $b_y \in A_y$ and $c_z \in A_z$ we have

$$\begin{aligned}(a_x \cdot b_y) \cdot c_z &= \eta_y(a_x)b_y \cdot c_z \\ &= \eta_z(\eta_y(a_x)b_y)c_z \\ &= \eta_{yz}(a_x)\eta_z(b_y)c_z \\ &= a_x \cdot (b_y \cdot c_z).\end{aligned}$$

Also, for any $a_x \in A_x$ we have $v_e \cdot a_x = \eta_x(v_e)a_x = v_x a_x = a_x$ and $a_x \cdot v_e = \eta_e(a_x)v_e = a_x v_e = a_x$.

(2) Obviously, $A_x A_y \subseteq A_{xy}$ for all $x, y \in G$. □

In the following, for each $x \in G$ we denote by $\delta_x : A \to A_x$ the projection. Then, setting $\lambda_{x,y} = v_x \lambda v_y$ for $\lambda \in \Lambda$ and $x, y \in G$, we have a mapping $\varphi : A \to \Lambda$ such that $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a))$ for all $a \in A$ and $x, y \in G$.

**Proposition 3.3.3.** *The following hold.*

*(1) $\varphi : A \to \Lambda$ is an injective ring homomorphism with* $\mathrm{Im}\ \varphi = \Lambda^G$.

*(2) $v_x \Lambda v_y = v_x \varphi(A_{xy^{-1}})$ for all $x, y \in G$.*

*(3) $\{v_x\}_{x\in G}$ is a basis for the right $A$-module $\Lambda$.*

*(4) $\varphi(a)v_x = \sum_{y\in G} v_y \varphi(\delta_{yx^{-1}}(a))$ for all $a \in A$ and $x \in G$.*

*(5) $v_x\varphi(a)v_y\varphi(b) = v_x\varphi(ab)$ for all $x, y, z \in G$ and $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.*

*Proof.* (1) Obviously, $\varphi$ is a monomorphism of additive groups. Also, we have

$$\varphi(v_e)_{x,y} = \begin{cases} v_x & \text{if } x = y, \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi(1_A) = 1_\Lambda$. Let $a_x \in A_x, b_y \in A_y$ and $z, w \in G$. Since $\varphi(a_x \cdot b_y)_{z,w} = \varphi(\eta_y(a_x)b_y)_{z,w} = \eta_w(\delta_{zw^{-1}}(\eta_y(a_x)b_y))$, $\varphi(a_x \cdot b_y)_{z,w} = 0$ unless $xy = zw^{-1}$. If $xy = zw^{-1}$, then $\eta_w(\delta_{zw^{-1}}(\eta_y(a_x)b_y)) = \eta_{yw}(a_x)\eta_w(b_y)$. On the other hand,

$$\begin{aligned}(\varphi(a_x)\varphi(b_y))_{z,w} &= \sum_{u\in G} \varphi(a_x)_{z,u}\varphi(b_y)_{u,w} \\ &= \sum_{u\in G} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)).\end{aligned}$$

Thus $(\varphi(a_x)\varphi(b_y))_{z,w} = 0$ unless $zu^{-1} = x$ and $uw^{-1} = y$, i.e., $zw^{-1} = xy$. If $zw^{-1} = xy$, then $\sum_{u\in G} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)) = \eta_{yw}(a_x)\eta_w(b_y)$. As a consequence, $\varphi(a_x \cdot b_y)_{z,w} = (\varphi(a_x)\varphi(b_y))_{z,w}$. The first assertion follows.

Next, for any $a \in A$ and $x, y, z \in G$ we have

$$\begin{aligned}\eta_x(\varphi(a))_{y,z} &= v_y\eta_x(\varphi(a))v_z \\ &= \eta_x(v_{yx^{-1}}\varphi(a)v_{zx^{-1}}) \\ &= \eta_x(\varphi(a)_{yx^{-1},zx^{-1}}) \\ &= \eta_x(\eta_{zx^{-1}}(\delta_{yz^{-1}}(a))) \\ &= \eta_z(\delta_{yz^{-1}}(a)) \\ &= \varphi(a)_{y,z},\end{aligned}$$

so that $\mathrm{Im}\ \varphi \subseteq \Lambda^G$. Conversely, let $\lambda \in \Lambda^G$. Then $\lambda_{x,y} = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e}$ for all $x, y \in G$. Thus, setting $a = \sum_{x\in G} \lambda_{x,e}$, we have $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a)) = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e} = \lambda_{x,y}$ for all $x, y \in G$ and $\varphi(a) = \lambda$.

(2) Let $x, y \in G$ and $a \in A_{xy^{-1}}$. For any $z \neq y$ we have $\delta_{xz^{-1}}(a) = 0$ and hence $v_x\varphi(a)v_z = \varphi(a)_{x,z} = \eta_z(\delta_{xz^{-1}}(a)) = 0$. Thus $v_x\varphi(a) = \varphi(a)_{x,y} = \eta_y(a)$. It follows that $v_x\Lambda v_y = \eta_y(v_{xy^{-1}}\Lambda v_e) = \eta_y(A_{xy^{-1}}) = v_x\varphi(A_{xy^{-1}})$.

(3) This follows by (2).

(4) Note that $\eta_x(\delta_{yx^{-1}}(a)) = v_y\eta_x(\delta_{yx^{-1}}(a))$ for all $y \in G$. Thus $\varphi(a)v_x = \sum_{y\in G} v_y\varphi(a)v_x = \sum_{y\in G}\eta_x(\delta_{yx^{-1}}(a)) = \sum_{y\in G} v_y\eta_x(\delta_{yx^{-1}}(a))$. Also,

$$\begin{aligned} v_y\varphi(\delta_{yx^{-1}}(a)) &= \sum_{z\in G} v_y\varphi(\delta_{yx^{-1}}(a))v_z \\ &= \sum_{z\in G} v_y\eta_z(\delta_{yz^{-1}}(\delta_{yx^{-1}}(a))) \\ &= v_y\eta_x(\delta_{yx^{-1}}(a)) \end{aligned}$$

for all $y \in G$.

(5) This follows by (2) and (4). □

Let us call the $G$-bigraded ring constructed in Section 2 standard. Then the proposition above asserts that every $G$-bigraded ring is isomorphic to a standard one. Namely, according to Lemma 3.1.4, $\varphi : A \to \Lambda$ can be extended to an isomorphism of $G$-bigraded rings.

## 3.4 Examples

In this section, we will provide a systematic construction of $G$-graded rings $A$ such that $A/A_e$ is a Frobenius extension of second kind.

Let $(s, \chi)$ be a pair of an element $s \in G$ and a mapping $\chi : G \to \mathbb{Z}$ satisfying the following conditions:

(X1) $\chi(x) + \chi(y) \geq \chi(xy)$ for all $x, y \in G$;

(X2) $\chi(x) + \chi(x^{-1}s) = \chi(s)$ for all $x \in G$.

These are obviously satisfied if $s$ is arbitrary and $\chi(x) = 0$ for all $x \in G$. We set

$$\omega(x, y) = \chi(x) + \chi(y) - \chi(xy)$$

for $x, y \in G$.

**Lemma 3.4.1.** *The following hold.*

*(1) $\omega(x, y) \geq 0$ for all $x, y \in G$.*

*(2) $\omega(e, x) = \omega(x, e) = \chi(e) = 0$ for all $x \in G$.*

*(3) $\chi(x) + \chi(y) = \omega(x, y) + \chi(xy)$ for all $x, y \in G$.*

*(4) $\omega(xy, z) + \omega(x, y) = \omega(x, yz) + \omega(y, z)$ for all $x, y, z \in G$.*

*(5) $\omega(x, x^{-1}s) = 0$ for all $x \in G$.*

*Proof.* It follows by (X2) that $\chi(e) = 0$. The other assertions are obvious. □

In the following, we fix a ring $R$ together with a pair $(\sigma, c)$ of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition:

$$(*) \quad \sigma(c) = c \quad \text{and} \quad ac = c\sigma(a) \text{ for all } a \in R.$$

This is obviously satisfied if either $\sigma = \mathrm{id}_R$ and $c \in \mathrm{Z}(R)$, or $\sigma$ is arbitrary and $c = 0$. As usual, we require $c^0 = 1$ even if $c = 0$.

Let $A$ be a free right $R$-module with a basis $\{u_x\}_{x \in G}$. By abuse of notation we denote by $\{\delta_x\}_{x \in G}$ the dual basis of $\{u_x\}_{x \in G}$ for the free left $R$-module $\mathrm{Hom}_R(A, R)$, i.e., $a = \sum_{x \in G} u_x \delta_x(a)$ for all $a \in A$. According to Lemma 3.4.1(1), we can define a multiplication on $A$ subject to the following axioms:

(M1) $u_x u_y = u_{xy} c^{\omega(x,y)}$ for all $x, y \in G$;

(M2) $a u_x = u_x \sigma^{\chi(x)}(a)$ for all $a \in R$ and $x \in G$.

**Proposition 3.4.2.** *The following hold.*

*(1) $A$ is a $G$-graded ring with $A_e \cong R$.*

*(2) $A/A_e$ is a Frobenius extension of second kind.*

*(3) If $c \in \mathrm{rad}(R)$, then $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ with $\omega(x, x^{-1}) > 0$.*

*Proof.* (1) It follows by Lemma 3.4.1(2) that $u_e \cdot u_x a = u_x a = u_x a \cdot u_e$ for all $x \in G$ and $a \in R$. For any $x, y, z \in G$ and $a_x, a_y, a_z \in R$ we have

$$\begin{aligned}
(u_x a_x \cdot u_y a_y) \cdot u_z a_z &= u_{xy} c^{\omega(x,y)} \sigma^{\chi(y)}(a_x) a_y \cdot u_z a_z \\
&= u_{xyz} c^{\omega(xy,z)} \sigma^{\chi(z)}(c^{\omega(x,y)} \sigma^{\chi(y)}(a_x) a_y) a_z \\
&= u_{xyz} c^{\omega(xy,z)} c^{\omega(x,y)} \sigma^{\chi(z)+\chi(y)}(a_x) \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(xy,z)+\omega(x,y)} \sigma^{\chi(z)+\chi(y)}(a_x) \sigma^{\chi(z)}(a_y) a_z, \\
u_x a_x \cdot (u_y a_y \cdot u_z a_z) &= u_x a_x \cdot u_{yz} c^{\omega(y,z)} \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(x,yz)} \sigma^{\chi(yz)}(a_x) c^{\omega(y,z)} \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(x,yz)} c^{\omega(y,z)} \sigma^{\omega(y,z)}(\sigma^{\chi(yz)}(a_x)) \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(x,yz)+\omega(y,z)} \sigma^{\omega(y,z)+\chi(yz)}(a_x) \sigma^{\chi(z)}(a_y) a_z
\end{aligned}$$

and by (3), (4) of Lemma 3.4.1 $(u_x a_x \cdot u_y a_y) \cdot u_z a_z = u_x a_x \cdot (u_y a_y \cdot u_z a_z)$. Thus $A$ is an associative ring with $1 = u_e$. Obviously, $A$ contains $R$ as a subring via the injective ring homomorphism $R \to A, a \mapsto u_e a$, i.e., setting $A_x = u_x R$ for $x \in G$, $A = \oplus_{x \in G} A_x$ is a $G$-graded ring with $A_e = R$.

(2) It follows by (M2) that $\delta_x a = \sigma^{\chi(x)}(a) \delta_x$ for all $a \in R$ and $x \in G$. In particular, $\{\delta_x\}_{x \in G}$ is a basis for the right $R$-module $\mathrm{Hom}_R(A, R)$. Also, for any $x \in G$ by Lemma 3.4.1(5) $u_x u_{x^{-1}s} = u_s$ and hence $\delta_s u_x = \delta_{x^{-1}s}$. It follows that $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_s a$ in Mod-$A$. Obviously, $A$ is a free left $R$-module with a basis $\{u_x\}_{x \in G}$. Thus, since $\delta_s a = \sigma^{\chi(s)}(a) \delta_s$ for all $a \in R$, $A/R$ is a Frobenius extension of second kind.

(3) Immediate by (M1). $\square$

**Example 3.4.3.** For any $s \in G \setminus \{e\}$, setting

$$\chi(x) = \begin{cases} 0 & \text{if } x = e, \\ 2 & \text{if } x = s, \\ 1 & \text{otherwise,} \end{cases}$$

we have a pair $(s, \chi)$ satisfying the conditions (X1), (X2).

**Example 3.4.4.** Consider the case where $G = G_1 \times \cdots \times G_n$ with the $G_k$ cyclic. For each $1 \le k \le n$, fix a generator $x_k \in G_k$ and set $m_k = |G_k|$. Set $s = (x_1^{m_1-1}, \ldots, x_n^{m_n-1})$ and $\chi((x_1^{i_1}, \ldots, x_n^{i_n})) = i_1 + \cdots + i_n$, where $0 \le i_k \le m_k - 1$ for all $1 \le k \le n$. Then the pair $(s, \chi)$ satisfies the conditions (X1), (X2).

*Remark* 3.4.5. The following hold.

(1) $0 \le \chi(x) \le \chi(s)$ for all $x \in G$.

(2) $G_0 = \chi^{-1}(0)$ is a subgroup of $G$ with $sG_0 = G_0 s$.

(3) $\chi$ takes the constant value $\chi(x)$ on $G_0 x G_0$ for all $x \in G$.

(4) $\omega(x, x^{-1}) > 0$ for all $x \neq e$ if and only if $G_0 = \{e\}$.

*Proof.* (1) For any $x \in G$, since $x^m = e$ for some $m > 0$, it follows by (X1) that $m\chi(x) \ge \chi(x^m) = \chi(e) = 0$ and $\chi(x) \ge 0$. It then follows by (X2) that $\chi(x) \le \chi(s)$ for all $x \in G$.

(2) We have $e \in G_0$ and by (X1) $xy \in G_0$ for all $x, y \in G_0$. Also, by (X2) we have $sG_0 = \chi^{-1}(\chi(s)) = G_0 s$.

(3) It follows by (X1) that $\chi(x) \ge \chi(xy)$ for all $x \in G$ and $y \in G_0$. It then follows that $\chi(xy) \ge \chi(xyy^{-1}) = \chi(x)$ for all $x \in G$ and $y \in G_0$. Similarly, $\chi(x) = \chi(yx)$ for all $x \in G$ and $y \in G_0$.

(4) By the fact that $G_0$ is a subgroup of $G$. $\square$

*Remark* 3.4.6. Set $A_0 = \oplus_{x \in G_0} A_x$, which is the group ring of $G_0$ over $R$. It follows by Remark 3.4.5(3) that $A$ is free as a right (resp., left) $A_0$-module. Next, define mappings $\delta_0 : A \to A_0$ and $\theta : A_0 \to A_0$ as follows:

$$\delta_0(a) = \sum_{x \in G_0} u_x \delta_{sx}(a) \quad \text{and} \quad \theta(b) = \sum_{x \in G_0} u_x \sigma^{\chi(s)}(\delta_{sxs^{-1}}(b))$$

for $a \in A$ and $b \in A_0$, respectively. Then $\delta_0 \in \mathrm{Hom}_{A_0}(A, A_0)$ and $\theta \in \mathrm{Aut}(A_0)$. Furthermore, $A \xrightarrow{\sim} \mathrm{Hom}_{A_0}(A, A_0), a \mapsto \delta_0 a$ in Mod-$A$ and $\delta_0 b = \theta(b)\delta_0$ for all $b \in A_0$. Consequently, $A/A_0$ is a Frobenius extension of second kind.

*Remark* 3.4.7. Consider the case where $R$ is commutative, $\sigma = \mathrm{id}_R$ and $s$ lies in the center of $G$. Then $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_s a$ as $A$-$A$-bimodules.

*Proof.* Note first that $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_s a$ in Mod-$A$, which we have shown in the proof of Proposition 3.4.2(2). Next, for any $a, b \in A$ we have

$$\begin{aligned}\delta_s(ab) &= \sum_{x\in G} \delta_x(a)\delta_{x^{-1}s}(b)\\ &= \sum_{x\in G} \delta_{sx^{-1}}(b)\delta_x(a)\\ &= \sum_{y\in G} \delta_y(b)\delta_{y^{-1}s}(a)\\ &= \delta_s(ba),\end{aligned}$$

so that $\delta_s a = a\delta_s$ for all $a \in G$. $\square$

# Chapter4

# Constructions of Auslander-Gorenstein local rings

## 4.1 Construction

Throughout this chapter, we fix a set of integers $G = \{0, 1, \ldots, n-1\}$ with $n \geq 2$ and a cyclic permutation

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

of $G$. Note that the law of composition $G \times G \to G, (i,j) \mapsto \pi^j(i)$ makes $G$ a cyclic group with 0 the unit element. Note also that if $A = F[X]$ is a polynomial ring in one variable $X$ over a ring $F$ and $R = F[X^n]$ is a subring of $A$ then $A$ can be considered as a $G$-graded ring over $R$. In the following, we will formulate this example and provide a systematic construction of $G$-graded local rings starting from an arbitrary local ring.

Also, throughout this chapter, we fix a pair $(q, \chi)$ of an integer $q$ and a mapping $\chi : G \to \mathbb{Z}$ satisfying the following conditions:

(X1) $q - \chi(n-j+i) \leq \chi(j) - \chi(i) \leq \chi(j-i)$ for all $i, j \in G$ with $i < j$;

(X2) $\chi(i) + \chi(n-i-1) = \chi(n-1)$ for all $i \in G$.

These are obviously satisfied if $q \leq n$ and $\chi(i) = i$ for all $i \in G$. We set

$$\omega(i,j) = \begin{cases} \chi(i) + \chi(j) - \chi(\pi^j(i)) & \text{if } i+j < n, \\ \chi(i) + \chi(j) - \chi(\pi^j(i)) - q & \text{if } i+j \geq n \end{cases}$$

for $i, j \in G$.

**Lemma 4.1.1.** *The following hold.*

*(1)* $\omega(i,j) \geq 0$ *for all* $i, j \in G$.

*(2)* $\omega(0,i) = \omega(i,0) = \chi(0) = 0$ *for all* $i \in G$.

*(3)* $\omega(i, n-i-1) = 0$ *for all* $i \in G$.

*Proof.* (1) If $i + j < n$, setting $j' = i + j$, we have $i, j' \in G$ with $i < j'$ and

$$\omega(i, j) = \chi(j' - i) - \{\chi(j') - \chi(i)\}.$$

If $i + j \geq n$, setting $i' = i + j - n$, we have $i', j \in G$ with $i' < j$ and

$$\omega(i, j) = \{\chi(j) - \chi(i')\} - \{q - \chi(n - j + i')\}.$$

Consequently, the assertion follows by (X1).

(2) By definition we have $\omega(0, i) = \omega(i, 0) = \chi(0)$ and by (X2) $\chi(0) = 0$.

(3) Immediate by (X2). □

In the following, we fix a ring $R$ together with a triple $(\sigma, c, t)$ of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R$ satisfying the following condition:

$$(*) \quad c, t \in R^\sigma \quad \text{and} \quad xc = c\sigma(x), xt = t\sigma^q(x) \text{ for all } x \in R.$$

This is obviously satisfied if either $\sigma = \mathrm{id}_R$ and $c, t \in \mathrm{Z}(R)$, or $\sigma$ is arbitrary and $c = t = 0$. Note also that $ct = tc$. As usual, we require $c^0 = 1$ even if $c = 0$.

Let $A$ be a free right $R$-module with a basis $\{e_i\}_{i \in G}$ and $\{\delta_i\}_{i \in G}$ the dual basis of $\{e_i\}_{i \in G}$ for the free left $R$-module $\mathrm{Hom}_R(A, R)$, i.e., $a = \sum_{i \in G} e_i \delta_i(a)$ for all $a \in A$. According to Lemma 4.1.1(1), we can define a multiplication on $A$ subject to the following axioms:

(M1) $e_i e_j = e_{\pi^j(i)} c^{\omega(i,j)}$ if $i + j < n$ and $e_i e_j = e_{\pi^j(i)} t c^{\omega(i,j)}$ if $i + j \geq n$;

(M2) $x e_i = e_i \sigma^{\chi(i)}(x)$ for all $x \in R$ and $i \in G$.

We will see that $A$ is an associative ring with $1 = e_0$ and the mapping

$$\phi : A \to \mathrm{Hom}_R(A, R), a \mapsto \delta_{n-1} a$$

is an isomorphism in Mod-$A$ with $\sigma^{\chi(n-1)}(x)\phi(1) = \phi(1)x$ for all $x \in R$.

**Lemma 4.1.2.** *The following hold.*

*(1) For any* $a, b \in A$ *we have*

$$\begin{aligned} ab = &\sum_{i+j<n} e_{\pi^j(i)} c^{\omega(i,j)} \sigma^{\chi(j)}(\delta_i(a)) \delta_j(b) \\ &+ \sum_{i+j \geq n} e_{\pi^j(i)} t c^{\omega(i,j)} \sigma^{\chi(j)}(\delta_i(a)) \delta_j(b) \end{aligned}$$

*and* $\delta_0(ab) = \delta_0(a)\delta_0(b) + \sum_{i \neq 0} t c^{\omega(i,n-i)} \sigma^{\chi(n-i)}(\delta_i(a)) \delta_{n-i}(b)$.

*(2) For any* $a \in A$ *and* $i, j \in G$ *we have*

$$\delta_i(a e_j) = \begin{cases} c^{\omega(\pi^{-j}(i), j)} \sigma^{\chi(j)}(\delta_{\pi^{-j}(i)}(a)) & \text{if } i \geq j, \\ t c^{\omega(\pi^{-j}(i), j)} \sigma^{\chi(j)}(\delta_{\pi^{-j}(i)}(a)) & \text{if } i < j. \end{cases}$$

*Proof.* (1) Straightforward.

(2) Obviously, the equality holds for $j = 0$. Let $j \neq 0$. For any $a \in A$ and $k \in G$ we have

$$e_k\delta_k(a) \cdot e_j = \begin{cases} e_{\pi^j(k)}c^{\omega(k,j)}\sigma^{\chi(j)}(\delta_k(a)) & \text{if } k+j < n, \\ e_{\pi^j(k)}tc^{\omega(k,j)}\sigma^{\chi(j)}(\delta_k(a)) & \text{if } k+j \geq n. \end{cases}$$

If $k + j < n$, setting $i = k + j$, we have

$$e_{i-j}\delta_{i-j}(a) \cdot e_j = e_i c^{\omega(i-j,j)}\sigma^{\chi(j)}(\delta_{i-j}(a))$$

and $\delta_i(ae_j) = c^{\omega(i-j,j)}\sigma^{\chi(j)}(\delta_{i-j}(a))$. If $k+j \geq n$, setting $i = k+j-n$, we have

$$e_{i-j+n}\delta_{i-j+n}(a) \cdot e_j = e_i tc^{\omega(i-j+n,j)}\sigma^{\chi(j)}(\delta_{i-j+n}(a))$$

and $\delta_i(ae_j) = tc^{\omega(i-j+n,j)}\sigma^{\chi(j)}(\delta_{i-j+n}(a))$. $\square$

In the following, we set

$$e_{i+kn} = e_i t^k \quad \text{and} \quad \chi_q(i+kn) = \chi(i) + kq$$

for $i \in G$ and $k \in \mathbb{Z}_+$, the set of non-negative integers, and set

$$\omega_q(k,l) = \chi_q(k) + \chi_q(l) - \chi_q(k+l)$$

for $k, l \in \mathbb{Z}_+$. Obviously, $\chi_q|_G = \chi$ and $\omega_q|_{G\times G} = \omega$. Also, it is not difficult to check the following:

(a) $e_ie_j = e_{i+j}c^{\omega_q(i,j)}$ for all $i, j \in G$;
(b) $xe_k = e_k\sigma^{\chi_q(k)}(x)$ for all $x \in R$ and $k \in \mathbb{Z}_+$;
(c) $\omega_q(i,j) = \chi_q(i) + \chi_q(j) - \chi_q(i+j)$ for all $i, j \in G$;
(d) $\omega_q(i+j,k) + \omega_q(i,j) = \omega_q(i,j+k) + \omega_q(j,k)$ for all $i, j, k \in G$.

**Proposition 4.1.3.** *The following hold.*

*(1) $A$ is an associative ring with $1 = e_0$ and contains $R$ as a subring via the injective ring homomorphism $R \to A, x \mapsto e_0x$, i.e., setting $A_i = e_iR$ for $i \in G$, $A = \oplus_{i\in G}A_i$ is an $G$-graded ring with $A_0 = R$.*

*(2) $\phi$ is an isomorphism in* Mod-$A$ *with $\sigma^{\chi(n-1)}(x)\phi(1) = \phi(1)x$ for all $x \in R$, i.e., $A/R$ is a split Frobenius extension of second kind.*

*Proof.* (1) It follows by Lemma 4.1.1(2) that $e_0 \cdot e_ix = e_ix = e_ix \cdot e_0$ for all $i \in G$ and $x \in R$. Let $i, j, k \in G$ and $x, y, z \in R$. By (a), (b) we have

$$\begin{aligned}(e_ix \cdot e_jy) \cdot e_kz &= e_{i+j}c^{\omega_q(i,j)}\sigma^{\chi_q(j)}(x)y \cdot e_kz \\ &= e_{i+j+k}c^{\omega_q(i+j,k)}\sigma^{\chi_q(k)}(c^{\omega_q(i,j)}\sigma^{\chi_q(j)}(x)y)z \\ &= e_{i+j+k}c^{\omega_q(i+j,k)}c^{\omega_q(i,j)}\sigma^{\chi_q(k)+\chi_q(j)}(x)\sigma^{\chi_q(k)}(y)z \\ &= e_{i+j+k}c^{\omega_q(i+j,k)+\omega_q(i,j)}\sigma^{\chi_q(k)+\chi_q(j)}(x)\sigma^{\chi_q(k)}(y)z,\end{aligned}$$

$$
\begin{aligned}
e_i x \cdot (e_j y \cdot e_k z) &= e_i x \cdot e_{j+k} c^{\omega_q(j,k)} \sigma^{\chi_q(k)}(y) z \\
&= e_{i+j+k} c^{\omega_q(i,j+k)} \sigma^{\chi_q(j+k)}(x) c^{\omega_q(j,k)} \sigma^{\chi_q(k)}(y) z \\
&= e_{i+j+k} c^{\omega_q(i,j+k)} c^{\omega_q(j,k)} \sigma^{\omega_q(j,k)}(\sigma^{\chi_q(j+k)}(x)) \sigma^{\chi_q(k)}(y) z \\
&= e_{i+j+k} c^{\omega_q(i,j+k)+\omega_q(j,k)} \sigma^{\omega_q(j,k)+\chi_q(j+k)}(x) \sigma^{\chi_q(k)}(y) z.
\end{aligned}
$$

It then follows by (c), (d) that $(e_i x \cdot e_j y) \cdot e_k z = e_i x \cdot (e_j y \cdot e_k z)$. The last assertion is obvious.

(2) It follows by (M2) that $\delta_i x = \sigma^{\chi(i)}(x)\delta_i$ for all $x \in R$ and $i \in G$. In particular, $\{\delta_i\}_{i \in G}$ is a basis for the right $R$-module $\mathrm{Hom}_R(A, R)$. Also, for any $i \in G$ by Lemma 4.1.1(3) $e_i e_{n-i-1} = e_{n-1}$ and hence $\delta_{n-1} e_i = \delta_{n-i-1}$. It follows that $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_{n-1} a$ in Mod-$A$. Obviously, $A$ is a free left $R$-module with a basis $\{e_i\}_{i \in G}$. Thus, since $\delta_{n-1} x = \sigma^{\chi(n-1)}(x)\delta_{n-1}$ for all $x \in R$, the associated ring homomorphism is just $\sigma^{-\chi(n-1)} : R \xrightarrow{\sim} R$ and hence $A/R$ is a Frobenius extension of second kind. Also, by (1) $A/R$ is split. □

In the following, we set

$$
\varepsilon(i) = \sum_{k=1}^{n-1} \omega_q(i, ki)
$$

for $i \in G$. By Lemma 4.1.1(1) $\varepsilon(i) \geq 0$ for all $i \in G$. Also, for any $i \in G$ we have $\chi_q(in) = iq$ and hence

$$
\begin{aligned}
\varepsilon(i) &= \sum_{k=1}^{n-1} \{\chi_q(i) + \chi_q(ki) - \chi_q((k+1)i)\} \\
&= n\chi_q(i) - \chi_q(ni) \\
&= n\chi(i) - iq.
\end{aligned}
$$

**Lemma 4.1.4.** *The following hold.*

*(1)* $e_i e_j = e_j e_i$ *for all* $i, j \in G$ *and* $e_i^n = e_0 t^i c^{\varepsilon(i)}$ *for all* $i \in G$.

*(2) If* $t \in \mathrm{rad}(R)$, *then* $\delta_0(a) \in R^\times$ *for all* $a \in A^\times$.

*Proof.* (1) For any $i, j \in G$ by (a) we have $\omega_q(i, j) = \omega_q(j, i)$ and $e_i e_j = e_j e_i$. Next, by induction we have $e_i^r = e_{ir} c^{\omega_q(i,i)+\cdots+\omega_q(i,(r-1)i)}$ for all $r \geq 2$, so that $e_i^n = e_{in} c^{\varepsilon(i)} = e_0 t^i c^{\varepsilon(i)}$.

(2) Let $a \in A^\times$. By Lemma 4.1.2(1) we have

$$
\delta_0(aa^{-1}) = \delta_0(a)\delta_0(a^{-1}) + \sum_{i \neq 0} t c^{\omega(i,n-i)} \sigma^{\chi(n-i)}(\delta_i(a))\delta_{n-i}(a^{-1}).
$$

Since $tc^{\omega(i,n-i)} \in \mathrm{rad}(R)$ for all $i \neq 0$, and since $\delta_0(aa^{-1}) = 1$, $\delta_0(a)\delta_0(a^{-1}) \in R^\times$ and $\delta_0(a)$ has a right inverse. Similarly, $\delta_0(a)$ has a left inverse. □

**Proposition 4.1.5.** *If* $t \in \mathrm{rad}(R)$, *then* $R/\mathrm{rad}(R) \xrightarrow{\sim} A/\mathrm{rad}(A)$ *canonically.*

*Proof.* Setting $\mathfrak{m} = \mathrm{rad}(R)$, we will see that $\mathrm{rad}(A) = e_0\mathfrak{m} \oplus (\oplus_{i\neq 0} e_i R)$. We divide the proof into several steps.

*Claim 1:* There exists an injective ring homomorphism

$$\rho : A \to \mathrm{M}_n(R), a \mapsto (\delta_i(ae_j))_{i,j\in G}$$

such that for any $a \in A$ if $\rho(a) \in \mathrm{M}_n(R)^\times$ then $a \in A^\times$.

*Proof.* We have an injective ring homomorphism $A \to \mathrm{End}_R(A), a \mapsto (b \mapsto ab)$ and a ring isomorphism $\varphi : \mathrm{End}_R(A) \xrightarrow{\sim} \mathrm{M}_n(R), f \mapsto (\delta_i(f(e_j)))_{i,j\in G}$, so that as the composite of them we have an injective ring homomorphism $\rho : A \to \mathrm{M}_n(R)$ such that $\rho(a) = (\delta_i(ae_j))_{i,j\in G}$ for all $a \in A$. Next, for any $a \in A$ with $\rho(a) \in \mathrm{M}_n(R)^\times$, since $(b \mapsto ab) = \varphi^{-1}(\rho(a)) \in \mathrm{End}_R(A)^\times$, we have $A \xrightarrow{\sim} A, b \mapsto ab$ and hence $a \in A^\times$. □

*Claim 2:* $A\mathfrak{m} = \oplus_{i\in G} e_i\mathfrak{m}$ is a two-sided ideal of $A$ with $A\mathfrak{m} \subseteq \mathrm{rad}(A)$.

*Proof.* Obviously, $A\mathfrak{m}$ is a left ideal. Since $A\mathfrak{m}$ consists of $a \in A$ with $\delta_i(a) \in \mathfrak{m}$ for all $i \in G$, and since $\sigma(\mathfrak{m}) = \mathfrak{m}$, it follows by Lemma 4.1.2(1) that $A\mathfrak{m}$ is a two-sided ideal. Let $a \in A\mathfrak{m}$. We claim that $a \in \mathrm{rad}(A)$. Since $\delta_i(1 - a) = -\delta_i(a) \in \mathfrak{m}$ for $i \neq 0$ and $\delta_0(1 - a) = 1 - \delta_0(a) \in R^\times$, it follows by Lemmas 4.1.1(2) and 4.1.2(2) that $\rho(1 - a)_{ii} \in R^\times$ for all $i$ and $\rho(1 - a)_{ij} \in \mathfrak{m}$ unless $i = j$. Note that $\mathrm{rad}(\mathrm{M}_n(R))$ consists of all matrices with the entries in $\mathfrak{m}$ (see e.g. [15, Chapter 1, Proposition 7.22]). Thus $\rho(1 - a) \in \mathrm{M}_n(R)^\times$ and by Claim 1 we have $1 - a \in A^\times$, so that $a \in \mathrm{rad}(A)$. □

*Claim 3:* $\mathfrak{n} = e_0\mathfrak{m} \oplus (\oplus_{i\neq 0} e_i R)$ is a two-sided ideal of $A$ with $\mathfrak{n} \subseteq \mathrm{rad}(A)$.

*Proof.* Obviously, $\mathfrak{n}$ is a subgroup of $A$. It then follows by Lemma 4.1.2(1) that $\mathfrak{n}$ is a two-sided ideal of $A$. Next, since $t^i c^{\varepsilon(i)} \in \mathfrak{m}$ for all $i \neq 0$, by Lemma 4.1.4(1) there exists $m \geq 1$ such that $a^m \in A\mathfrak{m}$ for all $a \in \mathfrak{n}$, i.e., $\mathfrak{n}/A\mathfrak{m}$ is a two-sided ideal of $A/A\mathfrak{m}$ consisting only of nilpotent elements. Thus $\mathfrak{n}/A\mathfrak{m} \subseteq \mathrm{rad}(A/A\mathfrak{m})$. It follows by Claim 2 that $\mathfrak{n} \subseteq \mathrm{rad}(A)$. □

*Claim 4:* $\mathrm{rad}(A) \subseteq \mathfrak{n}$.

*Proof.* Let $a \in \mathrm{rad}(A)$. For any $x \in R$ we have $1 - a(e_0x) \in A^\times$ and by Lemma 4.1.4(2) $1 - \delta_0(a)x = \delta_0(1 - a(e_0x)) \in R^\times$. Thus $\delta_0(a) \in \mathfrak{m}$ and $a \in \mathfrak{n}$. □

This finishes the proof of Proposition 4.1.5. □

Now, by Propositions 2.1.6, 2.1.9, 4.1.3 and 4.1.5 we have the following.

**Theorem 4.1.6** ([12] Theorem 2.6)**.** *Assume $t \in \mathrm{rad}(R)$. Then $A$ is an Auslander-Gorenstein local ring if and only if so is $R$.*

In Lemma 4.1.4(2) the assumption $t \in \mathrm{rad}(R)$ can be replaced by the condition that $c \in \mathrm{rad}(R)$ and $\omega(i, n-i) > 0$ for all $i \neq 0$. Similarly, in Claim 3 in the proof of Proposition 4.1.5 the assumption $t \in \mathrm{rad}(R)$ can be replaced by the condition that $c \in \mathrm{rad}(R)$ and $\varepsilon(i) > 0$ for all $i \neq 0$. Note also that

$$\varepsilon(i) + \varepsilon(n-i) = n\omega(i, n-i)$$

for all $i \neq 0$. Consequently, we have the following.

**Theorem 4.1.7** ([12] Theorem 2.7)**.** *Assume $c \in \mathrm{rad}(R)$ and $n\chi(i) > iq$ for all $i \neq 0$. Then $A$ is an Auslander-Gorenstein local ring if and only if so is $R$.*

## 4.2 Matrix rings

In this section, we will construct an extension ring $\Lambda$ of $A$ such that $\Lambda/R$ is a split Frobenius extension of second kind and $\Lambda \cong \mathrm{M}_n(R)$ as right $R$-modules (see Remark 4.2.5 below). We also show that the following are equivalent: (1) $R$ is an Auslander-Gorenstein ring; (2) $A$ is an Auslander-Gorenstein ring; (3) $\Lambda$ is an Auslander-Gorenstein ring. We refer to [1] and [9] for similar constructions of matrix rings.

We denote by $\hat{\delta}_i : A \to A_i, a \mapsto e_i\delta_i(a)$ the projection for each $i \in G$. Then the following conditions are satisfied:

(D1) $\hat{\delta}_i\hat{\delta}_j = 0$ unless $i = j$ and $\sum_{i \in G} \hat{\delta}_i = \mathrm{id}_A$;

(D2) $\hat{\delta}_i(a)\hat{\delta}_j(b) = \hat{\delta}_{\pi^j(i)}(\hat{\delta}_i(a)b)$ for all $a, b \in A$ and $i, j \in G$.

Let $\Lambda$ be a free right $A$-module with a basis $\{v_i\}_{i \in G}$ and define a multiplication on $\Lambda$ subject to the following axioms:

(L1) $v_iv_j = 0$ unless $i = j$ and $v_iv_i = v_i$ for all $i \in G$;

(L2) $av_i = \sum_{j \in G} v_j\hat{\delta}_{\pi^{-i}(j)}(a)$ for all $a \in A$ and $i \in G$.

We denote by $\{\gamma_i\}_{i \in G}$ the dual basis of $\{v_i\}_{i \in G}$ for the free left $A$-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{i \in G} v_i\gamma_i(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{i,j \in G} v_i\hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda))\gamma_j(\mu)$$

for all $\lambda, \mu \in \Lambda$. Also, setting $\gamma = \sum_{i \in G} \gamma_i$, we define a mapping

$$\psi : \Lambda \to \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda.$$

**Proposition 4.2.1.** *The following hold.*

*(1) $\Lambda$ is an associative ring with $1 = \sum_{i \in G} v_i$ and contains $A$ as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{i \in G} v_ia$.*

*(2) $\psi$ is an isomorphism of $A$-$\Lambda$-bimodules, i.e., $\Lambda/A$ is a Frobenius extension of first kind.*

*Proof.* (1) Let $\lambda \in \Lambda$. Obviously, $\sum_{i \in G} v_i \cdot \lambda = \lambda$. Also, by (D1) we have

$$\begin{aligned}\lambda \cdot \sum_{j \in G} v_j &= \sum_{i,j \in G} v_i \hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda)) \\ &= \sum_{i \in G} v_i \gamma_i(\lambda) \\ &= \lambda.\end{aligned}$$

Next, for any $\lambda, \mu, \nu \in \Lambda$ by (D2) we have

$$\begin{aligned}(\lambda\mu)\nu &= \sum_{i,j,k \in G} v_i \hat{\delta}_{\pi^{-k}(i)}(\hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda))\gamma_j(\mu))\gamma_k(\nu) \\ &= \sum_{i,j,k \in G} v_i \hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda))\hat{\delta}_{\pi^{-k}(j)}(\gamma_j(\mu))\gamma_k(\nu) \\ &= \lambda(\mu\nu).\end{aligned}$$

The remaining assertions are obvious.

(2) Let $\lambda \in \operatorname{Ker} \psi$. For any $j \in G$, $0 = \gamma(\lambda v_j) = \sum_{i \in G} \hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda))$ and hence $\hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda)) = 0$ for all $i \in G$. Thus for any $i \in G$, $\hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda)) = 0$ for all $j \in G$ and by (D1) $\gamma_i(\lambda) = 0$, so that $\lambda = 0$. Next, for any $f = \sum_{i \in G} a_i \gamma_i \in \operatorname{Hom}_A(\Lambda, A)$, setting $\lambda = \sum_{i,k \in G} v_i \hat{\delta}_{\pi^{-k}(i)}(a_k)$, by (D1) we have

$$\begin{aligned}(\gamma\lambda)(v_j) &= \gamma(\lambda v_j) \\ &= \sum_{i \in G} \hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda)) \\ &= \sum_{i,k \in G} \hat{\delta}_{\pi^{-j}(i)}(\hat{\delta}_{\pi^{-k}(i)}(a_k)) \\ &= a_j \\ &= f(v_j)\end{aligned}$$

for all $j \in G$ and $f = \gamma\lambda$. Finally, for any $a \in A$, by (D1) we have

$$\begin{aligned}(\gamma a)(\lambda) &= \gamma(a\lambda) \\ &= \sum_{i,j \in G} \hat{\delta}_{\pi^{-i}(j)}(a)\gamma_i(\lambda) \\ &= a\gamma(\lambda)\end{aligned}$$

for all $\lambda \in \Lambda$ and $\gamma a = a\gamma$. □

*Remark* 4.2.2. If $n \cdot 1_A \in A^\times$, then $\Lambda/A$ is a split ring extension.

Setting $\lambda_{ij} = \hat{\delta}_{\pi^{-j}(i)}(\gamma_i(\lambda)) \in A_{\pi^{-j}(i)}$ for $\lambda \in \Lambda$ and $i, j \in G$, we have a ring automorphism $\eta \in \operatorname{Aut}(\Lambda)$ such that $\eta(\lambda)_{ij} = \lambda_{\pi^{-1}(i),\pi^{-1}(j)}$ for all $\lambda \in \Lambda$ and $i, j \in G$. Obviously, $\eta^n = \mathrm{id}_\Lambda$ and the mapping $G \to \operatorname{Aut}(\Lambda), i \mapsto \eta^i$ is a group homomorphism.

Note that $\Lambda$ is a free right $R$-module with a basis $\{v_i e_j\}_{i,j\in G}$ and $\{\delta_j \gamma_i\}_{i,j\in G}$ is the dual basis of $\{v_i e_j\}_{i,j\in G}$ for the left $R$-module $\mathrm{Hom}_R(\Lambda, R)$, and that for any $i \in G$ by (L2) $xv_i = v_i x$ for all $x \in R$ and $\Lambda v_i$ is a $\Lambda$-$R$-bimodule

**Proposition 4.2.3.** *The following hold.*

*(1)* $\eta^j(v_i) = v_{\pi^j(i)}$ *for all* $i, j \in G$.

*(2)* $\Lambda^\eta = A$.

*(3)* $v_i\Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda v_{\pi(i)}, R), \lambda \mapsto \delta_{n-1}\gamma_i\lambda$ *in* Mod-$\Lambda$ *for all* $i \in G$.

*Proof.* (1) Let $i, j \in G$. For any $k, l \in G$, since $\eta^j(v_i)_{kl} = \hat{\delta}_{\pi^{-l}(k)}(\gamma_{\pi^{-j}(k)}(v_i))$, we have

$$\eta^j(v_i)_{kl} = \begin{cases} 1 & \text{if } k = l \text{ and } i = \pi^{-j}(k), \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $a \in A$, since $\eta^i(a)_{jk} = a_{\pi^{-i}(j),\pi^{-i}(k)} = \hat{\delta}_{\pi^{-\pi^{-i}(k)}(\pi^{-i}(j))}(a) = \hat{\delta}_{\pi^{-k}(j)}(a) = a_{jk}$ for all $i, j, k \in G$, we have $a \in \Lambda^\eta$. Conversely, for any $\lambda \in \Lambda^\eta$ we have $\hat{\delta}_{n-j}(\gamma_i(\lambda)) = \lambda_{i,\pi^j(i)} = \eta^{n-i}(\lambda)_{0,j} = \lambda_{0,j} = \hat{\delta}_{n-j}(\gamma_0(\lambda))$ for all $i, j \in G$, so that $\gamma_i(\lambda) = \gamma_0(\lambda)$ for all $i \in G$.

(3) Let $i \in G$. Since by (L2) $e_i v_j = v_{\pi^i(j)} e_i$ for all $j \in G$, it follows that

$$v_k e_j v_{\pi(i)} = \begin{cases} v_{\pi^{j+1}(i)} e_j & \text{if } k = \pi^{j+1}(i), \\ 0 & \text{otherwise} \end{cases}$$

for all $j, k \in G$ and hence $\mathrm{Hom}_R(\Lambda v_{\pi(i)}, R)$ has a basis $\{\delta_j \gamma_{\pi^{j+1}(i)}\}_{j\in G}$ as a left $R$-module. Also, it follows by (M2) and (L2) that $\delta_j \gamma_{\pi^{j+1}(i)} x = \sigma^{\chi(j)}(x) \delta_j \gamma_{\pi^{j+1}(i)}$ for all $x \in R$ and $j \in G$. Thus $\{\delta_j \gamma_{\pi^{j+1}(i)}\}_{j\in G}$ is a basis for the right $R$-module $\mathrm{Hom}_R(\Lambda v_{\pi(i)}, R)$. Since $v_i\Lambda$ has a basis $\{v_i e_j\}_{j\in G}$ as a right $R$-module, it suffices to show that $\delta_{n-1}\gamma_i \cdot v_i e_j = \delta_k \gamma_{\pi^{k+1}(i)}$ with $k = \pi^{-j}(n-1)$ for all $j \in G$. Let $j \in G$. For any $r, s \in G$, since $e_i v_j = v_{\pi^i(j)} e_i$, we have

$$v_i e_j v_r e_s = v_i v_{\pi^j(r)} e_j e_s = \begin{cases} v_i e_j e_s & \text{if } i = \pi^j(r), \\ 0 & \text{otherwise.} \end{cases}$$

It follows by (M1) that $(\delta_{n-1}\gamma_i \cdot v_i e_j)(v_r e_s) = \delta_{n-1}\gamma_i(v_i e_j v_r e_s) \neq 0$ if and only if $r = \pi^{-j}(i)$ and $s = n - j - 1$. It then follows by (X2) that $\delta_{n-1}\gamma_i \dot{v}_i e_j = \delta_{\pi^{-j}(n-1)}\gamma_{\pi^{-j}(i)}$ and, setting $k = \pi^{-j}(n-1)$, $\delta_{n-1}\gamma_i \cdot v_i e_j = \delta_k \gamma_{\pi^{k+1}(i)}$. $\square$

**Theorem 4.2.4.** *The following are equivalent.*

*(1) $R$ is an Auslander-Gorenstein ring.*

*(2) $A$ is an Auslander-Gorenstein ring.*

*(3) $\Lambda$ is an Auslander-Gorenstein ring.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3). By Propositions 2.1.9(1), 4.1.3(2) and 4.2.1(2).

(3) $\Rightarrow$ (1). Since by (L2) $xv_i = v_i x$ for all $x \in R$ and $i \in G$, $\delta_0\gamma_0 : \Lambda \to R$ is a homomorphism of $R$-$R$-bimodules with $\delta_0\gamma_0|_R = \mathrm{id}_R$. It follows by Propositions 2.1.7, 4.1.3(2) and 4.2.1(2) that $\Lambda/R$ is a split Frobenius extension of second kind. The assertion now follows by Propositions 2.1.6 and 2.1.9(2). $\square$

*Remark* 4.2.5. Set $e_{ij} = v_i e_{\pi^{-j}(i)}$ for $i, j \in G$. Then $\{e_{ij}\}_{i,j\in G}$ is a basis for the right $R$-module $\Lambda$ and the multiplication in $\Lambda$ can be defined subject to the following axioms:

(N1) $e_{ij}e_{kl} = 0$ unless $j = k$ and

$$e_{ij}e_{jk} = \begin{cases} e_{ik}c^{\omega(\pi^{-j}(i),\pi^{-k}(j))} & \text{if } \pi^{-j}(i) + \pi^{-k}(j) < n, \\ e_{ik}tc^{\omega(\pi^{-j}(i),\pi^{-k}(j))} & \text{if } \pi^{-j}(i) + \pi^{-k}(j) \geq n; \end{cases}$$

(N2) $xe_{ij} = e_{ij}\sigma^{\chi(\pi^{-j}(i))}(x)$ for all $x \in R$ and $i, j \in G$.

*Remark* 4.2.6. Set $\Delta_k = \{(i, \pi^{-k}(i)) \mid i \in G\}$ for $k \in G$. Then $G\times G = \cup_{k\in G}\Delta_k$, which is a disjoint union. The mapping $G \times G \to \mathbb{Z}, (i, j) \mapsto \chi(\pi^{-j}(i))$ takes the constant value $\chi(k)$ on $\Delta_k$ and $1_\Lambda \cdot e_k = \sum_{(i,j)\in\Delta_k} e_{ij}$ for all $k \in G$.

## 4.3 Classification

In this section, we will provide a way to obtain every pair $(q, \chi)$ satisfying the conditions (X1) and (X2).

Let $q$ be an integer and $\chi : G \to \mathbb{Z}$ a mapping satisfying the condition (X2).

**Lemma 4.3.1.** *The following hold.*

*(1)* $\chi(i) - \chi(i-1) = \chi(n-i) - \chi(n-i-1)$ *for all* $1 \leq i \leq n-1$.

*(2) If* $n = 2m$ *with* $m \geq 1$, *then there exist* $p_1, \dots, p_m, p_{m+1} \in \mathbb{Z}$ *such that*

$$\chi(i) = \begin{cases} 0 & \textit{if } i = 0, \\ p_1 + \cdots + p_i & \textit{if } 1 \leq i \leq m, \\ p_1 + \cdots + p_m + p_{m-1} + \cdots + p_{n-i} & \textit{if } m+1 \leq i \leq n-1 \end{cases}$$

*and* $q = 2\{p_1 + \cdots + p_{m-1}\} + p_m + p_{m+1}$.

*(3) If* $n = 2m+1$ *with* $m \geq 1$, *then there exist* $p_1, \dots, p_m, p_{m+1} \in \mathbb{Z}$ *such that*

$$\chi(i) = \begin{cases} 0 & \textit{if } i = 0, \\ p_1 + \cdots + p_i & \textit{if } 1 \leq i \leq m, \\ p_1 + \cdots + p_m + p_m + \cdots + p_{n-i} & \textit{if } m+1 \leq i \leq n-1 \end{cases}$$

*and* $q = 2\{p_1 + \cdots + p_m\} + p_{m+1}$.

*Proof.* (1) For any $1 \le i \le n-1$, $\chi(i)+\chi(n-i-1) = \chi(n-1) = \chi(i-1)+\chi(n-i)$ and hence $\chi(i) - \chi(i-1) = \chi(n-i) - \chi(n-i-1)$.

(2) and (3) Since $\chi(0) = 0$, $\chi(i) = \sum_{k=1}^{i}\{\chi(k)-\chi(k-1)\}$ for all $1 \le i \le n-1$. Thus, setting $p_i = \chi(i) - \chi(i-1)$ for $1 \le i \le m$ and $p_{m+1} = q - \chi(n-1)$, the assertions follow by (1). □

**Proposition 4.3.2.** *Let $n = 2m$ with $m \ge 1$. Then $(q, \chi)$ satisfies the condition (X1) if and only if the following hold.*

*(1) $p_i \ge p_{m+1}$ for all $1 \le i \le m$.*

*(2) If $m \ge 2$, and if $1 \le r \le \frac{m}{2}$ and $1 \le s \le m - 2r + 1$, then*

$$(p_1 + \cdots + p_r) - (p_{s+r} + \cdots + p_{s+2r-1}) \ge 0.$$

*(3) If $m \ge 5$, and if $3 \le r \le \frac{2m-1}{3}$ and $1 \le s \le \frac{r-1}{2}$, then*

$$(p_1 + \cdots + p_r) - (p_{m-s} + \cdots + p_m + p_{m-1} + \cdots + p_{m+s-r+1}) \ge 0.$$

*(4) If $m \ge 3$, and if $1 \le r \le \frac{m-1}{2}$ and $1 \le s \le m - 2r$, then*

$$-(p_1 + \cdots + p_r) + (p_{s+r} + \cdots + p_{s+2r}) \ge p_{m+1}.$$

*(5) If $m \ge 4$, and if $2 \le r \le \frac{2m-2}{3}$ and $1 \le s \le \frac{r}{2}$, then*

$$-(p_1 + \cdots + p_r) + (p_{m-s} + \cdots + p_m + p_{m-1} + \cdots + p_{m+s-r}) \ge p_{m+1}.$$

*Proof.* For the sake of convenience, we set $\chi(n) = q$. Then the condition (X1) is equivalent to that

$$\chi(j-i) - \{\chi(j) - \chi(i)\} \ge 0 \quad \text{and} \quad \{\chi(j) - \chi(i)\} - \{\chi(n) - \chi(n-j+i)\} \ge 0$$

for all $0 \le i < j \le n-1$. In case $i = 0$, the first inequality is trivial and

$$\begin{aligned} \chi(j) - \{\chi(n) - \chi(n-j)\} &= \chi(j) + \chi(n-j) - \chi(n) \\ &= \chi(j) + \{\chi(n-1) - \chi(j-1)\} - \chi(n) \\ &= \{\chi(j) - \chi(j-1)\} - \{\chi(n) - \chi(n-1)\}. \end{aligned}$$

for all $1 \le j \le n-1$. Let $1 \le i < j \le n-1$. Setting $r = j - i$ and $s = i$, we have $r, s \ge 1$ with $r + s \le n-1$ and

$$\chi(j-i) - \{\chi(j) - \chi(i)\} = \sum_{k=1}^{r}\{\chi(k) - \chi(k-1)\} - \sum_{l=s+1}^{r+s}\{\chi(l) - \chi(l-1)\},$$

$$\begin{aligned} &\{\chi(j) - \chi(i)\} - \{q - \chi(n-j+i)\} \\ &\qquad = \sum_{k=s+1}^{r+s}\{\chi(k) - \chi(k-1)\} - \sum_{l=n-r+1}^{n}\{\chi(l) - \chi(l-1)\}. \end{aligned}$$

Consequently, canceling common terms, the assertion follows. □

**Proposition 4.3.3.** *Let $n = 2m + 1$ with $m \geq 1$. Then $(q, \chi)$ satisfies the condition (X1) if and only if the following hold.*

*(1) $p_i \geq p_{m+1}$ for all $1 \leq i \leq m$.*

*(2) If $m \geq 2$, and if $1 \leq r \leq \frac{m}{2}$ and $1 \leq s \leq m - 2r + 1$, then*

$$(p_1 + \cdots + p_r) - (p_{s+r} + \cdots + p_{s+2r-1}) \geq 0.$$

*(3) If $m \geq 3$, and if $2 \leq r \leq \frac{2m}{3}$ and $1 \leq s \leq \frac{r}{2}$, then*

$$(p_1 + \cdots + p_r) - (p_{m-s+1} + \cdots + p_m + p_m + \cdots + p_{m+s-r+1}) \geq 0.$$

*(4) If $m \geq 3$, and if $1 \leq r \leq \frac{m-1}{2}$ and $1 \leq s \leq m - 2r$, then*

$$-(p_1 + \cdots + p_r) + (p_{s+r} + \cdots + p_{s+2r}) \geq p_{m+1}.$$

*(5) If $m \geq 2$, and if $1 \leq r \leq \frac{2m-1}{3}$ and $1 \leq s \leq \frac{r+1}{2}$, then*

$$-(p_1 + \cdots + p_r) + (p_{m-s+1} + \cdots + p_m + p_m + \cdots + p_{m+s-r}) \geq p_{m+1}.$$

*Proof.* Similar to Proposition 4.3.2. □

## 4.4 Examples

In this final section, we will provide some examples of $A$ and $\Lambda$ which are constructed from an arbitrary ring $R$ (see Section 1 and Section 2, respectively).

**Lemma 4.4.1.** *For the ring $\Lambda$, we have $1_\Lambda = \sum_{i \in G} e_{ii}$*

*Proof.* It follows by Proposition 4.2.1 and Remark 4.2.5 that

$$\sum_{i \in G} e_{ii} = \sum_{i \in G} v_i e_{\pi^{-i}(i)} = \sum_{i \in G} v_i e_0 = \sum_{i \in G} v_i = 1_\Lambda.$$

□

**Example 4.4.2.** Let $n = 2$, $q = \chi(1) = 1$, $R = k[[X, Y]]$ a formal power series ring over a field $k$ and $c = t = 0$. We take $\sigma$ as a $k$-algebra automorphism of $R$ such that $\sigma(X) = Y$ and $\sigma(Y) = X$. Then $(\sigma, c, t)$ satisfies the condition $(*)$ and the pair $(q, \chi)$ satisfies the conditions (X1) and (X2). It follows by (M1) and (M2) that $e_1 e_1 = e_0 t c^{\omega(1,1)} = 0$ and $x e_1 = e_1 \sigma^{\chi(1)}(x) = e_1 \sigma(x)$ for any $x \in R$. Setting $e_1 = Z$, we have

$$A \cong k\langle\langle X, Y, Z \rangle\rangle / (XY - YX, Z^2, XZ - ZY, YZ - ZX),$$

where $k\langle\langle X, Y, Z \rangle\rangle$ is a noncommutative power series ring over $k$. By Theorem 4.1.6 $A$ is an Auslander-Gorenstein local ring with inj dim $A = 2$, which is

neither quasi-Frobenius nor commutative Gorenstein. Also, by Remark 4.2.5 and Lemma 4.4.1 we have $1_\Lambda = e_{00} + e_{11}$ and the following relations:
and

$$\begin{cases} xe_{00} = e_{00}\sigma^{\chi(0)}(x) = e_{00}x, \\ xe_{01} = e_{01}\sigma^{\chi(1)}(x) = e_{01}\sigma(x), \\ xe_{10} = e_{10}\sigma^{\chi(1)}(x) = e_{10}\sigma(x), \\ xe_{11} = e_{11}\sigma^{\chi(0)}(x) = e_{11}x. \end{cases}$$

for all $x \in R$. Setting $e_{00} = S, e_{01} = T, e_{10} = U$ and $e_{11} = V$, we have

$$\Lambda \cong k\langle\langle X, Y, S, T, U, V\rangle\rangle/I,$$

where $I$ is an ideal generated by the following elements:

$$\begin{cases} S^2 - S,\ T^2,\ U^2,\ V^2 - V, \\ ST - T,\ TV - T,\ US - U,\ VU - U, \\ SU,\ SV,\ TS,\ TU,\ UT,\ UV,\ VS,\ VT,\ 1 - S - T \\ XY - YX,\ XS - SX,\ YS - SY,\ XT - TY,\ YT - TX, \\ XU - UY,\ YU - UX,\ XV - VX,\ YV - VY. \end{cases}$$

By Theorem 4.2.4 $\Lambda$ is an Auslander-Gorenstein ring with inj dim $\Lambda = 2$.

Throughout the rest of this section, we restrict ourselves to the case where $\sigma = \mathrm{id}_R$.

**Example 4.4.3.** Let $n = 2$, $(q, \chi)$ an arbitrary pair satisfying the conditions (X1) and (X2) and $R$ an arbitrary ring. Since $\omega(1, 1) = \chi(1)+\chi(1)-\chi(0)-q = 2\chi(1) - q$, it follows that $e_1e_1 = e_0tc^{\omega(1,1)} = tc^{2\chi(1)-q}$. Setting $e_1 = X$, we have

$$A \cong R[X]/(X^2 - tc^{2\chi(1)-q}),$$

where $R[X]$ is a polynomial ring over $R$. Also, by Remark 4.2.5 and Lemma 4.4.1 we have $1_\Lambda = e_{00} + e_{11}$ and the following relations:
Setting $e_{00} = X, e_{01} = Y, e_{10} = Z$ and $e_{11} = W$, we have

$$\Lambda \cong R\langle X, Y, Z, W\rangle/I,$$

| | $e_{00}$ | $e_{01}$ | $e_{10}$ | $e_{11}$ |
|---|---|---|---|---|
| $e_{00}$ | $e_{00}$ | $e_{01}$ | 0 | 0 |
| $e_{01}$ | 0 | 0 | 0 | $e_{01}$ |
| $e_{10}$ | $e_{10}$ | 0 | 0 | 0 |
| $e_{11}$ | 0 | 0 | $e_{10}$ | $e_{11}$ |

| | $e_{00}$ | $e_{01}$ | $e_{10}$ | $e_{11}$ |
|---|---|---|---|---|
| $e_{00}$ | $e_{00}$ | $e_{01}$ | $0$ | $0$ |
| $e_{01}$ | $0$ | $0$ | $e_{00}tc^{2\chi(1)-q}$ | $e_{01}$ |
| $e_{10}$ | $e_{10}$ | $e_{11}tc^{2\chi(1)-q}$ | $0$ | $0$ |
| $e_{11}$ | $0$ | $0$ | $e_{10}$ | $e_{11}$ |

where $R\langle X, Y, Z, W\rangle$ is a noncommutative polynomial ring over $R$ and $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - X,\ Y^2,\ Z^2,\ W^2 - W, \\ XY - Y,\ YW - Y,\ WZ - Z,\ ZX - Z, \\ XZ,\ XW,\ YX,\ ZW,\ WX,\ WY, \\ YZ - Xtc^{2\chi(1)-q},\ ZY - Wtc^{2\chi(1)-q},\ 1 - X - W. \end{cases}$$

In particular, if $t = 1$ and $2\chi(1) = q$ then $\Lambda$ is isomorphic to $\mathrm{M}_2(R)$ as rings.

**Example 4.4.4** ([9, Section 3])**.** Let $n \geq 2$, $t = 1$, $q = n\chi(1) - 1$ and $\chi(i) = i\chi(1)$ for all $i \in G$. Then $\Lambda$ is isomorphic to $\mathrm{M}_n(R; \mathrm{id}_R, c)$ in the sense of Hoshino [9, Section 3] as rings.

# Chapter5

# Clifford extensions

## 5.1 Graded rings

Throughout this chapter, we fix a set of integers $G = \{0, 1, \ldots, n-1\}$ with $n \geq 2$ and a ring $R$ together with a pair $(\sigma, c)$ of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition:

$$(*) \quad \sigma^n = \mathrm{id}_R \quad \text{and} \quad c \in R^\sigma \cap \mathrm{Z}(R).$$

This is obviously satisfied if $\sigma = \mathrm{id}_R$ and $c \in \mathrm{Z}(R)$.

Let $\Lambda$ be a free right $R$-module with a basis $\{v_i\}_{i \in G}$ and $\{\delta_i\}_{i \in G}$ the dual basis of $\{v_i\}_{i \in G}$ for the free left $R$-module $\mathrm{Hom}_R(\Lambda, R)$, i.e., $\lambda = \sum_{i \in G} v_i \delta_i(\lambda)$ for all $\lambda \in \Lambda$. We set

$$v_{i+kn} = v_i c^k$$

for $i \in G$ and $k \in \mathbb{Z}_+$, the set of non-negative integers, and define a multiplication on $\Lambda$ subject to the following axioms:

(L1) $v_i v_j = v_{i+j}$ for all $i, j \in G$;

(L2) $a v_i = v_i \sigma^i(a)$ for all $a \in R$ and $i \in G$.

**Lemma 5.1.1.** *The following hold.*

*(1) $v_i v_j = v_j v_i$ for all $i, j \in G$ and $v_i^n = v_0 c^i$ for all $i \in G$.*

*(2) For any $\lambda, \mu \in \Lambda$ we have $\lambda\mu = \sum_{i,j \in G} v_{i+j} \sigma^j(\delta_i(\lambda))\delta_j(\mu)$ and hence*

$$\delta_0(\lambda\mu) = \delta_0(\lambda)\delta_0(\mu) + \sum_{i \in G \setminus \{0\}} \sigma^{n-i}(\delta_i(\lambda))\delta_{n-i}(\mu)c.$$

*(3) For any $\lambda \in \Lambda$ and $i, j \in G$ we have*

$$\delta_i(\lambda v_j) = \begin{cases} \sigma^j(\delta_{i-j}(\lambda)) & \text{if } i \geq j, \\ \sigma^j(\delta_{i-j+n}(\lambda))c & \text{if } i < j. \end{cases}$$

*Proof.* (1) The first assertion is obvious. It follows by induction that $v_i^r = v_{ir}$ for all $r \geq 1$. In particular, $v_i^n = v_{in} = v_0 c^i$.

(2) Straightforward.

(3) Obviously, the equality holds for $j = 0$. Let $j \in G \setminus \{0\}$. Then for any $\lambda \in \Lambda$ and $k \in G$ we have

$$v_k \delta_k(\lambda) \cdot v_j = \begin{cases} v_{j+k} \sigma^j(\delta_k(\lambda)) & \text{if } k + j < n, \\ v_{j+k-n} \sigma^j(\delta_k(\lambda)) c & \text{if } k + j \geq n. \end{cases}$$

If $k + j < n$ then, setting $i = k + j$, we have $v_{i-j} \delta_{i-j}(\lambda) \cdot v_j = v_i \sigma^j(\delta_{i-j}(\lambda))$ and $\delta_i(\lambda v_j) = \sigma^j(\delta_{i-j}(\lambda))$. Assume $k + j \geq n$. Then, setting $i = k + j - n$, we have $v_{i-j+n} \delta_{i-j+n}(\lambda) \cdot v_j = v_i \sigma^j(\delta_{i-j+n}(\lambda)) c$ and $\delta_i(\lambda v_j) = \sigma^j(\delta_{i-j+n}(\lambda)) c$. □

**Proposition 5.1.2.** *The following hold.*

*(1)* $\Lambda$ *is an associative ring with* $1 = v_0$ *and contains* $R$ *as a subring via the injective ring homomorphism* $R \to \Lambda, a \mapsto v_0 a$.

*(2)* $\Lambda / R$ *is a split Frobenius extension of second kind.*

*(3)* $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda/\mathrm{rad}(\Lambda)$ *canonically if* $c \in \mathrm{rad}(R)$.

*(4) For any* $\varepsilon \in R^\sigma \cap \mathrm{Z}(R)$ *with* $\varepsilon^n = 1$ *there exists* $\tilde{\sigma} \in \mathrm{Aut}(\Lambda)$ *such that* $\delta_i(\tilde{\sigma}(\lambda)) = \sigma(\delta_i(\lambda)) \varepsilon^i$ *for all* $\lambda \in \Lambda$ *and* $i \in G$, *and for any* $c' \in R^\sigma \cap \mathrm{Z}(R)$ *the pair* $(\tilde{\sigma}, c')$ *satisfies the condition (*).*

*Proof.* (1) Obviously, $v_0 \cdot v_i a = v_i a = v_i a \cdot v_0$ for all $i \in G$ and $a \in R$. Since $\sigma^{i+kn} = \sigma^i$ for all $i \in G$ and $k \in \mathbb{Z}_+$, it is easy to check that $a v_i = v_i \sigma^i(a)$ for all $a \in R$ and $i \in \mathbb{Z}_+$. Thus for any $i, j, k \in G$ and $\lambda_i, \mu_j, \nu_k \in R$ we have

$$\begin{aligned} (v_i \lambda_i \cdot v_j \mu_j) \cdot v_k \nu_k &= v_{i+j} \sigma^j(\lambda_i) \mu_j \cdot v_k \nu_k \\ &= v_{i+j+k} \sigma^k(\sigma^j(\lambda_i) \mu_j) \nu_k \\ &= v_{i+j+k} \sigma^{j+k}(\lambda_i) \sigma^k(\mu_j) \nu_k, \\ v_i \lambda_i \cdot (v_j \mu_j \cdot v_k \nu_k) &= v_i \lambda_i \cdot v_{j+k} \sigma^k(\mu_j) \nu_k \\ &= v_{i+j+k} \sigma^{j+k}(\lambda_i) \sigma^k(\mu_j) \nu_k \end{aligned}$$

and $(v_i \lambda_i \cdot v_j \mu_j) \cdot v_k \nu_k = v_i \lambda_i \cdot (v_j \mu_j \cdot v_k \nu_k)$. The last assertion is obvious.

(2) It follows by (L2) that $\delta_i a = \sigma^i(a) \delta_i$ for all $a \in R$ and $i \in G$. In particular, $\{\delta_i\}_{i \in G}$ is a basis for the right $R$-module $\mathrm{Hom}_R(\Lambda, R)$. Also, $v_i v_{n-i-1} = v_{n-1}$ implies $\delta_{n-1} v_i = \delta_{n-i-1}$ for all $i \in G$. Thus $\Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto \delta_{n-1} \lambda$ in Mod-$\Lambda$. Obviously, $\Lambda$ is generated by $\{v_i\}_{i \in G}$ as a left $R$-module. Consequently, since $\delta_{n-1} a = \sigma^{n-1}(a) \delta_{n-1}$ and $\delta_0 a = a \delta_0$ for all $a \in R$, the assertion follows.

(3) Set $\mathfrak{m} = \mathrm{rad}(R)$ and $\mathfrak{n} = v_0 \mathfrak{m} \oplus (\oplus_{i \in G \setminus \{0\}} v_i R) \subseteq \Lambda$. It suffices to show that $\mathfrak{n} = \mathrm{rad}(\Lambda)$. We divide the proof into several steps.

*Claim 1:* $\delta_0(\lambda) \in R^\times$ for all $\lambda \in \Lambda^\times$.

*Proof.* Let $\lambda \in \Lambda^\times$. Since $\delta_0(\lambda\lambda^{-1}) = 1$, and since $c \in \mathrm{rad}(R)$, by Lemma 5.1.1(2) $\delta_0(\lambda)\delta_0(\lambda^{-1}) = 1 - \sum_{i \in G \setminus \{0\}} \sigma^{n-i}(\delta_i(\lambda))\delta_{n-i}(\lambda^{-1})c \in R^\times$ and $\delta_0(\lambda)$ has a right inverse. Similarly, $\delta_0(\lambda^{-1})\delta_0(\lambda) \in R^\times$ and $\delta_0(\lambda)$ has a left inverse. □

*Claim 2:* There exists a ring homomorphism $\rho : \Lambda \to \mathrm{M}_n(R), \lambda \mapsto (\delta_i(\lambda v_j))_{i,j \in G}$ and $\lambda \in \Lambda^\times$ for all $\lambda \in \Lambda$ with $\rho(\lambda) \in \mathrm{M}_n(R)^\times$.

*Proof.* Since we have a ring homomorphism $\Lambda \to \mathrm{End}_R(\Lambda), \lambda \mapsto (\mu \mapsto \lambda\mu)$ and a ring isomorphism $\mathrm{End}_R(\Lambda) \xrightarrow{\sim} \mathrm{M}_n(R), f \mapsto (\delta_i(f(v_j)))_{i,j \in G}$, as the composite of them we have a desired ring homomorphism $\rho : \Lambda \to \mathrm{M}_n(R)$. If $\lambda \in \Lambda$ with $\rho(\lambda) \in \mathrm{M}_n(R)^\times$, then $\Lambda \xrightarrow{\sim} \Lambda, \mu \mapsto \lambda\mu$ and $\lambda \in \Lambda^\times$. □

*Claim 3:* $\Lambda\mathfrak{m} = \oplus_{i \in G} v_i\mathfrak{m}$ is a two-sided ideal of $\Lambda$ with $\Lambda\mathfrak{m} \subseteq \mathrm{rad}(\Lambda)$.

*Proof.* Obviously, $\Lambda\mathfrak{m}$ is a left ideal. Since $\Lambda\mathfrak{m}$ consists of $\lambda \in \Lambda$ with $\delta_i(\lambda) \in \mathfrak{m}$ for all $i \in G$, and since $\sigma(\mathfrak{m}) = \mathfrak{m}$, it follows that $\Lambda\mathfrak{m}$ is a two-sided ideal. Let $\lambda \in \Lambda\mathfrak{m}$. We claim $\lambda \in \mathrm{rad}(\Lambda)$. Since $\delta_i(1-\lambda) = -\delta_i(\lambda) \in \mathfrak{m}$ for $i \in G \setminus \{0\}$ and $\delta_0(1-\lambda) = 1-\delta_0(\lambda) \in R^\times$, it follows by Lemma 5.1.1(3) that $\rho(1-\lambda)_{ii} \in R^\times$ for all $i \in G$ and $\rho(1-\lambda)_{ij} \in \mathfrak{m}$ unless $i = j$. Note that $\mathrm{rad}(\mathrm{M}_n(R))$ consists of all matrices with the entries in $\mathfrak{m}$ (see e.g. [15, Chapter 1, Proposition 7.22]). Thus $\rho(1-\lambda) \in \mathrm{M}_n(R)^\times$ and by Claim 2 we have $1-\lambda \in \Lambda^\times$, so that $\lambda \in \mathrm{rad}(\Lambda)$. □

*Claim 4:* $\mathfrak{n}$ is a two-sided ideal of $\Lambda$ with $\mathfrak{n} \subseteq \mathrm{rad}(\Lambda)$.

*Proof.* Obviously, $\mathfrak{n}$ is a subgroup. It then follows by Lemma 5.1.1(2) that $\mathfrak{n}$ is a two-sided ideal. Next, by Lemma 5.1.1(1) there exists $m \geq 1$ such that $\lambda^m \in \Lambda\mathfrak{m}$ for all $\lambda \in \mathfrak{n}$, i.e., $\mathfrak{n}/\Lambda\mathfrak{m}$ is a two-sided ideal of $\Lambda/\Lambda\mathfrak{m}$ consisting only of nilpotent elements. Thus $\mathfrak{n}/\Lambda\mathfrak{m} \subseteq \mathrm{rad}(\Lambda/\Lambda\mathfrak{m})$. It follows by Claim 3 that $\mathfrak{n} \subseteq \mathrm{rad}(\Lambda)$. □

*Claim 5:* $\mathrm{rad}(\Lambda) \subseteq \mathfrak{n}$.

*Proof.* Let $\lambda \in \mathrm{rad}(\Lambda)$. For any $a \in R$, since $1 - \lambda a \in \Lambda^\times$, by Claim 1 we have $1 - \delta_0(\lambda)a = \delta_0(1 - \lambda a) \in R^\times$, so that $\delta_0(\lambda) \in \mathfrak{m}$ and $\lambda \in \mathfrak{n}$. □

This finishes the proof of (3).

(4) Obviously, $\tilde{\sigma}$ is an automorphism of the additive group $\Lambda$ satisfying $\tilde{\sigma}(1) = 1$ and $\tilde{\sigma}^n = \mathrm{id}_\Lambda$. Since $\varepsilon^{i+kn} = \varepsilon^i$ for all $i \in G$ and $k \in \mathbb{Z}_+$, it follows that for any $\lambda, \mu \in \Lambda$, setting $\delta_i(\lambda) = \lambda_i$ and $\delta_i(\mu) = \mu_i$ for $i \in G$, we have $\tilde{\sigma}(\lambda\mu) = \sum_{i,j \in G} v_{i+j}\sigma(\sigma^j(\lambda_i)\mu_j)\varepsilon^{i+j} = \sum_{i,j \in G} v_{i+j}\sigma^j(\sigma(\lambda_i)\varepsilon^i)\sigma(\mu_j)\varepsilon^j = \tilde{\sigma}(\lambda)\tilde{\sigma}(\mu)$. □

*Remark* 5.1.3. According to Proposition 5.1.2(4), the construction above can be iterated arbitrary times.

*Remark* 5.1.4. Let $R[t;\sigma]$ be a right skew polynomial ring with trivial derivation, i.e., $R[t;\sigma]$ consists of all polynomials in an indeterminate $t$ with right-hand coefficients in $R$ and the multiplication is defined by the following rule: $at = t\sigma(a)$ for all $a \in R$. Then $(t^n - c) = (t^n - c)R[t;\sigma]$ is a two-sided ideal and the residue ring $R[t;\sigma]/(t^n - c)$ is isomorphic to $\Lambda$ as extension rings of $R$.

In the next section, we will deal with the case where $n = 2$ and denote by $Cl_1(R; \sigma, c)$ the ring $\Lambda$ constructed above.

## 5.2 Clifford extensions

In this section, we fix a set of integers $G = \{0, 1\}$ and a ring $R$ together with a sequence of elements $c_1, c_2, \dots$ in $\mathrm{Z}(R)$.

We may consider $G$ as a cyclic group of order 2 with 0 the unit element, i.e., we set $0 + i = i + 0 = i$ for all $i \in G$ and $1 + 1 = 0$. For any $n \geq 1$ we denote by $G^n$ the direct product of $n$ copies of $G$ and by 0 the unit element $(0, \dots, 0)$ of $G^n$. We consider $G^{n-1}$ as a subgroup of $G^n$ via the injective group homomorphism

$$G^{n-1} \to G^n, (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0),$$

where $G^0$ denotes the trivial group $\{0\}$. Although Proposition 5.1.2(4) enables us to construct inductively various $G^n$-graded rings which are Frobenius extensions of $R$, in this paper we restrict ourselves to the following particular case (see the proof of Theorem 5.2.1(1) below).

Let $n \geq 1$. For each $x = (x_1, \dots, x_n) \in G^n$ we set $S(x) = \{i \mid x_i = 1\}$. Note that $S(x + y) = S(x) + S(y)$, the symmetric difference of $S(x)$ and $S(y)$, for all $x, y \in G^n$. We set

$$\varepsilon(i, j) = \begin{cases} +1 & \text{if } i \leq j, \\ -1 & \text{if } i > j \end{cases}$$

for $1 \leq i, j \leq n$ and

$$c(x, y) = \prod_{(i,j) \in S(x) \times S(y)} \varepsilon(i, j) \prod_{k \in S(x) \cap S(y)} c_k$$

for $x, y \in G^n$. We denote by $s$ the element $x \in G^n$ with $S(x) = \{1, \dots, n\}$.

Let $\Lambda_n$ be a free right $R$-module with a basis $\{v_x\}_{x \in G^n}$. We denote by $\{\delta_x\}_{x \in G^n}$ the dual basis of $\{v_x\}_{x \in G^n}$ for the free left $R$-module $\mathrm{Hom}_R(\Lambda_n, R)$, i.e., $\lambda = \sum_{x \in G^n} v_x \delta_x(\lambda)$ for all $\lambda \in \Lambda_n$. We define a multiplication on $\Lambda_n$ subject to the following axioms:
(M1) $v_x v_y = v_{x+y} c(x, y)$ for all $x, y \in G^n$;
(M2) $a v_x = v_x a$ for all $x \in G^n$ and $a \in R$.

In the following, we set $v_x = t_i$ for $x \in G^n$ with $S(x) = \{i\}$. It is easy to see the following:
(C1) $t_i^2 = v_0 c_i$ for all $1 \leq i \leq n$;
(C2) $t_i t_j + t_j t_i = 0$ unless $i = j$;
(C3) $v_x = t_{i_1} \cdots t_{i_r}$ if $S(x) = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$.

**Theorem 5.2.1** ([11] Theorem 3.1)**.** *For any $n \geq 1$ the following hold.*

*(1) $\Lambda_n$ is an associative ring with $1 = v_0$ and contains $R$ as a subring via the injective ring homomorphism $R \to \Lambda_n, a \mapsto v_0 a$.*

*(2) $\Lambda_n/R$ is a split Frobenius extension of first kind.*

*(3) $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_n/\mathrm{rad}(\Lambda_n)$ canonically if $c_i \in \mathrm{rad}(R)$ for all $1 \le i \le n$.*

*Proof.* (1) We will make use of induction on $n \ge 1$. Since $\Lambda_1 = Cl_1(R; \mathrm{id}_R, c_1)$, the assertion holds true if $n = 1$. Let $n > 1$ and assume that the assertion holds true for $n-1$. Then there exists $\sigma_{n-1} \in \mathrm{Aut}(\Lambda_{n-1})$ such that

$$\sigma_{n-1}(v_x\lambda_x) = v_x\lambda_x(-1)^{|S(x)|}$$

for all $x \in G^{n-1}$ and $\lambda_x \in R$, where $|S(x)|$ denotes the order of $S(x)$. It is not difficult to see that $\Lambda_n \cong Cl_1(\Lambda_{n-1}; \sigma_{n-1}, c_n)$.

(2) By (M2) $a\delta_x = \delta_x a$ for all $x \in G^n$ and $a \in R$. In particular, $\{\delta_x\}_{x\in G^n}$ is a basis for the right $R$-module $\mathrm{Hom}_R(\Lambda_n, R)$. Also, for any $x \in G^n$, by (M1) $v_x v_{x+s} = v_s c(x, x+s)$ and hence $\delta_s v_x = c(x, x+s)\delta_{x+s}$. Since $c(x, x+s) \in \mathbb{Z}^\times$ for all $x \in G^n$, it follows that $\Lambda_n \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda_n, R), \lambda \mapsto \delta_s\lambda$ as $R$-$\Lambda_n$-bimodules. Obviously, $\delta_0$ is a homomorphism of $R$-$R$-bimodules with $\delta_0|_R = \mathrm{id}_R$.

(3) Setting $\Lambda_0 = R$ and $\sigma_0 = \mathrm{id}_R$, we have $\Lambda_k \cong Cl_1(\Lambda_{k-1}; \sigma_{k-1}, c_k)$ for all $1 \le k \le n$. The assertion follows by Proposition 5.1.2(3). □

*Remark* 5.2.2. If $d(x,y) = |S(x)\times S(y)| - |S(x)\cap S(y)|$ is even, then $v_x v_y = v_y v_x$. In particular, $v_s \in \mathrm{Z}(\Lambda_n)$ whenever $n$ is odd.

*Proof.* We have $\prod_{(i,j)\in S(x)\times S(y)} \varepsilon(i,j) \prod_{(j,i)\in S(y)\times S(x)} \varepsilon(j,i) = (-1)^{d(x,y)}$. □

Let $H^n$ denote the subset of $G^n$ consisting of all $x \in G^n$ with $|S(x)|$ even. Obviously, $H^n$ is a subgroup of $G^n$ and $\Lambda_n^0 = \oplus_{x\in H^n} v_x R$ is a subring of $\Lambda_n$.

**Proposition 5.2.3.** *Assume $n$ is even. Then $v_s \in \Lambda_n^0$ and the following hold.*

*(1) $\Lambda_n^0/R$ is a split Frobenius extension of first kind.*

*(2) $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_n^0/\mathrm{rad}(\Lambda_n^0)$ canonically if $c_i \in \mathrm{rad}(R)$ for all $1 \le i \le n$.*

*Proof.* The first assertion is obvious.

(1) By the same argument as in the proof of Theorem 5.2.1(2).

(2) Set $\mathfrak{m} = \mathrm{rad}(R)$ and $\mathfrak{n} = v_0\mathfrak{m} \oplus (\oplus_{x\in H^n\setminus\{0\}} v_x R)$. Then for $m \in \mathbb{Z}$ with $2m = n$ we have $\lambda^m \in \Lambda_n^0\mathfrak{m}$ for all $\lambda \in \mathfrak{n}$, so that the assertion follows by the same argument as in the proof of Proposition 5.1.2(3). □

We denote by $Cl_n(R; c_1, \dots, c_n)$ (resp., $Cl_n^0(R; c_1, \dots, c_n)$) the ring $\Lambda_n$ (resp., $\Lambda_n^0$) constructed above.

*Remark* 5.2.4. Let $K$ be a commutative field and $V$ a 3-dimensional $K$-space. Then $Cl_3^0(K; 0, 0, 0) \cong K \ltimes V$, the trivial extension of $K$ by $V$ (see e.g. [7]), which is not a Frobenius $K$-algebra. Thus in Proposition 5.2.3 the assumption can not be removed in general.

*Remark* 5.2.5. If $c_n \in R^\times$, then

$$Cl_{n-1}(R; -c_1c_n, \dots, -c_{n-1}c_n) \xrightarrow{\sim} Cl_n^0(R; c_1, \dots, c_n), t_i \mapsto t_i t_n$$

as extension rings of $R$.

We will call those rings $Cl_n(R; c_1, \ldots, c_n)$ Clifford extensions of $R$. Similarly, we will call those rings $Cl_n^0(R; c_1, \ldots, c_n)$ with $n$ even Clifford extensions of $R$.

# Appendix

# The case of commutative rings

## 6.1 The case of commutative rings

The rest of this paper, we provide some examples of ring $A$ which are constructed from an arbitrary ring $R$ (cf. Chapter 4). It follows by Theorem 4.1.6 that if $R$ is an Auslander-Gorenstein local ring then $A$ is also an Auslander-Gorenstein local ring. Moreover, the ring $A$ is isomorphic to the $n \times n$ matrix for each $n$ where the $(i,\ j)$ element corresponds to $(\omega(i, n-j-1))$. There are too many examples to list in this paper, we choose from examples where $R$ is a commutative ring, $4 \leq n \leq 20$, $t = 1$ and $\sigma = \mathrm{id}_R$(cf. Chapter 4). For $4 \leq n \leq 10$, we will write down all types of relations of each algebra. For $11 \leq n \leq 20$, we do the same things for two variables.

The case of $n = 4$.

The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 3}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & 0 \\ p_2 & p_2 & 0 & 0 \\ p_1 & p_2 & p_1 & 0 \end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where $f_1 = p_1 - p_2$, and one of the following cases occurs:

1. $f_1 = 0$: Setting $p = p_2$, we have $(p_1, p_2) = (p, p)$ and

$$A \cong R[X]/(X^4 - c^p)$$

2. $f_1 \geq 1$: Setting $p = p_2$ and $q = f_1$, we have $(p_1, p_2) = (p + q, p)$ and

$$A \cong R[X, Y]/(X^2 - c^p, Y^2 - Xc^q)$$

The case of $n = 5$.
The integral matrix $(\omega(i, n-j-1))_{0 \le i,j \le 4}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & f_1 & 0 \\ p_2 & p_2 & 0 & f_1 & 0 \\ p_2 & f_2 & p_2 & 0 & 0 \\ p_1 & p_2 & p_2 & p_1 & 0 \end{pmatrix}$$

with the $p_i \ge 1$ and the $f_j \ge 0$, where $f_1 = p_1 - p_2$, $f_2 = -p_1 + 2p_2$, and one of the following cases occurs:

1. $f_1 = 0$, $f_2 \ge 1$: Setting $p = p_2$, we have $(p_1, p_2) = (p, p)$ and

$$A \cong R[X]/(X^5 - c^p)$$

2. $f_1 \ge 1$, $f_2 = 0$: Setting $p = p_2$, we have $(p_1, p_2) = (2p, p)$ and

$$A \cong R[X, Y]/(XY - c^p, X^3 - Y^2)$$

3. $f_1 \ge 1$, $f_2 \ge 1$: Setting $p = f_1$ and $q = f_2$, we have $(p_1, p_2) = (2p+q, p+q)$ and

$$A \cong R[X, Y, Z]/(YZ - Xc^p, Z^2 - Yc^p, X^2 - Zc^q, XY - c^{p+q}, XZ - Y^2)$$

The case of $n = 6$.
The integral matrix $(\omega(i, n-j-1))_{0 \le i,j \le 5}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & f_2 & f_1 & 0 \\ p_2 & p_2 & 0 & f_2 & f_2 & 0 \\ p_3 & f_3 & p_3 & 0 & f_1 & 0 \\ p_2 & f_3 & f_3 & p_2 & 0 & 0 \\ p_1 & p_2 & p_3 & p_2 & p_1 & 0 \end{pmatrix}$$

with the $p_i \ge 1$ and the $f_j \ge 0$, where $f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = -p_1 + p_2 + p_3$, and one of the following cases occurs:

1. $f_1 = 0$, $f_2 = 0$, $f_3 \ge 1$: Setting $p = p_3$, we have $(p_1, p_2, p_3) = (p, p, p)$ and

$$A \cong R[X]/(X^6 - c^p)$$

2. $f_1 = 0$, $f_2 \ge 1$, $f_3 \ge 1$: Setting $p = p_3$ and $q = f_2$, we have $(p_1, p_2, p_3) = (p+q, p+q, p)$ and

$$A \cong R[X, Y]/(X^2 - c^p, Y^3 - Xc^q)$$

3. $f_1 \geq 1$, $f_2 = 0$, $f_3 \geq 1$: Setting $p = p_2$ and $q = f_1$, we have $(p_1, p_2, p_3) = (p+q, p, p+q)$ and
$$A \cong R[X,Y]/(X^3 - c^p, Y^2 - Xc^q)$$
4. $f_1 \geq 1$, $f_2 \geq 1$, $f_3 = 0$: Setting $p = p_2$ and $q = p_3$, we have $(p_1, p_2, p_3) = (p+q, p, q)$ and
$$A \cong R[X,Y]/(X^3 - c^p, Y^2 - c^q)$$
5. $f_1 \geq 1$, $f_2 \geq 1$, $f_3 \geq 1$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - Zc^{f_3}, XY - Wc^{f_3}, XZ - c^{p_2}, Y^2 - c^{p_3}, YW - Xc^{f_1}, \\ Z^2 - Xc^{f_2}, ZW - Yc^{f_2}, W^2 - Zc^{f_1}, XW - YZ. \end{cases}$$

<u>The case of $n = 7$.</u>
The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 6}$ is of the form
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & f_2 & f_2 & f_1 & 0 \\ p_2 & p_2 & 0 & f_2 & f_5 & f_2 & 0 \\ p_3 & f_3 & p_3 & 0 & f_2 & f_2 & 0 \\ p_3 & f_4 & f_4 & p_3 & 0 & f_1 & 0 \\ p_2 & f_3 & f_4 & f_3 & p_2 & 0 & 0 \\ p_1 & p_2 & p_3 & p_3 & p_2 & p_1 & 0 \end{pmatrix}$$
with the $p_i \geq 1$ and the $f_j \geq 0$, where
$$\begin{aligned} &f_1 = p_1 - p_2, f_2 = p_1 - p_3, \\ &f_3 = -p_1 + p_2 + p_3, f_4 = -p_1 + 2p_3, \\ &f_5 = p_1 + p_2 - 2p_3, \end{aligned}$$
and one of the the following cases occurs:

1. $f_1 = f_2 = f_5 = 0$ and $f_i \geq 1$ for $i = 3, 4$: Setting $p = p_3$, we have $(p_1, p_2, p_3) = (p, p, p)$ and
$$A \cong R[X]/(X^7 - c^p)$$
2. $f_1 = f_4 = 0$ and $f_i \geq 1$ for $i = 2, 3, 5$: Setting $p = p_3$, we have $(p_1, p_2, p_3) = (2p, 2p, p)$ and
$$A \cong R[X,Y]/(XY - c^p, X^5 - Y^2)$$
3. $f_3 = f_4 = 0$ and $f_i \geq 1$ for $i = 1, 2, 5$: Setting $p = p_3$, we have $(p_1, p_2, p_3) = (2p, p, p)$ and
$$A \cong R[X,Y]/(XY - c^p, X^4 - Y^3)$$

4. $f_3 = f_5 = 0$ and $f_i \geq 1$ for $i = 1, 2, 4$: Setting $p = p_2$, we have $(p_1, p_2, p_3) = (3p, p, 2p)$ and
$$A \cong R[X, Y]/(X^2Y - c^p, X^3 - Y^2)$$

5. $f_1 = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5$: Setting $p = f_2$ and $q = f_4$, we have $(p_1, p_2, p_3) = (2p + q, 2p + q, p + q)$ and
$$A \cong R[X, Y, Z]/(X^2 - Zc^q, XY - c^{p+q}, YZ - Xc^p, Z^3 - Yc^p, Y^2 - XZ^2)$$

6. $f_3 = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5$: Setting $p = f_4$ and $q = f_5$, we have $(p_1, p_2, p_3) = (3p + 2q, p + q, 2p + q)$ and
$$A \cong R[X, Y, Z]/(XY - Zc^p, XZ - c^{p+q}, Y^2 - X^3c^p, Z^2 - Yc^q, X^4 - YZ)$$

7. $f_4 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5$: Setting $p = f_1$ and $q = f_3$, we have $(p_1, p_2, p_3) = (2p + 2q, p + 2q, p + q)$ and
$$A \cong R[X, Y, Z]/(X^2 - Zc^q, XZ - Y^2c^q, Y^3 - Xc^p, YZ - c^{p+q}, X^2Y - Z^2)$$

8. $f_5 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4$: Setting $p = f_2$ and $q = f_3$, we have $(p_1, p_2, p_3) = (3p + q, p + q, 2p + q)$ and $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - Yc^q, XY - Wc^q, XZ - c^{p+q}, YZ - Xc^p, YW - Z^2c^p, \\ ZW - Yc^p, W^2 - Zc^p, XW - Y^2, Y^2 - Z^3. \end{cases}$$

9. $f_i \geq 1$ for $i = 1, 2, 3, 4, 5$: We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - Zc^{f_3}, XY - Wc^{f_4}, XZ - Uc^{f_3}, XW - c^{p_2}, Y^2 - Uc^{f_4}, \\ YZ - c^{p_3}, YU - Xc^{f_1}, ZW - Xc^{f_2}, ZU - Yc^{f_2}, W^2 - Yc^{f_5}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XU - YW, YW - Z^2. \end{cases}$$

<u>The case of $n = 8$.</u>
The integral matrix $(\omega(i, n - j - 1))_{0 \leq i,j \leq 7}$ is of the form
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & f_2 & f_3 & f_2 & f_1 & 0 \\ p_2 & p_2 & 0 & f_2 & f_6 & f_6 & f_2 & 0 \\ p_3 & f_4 & p_3 & 0 & f_3 & f_6 & f_3 & 0 \\ p_4 & f_5 & f_5 & p_4 & 0 & f_2 & f_2 & 0 \\ p_3 & f_5 & f_7 & f_5 & p_3 & 0 & f_1 & 0 \\ p_2 & f_4 & f_5 & f_5 & f_4 & p_2 & 0 & 0 \\ p_1 & p_2 & p_3 & p_4 & p_3 & p_2 & p_1 & 0 \end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$$\begin{aligned}
&f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4,\\
&f_4 = -p_1 + p_2 + p_3, f_5 = -p_1 + p_3 + p_4,\\
&f_6 = p_1 + p_2 - p_3 - p_4,\\
&f_7 = -p_1 - p_2 + 2p_3 + p_4,
\end{aligned}$$

and one of the following cases occurs:

1. $f_1 = f_2 = f_3 = f_6 = 0$ and $f_i \geq 1$ for $i = 4, 5, 7$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4) = (p, p, p, p)$ and
$$A \cong R[X]/(X^8 - c^p)$$

2. $f_1 = f_2 = 0$ and $f_i \geq 1$ for $i = 3, 4, 5, 6, 7$: Setting $p = p_4$ and $q = f_3$, we have $(p_1, p_2, p_3, p_4) = (p + q, p + q, p + q, p)$ and
$$A \cong R[X, Y]/(X^2 - c^p, Y^4 - Xc^q)$$

3. $f_2 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 7$: Setting $p = p_2$ and $q = f_1$, we have $(p_1, p_2, p_3, p_4) = (p + q, p, p + q, p)$ and
$$A \cong R[X, Y]/(X^4 - c^p, Y^2 - Xc^q)$$

4. $f_4 = f_5 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 6, 7$: Setting $p = p_4$ and $q = f_7$, we have $(p_1, p_2, p_3, p_4) = (2p + q, p, p + q, p)$ and
$$A \cong R[X, Y]/(X^4 - c^p, Y^2 - X^3c^q)$$

5. $f_3 = f_4 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 5, 7$: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4) = (2p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^2Y - c^p, X^4 - Y^2)$$

6. $f_3 = f_4 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 5, 6$: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4) = (3p, 2p, p, 3p)$ and
$$A \cong R[X, Y]/(XY - c^p, X^3 - Y^5)$$

7. $f_4 = f_5 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 6$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4) = (2p, p, p, p)$ and
$$A \cong R[X, Y]/(X^4 - c^p, X^3 - Y^2)$$

8. $f_2 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 7$: Setting $p = p_4$, $q = f_1$ and $r = f_6$, we have $(p_1, p_2, p_3, p_4) = (p + q + r, p + r, p + q + r, p)$ and
$$A \cong R[X, Y, Z]/(X^2 - c^p, Z^2 - Yc^q, Y^2 - Xc^r)$$

9. $f_5 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7$: Setting $p = p_4$, $q = f_4$ and $r = f_7$, we have $(p_1, p_2, p_3, p_4) = (2p + q + r, p + q, p + q + r, p)$ and

$$A \cong R[X, Y, Z]/(Z^2 - c^p, X^2 - Zc^q, Y^2 - XZc^r)$$

10. $f_4 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 7$: Setting $p = p_2$ and $q = f_3$, we have $(p_1, p_2, p_3, p_4) = (2p + q, p, p + q, 2p)$ and

$$A \cong R[X, Y, Z]/(XY - c^p, Z^2 - Xc^q, X^2 - Y^2)$$

11. $f_5 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6$: Setting $p = p_4$ and $q = f_4$, we have $(p_1, p_2, p_3, p_4) = (2p + q, p + q, p + q, p)$ and

$$A \cong R[X, Y, Z]/(Z^2 - c^p, X^2 - Zc^q, XZ - Y^2)$$

12. $f_1 = f_3 = f_7 = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6$: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4) = (2p, 2p, p, 2p)$ and

$$A \cong R[X, Y, Z]/(XY - c^p, X^2 - Z^2, XZ - Y^2)$$

13. $f_1 = f_3 = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6, 7$: We have

$$A \cong R[X, Y, Z]/(XY - c^{p_3}, YZ^2 - Xc^{f_6}, Z^3 - Yc^{f_2}, X^2 - Z^2c^{f_7}, XZ - Y^2)$$

14. $f_3 = f_4 = 0$ and $f_i \geq 1$ for $i = 1, 2, 5, 6, 7$: We have

$$A \cong R[X, Y, Z]/(XY - c^{p_3}, YZ - Xc^{f_6}, Z^2 - Y^4c^{f_6}, XZ - Y^5, X^2 - Zc^{f_7})$$

15. $f_3 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 7$: We have

$$A \cong R[X, Y, Z]/(X^2Y - c^{p_3}, X^3 - Zc^{f_4}, YZ - Xc^{f_2}, Z^2 - Yc^{f_1}, XZ - Y^2)$$

16. $f_3 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6$: We have

$$A \cong R[X, Y, Z]/(XY - c^{p_3}, XZ - Y^2c^{f_1}, Y^3 - Zc^{f_4}, Z^2 - X^2c^{f_1}, X^3 - Y^2Z)$$

17. $f_4 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6$: We have

$$A \cong R[X, Y, Z]/(XY - Zc^{f_5}, X^3 - Y^2c^{f_5}, YZ - c^{p_3}, Z^2 - Xc^{f_3}, Y^3 - X^2Z)$$

18. $f_1 = f_7 = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5, 6$: We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ - c^{p_3}, XY - Wc^{f_7}, Y^2 - c^{p_4}, YW - Xc^{f_2}, Z^2 - XWc^{f_3}, \\ ZW - Yc^{f_3}, X^2W - Zc^{f_5}, XW^2 - YZ, X^2 - W^2. \end{cases}$$

19. $f_1 = 0$ and $f_i \geq 1$ for $i = 2,3,4,5,6,7$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - W^2c^{f_7}, XY - Wc^{f_5}, XZ - c^{p_3}, Y^2 - c^{p_4}, YW - Xc^{f_2}, \\ Z^2 - XWc^{f_3}, ZW - Yc^{f_3}, W^3 - Zc^{f_2}, XW^2 - YZ. \end{cases}$$

20. $f_3 = 0$ and $f_i \geq 1$ for $i = 1,2,4,5,6,7$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Z^2c^{f_7}, XY - c^{p_3}, XW - Y^2c^{f_1}, Y^3 - Wc^{f_4}, YZ - Xc^{f_6}, \\ Z^2 - YWc^{f_6}, ZW - Yc^{f_2}, W^2 - Zc^{f_1}, XZ - Y^2W. \end{cases}$$

21. $f_4 = 0$ and $f_i \geq 1$ for $i = 1,2,3,5,6,7$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Zc^{f_5}, X^3 - Wc^{f_5}, XW - c^{p_2}, Y^2 - Wc^{f_7}, YZ - c^{p_3}, \\ Z^2 - Xc^{f_3}, ZW - Yc^{f_6}, W^2 - X^2c^{f_6}, X^2Z - YW. \end{cases}$$

22. $f_6 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,5,7$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Z^2c^{f_4}, XY - Wc^{f_4}, XZ - c^{p_2}, Z^3 - Xc^{f_2}, Y^2 - Xc^{f_3}, \\ YW - Z^2c^{f_3}, ZW - Yc^{f_2}, W^2 - Zc^{f_1}, XW - YZ^2. \end{cases}$$

23. $f_7 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,5,6$: We have $A \cong R[X,Y,Z,W,U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_4}, XY - Zc^{f_5}, XZ - Y^2c^{f_5}, XW - Uc^{f_4}, YZ - Uc^{f_5}, \\ YW - c^{p_3}, YU - Xc^{f_1}, Z^2 - c^{p_4}, ZU - Yc^{f_2}, W^2 - Xc^{f_3}, \\ WU - Zc^{f_3}, U^2 - Y^2c^{f_1}, XU - ZW, ZW - Y^3. \end{cases}$$

24. $f_i \geq 1$ for $i = 1,2,3,4,5,6,7$: We have $A \cong R[X,Y,Z,U,V,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_4}, XY - Uc^{f_5}, XZ - Vc^{f_5}, XU - Wc^{f_4}, XV - c^{p_2}, \\ Y^2 - Vc^{f_7}, YZ - Wc^{f_5}, YU - c^{p_3}, YW - Xc^{f_1}, Z^2 - c^{p_4}, \\ ZV - Xc^{f_2}, ZW - Yc^{f_2}, U^2 - Xc^{f_3}, UV - Yc^{f_6}, UW - Zc^{f_3}, \\ V^2 - Zc^{f_6}, VW - c^{f_2}, W^2 - Vc^{f_1}, XW - YV, YV - ZU. \end{cases}$$

The case of $n = 9$.

The integral matrix $(\omega(i, n-j-1))_{0\leq i,j\leq 8}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & f_2 & f_3 & f_3 & f_2 & f_1 & 0 \\ p_2 & p_2 & 0 & f_2 & f_7 & f_8 & f_7 & f_2 & 0 \\ p_3 & f_4 & p_3 & 0 & f_3 & f_8 & f_8 & f_3 & 0 \\ p_4 & f_5 & f_5 & p_4 & 0 & f_3 & f_7 & f_3 & 0 \\ p_4 & f_6 & f_9 & f_6 & p_4 & 0 & f_2 & f_2 & 0 \\ p_3 & f_5 & f_9 & f_9 & f_5 & p_3 & 0 & f_1 & 0 \\ p_2 & f_4 & f_5 & f_6 & f_5 & f_4 & p_2 & 0 & 0 \\ p_1 & p_2 & p_3 & p_4 & p_4 & p_3 & p_2 & p_1 & 0 \end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$$\begin{aligned} &f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, \\ &f_4 = -p_1 + p_2 + p_3, f_5 = -p_1 + p_3 + p_4, f_6 = -p_1 + 2p_4, \\ &f_7 = p_1 + p_2 - p_3 - p_4, f_8 = p_1 + p_2 - 2p_4, \\ &f_9 = -p_1 - p_2 + p_3 + 2p_4, \end{aligned}$$

and the following cases are possible:

1. $f_1 = f_2 = f_3 = f_7 = f_8 = 0$ and $f_i \geq 1$ for $i = 4, 5, 6, 9$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4) = (p, p, p, p)$ and
$$A \cong R[X]/(X^9 - c^p)$$

2. $f_1 = f_3 = f_8 = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6, 7, 9$: Setting $p = p_3$ and $q = f_2$, we have $(p_1, p_2, p_3, p_4) = (p+q, p+q, p, p+q)$ and
$$A \cong R[X, Y]/(X^3 - c^p, Y^3 - Xc^q)$$

3. $f_2 = f_6 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 8, 9$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4) = (2p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^2Y - c^p, X^3 - Y^3)$$

4. $f_4 = f_5 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 7, 8, 9$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4) = (2p, p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - c^p, Y^3 - X^2)$$

5. $f_4 = f_5 = f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 6, 7, 8$: Setting $p = p_3$ and $q = f_6$, we have $(p_1, p_2, p_3, p_4) = (2p+q, p+q, p, p+q)$ and
$$A \cong R[X, Y]/(X^3 - c^p, Y^3 - X^2c^q)$$

6. $f_4 = f_6 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 8, 9$: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4) = (4p, p, 3p, 2p)$ and

$$A \cong R[X, Y]/(XY - c^p, X^5 - Y^4)$$

7. $f_1 = f_2 = f_6 = f_9 = 0$ and $f_i \geq 1$ for $i = 3, 4, 5, 7, 8$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4) = (2p, 2p, 2p, p)$ and

$$A \cong R[X, Y]/(XY - c^p, X^7 - Y^2)$$

8. $f_4 = f_7 = f_8 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 9$: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4) = (3p, p, 2p, 2p)$ and

$$A \cong R[X, Y, Z]/(XZ - c^p, X^2 - YZ, Y^2 - Z^3)$$

9. $f_1 = f_2 = 0$ and $f_i \geq 1$ for $i = 3, 4, 5, 6, 7, 8, 9$: We have

$$A \cong R[X, Y, Z]/(X^2 - Zc^{f_6}, XY - c^{p_4}, YZ - Xc^{f_3}, Z^4 - Yc^{f_3}, XZ^3 - Y^2)$$

10. $f_2 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 7, 8, 9$: We have

$$A \cong R[X, Y, Z]/(X^2 - Yc^{f_7}, XY - Z^3c^{f_7}, YZ - c^{p_4}, Z^4 - Xc^{f_1}, XZ^3 - Y^2)$$

11. $f_2 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 8, 9$: We have

$$A \cong R[X, Y, Z]/(XY^2 - c^{p_2}, X^2 - Zc^{f_6}, Y^2Z - Xc^{f_3}, Z^2 - Yc^{f_1}, XZ - Y^3)$$

12. $f_4 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 8, 9$: We have

$$A \cong R[X, Y, Z]/(X^3 - Yc^{f_6}, XZ - c^{p_2}, Y^2 - Z^3c^{f_8}, YZ - X^2c^{f_9}, X^2Y - Z^4)$$

13. $f_6 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 8, 9$: We have

$$A \cong R[X, Y, Z]/(X^2 - Yc^{f_4}, XZ - c^{p_2}, YZ - Xc^{f_2}, Y^3 - Z^3c^{f_2}, XY^2 - Z^4)$$

14. $f_2 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 7, 8, 9$: We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Wc^{f_6}, XY - c^{p_4}, YZ - XWc^{f_7}, YW - Xc^{f_3}, Z^2 - Yc^{f_7}, \\ W^2 - Zc^{f_1}, XZW - Y^2. \end{cases}$$

15. $f_4 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 7, 8, 9$: We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Zc^{f_5}, XZ - Wc^{f_5}, XW - c^{p_2}, Y^2 - X^3c^{f_9}, YZ - X^4c^{f_5}, \\ ZW - Yc^{f_7}, W^2 - Zc^{f_7}, X^2 - Z^5, X^2 - YW. \end{cases}$$

16. $f_1 = f_9 = 0$ and $f_i \geq 1$ for $i = 2,3,4,5,6,7,8$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ - Wc^{f_5}, X^3 - c^{p_3}, Y^2 - Wc^{f_6}, YZ - c^{p_4}, YW - Xc^{f_2}, \\ ZW - Yc^{f_3}, X^2W - Zc^{f_3}, XY - W^2, Z^2 - X^2Y. \end{cases}$$

17. $f_4 = f_8 = 0$ and $f_i \geq 1$ for $i = 1,2,3,5,6,7,9$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ^2 - Yc^{f_5}, XY - Wc^{f_5}, XW - c^{p_2}, Z^3 - c^{p_3}, YZ - Xc^{f_3}, \\ YW - Z^2c^{f_7}, W^2 - Yc^{f_7}, X^2 - ZW, Y^2 - Z^2W. \end{cases}$$

18. $f_6 = f_9 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,5,7,8$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_4}, XY - Wc^{f_5}, XW - YZc^{f_5}, YZ^2 - Xc^{f_2}, YW - Z^2c^{f_5}, \\ ZW - c^{p_4}, Z^3 - Yc^{f_2}, XZ - Y^2, XZ^2 - W^2. \end{cases}$$

19. $f_7 = f_8 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,5,6,9$: We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - YZc^{f_4}, XY - Wc^{f_4}, XZ - c^{p_2}, YZ^2 - Xc^{f_2}, YW - Z^2c^{f_3}, \\ ZW - Yc^{f_2}, W^2 - Zc^{f_1}, Y^2 - Z^3, XW - Z^4. \end{cases}$$

20. $f_1 = 0$ and $f_i \geq 1$ for $i = 2,3,4,5,6,7,8,9$: We have $A \cong R[X,Y,Z,W,U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Wc^{f_9}, XY - U^2c^{f_5}, XZ - Uc^{f_5}, XW - c^{p_3}, Y^2 - Uc^{f_6}, \\ YZ - c^{p_4}, YU - Xc^{f_2}, ZW - XUc^{f_3}, ZU - Yc^{f_3}, W^2 - Xc^{f_8}, \\ WU - Zc^{f_3}, U^3 - Wc^{f_2}, XU^2 - Z^2, YW - Z^2. \end{cases}$$

21. $f_4 = 0$ and $f_i \geq 1$ for $i = 1,2,3,5,6,7,8,9$: We have $A \cong R[X,Y,Z,W,U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Zc^{f_5}, X^3 - Wc^{f_6}, XZ - Uc^{f_5}, XU - c^{p_2}, Y^2 - Wc^{f_9}, \\ YZ - XWc^{f_5}, YW - c^{p_3}, ZW - Xc^{f_3}, ZU - Yc^{f_7}, W^2 - Yc^{f_8}, \\ WU - X^2c^{f_8}, U^2 - Zc^{f_7}, X^2W - Z^2, YU - Z^2. \end{cases}$$

22. $f_6 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,5,7,8,9$: We have $A \cong R[X,Y,Z,W,U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_4}, XY - Wc^{f_5}, XW - Uc^{f_5}, XU - c^{p_2}, Y^2 - XZc^{f_9}, \\ YZ - Uc^{f_9}, YW - Z^2c^{f_5}, ZW - c^{p_4}, ZU - Xc^{f_2}, Z^3 - Yc^{f_2}, \\ WU - Yc^{f_7}, U^2 - Wc^{f_7}, XZ^2 - W^2, YU - W^2. \end{cases}$$

23. $f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 8, 9$: We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Yc^{f_4}, XY - Zc^{f_6}, XZ - Uc^{f_4}, XW - c^{p_2}, Y^2 - Uc^{f_6}, \\ YW - Xc^{f_2}, YU - W^3c^{f_2}, Z^2 - W^3c^{f_8}, ZW - Yc^{f_8}, ZU - W^2c^{f_3}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XU - W^4, YZ - W^4. \end{cases}$$

24. $f_8 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 9$: We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - ZWc^{f_4}, XY - Wc^{f_5}, XZ - Uc^{f_4}, XW - c^{p_2}, Z^3 - c^{p_3}, \\ YZ - Xc^{f_3}, YW - Z^2c^{f_7}, YU - ZWc^{f_3}, ZU - Yc^{f_3}, W^2 - Yc^{f_7}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XU - Y^2, Z^2W - Y^2. \end{cases}$$

25. $f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 8$: We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_4}, XY - Wc^{f_5}, XZ - Y^2c^{f_6}, XW - YZc^{f_5}, YW - Uc^{f_5}, \\ Y^3 - c^{p_3}, YU - Xc^{f_1}, Z^2 - Uc^{f_6}, ZW - c^{p_4}, ZU - Yc^{f_2}, \\ WU - Zc^{f_3}, U^2 - YZc^{f_1}, XU - W^2, Y^2Z - W^2. \end{cases}$$

26. $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$: We have $A \cong R[X, Y, Z, W, U, V, S]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_4}, XY - Wc^{f_5}, XZ - Uc^{f_6}, XW - Vc^{f_5}, XU - Sc^{f_4}, \\ XV - c^{p_2}, Y^2 - Uc^{f_9}, YZ - Vc^{f_9}, YW - Sc^{f_5}, YU - c^{p_3}, \\ YS - Xc^{f_1}, Z^2 - S^{f_6}, ZW - c^{p_4}, ZV - Xc^{f_2}, ZS - Yc^{f_2}, \\ WU - Xc^{f_3}, WV - Yc^{f_7}, WS - Zc^{p-s}, U^2 - Yc^{f_8}, UV - Zc^{f_8}, \\ US - Wc^{f_3}, V^2 - Wc^{f_7}, VS - Uc^{f_2}, S^2 - Vc^{f_1}, XS - YV, \\ YV - ZU, ZU - W^2. \end{cases}$$

<u>The case of $n = 10$.</u>
The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 9}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_3 & f_2 & f_1 & 0 \\ p_2 & p_2 & 0 & f_2 & f_8 & f_9 & f_9 & f_8 & f_2 & 0 \\ p_3 & f_5 & p_3 & 0 & f_3 & f_9 & f_{12} & f_9 & f_3 & 0 \\ p_4 & f_6 & f_6 & p_4 & 0 & f_4 & f_9 & f_9 & f_4 & 0 \\ p_5 & f_7 & f_{10} & f_7 & p_5 & 0 & f_3 & f_8 & f_3 & 0 \\ p_4 & f_7 & f_{11} & f_{11} & f_7 & p_4 & 0 & f_2 & f_2 & 0 \\ p_3 & f_6 & f_{10} & f_{11} & f_{10} & f_6 & p_3 & 0 & f_1 & 0 \\ p_2 & f_5 & f_6 & f_7 & f_7 & f_6 & f_5 & p_2 & 0 & 0 \\ p_1 & p_2 & p_3 & p_4 & p_5 & p_4 & p_3 & p_2 & p_1 & 0 \end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$$\begin{aligned}
&f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5,\\
&f_5 = -p_1 + p_2 + p_3, f_6 = -p_1 + p_3 + p_4, f_7 = -p_1 + p_4 + p_5,\\
&f_8 = p_1 + p_2 - p_3 - p_4, f_9 = p_1 + p_2 - p_4 - p_5,\\
&f_{10} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{11} = -p_1 - p_2 + p_4 + p_4 + p_5,\\
&f_{12} = p_1 + p_2 + p_3 - p_4 - p_4 - p_5,
\end{aligned}$$

and the following cases are possible:

1. $f_1 = f_2 = f_3 = f_4 = f_8 = f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 5, 6, 7, 10, 11$: Setting $p = p_1$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, p, p, p, p)$ and
$$A \cong R[X]/(X^{10} - c^p)$$

2. $f_2 = f_4 = f_8 = f_9 = 0$ and $f_i \geq 1$ for $i = 1, 3, 5, 6, 7, 10, 11, 12$: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p, q, p)$ and
$$A \cong R[X, Y]/(X^5 - c^q, Y^2 - Xc^{p-q})$$

3. $f_1 = f_2 = f_3 = f_8 = 0$ and $f_i \geq 1$ for $i = 4, 5, 6, 7, 9, 10, 11, 12$: Setting $p = p_1$ and $q = p_5$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, p, p, p, q)$ and
$$A \cong R[X, Y]/(X^2 - c^q, Y^5 - Xc^{p-q})$$

4. $f_4 = f_5 = f_6 = f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 7, 8, 10, 11$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^2 - Y^4, X^2Y - c^p)$$

5. $f_4 = f_5 = f_6 = f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 7, 8, 10, 11, 12$: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p - q, q, p)$ and
$$A \cong R[X, Y]/(X^5 - c^q, Y^2 - X^4c^{p-2q})$$

6. $f_1 = f_4 = f_6 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 2, 3, 5, 7, 8, 9, 11$: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5) = (3p, 3p, p, 2p, 3p)$ and
$$A \cong R[X, Y]/(X^7 - Y^3, XY - c^p)$$

7. $f_2 = f_7 = f_8 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 9, 10, 12$: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p, q, p - q)$ and
$$A \cong R[X, Y]/(X^5 - c^q, Y^2 - c^{p-q})$$

8. $f_1 = f_4 = f_6 = f_{10} = f_{11} = 0$ and $f_i \geq 1$ for $i = 2, 3, 5, 7, 8, 9, 12$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, 2p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^6 - Y^2, X^2Y - c^p)$$
9. $f_5 = f_6 = f_7 = f_{10} = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 8, 9, 12$: Setting $p = p_4$ , we have $(p_1, p_2, p_3, p_4, p_5) = (2p, p, p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^2, X^2Y^2 - c^p)$$
10. $f_5 = f_6 = f_7 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 8, 9, 10, 12$: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p - q, q, p - q)$ and
$$A \cong R[X, Y]/(X^5 - c^q, Y^2 - X^3c^{p-2q})$$
11. $f_1 = f_2 = f_4 = f_{11} = 0$ and $f_i \geq 1$ for $i = 3, 5, 6, 7, 8, 9, 10, 12$: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, 2p, 2p, p, 2p)$ and
$$A \cong R[X, Y, Z]/(XY - c^p, X^2 - Z^2, X^3 - Y^2)$$
12. $f_1 = f_2 = f_4 = 0$ and $f_i \geq 1$ for $i = 3, 5, 6, 7, 8, 9, 10, 11, 12$: Setting $p = p_1$ and $q = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, p, p, q, p)$ and
$$A \cong R[X, Y, Z]/(XY - c^q, X^2 - Z^2c^{2q-p}, YZ^2 - Xc^{p-q}, Z^4 - Yc^{p-q}, Y^2 - XZ^2)$$
13. $f_2 = f_4 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 3, 5, 6, 7, 8, 9, 10, 12$: Setting $p = p_1$ and $q = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, 2q, p, q, p)$ and
$$A \cong R[X, Y, Z]/(XY - c^q, X^3 - Y^2, Z^2 - X^2c^{p-q})$$
14. $f_4 = f_6 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 7, 8, 9, 10, 11$: Setting $p = p_3$ and $q = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (p + q, 2q - p, p, q, p + q)$ and
$$A \cong R[X, Y, Z]/(XZ - c^p, X^2 - Yc^{2p-q}, YZ - Xc^{q-p}, Y^4 - Z^2c^{q-p}, XY^3 - Z^3)$$
15. $f_1 = f_4 = f_6 = f_{10} = 0$ and $f_i \geq 1$ for $i = 2, 3, 5, 7, 8, 9, 11, 12$: Setting $p = p_1$ and $q = p_3$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, p, q, p - q, p)$ and
$$A \cong R[X, Y, Z]/(X^8 - Yc^q, XY - Zc^{p-2q}, XZ - c^q, Y^2 - X^6c^{p-2q}, Z^2 - Yc^{3q-p})$$
16. $f_4 = f_6 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 7, 8, 9, 10, 12$: Setting $p = p_1$ and $q = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, 2q, p - q, q, p)$ and
$$A \cong R[X, Y, Z]/(XY - c^q, Z^2 - Yc^{p-2q}, X^2 - Y^3)$$
17. $f_5 = f_8 = f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7, 10, 11$: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (3p, p, 2p, 2p, 2p)$ and
$$A \cong R[X, Y, Z]/(XY - c^p, X^2 - Y^3, Y^3 - Z^2)$$

18. $f_5 = f_8 = f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7, 10, 11, 12$:
Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p-q, 2q, p-q)$ and
$$A \cong R[X, Y, Z]/(XY - c^q, Z^2 - X^2c^{p-3q}, X^2 - Y^3)$$

19. $f_5 = f_7 = f_8 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 9, 10, 11, 12$:
Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p - q, 2q, p - 2q)$ and
$$A \cong R[X, Y, Z]/(XY - c^q, Z^2 - c^{p-2q}, X^3 - Y^2)$$

20. $f_5 = f_6 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 7, 8, 9, 11$:
Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5) = (3p, 2p, p, 2p, 2p)$ and
$$A \cong R[X, Y, Z]/(XY - c^p, Y^2 - Z^2, X^3 - YZ)$$

21. $f_5 = f_6 = f_{10} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 7, 8, 9, 11, 12$:
Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (p, q, p - q, q, q)$ and
$$A \cong R[X, Y, Z]/(XY - c^{p-q}, Z^3 - X^2c^{2q-p}, XZ^2 - Yc^{2q-p}, Y^2 - Z^2c^{2p-3q}, X^3 - YZ)$$

22. $f_6 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 7, 8, 9, 11$:
Setting $p = p_3$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (3p, q, p, 2p, q)$ and
$$A \cong R[X, Y, Z]/(XY - c^p, X^4 - Zc^{3p-q}, YZ - X^3c^{q-2p}, Z^2 - Y^2c^{q-2p}, X^3Z - Y^3)$$

23. $f_5 = f_7 = f_{10} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 8, 9, 11, 12$:
Setting $p = p_4$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (2q, q, q, p, 2q - p)$ and
$$A \cong R[X, Y, Z]/(XY - Zc^{p-q}, Z^2 - c^{2q-p}, X^3 - Y^2)$$

24. $f_6 = f_{10} = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 7, 8, 9, 12$:
Setting $p = p_4$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, q, p, p, q)$ and
$$A \cong R[X, Y, Z]/(X^2Y - c^p, X^4 - Zc^{2p-q}, YZ - X^2c^{q-p}, Z^2 - Yc^{q-p}, X^2Z - Y^2)$$

25. $f_4 = f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 7, 8, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^3 - Zc^{f_6}, X^2Y - Wc^{f_5}, X^2Z - c^{p_2}, Y^2 - XZc^{f_{12}}, YW - Xc^{f_3}, \\ Z^2 - Xc^{f_8}, ZW - Yc^{f_2}, W^2 - Zc^{f_1}, XW - YZ. \end{cases}$$

26. $f_4 = f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 7, 8, 10, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^3 - Zc^{f_6}, X^2Y - Wc^{f_6}, X^2Z - c^{p_2}, YW - Xc^{f_3}, Z^2 - Xc^{f_8}, \\ ZW - Yc^{f_2}, W^2 - Zc^{f_1}, XZ - Y^2, XW - YZ. \end{cases}$$

27. $f_3 = f_8 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6, 7, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - c^{p_5}, YZ - Xc^{f_9}, ZW - c^{f_2}, W^2 - Zc^{f_1}, \\ XW - Y^2, YW - Z^2. \end{cases}$$

28. $f_3 = f_8 = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6, 7, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - c^{p_5}, Y^2 - XWc^{f_{12}}, YZ - Xc^{p_9}, ZW - Yc^{f_2}, \\ W^2 - Zc^{f_1}, YW - Z^2. \end{cases}$$

29. $f_2 = f_4 = 0$ and $f_i \geq 1$ for $i = 1, 3, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - Zc^{f_{11}}, XY - c^{p_4}, YZ - Xc^{f_9}, Z^2 - Yc^{f_8}, \\ W^2 - Zc^{f_1}, Y^2 - XZ. \end{cases}$$

30. $f_2 = f_8 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 7, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - Zc^{f_{11}}, XY - Wc^{f_7}, XZ^2 - c^{p_4}, Y^2 - c^{p_5}, YW - Xc^{f_3}, \\ Z^3 - Xc^{f_9}, Z^2W - Yc^{f_4}, W^2 - Zc^{f_1}, XW - YZ. \end{cases}$$

31. $f_2 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 7, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^4 - Zc^{f_7}, XY - Wc^{f_7}, XZ - c^{p_4}, Y^2 - c^{p_5}, YW - Xc^{f_3}, \\ Z^2 - X^3c^{f_4}, ZW - Yc^{f_4}, W^2 - X^2c^{f_1}, X^3W - YZ. \end{cases}$$

32. $f_1 = f_2 = f_{11} = 0$ and $f_i \geq 1$ for $i = 3, 4, 5, 6, 7, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^4 - Zc^{f_7}, XY - Wc^{f_7}, XZ - c^{p_4}, Y^2 - c^{p_5}, YW - Xc^{f_3}, \\ Z^2 - X^3c^{f_4}, ZW - Yc^{f_4}, X^2 - W^2, XW^3 - YZ. \end{cases}$$

33. $f_1 = f_2 = 0$ and $f_i \geq 1$ for $i = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:
$$\begin{cases} X^2 - W^2c^{f_{11}}, XY - Wc^{f_7}, XZ - c^{p_4}, Y^2 - c^{p_5}, YW - Xc^{f_3}, \\ Z^2 - XW^2c^{f_4}, ZW - Yc^{f_4}, W^4 - Zc^{f_3}, XW^3 - YZ. \end{cases}$$

34. $f_1 = f_4 = f_{12} = 0$ and $f_i \geq 1$ for $i = 2, 3, 5, 6, 7, 8, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Yc^{f_{10}}, XY - Wc^{f_6}, XZ - c^{p_3}, YW^2 - Z^2c^{f_3}, YZ - Xc^{f_9}, \\ ZW - Yc^{f_3}, W^3 - Zc^{f_2}, XW - Y^2, XW^2 - Z^3. \end{cases}$$

35. $f_1 = f_3 = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - ZWc^{f_{10}}, XY - W^2c^{f_6}, XZ - c^{p_3}, Y^2 - c^{p_5}, YW^2 - Xc^{f_2}, \\ Z^2 - YWc^{f_{12}}, ZW^2 - Yc^{f_4}, W^3 - Zc^{f_2}, XW - YZ. \end{cases}$$

36. $f_3 = f_{10} = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6, 7, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ - c^{p_3}, Y^2 - c^{p_5}, XW - YZc^{f_1}, Z^2 - YWc^{f_{12}}, \\ W^2 - XYc^{f_1}, X^2 - ZW. \end{cases}$$

37. $f_1 = f_3 = f_{10} = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6, 7, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ - c^{p_3}, Y^2 - c^{p_5}, Z^2 - YWc^{f_{12}}, \\ XW - YZ, X^2 - ZW, XY - W^2. \end{cases}$$

38. $f_1 = f_3 = f_{12} = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6, 7, 8, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - ZWc^{f_{10}}, XY - W^2c^{f_{10}}, XZ - c^{p_3}, Y^2 - c^{p_5}, YW^2 - Xc^{f_2}, \\ ZW^2 - Yc^{f_4}, W^3 - Zc^{f_2}, XW - YZ, YW - Z^2. \end{cases}$$

39. $f_1 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5, 6, 7, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Wc^{f_6}, X^2W - Yc^{f_6}, XZ - c^{p_3}, Y^2 - c^{p_5}, YZ - XWc^{f_3}, \\ YW - Z^2c^{f_3}, ZW - X^2c^{f_3}, XY - W^2, X^2Y - Z^3. \end{cases}$$

40. $f_3 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6, 7, 8, 9, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ - c^{p_3}, XW - YZc^{f_1}, Y^2 - c^{p_5}, W^2 - XYc^{f_1}, \\ X^2 - ZW, Z^2 - YW. \end{cases}$$

41. $f_1 = f_3 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 2, 4, 5, 6, 7, 8, 9, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XZ - c^{p_3}, Y^2 - c^{p_5}, XW - YZ, X^2 - ZW, \\ Z^2 - YW, W^2 - XY. \end{cases}$$

42. $f_1 = f_4 = f_{11} = 0$ and $f_i \geq 1$ for $i = 2, 3, 5, 6, 7, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XZ - Wc^{p_6}, YZ - c^{p_4}, YW - Xc^{f_2}, \\ XW - Z^2, Y^2 - W^2. \end{cases}$$

43. $f_4 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 7, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XW - Z^2c^{f_1}, XZ - Wc^{f_6}, Y^3 - Z^2c^{f_2}, YW - Xc^{f_3}, \\ YZ - c^{p_4}, Z^3 - Y^2c^{f_6}, W^2 - Y^2c^{f_1}, XY^2 - Z^2W. \end{cases}$$

44. $f_4 = f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XY - Wc^{f_{11}}, XW - c^{p_3}, Y^2 - Z^3c^{f_{11}}, YZ - c^{p_4}, \\ Z^4 - Yc^{f_5}, ZW - Xc^{f_9}, W^2 - Yc^{f_{12}}, XZ^3 - YW. \end{cases}$$

45. $f_1 = f_{10} = f_{11} = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5, 6, 7, 8, 9, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Wc^{f_6}, X^2Y - c^{p_4}, X^2W - Zc^{f_6}, YZ - Wc^{f_7}, YW - Xc^{f_2}, \\ Z^2 - c^{p_5}, ZW - Yc^{f_3}, Y^2 - XZ, XZ - W^2. \end{cases}$$

46. $f_8 = f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 10, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Y^2c^{f_5}, XY - Wc^{f_5}, XZ - c^{p_2}, YW - Z^2c^{f_3}, Z^4 - Xc^{f_2} \\ ZW - Yc^{f_2}, W^2 - Zc^{f_1}, XW - YZ^3, Y^2 - Z^3. \end{cases}$$

47. $f_8 = f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Z^3c^{f_5}, XY - Wc^{f_5}, XZ - c^{p_2}, Y^2 - Z^3c^{f_{12}}, YW - Z^2c^{f_3}, \\ ZW - Yc^{f_2}, W^2 - Zc^{f_1}, Z^4 - Xc^{f_2}, XW - YZ^3. \end{cases}$$

48. $f_5 = f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,6,7,8,10,11$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Wc^{f_6}, XW - c^{p_2}, Y^2 - Xc^{f_4}, \\ W^2 - Yc^{f_8}, X^2 - Z^2, YW - Z^2. \end{cases}$$

49. $f_5 = f_9 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,6,7,8,10,11,12$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Wc^{f_6}, XW - c^{p_2}, Y^2 - Xc^{f_4}, Z^2 - YWc^{f_{12}}, \\ W^2 - Yc^{f_8}, X^2 - YW. \end{cases}$$

50. $f_5 = f_8 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,6,7,9,10,11$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - W^2c^{f_7}, XY - Zc^{f_7}, XW - c^{p_2}, Y^2 - c^{p_5}, YZ - Xc^{f_3}, \\ ZW - Yc^{f_9}, W^3 - X^2c^{f_9}, X^2 - Z^2, YW^2 - Z^3. \end{cases}$$

51. $f_5 = f_8 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,6,7,9,10,11,12$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - W^2c^{f_7}, XY - Zc^{f_7}, XW - c^{p_2}, Y^2 - c^{p_5}, YZ - Xc^{f_3}, \\ Z^2 - X^2c^{f_{12}}, ZW - Yc^{f_9}, W^3 - X^2c^{f_{12}}, X^2Z - YW^2. \end{cases}$$

52. $f_7 = f_8 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,5,6,9,10,11,12$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Yc^{f_5}, XW - c^{p_2}, Y^2 - Wc^{f_{11}}, \\ YW - Xc^{f_2}, Z^2 - c^{p_5}, XY - W^2. \end{cases}$$

53. $f_5 = f_6 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,7,8,9,10,11$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} Y^2 - Zc^{f_{10}}, YW - c^{p_3}, Z^2 - Xc^{f_4}, \\ ZW - Yc^{f_9}, X^2 - W^2, XW - YZ. \end{cases}$$

54. $f_5 = f_6 = 0$ and $f_i \geq 1$ for $i = 1,2,3,4,7,8,9,10,11,12$:
We have $A \cong R[X,Y,Z,W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Zc^{f_7}, X^2Y - Wc^{f_7}, X^2Z - c^{p_2}, Y^2 - Zc^{f_{10}}, YW - c^{p_3}, \\ Z^2 - Xc^{f_4}, ZW - Yc^{f_9}, W^2 - X^2c^{f_{12}}, XW - YZ. \end{cases}$$

55. $f_6 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - W^2 c^{f_5}, XW - YZc^{f_5}, Y^2 - Zc^{f_{10}}, YW - c^{p_3}, \\ Z^2 - Xc^{f_4}, ZW - Yc^{f_9}, XYZ - W^3. \end{cases}$$

56. $f_5 = f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Zc^{f_6}, X^4 - Wc^{f_6}, XW - c^{p_2}, Y^2 - X^3 c^{f_{10}}, YZ - Wc^{f_{10}}, \\ Z^2 - c^{p_5}, ZW - Yc^{f_8}, W^2 - X^3 c^{f_8}, X^3 Z - YW. \end{cases}$$

57. $f_5 = f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7, 8, 9, 11$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} XY - Zc^{f_6}, X^3 - Y^2 c^{f_7}, XZ - Wc^{f_7}, Y^3 - XWc^{f_6}, YW - c^{p_3}, \\ Z^2 - c^{p_5}, ZW - Xc^{f_3}, X^2 - W^2, X^2 W - Y^2 Z. \end{cases}$$

58. $f_5 = f_{10} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Y^2 c^{f_7}, XY - Zc^{f_6}, XZ - Wc^{f_7}, Y^3 - XWc^{f_6}, YW - c^{p_3}, \\ Z^2 - c^{p_5}, ZW - Xc^{f_3}, W^2 - X^2 c^{f_{12}}, X^2 W - Y^2 Z. \end{cases}$$

59. $f_6 = f_{10} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 7, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XZ - Y^2 c^{f_7}, XW - Y^3 c^{f_5}, Y^4 - Xc^{f_1}, YW - c^{p_3}, \\ YZ - Wc^{f_{11}}, Z^2 - XY^2 c^{f_{11}}, W^2 - Zc^{f_{12}}, XY^3 - ZW. \end{cases}$$

60. $f_6 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XZ - Wc^{f_7}, Y^2 - Wc^{f_{10}}, \\ ZW - c^{p_4}, W^2 - Xc^{f_4}, XW - Z^2. \end{cases}$$

61. $f_7 = f_{10} = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 8, 9, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, W^2 - c^{p_5}, \\ Y^2 - XZ, Z^2 - YW, XW - YZ. \end{cases}$$

62. $f_7 = f_{10} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, YZ - XWc^{f_{11}}, \\ \qquad\qquad\qquad\qquad Z^2 - YWc^{f_{11}}, W^2 - c^{p_5}, Y^2 - XZ. \end{cases}$$

63. $f_7 = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, X^2Z - c^{p_2}, Y^2 - XZc^{f_{10}}, YW - Z^2c^{f_{10}}, \\ Z^3 - Xc^{f_2}, Z^2W - Yc^{f_8}, W^2 - c^{p_5}, XW - YZ. \end{cases}$$

64. $f_2 = 0$ and $f_i \geq 1$ for $i = 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Wc^{f_{11}}, XY - Uc^{f_7}, XZ - c^{p_4}, Y^2 - c^{p_5}, YW - XUc^{f_8}, \\ YU - Xc^{f_3}, Z^2 - XWc^{f_4}, ZW - Xc^{f_9}, ZU - Yc^{f_4}, \\ W^2 - Zc^{f_8}, U^2 - Wc^{f_2}, XWU - YZ. \end{cases}$$

65. $f_4 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 7, 8, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Yc^{f_{10}}, XY - Uc^{f_6}, XZ - c^{p_3}, XU - Y^2c^{f_1}, Y^3 - Wc^{f_6}, \\ YZ - Xc^{f_9}, YW - Z^2c^{f_9}, ZW - YUc^{f_9}, ZU - Yc^{f_3}, W^2 - Yc^{f_8}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_2}, XW - Y^2U, Y^2U - Z^3. \end{cases}$$

66. $f_3 = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - ZUc^{f_{10}}, XY - Wc^{f_{10}}, XZ - c^{p_3}, XU - YZc^{f_1}, Y^2 - c^{p_5}, \\ YW - Xc^{f_8}, Z^2 - YUc^{f_{12}}, ZW - Yc^{f_9}, W^2 - ZUc^{f_8}, \\ U^2 - Wc^{f_2}, WU - Zc^{f_2}, XW - YZU. \end{cases}$$

67. $f_3 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 4, 5, 6, 7, 8, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - ZUc^{f_{10}}, XY - Wc^{f_{10}}, XZ - c^{p_3}, XU - YZc^{f_1}, \\ Y^2 - c^{p_5}, YW - Xc^{f_8}, ZW - Yc^{f_9}, W^2 - ZUc^{f_8}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XW - Z^3, YU - Z^2. \end{cases}$$

68. $f_1 = f_4 = 0$ and $f_i \geq 1$ for $i = 2, 3, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XY - Wc^{f_{11}}, XZ - Uc^{f_6}, XW - c^{p_3}, Y^2 - U^2c^{f_{11}}, \\ YZ - c^{p_4}, YU - Xc^{f_2}, ZW - Xc^{f_9}, ZU^2 - Yc^{f_9}, W^2 - Yc^{f_{12}}, \\ WU - Zc^{f_3}, U^3 - Wc^{f_2}, XU - Z^2, YW - Z^2U. \end{cases}$$

69. $f_1 = f_{10} = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5, 6, 7, 8, 9, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Uc^{f_6}, XY - Wc^{f_{11}}, XW - c^{p_3}, X^2U - Zc^{f_6}, Y^2 - U^2c^{f_{11}}, \\ YZ - Uc^{f_7}, YU - Xc^{f_2}, Z^2 - c^{p_5}, ZW - XUc^{f_3}, ZU - Yc^{f_3}, \\ W^2 - Yc^{f_{12}}, WU - X^2c^{f_3}, XZ - U^2, X^2Z - YW. \end{cases}$$

70. $f_1 = f_{11} = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XY - U^2c^{f_{10}}, XZ - Uc^{f_6}, Y^2 - c^{p_5}, YU - Wc^{f_3}, \\ Z^2 - XUc^{f_4}, ZU - Yc^{f_4}, YW - Uc^{f_7}, YZ - c^{p_4}, \\ WU - Xc^{f_2}, XU^2 - YZ, W^2 - U^2. \end{cases}$$

71. $f_9 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - YWc^{f_5}, XY - Wc^{f_6}, XZ - Uc^{f_5}, XW - c^{p_2}, Y^2 - Xc^{f_4}, \\ YU - ZWc^{f_4}, Z^2 - YWc^{f_{12}}, ZU - Yc^{f_3}, W^2 - Yc^{f_8}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XU - YZW. \end{cases}$$

72. $f_9 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 10, 11$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - YWc^{f_5}, XY - Wc^{f_6}, XZ - Uc^{f_5}, XU - c^{p_2}, Y^2 - Xc^{f_4}, \\ YU - ZWc^{f_4}, ZU - Yc^{f_3}, W^2 - Yc^{f_8}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XU - YZW, YW - Z^2. \end{cases}$$

73. $f_8 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Z^2c^{f_5}, XY - Zc^{f_7}, XZ - Uc^{f_5}, XW - c^{p_2}, Y^2 - c^{p_5}, \\ YZ - Xc^{f_3}, YU - Z^2c^{f_3}, ZW - Yc^{f_9}, ZU - W^2c^{f_3}, W^3 - Z^2c^{f_9}, \\ WU - Zc^{f_2}, U^2 - Wc^{f_1}, XU - Z^3, YW^2 - Z^3. \end{cases}$$

74. $f_{10} = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 11$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - W^2c^{f_5}, XY - Zc^{f_6}, XZ - Wc^{f_7}, XW - Uc^{f_5}, Y^3 - Uc^{f_6}, \\ YW - c^{p_3}, YZ - Xc^{f_1}, Z^2 - c^{p_5}, ZW - Xc^{f_3}, ZU - W^2c^{f_3}, \\ WU - Y^2c^{f_3}, U^2 - YZc^{f_1}, XU - Y^2Z, Y^2Z - W^3. \end{cases}$$

75. $f_6 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XZ - Wc^{f_7}, XU - YWc^{f_5}, Y^2 - Wc^{f_{10}}, \\ YZ - Uc^{f_{11}}, YU - c^{p_3}, Z^2 - XSc^{f_{11}}, ZW - c^{p_4}, W^2 - Xc^{f_4}, \\ WU - Yc^{f_9}, U^2 - Zc^{f_{12}}, XYW - ZU. \end{cases}$$

76. $f_{10} = f_{11} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, XZ - Y^2c^{f_7}, XW - YZc^{f_7}, Y^3 - Uc^{f_6}, \\ Y^2Z - c^{p_3}, YU - Xc^{f_1}, ZW - Uc^{f_7}, ZU - Yc^{f_2}, W^2 - c^{p_5}, \\ WU - Zc^{f_3}, U^2 - Z^2c^{f_1}, XU - Y^2W, Z^2 - YW. \end{cases}$$

77. $f_7 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, XU - c^{p_2}, Y^2 - XZc^{f_{10}}, \\ YZ - XWc^{f_{11}}, YW - Uc^{f_{10}}, Z^2 - Uc^{f_{11}}, ZU - Xc^{f_2}, W^2 - c^{p_5}, \\ WU - Yc^{f_8}, U^2 - XZc^{f_8}, XZW - YU. \end{cases}$$

78. $f_1 = f_{12} = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$:
We have $A \cong R[X, Y, Z, W, U]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XY - U^2c^{f_{10}}, XZ - Uc^{f_6}, XW - c^{p_3}, Y^2 - c^{p_5}, \\ YW - XUc^{f_3}, YU - W^2c^{f_3}, Z^2 - XUc^{f_4}, ZW - Xc^{f_9}, ZU - Yc^{f_4}, \\ WU - Zc^{f_3}, U^3 - Wc^{f_2}, XU^2 - W^3, W^3 - YZ. \end{cases}$$

79. $f_4 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12$:
We have $A \cong R[X, Y, Z, W, U, V]/I$, where $I$ is an ideal generated by the

following elements:

$$\begin{cases} X^2 - Zc^{f_{10}}, XY - Wc^{f_{11}}, XZ - Vc^{f_6}, XW - c^{p_3}, XV - Z^2c^{f_1}, \\ Y^2 - Uc^{f_{11}}, YZ - c^{p_4}, YU - Z^2c^{f_2}, YV - Xc^{f_2}, Z^3 - Uc^{f_6}, \\ ZW - Xc^{f_9}, ZU - Yc^{f_9}, W^2 - Zc^{f_{12}}, WU - ZVc^{f_9}, WV - Zc^{f_3}, \\ U^2 - Zc^{f_8}, UV - Wc^{f_2}, V^2 - Uc^{f_1}, XU - YW, YW - Z^2V. \end{cases}$$

80. $f_1 = 0$ and $f_i \geq 1$ for $i = 2, 3, 4, 6, 7, 8, 9, 10, 11, 12$:
    We have $A \cong R[X, Y, Z, W, U, V]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Wc^{f_{10}}, XY - Uc^{f_{11}}, XZ - V^2c^{f_{10}}, XW - Vc^{f_6}, XU - c^{p_3}, \\ Y^2 - V^2c^{f_{11}}, YZ - Vc^{f_7}, YW - c^{p_4}, YV - Xc^{f_2}, Z^2 - c^{p_5}, \\ ZU - XVc^{f_3}, ZV - Yc^{f_3}, W^2 - XVc^{f_4}, WU - Xc^{f_9}, WV - Zc^{f_4}, \\ U^2 - Yc^{f_{12}}, UV - Wc^{f_3}, V^3 - Uc^{f_2}, XV^2 - YU, YU - ZW. \end{cases}$$

81. $f_8 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12$:
    We have $A \cong R[X, Y, Z, W, U, V]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Yc^{f_5}, XY - U^2c^{f_7}, XZ - Wc^{f_7}, XW - Vc^{f_5}, XU - c^{p_2}, \\ Y^2 - Uc^{f_{11}}, YZ - Vc^{f_7}, YU - Xc^{f_2}, YV - ZUc^{f_2}, Z^2 - Uc^{p_5}, \\ ZW - Xc^{f_3}, ZV - Yc^{f_3}, W^2 - Yc^{f_{12}}, WU - Zc^{f_9}, WV - U^2c^{f_3}, \\ U^3 - Yc^{f_9}, UV - Wc^{f_2}, V^2 - Uc^{f_1}, XV - YW, YW - ZU^2. \end{cases}$$

82. $f_5 = f_{12} = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7, 8, 9, 10, 11$:
    We have $A \cong R[X, Y, Z, W, U, V]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Wc^{f_7}, XY - Zc^{f_6}, XZ - Uc^{f_6}, XW - Vc^{f_6}, XV - c^{p_2}, \\ Y^2 - Wc^{f_{10}}, YW - XUc^{f_6}, YZ - Vc^{f_{10}}, YU - c^{p_3}, Z^2 - c^{p_5}, \\ ZU - Xc^{f_3}, ZV - Yc^{f_8}, W^2 - Xc^{f_4}, WU - Yc^{f_9}, WV - X^2c^{f_9}, \\ UV - Zc^{f_9}, V^2 - Wc^{f_8}, V^2 - Uc^{f_1}, X^2U - YV, YV - ZW, X^2 - U^2. \end{cases}$$

83. $f_5 = 0$ and $f_i \geq 1$ for $i = 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12$:
    We have $A \cong R[X, Y, Z, W, U, V]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^3 - Wc^{f_7}, XY - Zc^{f_6}, XZ - Uc^{f_7}, XW - Vc^{f_6}, XV - c^{p_2}, \\ Y^2 - Wc^{f_{10}}, YZ - Vc^{f_{10}}, YW - XUc^{f_6}, YU - c^{p_3}, Z^2 - c^{p_5}, \\ ZU - Xc^{f_3}, ZV - Yc^{f_8}, W^2 - Xc^{f_4}, WU - Yc^{f_9}, WV - X^2c^{f_9}, \\ U^2 - X^2c^{f_{12}}, UV - Zc^{f_9}, V^2 - Wc^{f_8}, X^2U - YV, YV - ZW. \end{cases}$$

84. $f_{10}=0$ and $f_i \geq 1$ for $i=1,2,3,4,5,6,7,8,9,11,12$:
We have $A \cong R[X,Y,Z,W,U,V]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, XZ - Y^2c^{f_7}, XW - Uc^{f_7}, XU - Vc^{f_5}, \\ Y^3 - Vc^{f_6}, YZ - Uc^{f_{11}}, YU - c^{p_3}, YV - Xc^{f_1}, Z^2 - YWc^{f_{11}}, \\ ZW - Vc^{f_7}, ZV - Yc^{f_2}, W^2 - c^{p_5}, WU - Xc^{f_3}, WV - Zc^{f_3}, \\ U^2 - Zc^{f_{12}}, UV - Y^2c^{f_3}, V^2 - YWc^{f_1}, XV - Y^2W, Y^2W - ZU. \end{cases}$$

85. $f_{11}=0$ and $f_i \geq 1$ for $i=1,2,3,4,5,6,7,8,9,10,12$:
We have $A \cong R[X,Y,Z,W,U,V]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, XZ - Uc^{f_7}, XW - YZc^{f_7}, XU - Z^2c^{f_6}, \\ Y^2 - Uc^{f_{10}}, YW - Z^2c^{f_{10}}, YU - Vc^{f_6}, YV - Xc^{f_1}, Z^3 - Xc^{f_2}, \\ ZW - Vc^{f_7}, ZU - c^{p_4}, ZV - Yc^{f_2}, W^2 - c^{p_5}, WV - Zc^{f_3}, \\ U^2 - Xc^{f_4}, UV - Wc^{f_4}, V^2 - Z^2c^{f_1}, XV - YZ^2, YZ^2 - WU. \end{cases}$$

86. $f_{12}=0$ and $f_i \geq 1$ for $i=1,2,3,4,5,6,7,8,9,10,11$:
We have $A \cong R[X,Y,Z,W,U,V,S]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - U^2c^{f_5}, XY - Zc^{f_6}, XZ - Uc^{f_7}, XW - Vc^{f_6}, XU - Sc^{f_5}, \\ XV - c^{p_2}, Y^2 - Wc^{f_{10}}, YW - Sc^{f_6}, YZ - Vc^{f_{10}}, YU - c^{p_3}, \\ YS - Xc^{f_1}, Z^2 - c^{p_5}, ZU - Xc^{f_3}, ZV - Yc^{f_8}, ZS - U^2c^{f_3}, W^2 - Xc^{f_4}, \\ WU - Yc^{f_9}, WX - U^2c^{f_9}, WS - Zc^{f_4}, UV - Zc^{f_9}, US - Wc^{f_3}, \\ V^2 - Wc^{f_8}, VS - Uc^{f_2}, S^2 - Vc^{f_1}, XS - YV, YV - ZW, ZW - U^3. \end{cases}$$

87. $f_i \geq 1$ for $i=1,2,3,4,5,6,7,8,9,10,11,12$:
We have $A \cong R[X,Y,Z,W,U,V,S,T]/I$, where $I$ is an ideal generated by the following elements:

$$\begin{cases} X^2 - Zc^{f_5}, XY - Wc^{f_6}, XZ - Uc^{f_7}, XW - Vc^{f_7}, XU - Sc^{f_6}, XV - Tc^{f_5}, \\ XS - c^{p_2}, Y^2 - Uc^{f_{10}}, YZ - Vc^{f_{11}}, YW - Sc^{f_{10}}, YU - Tc^{p_3}, YV - c^{p_3} \\ YT - Xc^{f_1}, Z^2 - Sc^{f_{11}}, ZW - Tc^{f_{11}}, ZU - c^{p_4}, ZS - Xc^{f_2}, ZT - Yc^{f_2}, \\ W^2 - c^{p_5}, WV - Xc^{f_3}, WS - Yc^{f_8}, WT - Zc^{f_3}, U^2 - Xc^{f_4}, UV - Yc^{f_9}, \\ US - Zc^{f_9}, UT - Wc^{f_4}, V^2 - Zc^{f_{12}}, VS - Wc^{f_4}, VT - Uc^{f_3}, S^2 - Uc^{f_8}, \\ ST - Vc^{f_2}, T^2 - Sc^{f_1}, XT - YS, YS - ZV, ZV - WU. \end{cases}$$

<u>The case of $n=11$.</u>

The integral matrix $(\omega(i, n-j-1))_{0 \le i,j \le 10}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_9 & f_{10} & f_{11} & f_{10} & f_9 & f_2 & 0 \\
p_3 & f_5 & p_3 & 0 & f_3 & f_{10} & f_{14} & f_{14} & f_{10} & f_3 & 0 \\
p_4 & f_6 & f_6 & p_4 & 0 & f_4 & f_{11} & f_{14} & f_{11} & f_4 & 0 \\
p_5 & f_7 & f_{12} & f_7 & p_5 & 0 & f_4 & f_{10} & f_{10} & f_4 & 0 \\
p_5 & f_8 & f_{13} & f_{13} & f_8 & p_5 & 0 & f_3 & f_9 & f_3 & 0 \\
p_4 & f_7 & f_{13} & f_{15} & f_{13} & f_7 & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_6 & f_{12} & f_{13} & f_{13} & f_{12} & f_6 & p_3 & 0 & f_1 & 0 \\
p_2 & f_5 & f_6 & f_7 & f_8 & f_7 & f_6 & f_5 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \ge 1$ and the $f_j \ge 0$, where

$$\begin{aligned}
&f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5,\\
&f_5 = -p_1 + p_2 + p_3, f_6 = -p_1 + p_3 + p_4, f_7 = -p_1 + p_4 + p_5, f_8 = -p_1 + p_5 + p_5,\\
&f_9 = p_1 + p_2 - p_3 - p_4, f_{10} = p_1 + p_2 - p_4 - p_5, f_{11} = p_1 + p_2 - p_5 - p_5,\\
&f_{12} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{13} = -p_1 - p_2 + p_4 + p_5 + p_5,\\
&f_{14} = p_1 + p_2 + p_3 - p_4 - p_5 - p_5,\\
&f_{15} = -p_1 - p_2 - p_3 + p_4 + p_4 + p_5 + p_5,
\end{aligned}$$

and the following cases are possible:

1. $f_1 = f_3 = f_8 = f_{12} = f_{13} = 0$ and $f_i \ge 1$ otherwise: Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, 2p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^4, X^2Y - c^p)$$

2. $f_3 = f_5 = f_8 = f_{12} = f_{14} = 0$ and $f_i \ge 1$ otherwise : Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5) = (4p, 3p, p, 4p, 2p)$ and
$$A \cong R[X, Y]/(X^4 - Y^7, XY - c^p)$$

3. $f_5 = f_6 = f_7 = f_{11} = f_{15} = 0$ and $f_i \ge 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (3p, p, 2p, p, 2p)$ and
$$A \cong R[X, Y]/(X^5 - Y^3, X^2Y - c^p)$$

4. $f_5 = f_7 = f_9 = f_{11} = f_{15} = 0$ and $f_i \ge 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5) = (5p, p, 4p, 2p, 3p)$ and
$$A \cong R[X, Y]/(X^6 - Y^5, XY - c^p)$$

5. $f_1 = f_4 = f_6 = f_{11} = f_{12} = f_{14} = 0$ and $f_i \ge 1$ otherwise : Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, 2p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^7 - Y^2, X^2Y - c^p)$$

6. $f_1 = f_4 = f_6 = f_{11} = f_{12} = f_{15} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5) = (3p, 3p, 2p, p, 3p)$ and
$$A \cong R[X,Y]/(X^3 - Y^8, XY - c^p)$$

7. $f_3 = f_5 = f_8 = f_9 = f_{10} = f_{14} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, p, p, 2p, p)$ and
$$A \cong R[X,Y]/(X^5 - Y^2, X^3Y - c^p)$$

8. $f_5 = f_6 = f_7 = f_8 = f_{12} = f_{13} = f_{15} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, p, p, p, p)$ and
$$A \cong R[X,Y]/(X^3 - Y^2, X^4Y - c^p)$$

9. $f_1 = f_2 = f_3 = f_8 = f_9 = f_{13} = f_{15} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5) = (2p, 2p, 2p, 2p, p)$ and
$$A \cong R[X,Y]/(X^9 - Y^2, XY - c^p)$$

The case of $n = 12$.
The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 11}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{10} & f_{11} & f_{12} & f_{12} & f_{11} & f_{10} & f_2 & 0 \\
p_3 & f_6 & p_3 & 0 & f_3 & f_{11} & f_{16} & f_{17} & f_{16} & f_{11} & f_3 & 0 \\
p_4 & f_7 & f_7 & p_4 & 0 & f_4 & f_{12} & f_{17} & f_{17} & f_{12} & f_4 & 0 \\
p_5 & f_8 & f_{13} & f_8 & p_5 & 0 & f_5 & f_{12} & f_{16} & f_{12} & f_5 & 0 \\
p_6 & f_9 & f_{14} & f_{14} & f_9 & p_6 & 0 & f_4 & f_{11} & f_{11} & f_4 & 0 \\
p_5 & f_9 & f_{15} & f_{18} & f_{15} & f_9 & p_5 & 0 & f_3 & f_{10} & f_3 & 0 \\
p_4 & f_8 & f_{14} & f_{18} & f_{18} & f_{14} & f_8 & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_7 & f_{13} & f_{14} & f_{15} & f_{14} & f_{13} & f_7 & p_3 & 0 & f_1 & 0 \\
p_2 & f_6 & f_7 & f_8 & f_9 & f_9 & f_8 & f_7 & f_6 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6,$
$f_6 = -p_1 + p_2 + p_3, f_7 = -p_1 + p_3 + p_4, f_8 = -p_1 + p_4 + p_5, f_9 = -p_1 + p_5 + p_6,$
$f_{10} = p_1 + p_2 - p_3 - p_4, f_{11} = p_1 + p_2 - p_4 - p_5, f_{12} = p_1 + p_2 - p_5 - p_6,$
$f_{13} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{14} = -p_1 - p_2 + p_4 + p_5 + p_6,$
$f_{15} = -p_1 - p_2 + p_5 + p_5 + p_6,$
$f_{16} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6,$
$f_{17} = p_1 + p_2 + p_3 - p_5 - p_5 - p_6,$
$f_{18} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_5 + p_6,$

and the following cases are possible:

1. $f_1 = f_2 = f_3 = f_4 = f_{10} = f_{11} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, p, p, p, p, q)$ and
$$A \cong R[X, Y]/(X^2 - c^q, Y^6 - Xc^{p-q})$$

2. $f_1 = f_2 = f_4 = f_5 = f_{12} = f_{17} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, p, p, q, p, p)$ and
$$A \cong R[X, Y]/(X^3 - c^q, Y^4 - Xc^{p-q})$$

3. $f_1 = f_3 = f_4 = f_{11} = f_{16} = f_{17} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, p, q, p, p, q)$ and
$$A \cong R[X, Y]/(X^4 - c^q, Y^3 - Xc^{p-q})$$

4. $f_1 = f_4 = f_7 = f_{13} = f_{14} = f_{17} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, p, q, p - q, p, q)$ and
$$A \cong R[X, Y]/(X^4 - c^q, Y^3 - c^{p-q})$$

5. $f_2 = f_4 = f_{10} = f_{11} = f_{12} = f_{17} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, q, p, q, p, q)$ and
$$A \cong R[X, Y]/(X^6 - c^q, Y^2 - Xc^{p-q})$$

6. $f_3 = f_6 = f_9 = f_{10} = f_{11} = f_{15} = f_{16} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, p, p, 2p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^3, X^3Y - c^p)$$

7. $f_3 = f_6 = f_9 = f_{11} = f_{15} = f_{16} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, q, p - q, p, q, p - q)$ and
$$A \cong R[X, Y]/(X^4 - c^{p-q}, Y^3 - X^3c^{2q-p})$$

8. $f_5 = f_7 = f_8 = f_{12} = f_{13} = f_{17} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, pq, q, p - q, q, p)$ and
$$A \cong R[X, Y]/(X^3 - c^{p-q}, Y^4 - X^2c^{2q-p})$$

9. $f_5 = f_6 = f_8 = f_{10} = f_{12} = f_{16} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (3p, p, 2p, 2p, p, 3p)$ and
$$A \cong R[X, Y]/(X^6 - Y^3, X^2Y - c^p)$$

10. $f_5 = f_6 = f_8 = f_{10} = f_{15} = f_{16} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (5p, 2p, 3p, 4p, p, 5p)$ and
$$A \cong R[X, Y]/(X^5 - Y^7, XY - c^p)$$

11. $f_6 = f_7 = f_8 = f_9 = f_{13} = f_{14} = f_{15} = f_{18} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, p, p, p, p, p)$ and

$$A \cong R[X,Y]/(X^3 - Y^2, X^3Y^2 - c^p)$$

12. $f_6 = f_7 = f_8 = f_9 = f_{14} = f_{18} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, q, p-q, q, p-q, q)$ and

$$A \cong R[X,Y]/(X^6 - c^q, Y^2 - X^3c^{p-2q})$$

13. $f_5 = f_6 = f_7 = f_8 = f_{12} = f_{13} = f_{16} = f_{17} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, p, p, p, p, 2p)$ and

$$A \cong R[X,Y]/(X^4 - Y^2, X^4Y - c^p)$$

14. $f_6 = f_7 = f_9 = f_{13} = f_{14} = f_{15} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (p, q, p-q, q, q, p-q)$ and

$$A \cong R[X,Y]/(X^4 - c^{p-q}, Y^3 - X^2c^{2q-p})$$

15. $f_6 = f_8 = f_9 = f_{10} = f_{15} = f_{18} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (3p, p, 2p, 2p, p, 2p)$ and

$$A \cong R[X,Y]/(X^4 - Y^4, X^2Y - c^p)$$

The case of $n = 13$.
The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 12}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{11} & f_{12} & f_{13} & f_{14} & f_{13} & f_{12} & f_{11} & f_2 & 0 \\
p_3 & f_6 & p_3 & 0 & f_3 & f_{12} & f_{18} & f_{19} & f_{19} & f_{18} & f_{12} & f_3 & 0 \\
p_4 & f_7 & f_7 & p_4 & 0 & f_4 & f_{13} & f_{19} & f_{22} & f_{19} & f_{13} & f_4 & 0 \\
p_5 & f_8 & f_{15} & f_8 & p_5 & 0 & f_5 & f_{14} & f_{19} & f_{19} & f_{14} & f_5 & 0 \\
p_6 & f_9 & f_{16} & f_{16} & f_9 & p_6 & 0 & f_5 & f_{13} & f_{18} & f_{13} & f_5 & 0 \\
p_6 & f_{10} & f_{17} & f_{20} & f_{17} & f_{10} & p_6 & 0 & f_4 & f_{12} & f_{12} & f_4 & 0 \\
p_5 & f_9 & f_{17} & f_{21} & f_{21} & f_{17} & f_9 & p_5 & 0 & f_3 & f_{11} & f_3 & 0 \\
p_4 & f_8 & f_{16} & f_{20} & f_{21} & f_{20} & f_{16} & f_8 & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_7 & f_{15} & f_{16} & f_{17} & f_{17} & f_{16} & f_{15} & f_7 & p_3 & 0 & f_1 & 0 \\
p_2 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_9 & f_8 & f_7 & f_6 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$$
\begin{aligned}
&f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6,\\
&f_6 = -p_1 + p_2 + p_3, f_7 = -p_1 + p_3 + p_4, f_8 = -p_1 + p_4 + p_5,\\
&f_9 = -p_1 + p_5 + p_6, f_{10} = -p_1 + p_6 + p_6,\\
&f_{11} = p_1 + p_2 - p_3 - p_4, f_{12} = p_1 + p_2 - p_4 - p_5,\\
&f_{13} = p_1 + p_2 - p_5 - p_6, f_{14} = p_1 + p_2 - p_6 - p_6,\\
&f_{15} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{16} = -p_1 - p_2 + p_4 + p_5 + p_6,\\
&f_{17} = -p_1 - p_2 + p_5 + p_6 + p_6,\\
&f_{18} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{19} = p_1 + p_2 + p_3 - p_5 - p_6 - p_6,\\
&f_{20} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_6, f_{21} = -p_1 - p_2 - p_3 + p_5 + p_5 + p_6 + p_6,\\
&f_{22} = p_1 + p_2 + p_3 + p_4 - p_5 - p_5 - p_6 - p_6,
\end{aligned}
$$

and the following cases are possible:

1. $f_2 = f_8 = f_{11} = f_{13} = f_{14} = f_{20} = f_{22} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (3p, p, 3p, p, 2p, 2p)$ and
$$A \cong R[X,Y]/(X^5 - Y^4, X^2Y - c^p)$$

2. $f_1 = f_3 = f_4 = f_{10} = f_{12} = f_{17} = f_{18} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, 2p, p, 2p, 2p, p)$ and
$$A \cong R[X,Y]/(X^3 - Y^5, X^2Y - c^p)$$

3. $f_6 = f_8 = f_{10} = f_{11} = f_{13} = f_{20} = f_{22} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (6p, p, 5p, 2p, 4p, 3p)$ and
$$A \cong R[X,Y]/(X^7 - Y^6, XY - c^p)$$

4. $f_3 = f_6 = f_9 = f_{15} = f_{17} = f_{18} = f_{21} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (3p, 2p, p, 3p, p, 2p)$ and
$$A \cong R[X,Y]/(X^7 - Y^3, X^2Y - c^p)$$

5. $f_3 = f_6 = f_9 = f_{14} = f_{15} = f_{18} = f_{21} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (5p, 3p, 2p, 5p, p, 4p)$ and
$$A \cong R[X,Y]/(X^8 - Y^5, XY - c^p)$$

6. $f_2 = f_4 = f_{10} = f_{11} = f_{12} = f_{13} = f_{20} = f_{22} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, p, 2p, p, 2p, p)$ and
$$A \cong R[X,Y]/(X^4 - Y^3, X^3Y - c^p)$$

7. $f_1 = f_4 = f_7 = f_{10} = f_{15} = f_{17} = f_{18} = f_{22} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (4p, 4p, p, 3p, 4p, 2p)$ and
$$A \cong R[X,Y]/(X^9 - Y^4, XY - c^p)$$

8. $f_1 = f_2 = f_5 = f_8 = f_{14} = f_{15} = f_{16} = f_{20} = f_{21} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, 2p, 2p, p, p, 2p)$ and
$$A \cong R[X,Y]/(X^9 - Y^2, X^2Y - c^p)$$

9. $f_1 = f_2 = f_3 = f_8 = f_{14} = f_{15} = f_{16} = f_{20} = f_{22} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (3p, 3p, 3p, p, 2p, 3p)$ and
$$A \cong R[X,Y]/(X^1 0 - Y^3, XY - c^p)$$

10. $f_3 = f_6 = f_9 = f_{10} = f_{11} = f_{12} = f_{17} = f_{18} = f_{21} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, p, p, 2p, p, p)$ and
$$A \cong R[X,Y]/(X^5 - Y^2, X^4Y - c^p)$$

11. $f_1 = f_4 = f_7 = f_{10} = f_{15} = f_{16} = f_{17} = f_{20} = f_{22} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, 2p, p, p, 2p, p)$ and
$$A \cong R[X,Y]/(X^7 - Y^2, X^3Y - c^p)$$

12. $f_1 = f_2 = f_3 = f_4 = f_{10} = f_{11} = f_{12} = f_{17} = f_{20} = f_{21} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, 2p, 2p, 2p, 2p, p)$ and
$$A \cong R[X,Y]/(X^{11} - Y^2, XY - c^p)$$

13. $f_6 = f_7 = f_8 = f_9 = f_{10} = f_{15} = f_{16} = f_{17} = f_{20} = f_{21} = 0$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6) = (2p, p, p, p, p, p)$ and
$$A \cong R[X,Y]/(X^3 - Y^2, X^5Y - c^p)$$

<u>The case of $n = 14$.</u>

The integral matrix $(\omega(i, n-j-1))_{0 \le i,j \le 13}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{12} & f_{13} & f_{14} & f_{15} & f_{15} & f_{14} & f_{13} & f_{12} & f_2 & 0 \\
p_3 & f_7 & p_3 & 0 & f_3 & f_{13} & f_{20} & f_{21} & f_{22} & f_{21} & f_{20} & f_{13} & f_3 & 0 \\
p_4 & f_8 & f_8 & p_4 & 0 & f_4 & f_{14} & f_{21} & f_{25} & f_{25} & f_{21} & f_{14} & f_4 & 0 \\
p_5 & f_9 & f_{16} & f_9 & p_5 & 0 & f_5 & f_{15} & f_{22} & f_{25} & f_{22} & f_{15} & f_5 & 0 \\
p_6 & f_{10} & f_{17} & f_{17} & f_{10} & p_6 & 0 & f_6 & f_{15} & f_{21} & f_{21} & f_{15} & f_6 & 0 \\
p_7 & f_{11} & f_{18} & f_{23} & f_{18} & f_{11} & p_7 & 0 & f_5 & f_{14} & f_{20} & f_{14} & f_5 & 0 \\
p_6 & f_{11} & f_{19} & f_{24} & f_{24} & f_{19} & f_{11} & p_6 & 0 & f_4 & f_{13} & f_{13} & f_4 & 0 \\
p_5 & f_{10} & f_{18} & f_{24} & f_{26} & f_{24} & f_{18} & f_{10} & p_5 & 0 & f_3 & f_{12} & f_3 & 0 \\
p_4 & f_9 & f_{17} & f_{23} & f_{24} & f_{24} & f_{23} & f_{17} & f_9 & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_8 & f_{16} & f_{17} & f_{18} & f_{19} & f_{18} & f_{17} & f_{16} & f_8 & p_3 & 0 & f_1 & 0 \\
p_2 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{11} & f_{10} & f_9 & f_8 & f_7 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \ge 1$ and the $f_j \ge 0$, where

$$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6, f_6 = p_1 - p_7,$$
$$f_7 = -p_1 + p_2 + p_3, f_8 = -p_1 + p_3 + p_4, f_9 = -p_1 + p_4 + p_5,$$
$$f_{10} = -p_1 + p_5 + p_6, f_{11} = -p_1 + p_6 + p_7,$$
$$f_{12} = p_1 + p_2 - p_3 - p_4, f_{13} = p_1 + p_2 - p_4 - p_5, f_{14} = p_1 + p_2 - p_5 - p_6,$$
$$f_{15} = p_1 + p_2 - p_6 - p_7,$$
$$f_{16} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{17} = -p_1 - p_2 + p_4 + p_5 + p_6,$$
$$f_{18} = -p_1 - p_2 + p_5 + p_6 + p_7, f_{19} = -p_1 - p_2 + p_6 + p_6 + p_7$$
$$f_{20} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{21} = p_1 + p_2 + p_3 - p_5 - p_6 - p_7,$$
$$f_{22} = p_1 + p_2 + p_3 - p_6 - p_6 - p_7, f_{23} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_7,$$
$$f_{24} = -p_1 - p_2 - p_3 + p_5 + p_6 + p_6 + p_7,$$
$$f_{25} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_6 - p_7,$$
$$f_{26} = -p_1 - p_2 - p_3 - p_4 + p_5 + p_5 + p_6 + p_6 + p_7,$$

and the following cases are possible:

1. $f_i = 0$ for $i = 6, 7, 8, 10, 16, 17, 19, 21$ and $f_i \ge 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (3p, 2p, p, 2p, 2p, p, 3p)$ and
$$A \cong R[X, Y]/(X^4 - Y^5, X^4 Y - c^p)$$

2. $f_i = 0$ for $i = 1, 2, 3, 4, 5, 12, 13, 14, 20$ and $f_i \ge 1$ otherwise : Setting $p = p_1$ and $q = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, p, p, p, p, p, q)$ and
$$A \cong R[X, Y]/(X^2 - c^q, Y^7 - X c^{p-q})$$

3. $f_i = 0$ for $i = 2, 4, 11, 12, 13, 14, 19, 23, 24$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p, q, p, q, p - q)$ and
$$A \cong R[X, Y]/(X^7 - c^q, Y^2 - c^{p-q})$$

4. $f_i = 0$ for $i = 2, 4, 6, 12, 13, 14, 15, 21, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p, q, p, q, p)$ and
$$A \cong R[X, Y]/(X^7 - c^q, Y^2 - Xc^{p-q})$$

5. $f_i = 0$ for $i = 2, 5, 9, 12, 14, 15, 22, 23, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, 2p, p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^4 - Y^3, X^2Y^2 - c^p)$$

6. $f_i = 0$ for $i = 2, 6, 9, 10, 12, 15, 17, 23, 24$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p, q, p - q, q, p)$ and
$$A \cong R[X, Y]/(X^7 - c^q, Y^2 - X^6c^{p-2q})$$

7. $f_i = 0$ for $i = 4, 7, 8, 11, 13, 14, 19, 21, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p - q, q, p, q, p - q)$ and
$$A \cong R[X, Y]/(X^7 - c^q, Y^2 - X^5c^{p-2q})$$

8. $f_i = 0$ for $i = 6, 7, 8, 9, 10, 15, 17, 21, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p - q, q, p - q, q, p)$ and
$$A \cong R[X, Y]/(X^7 - c^q, Y^2 - X^4c^{p-2q})$$

9. $f_i = 0$ for $i = 7, 8, 9, 10, 11, 17, 19, 23, 24$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p-q, q, p-q, q, p-q)$ and
$$A \cong R[X, Y]/(X^7 - c^q, Y^2 - X^3c^{p-2q})$$

10. $f_i = 0$ for $i = 3, 6, 7, 10, 13, 19, 20, 21, 26$ and $f_i \geq 1$ otherwise : Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (3p, 2p, p, 3p, 2p, p, 3p)$ and
$$A \cong R[X, Y]/(X^8 - Y^3, X^2Y - c^p)$$

11. $f_i = 0$ for $i = 3, 6, 7, 10, 16, 19, 20, 21, 26$ and $f_i \geq 1$ otherwise : Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (5p, 4p, p, 5p, 3p, 2p, 5p)$ and
$$A \cong R[X, Y]/(X^5 - Y^9, XY - c^p)$$

12. $f_i = 0$ for $i = 2, 6, 9, 10, 12, 15, 17, 23, 24, 26$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, 2p, p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^2 - Y^6, X^2Y - c^p)$$

13. $f_i = 0$ for $i = 4, 7, 8, 11, 13, 14, 19, 20, 21, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, p, p, 2p, p, p)$ and

$$A \cong R[X, Y]/(X^5 - Y^2, X^2Y^2 - c^p)$$

14. $f_i = 0$ for $i = 1, 2, 5, 6, 9, 15, 16, 17, 22, 23, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, 2p, 2p, p, p, 2p, 2p)$ and

$$A \cong R[X, Y]/(X^{10} - Y^2, X^2Y - c^p)$$

15. $f_i = 0$ for $i = 1, 2, 5, 6, 9, 15, 16, 17, 22, 23, 26$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (3p, 3p, 3p, 2p, p, 3p, 3p)$ and

$$A \cong R[X, Y]/(X^3 - Y^{11}, XY - c^p)$$

16. $f_i = 0$ for $i = 3, 6, 7, 10, 12, 13, 15, 20, 21, 22, 26$ and $f_i \geq 1$ otherwise : Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, p, 2p, p, p, 2p)$ and

$$A \cong R[X, Y]/(X^6 - Y^2, X^4Y - c^p)$$

17. $f_i = 0$ for $i = 6, 7, 8, 9, 10, 15, 16, 17, 21, 22, 25$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, p, p, p, p, 2p)$ and

$$A \cong R[X, Y]/(X^4 - Y^2, X^3Y^2 - c^p)$$

18. $f_i = 0$ for $i = 7, 8, 9, 10, 11, 16, 17, 18, 19, 23, 24, 26$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, p, p, p, p, p)$ and

$$A \cong R[X, Y]/(X^2 - Y^3, X^4Y - c^p)$$

<u>The case of $n = 15$.</u>
The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 14}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{16} & f_{15} & f_{14} & f_{13} & f_2 & 0 \\
p_3 & f_7 & p_3 & 0 & f_3 & f_{14} & f_{22} & f_{23} & f_{24} & f_{24} & f_{23} & f_{22} & f_{14} & f_3 & 0 \\
p_4 & f_8 & f_8 & p_4 & 0 & f_4 & f_{15} & f_{23} & f_{28} & f_{29} & f_{28} & f_{23} & f_{15} & f_4 & 0 \\
p_5 & f_9 & f_{18} & f_9 & p_5 & 0 & f_5 & f_{16} & f_{24} & f_{29} & f_{29} & f_{24} & f_{16} & f_5 & 0 \\
p_6 & f_{10} & f_{19} & f_{19} & f_{10} & p_6 & 0 & f_6 & f_{17} & f_{24} & f_{28} & f_{24} & f_{17} & f_6 & 0 \\
p_7 & f_{11} & f_{20} & f_{25} & f_{20} & f_{11} & p_7 & 0 & f_6 & f_{16} & f_{23} & f_{23} & f_{16} & f_6 & 0 \\
p_7 & f_{12} & f_{21} & f_{26} & f_{26} & f_{21} & f_{12} & p_7 & 0 & f_5 & f_{15} & f_{22} & f_{15} & f_5 & 0 \\
p_6 & f_{11} & f_{21} & f_{27} & f_{30} & f_{27} & f_{21} & f_{11} & p_6 & 0 & f_4 & f_{14} & f_{14} & f_4 & 0 \\
p_5 & f_{10} & f_{20} & f_{26} & f_{30} & f_{30} & f_{26} & f_{20} & f_{10} & p_5 & 0 & f_3 & f_{13} & f_3 & 0 \\
p_4 & f_9 & f_{19} & f_{25} & f_{26} & f_{27} & f_{26} & f_{25} & f_{19} & f_9 & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_8 & f_{18} & f_{19} & f_{20} & f_{21} & f_{21} & f_{20} & f_{19} & f_{18} & f_8 & p_3 & 0 & f_1 & 0 \\
p_2 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} & f_{11} & f_{10} & f_9 & f_8 & f_7 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6, f_6 = p_1 - p_7,$$
$$f_7 = -p_1 + p_2 + p_3, f_8 = -p_1 + p_3 + p_4, f_9 = -p_1 + p_4 + p_5,$$
$$f_{10} = -p_1 + p_5 + p_6, f_{11} = -p_1 + p_6 + p_7, f_{12} = -p_1 + p_7 + p_7,$$
$$f_{13} = p_1 + p_2 - p_3 - p_4, f_{14} = p_1 + p_2 - p_4 - p_5, f_{15} = p_1 + p_2 - p_5 - p_6,$$
$$f_{16} = p_1 + p_2 - p_6 - p_7, f_{17} = p_1 + p_2 - p_7 - p_7,$$
$$f_{18} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{19} = -p_1 - p_2 + p_4 + p_5 + p_6,$$
$$f_{20} = -p_1 - p_2 + p_5 + p_6 + p_7, f_{21} = -p_1 - p_2 + p_6 + p_7 + p_7$$
$$f_{22} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{23} = p_1 + p_2 + p_3 - p_5 - p_6 - p_7,$$
$$f_{24} = p_1 + p_2 + p_3 - p_6 - p_7 - p_7,$$
$$f_{25} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_7,$$
$$f_{26} = -p_1 - p_2 - p_3 + p_5 + p_6 + p_7 + p_7,$$
$$f_{27} = -p_1 - p_2 - p_3 + p_6 + p_6 + p_7 + p_7,$$
$$f_{28} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_7,$$
$$f_{29} = p_1 + p_2 + p_3 + p_4 - p_6 - p_6 - p_7 - p_7,$$
$$f_{30} = -p_1 - p_2 - p_3 - p_4 + p_5 + p_6 + p_6 + p_7 + p_7,$$

and the following cases are possible:

1. $f_i = 0$ for $i = 7, 8, 9, 11, 12, 15, 25, 27, 28$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (4p, p, 3p, p, 3p, 2p, 2p)$ and
$$A \cong R[X, Y]/(X^7 - Y^4, X^2Y - c^p)$$

2. $f_i = 0$ for $i = 7, 9, 11, 13, 15, 17, 25, 27, 28$ and $f_i \geq 1$ otherwise : Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (7p, p, 6p, 2p, 5p, 3p, 4p)$ and
$$A \cong R[X, Y]/(X^8 - Y^7, XY - c^p)$$

3. $f_i = 0$ for $i = 1, 2, 4, 5, 12, 15, 21, 25, 26, 27$ and $f_i \geq 1$ otherwise : Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, 2p, 2p, p, 2p, 2p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^6, X^2Y - c^p)$$

4. $f_i = 0$ for $i = 1, 2, 3, 5, 6, 13, 16, 17, 24, 29$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, p, p, p, q, p, p)$ and
$$A \cong R[X, Y]/(X^3 - c^q, Y^5 - Xc^{p-q})$$

5. $f_i = 0$ for $i = 1, 3, 4, 6, 14, 17, 22, 23, 24, 28$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, p, q, p, p, q, p)$ and
$$A \cong R[X, Y]/(X^5 - c^q, Y^3 - Xc^{p-q})$$

6. $f_i = 0$ for $i = 1, 3, 6, 10, 17, 18, 19, 20, 24, 30$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, p, q, p, p-q, q, p)$ and
$$A \cong R[X, Y]/(X^5 - c^q, Y^3 - c^{p-q})$$

7. $f_i = 0$ for $i = 4, 7, 8, 11, 14, 21, 22, 23, 27, 28$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p-q, q, p, p-q, q)$ and
$$A \cong R[X, Y]/(X^5 - c^{p-q}, Y^3 - X^3 c^{2q-p})$$

8. $f_i = 0$ for $i = 7, 8, 10, 11, 18, 19, 20, 21, 27, 30$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (p, q, p-q, q, q, p-q, q)$ and
$$A \cong R[X, Y]/(X^5 - c^{p-q}, Y^3 - X^2 c^{2q-p})$$

9. $f_i = 0$ for $i = 7, 8, 9, 10, 11, 17, 19, 25, 27, 28$ and $f_i \geq 1$ otherwise : Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (3p, p, 2p, p, 2p, p, 2p)$ and
$$A \cong R[X, Y]/(X^6 - Y^3, X^3 Y - c^p)$$

10. $f_i = 0$ for $i = 1, 4, 5, 8, 12, 15, 18, 21, 25, 28, 29$ and $f_i \geq 1$ otherwise : Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (4p, 4p, 3p, p, 4p, 4p, 2p)$ and
$$A \cong R[X, Y]/(X^{11} - Y^4, XY - c^p)$$

11. $f_i = 0$ for $i = 1, 4, 5, 8, 12, 15, 18, 21, 22, 23, 28, 29$ and $f_i \geq 1$ otherwise : Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, 2p, p, p, 2p, 2p, p)$ and
$$A \cong R[X, Y]/(X^9 - Y^2, X^3 Y - c^p)$$

12. $f_i = 0$ for $i = 4, 7, 8, 11, 12, 14, 15, 21, 22, 23, 27, 28$ and $f_i \geq 1$ otherwise : Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, p, p, 2p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^3, X^2 Y^3 - c^p)$$

13. $f_i = 0$ for $i = 1, 2, 3, 4, 5, 12, 13, 14, 15, 21, 22, 26, 27, 30$ and $f_i \geq 1$ otherwise : Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, 2p, 2p, 2p, 2p, 2p, p)$ and
$$A \cong R[X, Y]/(X^{13} - Y^2, XY - c^p)$$

14. $f_i = 0$ for $i = 7, 8, 9, 10, 12, 18, 19, 20, 21, 25, 26, 27, 30$ and $f_i \geq 1$ otherwise : Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (2p, p, p, p, p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^2, X^6 Y - c^p)$$

The case of $n=16$.

The integral matrix $(\omega(i,n-j-1))_{0\le i,j\le 15}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0\\
p_2 & p_2 & 0 & f_2 & f_{14} & f_{15} & f_{16} & f_{17} & f_{18} & f_{18} & f_{17} & f_{16} & f_{15} & f_{14} & f_2 & 0\\
p_3 & f_8 & p_3 & 0 & f_3 & f_{15} & f_{24} & f_{25} & f_{26} & f_{27} & f_{26} & f_{25} & f_{24} & f_{15} & f_3 & 0\\
p_4 & f_9 & f_9 & p_4 & 0 & f_4 & f_{16} & f_{25} & f_{31} & f_{32} & f_{32} & f_{31} & f_{25} & f_{16} & f_4 & 0\\
p_5 & f_{10} & f_{19} & f_{10} & p_5 & 0 & f_5 & f_{17} & f_{26} & f_{32} & f_{35} & f_{32} & f_{26} & f_{17} & f_5 & 0\\
p_6 & f_{11} & f_{20} & f_{20} & f_{11} & p_6 & 0 & f_6 & f_{18} & f_{27} & f_{32} & f_{32} & f_{27} & f_{18} & f_6 & 0\\
p_7 & f_{12} & f_{21} & f_{28} & f_{21} & f_{12} & p_7 & 0 & f_7 & f_{18} & f_{26} & f_{31} & f_{26} & f_{18} & f_7 & 0\\
p_8 & f_{13} & f_{22} & f_{29} & f_{29} & f_{22} & f_{13} & p_8 & 0 & f_6 & f_{17} & f_{25} & f_{25} & f_{17} & f_6 & 0\\
p_7 & f_{13} & f_{23} & f_{30} & f_{33} & f_{30} & f_{23} & f_{13} & p_7 & 0 & f_5 & f_{16} & f_{24} & f_{16} & f_5 & 0\\
p_6 & f_{12} & f_{22} & f_{30} & f_{34} & f_{34} & f_{30} & f_{22} & f_{12} & p_6 & 0 & f_4 & f_{15} & f_{15} & f_4 & 0\\
p_5 & f_{11} & f_{21} & f_{29} & f_{33} & f_{34} & f_{33} & f_{29} & f_{21} & f_{11} & p_5 & 0 & f_3 & f_{14} & f_3 & 0\\
p_4 & f_{10} & f_{20} & f_{28} & f_{29} & f_{30} & f_{30} & f_{29} & f_{28} & f_{20} & f_{10} & p_4 & 0 & f_2 & f_2 & 0\\
p_3 & f_9 & f_{19} & f_{20} & f_{21} & f_{22} & f_{23} & f_{22} & f_{21} & f_{20} & f_{19} & f_9 & p_3 & 0 & f_1 & 0\\
p_2 & f_8 & f_9 & f_{10} & f_{11} & f_{12} & f_{13} & f_{13} & f_{12} & f_{11} & f_{10} & f_9 & f_8 & p_2 & 0 & 0\\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i\ge 1$ and the $f_j\ge 0$, where

$f_1=p_1-p_2, f_2=p_1-p_3, f_3=p_1-p_4, f_4=p_1-p_5, f_5=p_1-p_6, f_6=p_1-p_7,$
$f_7=p_1-p_8,$
$f_8=-p_1+p_2+p_3, f_9=-p_1+p_3+p_4, f_{10}=-p_1+p_4+p_5, f_{11}=-p_1+p_5+p_6,$
$f_{12}=-p_1+p_6+p_7, f_{13}=-p_1+p_7+p_8,$
$f_{14}=p_1+p_2-p_3-p_4, f_{15}=p_1+p_2-p_4-p_5, f_{16}=p_1+p_2-p_5-p_6,$
$f_{17}=p_1+p_2-p_6-p_7, f_{18}=p_1+p_2-p_7-p_8,$
$f_{19}=-p_1-p_2+p_3+p_4+p_5, f_{20}=-p_1-p_2+p_4+p_5+p_6,$
$f_{21}=-p_1-p_2+p_5+p_6+p_7, f_{22}=-p_1-p_2+p_6+p_7+p_8,$
$f_{23}=-p_1-p_2+p_7+p_7+p_8,$
$f_{24}=p_1+p_2+p_3-p_4-p_5-p_6, f_{25}=p_1+p_2+p_3-p_5-p_6-p_7,$
$f_{26}=p_1+p_2+p_3-p_6-p_7-p_8,$
$f_{27}=p_1+p_2+p_3-p_7-p_7-p_8,$
$f_{28}=-p_1-p_2-p_3+p_4+p_5+p_6+p_7,$
$f_{29}=-p_1-p_2-p_3+p_5+p_6+p_7+p_8,$
$f_{30}=-p_1-p_2-p_3+p_6+p_7+p_7+p_8,$
$f_{31}=p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8,$
$f_{32}=p_1+p_2+p_3+p_4-p_6-p_7-p_7-p_8,$
$f_{33}=-p_1-p_2-p_3-p_4+p_5+p_6+p_7+p_7+p_8,$
$f_{34}=-p_1-p_2-p_3-p_4+p_6+p_6+p_7+p_7+p_8,$
$f_{35}=p_1+p_2+p_3+p_4+p_5-p_6-p_6-p_7-p_7-p_8,$

and the following cases are possible:

1. $f_i = 0$ for $i = 8, 10, 12, 13, 14, 16, 23, 28, 30, 33$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (4p, p, 3p, 2p, 2p, 3p, p, 3p)$ and
$$A \cong R[X, Y]/(X^6 - Y^5, X^2Y - c^p)$$

2. $f_i = 0$ for $i = 4, 8, 9, 12, 15, 22, 24, 25, 27, 31, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, 2p, p, 2p, 3p, p, 2p, 2p)$ and
$$A \cong R[X, Y]/(X^4 - Y^6, X^2Y - c^p)$$

3. $f_i = 0$ for $i = 1, 2, 4, 5, 6, 16, 17, 25, 31, 32, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, p, p, q, p, p, p, q)$ and
$$A \cong R[X, Y]/(X^4 - c^q, Y^4 - Xc^{p-q})$$

4. $f_i = 0$ for $i = 2, 5, 10, 13, 16, 17, 23, 28, 29, 30, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, q, p, p - q, q, p, q, p - q)$ and
$$A \cong R[X, Y]/(X^4 - c^{p-q}, Y^4 - X^3c^{2q-p})$$

5. $f_i = 0$ for $i = 7, 8, 10, 12, 14, 16, 18, 26, 28, 31, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_2$ , we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (4p, p, 3p, 2p, 2p, 3p, p, 4p)$ and
$$A \cong R[X, Y]/(X^8 - Y^4, X^2Y - c^p)$$

6. $f_i = 0$ for $i = 7, 8, 10, 12, 14, 16, 23, 26, 28, 33, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (7p, 2p, 5p, 4p, 3p, 6p, p, 7p)$ and
$$A \cong R[X, Y]/(X^9 - Y^7, XY - c^p)$$

7. $f_i = 0$ for $i = 2, 6, 10, 11, 14, 17, 18, 20, 27, 28, 29, 32, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, 2p, p, p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^2 - Y^7, X^2Y - c^p)$$

8. $f_i = 0$ for $i = 2, 5, 10, 13, 14, 16, 17, 23, 28, 29, 30, 33, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, 2p, p, p, 2p, p, p)$ and
$$A \cong R[X, Y]/(X^4 - Y^3, X^4Y - c^p)$$

9. $f_i = 0$ for $i = 4, 8, 9, 12, 13, 15, 16, 22, 23, 24, 25, 30, 31, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, p, p, 2p, p, p, p)$ and
$$A \cong R[X, Y]/(X^5 - Y^2, X^3Y^2 - c^p)$$

10. $f_i = 0$ for $i = 1, 4, 7, 9, 12, 19, 20, 21, 22, 23, 28, 30, 31, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, p, p, 2p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^8 - Y^2, X^4Y - c^p)$$

11. $f_i = 0$ for $i = 1, 2, 3, 6, 7, 11, 14, 18, 20, 21, 27, 28, 29, 33, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, 2p, 2p, p, p, 2p, 2p)$ and
$$A \cong R[X, Y]/(X^{12} - Y^2, X^2Y - c^p)$$

12. $f_i = 0$ for $i = 1, 2, 3, 6, 7, 11, 14, 18, 20, 21, 27, 28, 29, 33, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, 3p, 3p, 3p, p, 2p, 3p, 3p)$ and
$$A \cong R[X, Y]/(X^{13} - Y^3, XY - c^p)$$

13. $f_i = 0$ for $i = 8, 9, 10, 11, 12, 13, 19, 20, 21, 22, 23, 28, 29, 30, 33, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, p, p, p, p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^2, X^2Y^4 - c^p)$$

14. $f_i = 0$ for $i = 7, 8, 9, 10, 11, 12, 18, 19, 20, 21, 26, 27, 28, 31, 32, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, p, p, p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^4 - Y^2, X^6Y - c^p)$$

15. $f_i = 0$ for $i = 1, 2, 3, 4, 5, 6, 14, 15, 16, 17, 24, 25$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_8$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, p, p, p, p, p, p, q)$ and
$$A \cong R[X, Y]/(X^2 - c^q, Y^8 - Xc^{p-q})$$

16. $f_i = 0$ for $i = 2, 4, 6, 14, 15, 16, 17, 18, 25, 27, 31, 32$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, q, p, q, p, q, p, q)$ and
$$A \cong R[X, Y]/(X^8 - c^q, Y^2 - Xc^{p-q})$$

17. $f_i = 0$ for $i = 2, 6, 10, 11, 14, 17, 18, 20, 27, 28, 29, 32$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, q, p, q, p - q, q, p, q)$ and
$$A \cong R[X, Y]/(X^8 - c^q, Y^2 - X^7c^{p-2q})$$

18. $f_i = 0$ for $i = 4, 8, 9, 12, 13, 15, 16, 22, 25, 30, 31, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, q, p - q, q, p, q, p - q, q)$ and
$$A \cong R[X, Y]/(X^8 - c^q, Y^2 - X^5c^{p-2q})$$

19. $f_i = 0$ for $i = 1, 3, 6, 11, 19, 20, 21, 22, 27, 29, 33, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, p, 2p, p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^5, X^2Y^2 - c^p)$$

20. $f_i = 0$ for $i = 1, 4, 7, 9, 12, 19, 20, 21, 22, 27, 31, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, 3p, p, 2p, 3p, p, 2p, 3p)$ and
$$A \cong R[X, Y]/(X^{10} - Y^3, X^2Y - c^p)$$

21. $f_i = 0$ for $i = 1, 4, 7, 9, 12, 19, 21, 22, 24, 27, 31, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (5p, 5p, p, 4p, 5p, 2p, 3p, 5p)$ and
$$A \cong R[X, Y]/(X^{11} - Y^5, XY - c^p)$$

22. $f_i = 0$ for $i = 7, 8, 10, 11, 12, 14, 18, 21, 26, 28, 33, 35$ and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, p, 2p, 2p, p, 2p, p, 3p)$ and
$$A \cong R[X, Y]/(X^7 - Y^3, X^3Y - c^p)$$

23. $f_i = 0$ for $i = 8, 9, 10, 11, 12, 13, 20, 22, 28, 29, 30, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (p, q, p - q, q, p - q, q, p - q, q)$ and
$$A \cong R[X, Y]/(X^8 - c^q, Y^2 - X^3c^{p-2q})$$

<u>The case of $n = 17$.</u>
The integral matrix $(\omega(i, n - j - 1))_{0 \leq i,j \leq 16}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{15} & f_{16} & f_{17} & f_{18} & f_{19} & f_{20} & f_{19} & f_{18} & f_{17} & f_{16} & f_{15} & f_2 & 0 \\
p_3 & f_8 & p_3 & 0 & f_3 & f_{16} & f_{26} & f_{27} & f_{28} & f_{29} & f_{29} & f_{28} & f_{27} & f_{26} & f_{16} & f_3 & 0 \\
p_4 & f_9 & f_9 & p_4 & 0 & f_4 & f_{17} & f_{27} & f_{34} & f_{35} & f_{36} & f_{35} & f_{34} & f_{27} & f_{17} & f_4 & 0 \\
p_5 & f_{10} & f_{21} & f_{10} & p_5 & 0 & f_5 & f_{18} & f_{28} & f_{35} & f_{39} & f_{39} & f_{35} & f_{28} & f_{18} & f_5 & 0 \\
p_6 & f_{11} & f_{22} & f_{22} & f_{11} & p_6 & 0 & f_6 & f_{19} & f_{29} & f_{36} & f_{39} & f_{36} & f_{29} & f_{19} & f_6 & 0 \\
p_7 & f_{12} & f_{23} & f_{30} & f_{23} & f_{12} & p_7 & 0 & f_7 & f_{20} & f_{29} & f_{35} & f_{35} & f_{29} & f_{20} & f_7 & 0 \\
p_8 & f_{13} & f_{24} & f_{31} & f_{31} & f_{24} & f_{13} & p_8 & 0 & f_7 & f_{19} & f_{28} & f_{34} & f_{28} & f_{19} & f_7 & 0 \\
p_8 & f_{14} & f_{25} & f_{32} & f_{37} & f_{32} & f_{25} & f_{14} & p_8 & 0 & f_6 & f_{18} & f_{27} & f_{27} & f_{18} & f_6 & 0 \\
p_7 & f_{13} & f_{25} & f_{33} & f_{38} & f_{38} & f_{33} & f_{25} & f_{13} & p_7 & 0 & f_5 & f_{17} & f_{26} & f_{17} & f_5 & 0 \\
p_6 & f_{12} & f_{24} & f_{32} & f_{38} & f_{40} & f_{38} & f_{32} & f_{24} & f_{12} & p_6 & 0 & f_4 & f_{16} & f_{16} & f_4 & 0 \\
p_5 & f_{11} & f_{23} & f_{31} & f_{37} & f_{38} & f_{38} & f_{37} & f_{31} & f_{23} & f_{11} & p_5 & 0 & f_3 & f_{15} & f_3 & 0 \\
p_4 & f_{10} & f_{22} & f_{30} & f_{31} & f_{32} & f_{33} & f_{32} & f_{31} & f_{30} & f_{22} & f_{10} & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_9 & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{25} & f_{24} & f_{23} & f_{22} & f_{21} & f_9 & p_3 & 0 & f_1 & 0 \\
p_2 & f_8 & f_9 & f_{10} & f_{11} & f_{12} & f_{13} & f_{14} & f_{13} & f_{12} & f_{11} & f_{10} & f_9 & f_8 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6, f_6 = p_1 - p_7,$
$f_7 = p_1 - p_8,$
$f_8 = -p_1 + p_2 + p_3, f_9 = -p_1 + p_3 + p_4, f_{10} = -p_1 + p_4 + p_5, f_{11} = -p_1 + p_5 + p_6,$
$f_{12} = -p_1 + p_6 + p_7, f_{13} = -p_1 + p_7 + p_8, f_{14} = -p_1 + p_8 + p_8,$
$f_{15} = p_1 + p_2 - p_3 - p_4, f_{16} = p_1 + p_2 - p_4 - p_5, f_{17} = p_1 + p_2 - p_5 - p_6,$
$f_{18} = p_1 + p_2 - p_6 - p_7, f_{19} = p_1 + p_2 - p_7 - p_8, f_{20} = p_1 + p_2 - p_8 - p_8,$
$f_{21} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{22} = -p_1 - p_2 + p_4 + p_5 + p_6,$
$f_{23} = -p_1 - p_2 + p_5 + p_6 + p_7, f_{24} = -p_1 - p_2 + p_6 + p_7 + p_8,$
$f_{25} = -p_1 - p_2 + p_7 + p_8 + p_8,$
$f_{26} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{27} = p_1 + p_2 + p_3 - p_5 - p_6 - p_7,$
$f_{28} = p_1 + p_2 + p_3 - p_6 - p_7 - p_8, f_{29} = p_1 + p_2 + p_3 - p_7 - p_8 - p_8,$
$f_{30} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_7, f_{31} = -p_1 - p_2 - p_3 + p_5 + p_6 + p_7 + p_8,$
$f_{32} = -p_1 - p_2 - p_3 + p_6 + p_7 + p_8 + p_8, f_{33} = -p_1 - p_2 - p_3 + p_7 + p_7 + p_8 + p_8,$
$f_{34} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8,$
$f_{35} = p_1 + p_2 + p_3 + p_4 - p_6 - p_7 - p_8 - p_8,$
$f_{36} = p_1 + p_2 + p_3 + p_4 - p_7 - p_7 - p_8 - p_8,$
$f_{37} = -p_1 - p_2 - p_3 - p_4 + p_5 + p_6 + p_7 + p_8 + p_8,$
$f_{38} = -p_1 - p_2 - p_3 - p_4 + p_6 + p_7 + p_7 + p_8 + p_8,$
$f_{39} = p_1 + p_2 + p_3 + p_4 + p_5 - p_6 - p_7 - p_7 - p_8 - p_8,$
$f_{40} = -p_1 - p_2 - p_3 - p_4 - p_5 + p_6 + p_6 + p_7 + p_7 + p_8 + p_8,$

and the following cases are possible:

1. $f_i = 0$ for $i = 2, 10, 14, 15, 17, 18, 19, 30, 32, 34, 36$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (4p, p, 4p, p, 3p, 2p, 3p, 2p)$ and
$$A \cong R[X, Y]/(X^7 - Y^5, X^2Y - c^p)$$
2. $f_i = 0$ for $i = 5, 8, 10, 13, 20, 21, 23, 26, 28, 33, 35, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (4p, 2p, 2p, 3p, p, 4p, p, 3p)$ and
$$A \cong R[X, Y]/(X^9 - Y^4, X^2Y - c^p)$$
3. $f_i = 0$ for $i = 5, 8, 10, 13, 15, 20, 23, 26, 28, 33, 37, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (7p, 3p, 4p, 6p, p, 7p, 2p, 5p)$ and
$$A \cong R[X, Y]/(X^7 - Y^{10}, XY - c^p)$$
4. $f_i = 0$ for $i = 1, 3, 6, 11, 21, 22, 23, 24, 29, 36, 37, 40$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, 3p, p, 3p, 2p, p, 3p, 2p)$ and
$$A \cong R[X, Y]/(X^5 - Y^6, X^2Y - c^p)$$

5. $f_i = 0$ for $i = 8, 10, 12, 14, 15, 17, 19, 30, 32, 34, 36, 40$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (8p, p, 7p, 2p, 6p, 3p, 5p, 4p)$ and
$$A \cong R[X, Y]/(X^9 - Y^8, XY - c^p)$$

6. $f_i = 0$ for $i = 1, 2, 4, 5, 6, 14, 17, 18, 25, 27, 32, 33, 34$ and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, 2p, p, 2p, 2p, 2p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^7, X^2Y - c^p)$$

7. $f_i = 0$ for $i = 5, 8, 10, 13, 15, 17, 18, 20, 26, 28, 33, 35, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, p, 2p, 2p, p, 3p, p, 2p)$ and
$$A \cong R[X, Y]/(X^8 - Y^3, X^3Y - c^p)$$

8. $f_i = 0$ for $i = 3, 6, 8, 11, 14, 21, 24, 26, 27, 29, 36, 37, 40$ and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (6p, 5p, p, 6p, 4p, 2p, 6p, 3p)$ and
$$A \cong R[X, Y]/(X^6 - Y^{11}, XY - c^p)$$

9. $f_i = 0$ for $i = 8, 10, 11, 12, 13, 15, 20, 23, 30, 31, 37, 38$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, p, 2p, 2p, p, 2p, p, 2p)$ and
$$A \cong R[X, Y]/(X^5 - Y^4, X^3Y - c^p)$$

10. $f_i = 0$ for $i = 2, 4, 6, 14, 15, 16, 17, 18, 19, 27, 32, 34, 36, 40$ and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, 2p, p, 2p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^5 - Y^3, X^4Y - c^p)$$

11. $f_i = 0$ for $i = 1, 4, 7, 9, 12, 20, 21, 23, 24, 26, 29, 34, 35, 40$ and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, 3p, p, 2p, 3p, 2p, p, 3p)$ and
$$A \cong R[X, Y]/(X^{11} - Y^3, X^2Y - c^p)$$

12. $f_i = 0$ for $i = 1, 4, 7, 9, 12, 20, 21, 23, 24, 26, 33, 34, 35, 40$ and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (5p, 5p, 2p, 3p, 5p, 4p, p, 5p)$ and
$$A \cong R[X, Y]/(X^{12} - Y^5, XY - c^p)$$

13. $f_i = 0$ for $i = 1, 2, 5, 6, 10, 14, 18, 21, 22, 25, 30, 32, 33, 34, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (4p, 4p, 4p, p, 3p, 4p, 4p, 2p)$ and
$$A \cong R[X, Y]/(X^{13} - Y^4, XY - c^p)$$

14. $f_i = 0$ for $i = 2, 6, 10, 11, 14, 15, 18, 19, 22, 30, 31, 32, 36, 37, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, 2p, p, p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^4 - Y^3, X^3Y^2 - c^p)$$

15. $f_i = 0$ for $i = 1, 2, 5, 6, 10, 14, 18, 21, 22, 25, 30, 31, 32, 33, 37, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, 2p, p, p, 2p, 2p, p)$ and
$$A \cong R[X, Y]/(X^{11} - Y^2, X^3Y - c^p)$$

16. $f_i = 0$ for $i = 3, 6, 8, 11, 14, 15, 16, 18, 19, 26, 27, 28, 29, 36, 37, 39$ and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, p, 2p, p, p, 2p, p)$ and
$$A \cong R[X, Y]/(X^7 - Y^2, X^5Y - c^p)$$

17. $f_i = 0$ for $i$ =1, 4, 7, 9, 12, 20, 21, 22, 23, 24, 29, 30, 34, 35, 36, 40 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, p, p, 2p, p, p, 2p)$ and
$$A \cong R[X, Y]/(X^9 - Y^2, X^4Y - c^p)$$

18. $f_i = 0$ for $i$ =5, 8, 9, 10, 13, 14, 17, 18, 21, 25, 26, 27, 28, 33, 34, 35, 39 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, p, p, p, 2p, p, p)$ and
$$A \cong R[X, Y]/(X^5 - Y^2, X^6Y - c^p)$$

19. $f_i = 0$ for $i$ = 1, 2, 3, 6, 7, 11, 15, 17, 20, 22, 23, 29, 30, 31, 36, 37, 39 and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, 2p, 2p, p, p, 2p, 2p)$ and
$$A \cong R[X, Y]/(X^{13} - Y^2, X^2Y - c^p)$$

20. $f_i = 0$ for $i$ =1, 2, 3, 6, 7, 11, 15, 19, 20, 22, 23, 29, 30, 31, 36, 37, 40 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3p, 3p, 3p, 3p, 2p, p, 3p, 3p)$ and
$$A \cong R[X, Y]/(X^{14} - Y^3, XY - c^p)$$

21. $f_i = 0$ for $i$ =1, 2, 3, 4, 5, 6, 14, 15, 16, 17, 18, 25, 26, 27, 32, 33, 37, 38 ,40 and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, 2p, 2p, 2p, 2p, 2p, 2p, p)$ and
$$A \cong R[X, Y]/(X^{15} - Y^2, XY - c^p)$$

22. $f_i = 0$ for $i$ =8, 9, 10, 11, 12, 13, 14, 21, 22, 23, 24, 25, 30, 31, 32, 33, 37, 38, 40 and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2p, p, p, p, p, p, p, p)$ and
$$A \cong R[X, Y]/(X^3 - Y^2, X^7Y - c^p)$$

<u>The case of $n = 18$.</u>
The integral matrix $(\omega(i, n-j-1))_{0 \le i,j \le 17}$ is of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{16} & f_{17} & f_{18} & f_{19} & f_{20} & f_{21} & f_{21} & f_{20} & f_{19} & f_{18} & f_{17} & f_{16} & f_2 & 0 \\
p_3 & f_9 & p_3 & 0 & f_3 & f_{17} & f_{28} & f_{29} & f_{30} & f_{31} & f_{32} & f_{31} & f_{30} & f_{29} & f_{28} & f_{17} & f_3 & 0 \\
p_4 & f_{10} & f_{10} & p_4 & 0 & f_4 & f_{18} & f_{29} & f_{37} & f_{38} & f_{39} & f_{39} & f_{38} & f_{37} & f_{29} & f_{18} & f_4 & 0 \\
p_5 & f_{11} & f_{22} & f_{11} & p_5 & 0 & f_5 & f_{19} & f_{30} & f_{38} & f_{43} & f_{44} & f_{43} & f_{38} & f_{30} & f_{19} & f_5 & 0 \\
p_6 & f_{12} & f_{23} & f_{23} & f_{12} & p_6 & 0 & f_6 & f_{20} & f_{31} & f_{39} & f_{44} & f_{44} & f_{39} & f_{31} & f_{20} & f_6 & 0 \\
p_7 & f_{13} & f_{24} & f_{33} & f_{24} & f_{13} & p_7 & 0 & f_7 & f_{21} & f_{32} & f_{39} & f_{43} & f_{39} & f_{32} & f_{21} & f_7 & 0 \\
p_8 & f_{14} & f_{25} & f_{34} & f_{34} & f_{25} & f_{14} & p_8 & 0 & f_8 & f_{21} & f_{31} & f_{38} & f_{38} & f_{31} & f_{21} & f_8 & 0 \\
p_9 & f_{15} & f_{26} & f_{35} & f_{40} & f_{35} & f_{26} & f_{15} & p_9 & 0 & f_7 & f_{20} & f_{30} & f_{37} & f_{30} & f_{20} & f_8 & 0 \\
p_8 & f_{15} & f_{27} & f_{36} & f_{41} & f_{41} & f_{36} & f_{27} & f_{15} & p_8 & 0 & f_6 & f_{19} & f_{29} & f_{29} & f_{19} & f_6 & 0 \\
p_7 & f_{14} & f_{26} & f_{36} & f_{42} & f_{45} & f_{42} & f_{36} & f_{26} & f_{14} & p_7 & 0 & f_5 & f_{18} & f_{28} & f_{18} & f_5 & 0 \\
p_6 & f_{13} & f_{25} & f_{35} & f_{41} & f_{45} & f_{45} & f_{41} & f_{35} & f_{25} & f_{13} & p_6 & 0 & f_4 & f_{17} & f_{17} & f_4 & 0 \\
p_5 & f_{12} & f_{24} & f_{34} & f_{40} & f_{41} & f_{42} & f_{41} & f_{40} & f_{34} & f_{24} & f_{12} & p_5 & 0 & f_3 & f_{16} & f_3 & 0 \\
p_4 & f_{11} & f_{23} & f_{33} & f_{34} & f_{35} & f_{36} & f_{36} & f_{35} & f_{34} & f_{33} & f_{23} & f_{11} & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_{10} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} & f_{27} & f_{26} & f_{25} & f_{24} & f_{23} & f_{22} & f_{10} & p_3 & 0 & f_1 & 0 \\
p_2 & f_9 & f_{10} & f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{15} & f_{14} & f_{13} & f_{12} & f_{11} & f_{10} & f_9 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{pmatrix}$$

with the $p_i \ge 1$ and the $f_j \ge 0$, where

$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6, f_6 = p_1 - p_7,$
$f_7 = p_1 - p_8, f_8 = p_1 - p_9,$
$f_9 = -p_1 + p_2 + p_3, f_{10} = -p_1 + p_3 + p_4, f_{11} = -p_1 + p_4 + p_5, f_{12} = -p_1 + p_5 + p_6,$
$f_{13} = -p_1 + p_6 + p_7, f_{14} = -p_1 + p_7 + p_8, f_{15} = -p_1 + p_8 + p_9,$
$f_{16} = p_1 + p_2 - p_3 - p_4, f_{17} = p_1 + p_2 - p_4 - p_5, f_{18} = p_1 + p_2 - p_5 - p_6,$
$f_{19} = p_1 + p_2 - p_6 - p_7, f_{20} = p_1 + p_2 - p_7 - p_8, f_{21} = p_1 + p_2 - p_8 - p_9,$
$f_{22} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{23} = -p_1 - p_2 + p_4 + p_5 + p_6,$
$f_{24} = -p_1 - p_2 + p_5 + p_6 + p_7, f_{25} = -p_1 - p_2 + p_6 + p_7 + p_8,$
$f_{26} = -p_1 - p_2 + p_7 + p_8 + p_9, f_{27} = -p_1 - p_2 + p_8 + p_8 + p_9,$
$f_{28} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{29} = p_1 + p_2 + p_3 - p_5 - p_6 - p_7,$
$f_{30} = p_1 + p_2 + p_3 - p_6 - p_7 - p_8, f_{31} = p_1 + p_2 + p_3 - p_7 - p_8 - p_9,$
$f_{32} = p_1 + p_2 + p_3 - p_8 - p_8 - p_9,$
$f_{33} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_7, f_{34} = -p_1 - p_2 - p_3 + p_5 + p_6 + p_7 + p_8,$
$f_{35} = -p_1 - p_2 - p_3 + p_6 + p_7 + p_8 + p_9, f_{36} = -p_1 - p_2 - p_3 + p_7 + p_8 + p_8 + p_9,$
$f_{37} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8,$
$f_{38} = p_1 + p_2 + p_3 + p_4 - p_6 - p_7 - p_8 - p_9,$
$f_{39} = p_1 + p_2 + p_3 + p_4 - p_7 - p_8 - p_8 - p_9,$
$f_{40} = -p_1 - p_2 - p_3 - p_4 + p_5 + p_6 + p_7 + p_8 + p_9,$
$f_{41} = -p_1 - p_2 - p_3 - p_4 + p_6 + p_7 + p_8 + p_8 + p_9,$
$f_{42} = -p_1 - p_2 - p_3 - p_4 + p_7 + p_7 + p_8 + p_8 + p_9,$
$f_{43} = p_1 + p_2 + p_3 + p_4 + p_5 - p_6 - p_7 - p_8 - p_8 - p_9,$
$f_{44} = p_1 + p_2 + p_3 + p_4 + p_5 - p_7 - p_7 - p_8 - p_8 - p_9,$
$f_{45} = -p_1 - p_2 - p_3 - p_4 - p_5 + p_6 + p_7 + p_7 + p_8 + p_8 + p_9,$

and the following cases are possible:

1. $f_i = 0$ for $i = 8, 9, 11, 12, 14, 19, 22, 27, 31, 34, 38, 43, 44$ and $f_i \ge 1$ other-

wise: Setting $p = p_1$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (4p, 2p, 2p, 3p, p, 3p, 3p, p, 4p)$$

and

$$A \cong R[X,Y]/(X^6 - Y^6, X^2Y - c^p)$$

2. $f_i = 0$ for $i = 3, 8, 9, 12, 14, 16, 19, 27, 28, 31, 34, 40, 41, 42$ and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (4p, 2p, 2p, 4p, p, 3p, 3p, p, 4p)$$

and

$$A \cong R[X,Y]/(X^{10} - Y^4, X^2Y - c^p)$$

3. $f_i = 0$ for $i = 3, 8, 9, 12, 14, 19, 22, 27, 28, 31, 34, 40, 42, 43$ and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (7p, 4p, 3p, 7p, p, 6p, 5p, 2p, 7p)$$

and

$$A \cong R[X,Y]/(X^{11} - Y^7, XY - c^p)$$

4. $f_i = 0$ for $i = 4, 8, 10, 13, 14, 18, 21, 25, 32, 33, 37, 38, 39, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 2p, 2p, p, 3p, 2p, p, 2p, 3p)$$

and

$$A \cong R[X,Y]/(X^4 - Y^7, X^2Y - c^p)$$

5. $f_i = 0$ for $i = 9, 10, 11, 13, 14, 15, 18, 26, 33, 35, 36, 37, 42, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, p, 2p, p, 2p, 2p, p, 2p, p)$$

and

$$A \cong R[X,Y]/(X^5 - Y^4, X^2Y^2 - c^p)$$

6. $f_i = 0$ for $i = 1, 2, 3, 4, 6, 7, 8, 16, 17, 20, 21, 31, 32, 39, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, p, p, p, p, q, p, p, p)$$

and

$$A \cong R[X,Y]/(X^3 - c^q, Y^6 - Xc^{p-q})$$

7. $f_i = 0$ for $i = 1, 3, 4, 6, 7, 17, 20, 28, 29, 30, 31, 32, 37, 39, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, p, q, p, p, q, p, p, q)$$

and

$$A \cong R[X, Y]/(X^6 - c^q, Y^3 - Xc^{p-q})$$

8. $f_i = 0$ for $i = 1, 4, 7, 10, 13, 22, 23, 24, 25, 26, 32, 35, 37, 39, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, p, q, p - q, p, q, p - q, p, q)$$

and

$$A \cong R[X, Y]/(X^6 - c^q, Y^3 - X^5c^{p-2q})$$

9. $f_i = 0$ for $i = 3, 6, 9, 12, 15, 17, 20, 27, 28, 29, 30, 31, 40, 41, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p - q, p, q, p - q, p, q, p - q)$$

and

$$A \cong R[X, Y]/(X^6 - c^{p-q}, Y^3 - X^4c^{2q-p})$$

10. $f_i = 0$ for $i = 9, 10, 12, 13, 15, 22, 23, 24, 25, 26, 27, 35, 40, 41, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p - q, q, q, p - q, q, q, p - q)$$

and

$$A \cong R[X, Y]/(X^6 - c^{p-q}, Y^3 - X^2c^{2q-p})$$

11. $f_i = 0$ for $i = 3, 8, 9, 12, 14, 22, 24, 25, 27, 28, 31, 34, 40, 41, 42$ and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 2p, p, 3p, p, 2p, 2p, p, 3p)$$

and

$$A \cong R[X, Y]/(X^9 - Y^3, X^3Y - c^p)$$

12. $f_i = 0$ for $i = 3, 8, 12, 13, 16, 17, 21, 24, 25, 31, 32, 34, 40, 41, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, q, p, q, p - q, q, q, p)$$

and

$$A \cong R[X, Y]/(X^3 - c^{p-q}, Y^6 - X^2c^{2q-p})$$

13. $f_i = 0$ for $i = 1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20, 28, 29, 30, 37$ and $f_i \geq 1$ otherwise : Setting $p = p_1$ and $q = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, p, p, p, p, p, p, p, q)$$

and

$$A \cong R[X, Y]/(X^2 - c^q, Y^9 - Xc^{p-q})$$

14. $f_i = 0$ for $i = 1, 4, 7, 10, 13, 22, 23, 24, 25, 26, 32, 33, 35, 37, 39, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, p, p, 2p, p, p, 2p, p)$$

and

$$A \cong R[X, Y]/(X^3 - Y^3, X^3Y - c^p)$$

15. $f_i = 0$ for $i = 1, 4, 5, 8, 10, 14, 18, 22, 25, 26, 32, 33, 37, 38, 42, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (5p, 5p, 3p, 2p, 5p, 5p, p, 4p, 5p)$$

and

$$A \cong R[X, Y]/(X^{13} - Y^5, XY - c^p)$$

16. $f_i = 0$ for $i = 1, 4, 5, 8, 10, 14, 18, 22, 25, 26, 33, 36, 37, 38, 42, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 3p, 2p, p, 3p, 3p, p, 2p, 3p)$$

and

$$A \cong R[X, Y]/(X^{12} - Y^3, X^2Y - c^p)$$

17. $f_i = 0$ for $i = 2, 6, 11, 12, 15, 16, 19, 20, 23, 27, 33, 34, 35, 36, 41, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p, q, p - q, q, p, q, p - q)$$

and

$$A \cong R[X, Y]/(X^9 - c^q, Y^2 - X^7c^{p-2q})$$

18. $f_i = 0$ for $i = 2, 4, 6, 8, 16, 17, 18, 19, 20, 21, 29, 31, 37, 38, 39, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p, q, p, q, p, q, p)$$

and

$$A \cong R[X, Y]/(X^9 - c^q, Y^2 - Xc^{p-q})$$

19. $f_i = 0$ for $i = 2, 4, 6, 15, 16, 17, 18, 19, 20, 27, 29, 35, 36, 37, 41, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p, q, p, q, p, q, p - q)$$

and

$$A \cong R[X, Y]/(X^9 - c^q, Y^2 - c^{p-q})$$

20. $f_i = 0$ for $i = 4, 8, 9, 10, 13, 14, 17, 18, 21, 25, 29, 31, 37, 38, 39, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p - q, q, p, q, p - q, q, p)$$

and

$$A \cong R[X, Y]/(X^9 - c^q, Y^2 - X^6 c^{p-2q})$$

21. $f_i = 0$ for $i = 8, 9, 10, 11, 12, 13, 14, 21, 23, 25, 31, 33, 34, 38, 39, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p - q, q, p - q, q, p - q, q, p)$$

and

$$A \cong R[X, Y]/(X^9 - c^q, Y^2 - X^4 c^{p-2q})$$

22. $f_i = 0$ for $i = 9, 10, 11, 12, 13, 14, 15, 23, 25, 27, 33, 34, 35, 36, 41, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (p, q, p - q, q, p - q, q, p - q, q, p - q)$$

and

$$A \cong R[X, Y]/(X^9 - c^q, Y^2 - X^3 c^{p-2q})$$

23. $f_i = 0$ for $i = 2, 6, 11, 12, 15, 16, 19, 20, 23, 27, 33, 34, 35, 36, 40, 41, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, 2p, p, p, p, 2p, p, p)$$

and

$$A \cong R[X, Y]/(X^7 - Y^2, X^2 Y^2 - c^p)$$

24. $f_i = 0$ for $i = 3, 6, 9, 12, 15, 16, 17, 19, 20, 27, 28, 29, 30, 31, 40, 41, 44$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, 2p, p, p, 2p, p, p)$$

and

$$A \cong R[X, Y]/(X^3 - Y^4, X^3 Y^2 - c^p)$$

25. $f_i = 0$ for $i = 1, 4, 5, 8, 10, 14, 18, 22, 25, 26, 27, 28, 29, 36, 37, 38, 42, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, p, p, 2p, 2p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^{10} - Y - 2, X^4Y - c^p)$$

26. $f_i = 0$ for $i = 4, 8, 9, 10, 13, 14, 17, 18, 21, 25, 28, 29, 31, 32, 37, 38, 39, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, p, 2p, p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^6 - Y^2, X^3Y^2 - c^p)$$

27. $f_i = 0$ for $i = 3, 8, 9, 12, 13, 14, 16, 17, 21, 24, 25, 28, 31, 32, 34, 40, 41, 42, 45$ and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, 2p, p, p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^6 - Y^2, X^6Y - c^p)$$

28. $f_i = 0$ for $i$ =5 , 9, 10, 11, 14, 15, 18, 19, 22, 26, 27, 28, 29, 30, 36, 37, 38, 42, 43 and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, p, p, 2p, p, p, p)$$

and

$$A \cong R[X, Y]/(X^3 - Y^3, X^5Y - c^p)$$

29. $f_i = 0$ for $i$ =8, 9, 10, 11, 12, 13, 14, 21, 22, 23, 24, 25, 31, 32, 33, 34, 38 ,39 ,43, 44 and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, p, p, p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^2 - Y^4, X^4Y - c^p)$$

30. $f_i = 0$ for $i$ =9, 10, 11, 12, 13, 14, 15, 22, 23, 24, 25, 26, 27, 33, 34, 35, 36, 40, 41, 42 ,45 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, p, p, p, p, p, p)$$

and

$$A \cong R[X, Y]/(X^3 - Y^2, X^6Y^2 - c^p)$$

The case of $n = 19$.
The integral matrix $(\omega(i, n-j-1))_{0 \le i,j \le 18}$ is of the form

$$
\left(\begin{array}{ccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{17} & f_{18} & f_{19} & f_{20} & f_{21} & f_{22} & f_{23} & f_{22} & f_{21} & f_{20} & f_{19} & f_{18} & f_{17} & f_2 & 0 \\
p_3 & f_9 & p_3 & 0 & f_3 & f_{18} & f_{30} & f_{31} & f_{32} & f_{33} & f_{34} & f_{34} & f_{33} & f_{32} & f_{31} & f_{30} & f_{18} & f_3 & 0 \\
p_4 & f_{10} & f_{10} & p_4 & 0 & f_4 & f_{19} & f_{31} & f_{40} & f_{41} & f_{42} & f_{43} & f_{42} & f_{41} & f_{40} & f_{31} & f_{19} & f_4 & 0 \\
p_5 & f_{11} & f_{24} & f_{11} & p_5 & 0 & f_5 & f_{20} & f_{32} & f_{41} & f_{47} & f_{48} & f_{48} & f_{47} & f_{41} & f_{32} & f_{20} & f_5 & 0 \\
p_6 & f_{12} & f_{25} & f_{25} & f_{12} & p_6 & 0 & f_6 & f_{21} & f_{33} & f_{42} & f_{48} & f_{51} & f_{48} & f_{42} & f_{33} & f_{21} & f_6 & 0 \\
p_7 & f_{13} & f_{26} & f_{35} & f_{26} & f_{13} & p_7 & 0 & f_7 & f_{22} & f_{34} & f_{43} & f_{48} & f_{48} & f_{43} & f_{34} & f_{22} & f_7 & 0 \\
p_8 & f_{14} & f_{27} & f_{36} & f_{36} & f_{27} & f_{14} & p_8 & 0 & f_8 & f_{23} & f_{34} & f_{42} & f_{47} & f_{42} & f_{34} & f_{23} & f_8 & 0 \\
p_9 & f_{15} & f_{28} & f_{37} & f_{44} & f_{37} & f_{28} & f_{15} & p_9 & 0 & f_8 & f_{22} & f_{33} & f_{41} & f_{41} & f_{33} & f_{22} & f_8 & 0 \\
p_9 & f_{16} & f_{29} & f_{38} & f_{45} & f_{45} & f_{38} & f_{29} & f_{16} & p_9 & 0 & f_7 & f_{21} & f_{32} & f_{40} & f_{32} & f_{21} & f_7 & 0 \\
p_8 & f_{15} & f_{29} & f_{39} & f_{46} & f_{49} & f_{46} & f_{39} & f_{29} & f_{15} & p_8 & 0 & f_6 & f_{20} & f_{31} & f_{31} & f_{20} & f_6 & 0 \\
p_7 & f_{14} & f_{28} & f_{38} & f_{46} & f_{50} & f_{50} & f_{46} & f_{38} & f_{28} & f_{14} & p_7 & 0 & f_5 & f_{19} & f_{30} & f_{19} & f_5 & 0 \\
p_6 & f_{13} & f_{27} & f_{37} & f_{45} & f_{49} & f_{50} & f_{49} & f_{45} & f_{37} & f_{27} & f_{13} & p_6 & 0 & f_4 & f_{18} & f_{18} & f_4 & 0 \\
p_5 & f_{12} & f_{26} & f_{36} & f_{44} & f_{45} & f_{46} & f_{46} & f_{45} & f_{44} & f_{36} & f_{26} & f_{12} & p_5 & 0 & f_3 & f_{17} & f_3 & 0 \\
p_4 & f_{11} & f_{25} & f_{35} & f_{36} & f_{37} & f_{38} & f_{39} & f_{38} & f_{37} & f_{36} & f_{35} & f_{25} & f_{11} & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_{10} & f_{24} & f_{25} & f_{26} & f_{27} & f_{28} & f_{29} & f_{29} & f_{28} & f_{27} & f_{26} & f_{25} & f_{24} & f_{10} & p_3 & 0 & f_1 & 0 \\
p_2 & f_9 & f_{10} & f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{15} & f_{14} & f_{13} & f_{12} & f_{11} & f_{10} & f_9 & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_9 & p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{array}\right)
$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6, f_6 = p_1 - p_7,$
$f_7 = p_1 - p_8, f_8 = p_1 - p_9,$
$f_9 = -p_1 + p_2 + p_3, f_{10} = -p_1 + p_3 + p_4, f_{11} = -p_1 + p_4 + p5, f_{12} = -p_1 + p_5 + p_6,$
$f_{13} = -p_1 + p_6 + p_7, f_{14} = -p_1 + p_7 + p_8, f_{15} = -p_1 + p_8 + p_9, f_{16} = -p_1 + p_9 + p_9,$
$f_{17} = p_1 + p_2 - p_3 - p_4, f_{18} = p_1 + p_2 - p_4 - p_5, f_{19} = p_1 + p_2 - p_5 - p_6,$
$f_{20} = p_1 + p_2 - p_6 - p_7, f_{21} = p_1 + p_2 - p_7 - p_8, f_{22} = p_1 + p_2 - p_8 - p_9,$
$f_{23} = p_1 + p_2 - p_9 - p_9,$
$f_{24} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{25} = -p_1 - p_2 + p_4 + p_5 + p_6,$
$f_{26} = -p_1 - p_2 + p_5 + p_6 + p_7, f_{27} = -p_1 - p_2 + p_6 + p_7 + p_8,$
$f_{28} = -p_1 - p_2 + p_7 + p_8 + p_9, f_{29} = -p_1 - p_2 + p_8 + p_9 + p_9,$
$f_{30} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{31} = p_1 + p_2 + p_3 - p_5 - p_6 - p_7,$
$f_{32} = p_1 + p_2 + p_3 - p_6 - p_7 - p_8, f_{33} = p_1 + p_2 + p_3 - p_7 - p_8 - p_9,$
$f_{34} = p_1 + p_2 + p_3 - p_8 - p_9 - p_9,$
$f_{35} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_7, f_{36} = -p_1 - p_2 - p_3 + p_5 + p_6 + p_7 + p_8,$
$f_{37} = -p_1 - p_2 - p_3 + p_6 + p_7 + p_8 + p_9, f_{38} = -p_1 - p_2 - p_3 + p_7 + p_8 + p_9 + p_9,$
$f_{39} = -p_1 - p_2 - p_3 + p_8 + p_8 + p_9 + p_9,$
$f_{40} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8,$
$f_{41} = p_1 + p_2 + p_3 + p_4 - p_6 - p_7 - p_8 - p_9,$
$f_{42} = p_1 + p_2 + p_3 + p_4 - p_7 - p_8 - p_9 - p_9,$
$f_{43} = p_1 + p_2 + p_3 + p_4 - p_8 - p_8 - p_9 - p_9,$
$f_{44} = -p_1 - p_2 - p_3 - p_4 + p_5 + p_6 + p_7 + p_8 + p_9,$
$f_{45} = -p_1 - p_2 - p_3 - p_4 + p_6 + p_7 + p_8 + p_9 + p_9,$
$f_{46} = -p_1 - p_2 - p_3 - p_4 + p_7 + p_8 + p_8 + p_9 + p_9,$
$f_{47} = p_1 + p_2 + p_3 + p_4 + p_5 - p_6 - p_7 - p_8 - p_9 - p_9,$
$f_{48} = p_1 + p_2 + p_3 + p_4 + p_5 - p_7 - p_8 - p_8 - p_9 - p_9,$
$f_{49} = -p_1 - p_2 - p_3 - p_4 - p_5 + p_6 + p_7 + p_8 + p_8 + p_9 + p_9,$
$f_{50} = -p_1 - p_2 - p_3 - p_4 - p_5 + p_7 + p_7 + p_8 + p_8 + p_9 + p_9,$
$f_{51} = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - p_7 - p_7 - p_8 - p_8 - p_9 - p_9,$

and the following cases are possible:

1. $f_i = 0$ for $i$ =9, 10, 11, 12, 13, 14, 15, 16, 24, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39, 44, 45, 46, 49, 50 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, p, p, p, p, p, p)$$

and

$$A \cong R[X, Y]/(X^3 - Y^2, X^8Y - c^p)$$

2. $f_i = 0$ for $i$ =5, 9, 10, 11, 14, 15, 16, 19, 20, 24, 28, 29, 30, 31, 32, 38, 39, 40, 41, 46, 47, 50 and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, p, p, 2p, p, p, p)$$

and

$$A \cong R[X, Y]/(X^5 - Y^2, X^7Y - c^p)$$

3. $f_i = 0$ for $i$ =3, 7, 9, 12, 13, 16, 17, 18, 21, 22, 26, 30, 32, 33, 34, 43, 44, 45, 48, 51 and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, 2p, p, p, p, 2p, p)$$

and

$$A \cong R[X, Y]/(X^4 - Y^3, X^5Y - c^p)$$

4. $f_i = 0$ for $i$ =3, 6, 9, 12, 15, 16, 17, 18, 20, 21, 29, 30, 31, 32, 33, 39, 44, 45, 46, 49, 51 and $f_i \geq 1$ otherwise: Setting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, p, 2p, p, p, 2p, p, p)$$

and

$$A \cong R[X, Y]/(X^7 - Y^2, X^6Y - c^p)$$

5. $f_i = 0$ for $i$ = 2, 5, 7, 11, 16, 17, 19, 20, 21, 22, 32, 35, 38, 40, 41, 43, 47, 48 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, p, 2p, p, p, 2p, p, 2p, p)$$

and

$$A \cong R[X, Y]/(X^5 - Y^3, X^3Y^2 - c^p)$$

6. $f_i = 0$ for $i$ =1, 4, 7, 10, 13, 16, 24, 25, 26, 27, 28, 29, 35, 37, 38, 40, 43, 45, 49 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, p, p, 2p, p, p, 2p, p)$$

and

$$A \cong R[X, Y]/(X^9 - Y^2, X^5Y - c^p)$$

7. $f_i = 0$ for $i$ =1, 4, 5, 8, 10, 14, 19, 23, 24, 27, 28, 30, 31, 34, 40, 41, 42, 43, 47, 50 and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, p, p, 2p, 2p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^{11} - Y^2, X^4Y - c^p)$$

8. $f_i = 0$ for $i$ =1, 3, 4, 6, 7, 16, 18, 21, 29, 30, 31, 32, 33, 40, 45, 49, 51 and $f_i \geq 1$: otherwiseSetting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, p, 2p, 2p, p, 2p, 2p, p)$$

and
$$A \cong R[X, Y]/(X^4 - Y^5, X^3Y - c^p)$$

9. $f_i = 0$ for $i$ =9, 10, 11, 12, 13, 14, 15, 23, 25, 27, 35, 36, 37, 39, 42, 49, 51 and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, p, 2p, p, 2p, p, 2p, p, 2p)$$

and
$$A \cong R[X, Y]/(X^7 - Y^3, X^4Y - c^p)$$

10. $f_i = 0$ for $i$ =1, 2, 5, 6, 7, 11, 16, 20, 21, 24, 25, 29, 32, 35, 38, 39, 40, 41, 47, 48, 51 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, 2p, p, p, 2p, 2p, 2p, p)$$

and
$$A \cong R[X, Y]/(X^{13} - Y^2, X^3Y - c^p)$$

11. $f_i = 0$ for $i$ =1, 2, 3, 5, 6, 7, 16, 17, 20, 21, 29, 32, 38, 39, 44, 45, 46 and $f_i \geq 1$ otherwise: Setting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, 2p, 2p, p, 2p, 2p, 2p, p)$$

and
$$A \cong R[X, Y]/(X^3 - Y^8, X^2Y - c^p)$$

12. $f_i = 0$ for $i$ =1, 2, 3, 4, 7, 8, 13 ,17, 18, 22, 23, 26, 27, 34, 35, 36, 37, 43, 44, 45, 49, 50 and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, 2p, 2p, 2p, p, p, 2p, 2p)$$

and
$$A \cong R[X, Y]/(X^{15} - Y^2, X^2Y - c^p)$$

13. $f_i = 0$ for $i$ =1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 29, 30, 31, 32, 38, 39, 40, 45, 46, 49, 50 and $f_i \geq 1$ otherwise: Setting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (2p, 2p, 2p, 2p, 2p, 2p, 2p, 2p, p)$$

and
$$A \cong R[X, Y]/(X^{17} - Y^2, XY - c^p)$$

14. $f_i = 0$ for $i$ =2, 4, 13, 17, 18, 19, 21, 22, 23, 35, 37, 40, 42, 43, 49, 51 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, p, 3p, p, 3p, p, 2p, 2p, 2p)$$

and
$$A \cong R[X, Y]/(X^7 - Y^4, X^3Y - c^p)$$

15. $f_i = 0$ for $i$ =3, 6, 9, 12, 15, 20, 24, 29, 30, 31, 32, 33, 39, 44, 46, 47, 51 and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 2p, p, 3p, p, 2p, 3p, p, 2p)$$

and
$$A \cong R[X, Y]/(X^{10} - Y^3, X^3Y - c^p)$$

16. $f_i = 0$ for $i$ =1, 3, 4, 7, 13, 18, 26, 27, 30, 33, 34, 43, 45, 49, 51 and otherwise $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 3p, p, 3p, 3p, p, 2p, 3p, 2p)$$

and
$$A \cong R[X, Y]/(X^5 - Y^7, X^2Y - c^p)$$

17. $f_i = 0$ for $i$ =1, 4, 5, 8, 10, 14, 19, 23, 24, 27, 28, 31, 39, 40, 41, 42, 47, 50 and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 3p, 2p, p, 3p, 3p, 2p, p, 3p)$$

and
$$A \cong R[X, Y]/(X^{13} - Y^3, X^2Y - c^p)$$

18. $f_i = 0$ for $i$ =5, 7, 11, 16, 17, 22, 26, 32, 35, 38, 43, 44, 47, 48 and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (4p, 2p, 3p, 3p, p, 4p, p, 4p, 2p)$$

and
$$A \cong R[X, Y]/(X^7 - Y^6, X^2Y - c^p)$$

19. $f_i = 0$ for $i$ =3, 6, 9, 12, 15, 24, 27, 29, 30, 31, 33, 39, 44, 46, 49, 51 and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (4p, 3p, p, 4p, 2p, 2p, 4p, p, 3p)$$

and
$$A \cong R[X, Y]/(X^{11} - Y^4, X^2Y - c^p)$$

20. $f_i = 0$ for $i$ =1, 2, 3, 4, 7, 8, 13, 17, 18, 22, 23, 26, 27, 34, 35, 36, 37, 43, 44, 45, 49, 51 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (3p, 3p, 3p, 3p, 3p, p, 2p, 3p, 3p)$$

and
$$A \cong R[X,Y]/(X^{16} - Y^3, XY - c^p)$$

21. $f_i = 0$ for $i$ =9, 10, 11, 13, 14, 15, 19, 23, 35, 37, 39, 40, 42, 49 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (5p, p, 4p, p, 4p, 2p, 3p, 2p, 3p)$$

and
$$A \cong R[X,Y]/(X^9 - Y^5, X^2Y - c^p)$$

22. $f_i = 0$ for $i$ =1, 2, 5, 6, 7, 11, 16, 20, 21, 24, 25, 29, 32, 35, 38, 39, 44, 47, 48, 51 and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (4p, 4p, 4p, 3p, p, 4p, 4p, 4p, 2p)$$

and
$$A \cong R[X,Y]/(X^{15} - Y^4, XY - c^p)$$

23. $f_i = 0$ for $i$ =1, 4, 5, 8, 10, 14, 19, 23, 24, 27, 28, 35, 39, 40, 41, 42, 47, 50 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (5p, 5p, 4p, p, 5p, 5p, 3p, 2p, 5p)$$

and
$$A \cong R[X,Y]/(X^{14} - Y^5, XY - c^p)$$

24. $f_i = 0$ for $i$ =1, 4, 7, 10, 13, 16, 24, 26, 27, 29, 30, 33, 40, 43, 45, 49, 51 and $f_i \geq 1$: otherwise Setting $p = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (6p, 6p, p, 5p, 6p, 2p, 4p, 6p, 3p)$$

and
$$A \cong R[X,Y]/(X^{13} - Y^6, XY - c^p)$$

25. $f_i = 0$ for $i$ =3, 6, 9, 12, 15, 23, 24, 27, 30, 31, 33, 39, 44, 46, 47, 51 and $f_i \geq 1$: otherwise Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (7p, 5p, 2p, 7p, 3p, 4p, 7p, p, 6p)$$

and
$$A \cong R[X,Y]/(X^{12} - Y^7, XY - c^p)$$

26. $f_i = 0$ for $i$ =5, 9, 11, 14, 16, 17, 22, 26, 30, 32, 38, 43, 44, 47, 50 and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (8p, 3p, 5p, 6p, 2p, 8p, p, 7p, 4p)$$

and

$$A \cong R[X, Y]/(X^{11} - Y^8, XY - c^p)$$

27. $f_i = 0$ for $i$ =9, 11, 13, 15, 17, 19, 21, 23, 35, 37, 39, 40, 42, 49, 51 and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9) = (9p, p, 8p, 2p, 7p, 3p, 6p, 4p, 5p)$$

and

$$A \cong R[X, Y]/(X^{10} - Y^9, XY - c^p)$$

The case of $n = 20$.
The integral matrix $(\omega(i, n-j-1))_{0 \leq i,j \leq 19}$ is of the form

$$\left(\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & 0 \\
p_2 & p_2 & 0 & f_2 & f_{18} & f_{19} & f_{20} & f_{21} & f_{22} & f_{23} & f_{24} & f_{24} & f_{23} & f_{22} & f_{21} & f_{20} & f_{19} & f_{18} & f_2 & 0 \\
p_3 & f_{10} & p_3 & 0 & f_3 & f_{19} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} & f_{37} & f_{36} & f_{35} & f_{34} & f_{33} & f_{32} & f_{19} & f_3 & 0 \\
p_4 & f_{11} & f_{11} & p_4 & 0 & f_4 & f_{20} & f_{33} & f_{43} & f_{44} & f_{45} & f_{46} & f_{46} & f_{45} & f_{44} & f_{43} & f_{33} & f_{20} & f_4 & 0 \\
p_5 & f_{12} & f_{25} & f_{12} & p_5 & 0 & f_5 & f_{21} & f_{34} & f_{44} & f_{51} & f_{52} & f_{53} & f_{52} & f_{51} & f_{44} & f_{34} & f_{21} & f_5 & 0 \\
p_6 & f_{13} & f_{26} & f_{26} & f_{13} & p_6 & 0 & f_6 & f_{22} & f_{35} & f_{45} & f_{52} & f_{56} & f_{56} & f_{52} & f_{45} & f_{35} & f_{22} & f_6 & 0 \\
p_7 & f_{14} & f_{27} & f_{38} & f_{27} & f_{14} & p_7 & 0 & f_7 & f_{23} & f_{36} & f_{46} & f_{53} & f_{56} & f_{53} & f_{46} & f_{36} & f_{23} & f_7 & 0 \\
p_8 & f_{15} & f_{28} & f_{39} & f_{39} & f_{28} & f_{15} & p_8 & 0 & f_8 & f_{24} & f_{37} & f_{46} & f_{52} & f_{52} & f_{46} & f_{37} & f_{24} & f_8 & 0 \\
p_9 & f_{16} & f_{29} & f_{40} & f_{47} & f_{40} & f_{29} & f_{16} & p_9 & 0 & f_9 & f_{24} & f_{36} & f_{45} & f_{51} & f_{45} & f_{36} & f_{24} & f_9 & 0 \\
p_{10} & f_{17} & f_{30} & f_{41} & f_{48} & f_{48} & f_{41} & f_{30} & f_{17} & p_{10} & 0 & f_8 & f_{23} & f_{35} & f_{44} & f_{44} & f_{35} & f_{23} & f_8 & 0 \\
p_9 & f_{17} & f_{31} & f_{42} & f_{49} & f_{54} & f_{49} & f_{42} & f_{31} & f_{17} & p_9 & 0 & f_7 & f_{22} & f_{34} & f_{43} & f_{34} & f_{22} & f_7 & 0 \\
p_8 & f_{16} & f_{30} & f_{42} & f_{50} & f_{55} & f_{55} & f_{50} & f_{42} & f_{30} & f_{16} & p_8 & 0 & f_6 & f_{21} & f_{33} & f_{33} & f_{21} & f_6 & 0 \\
p_7 & f_{15} & f_{29} & f_{41} & f_{49} & f_{55} & f_{57} & f_{55} & f_{49} & f_{41} & f_{29} & f_{15} & p_7 & 0 & f_5 & f_{20} & f_{32} & f_{20} & f_5 & 0 \\
p_6 & f_{14} & f_{28} & f_{40} & f_{48} & f_{54} & f_{55} & f_{55} & f_{54} & f_{48} & f_{40} & f_{28} & f_{14} & p_6 & 0 & f_4 & f_{19} & f_{19} & f_4 & 0 \\
p_5 & f_{13} & f_{27} & f_{39} & f_{47} & f_{48} & f_{49} & f_{50} & f_{49} & f_{48} & f_{47} & f_{39} & f_{27} & f_{13} & p_5 & 0 & f_3 & f_{18} & f_3 & 0 \\
p_4 & f_{12} & f_{26} & f_{38} & f_{39} & f_{40} & f_{41} & f_{42} & f_{42} & f_{41} & f_{40} & f_{39} & f_{38} & f_{26} & f_{12} & p_4 & 0 & f_2 & f_2 & 0 \\
p_3 & f_{11} & f_{25} & f_{26} & f_{27} & f_{28} & f_{29} & f_{30} & f_{31} & f_{30} & f_{29} & f_{28} & f_{27} & f_{26} & f_{25} & f_{11} & p_3 & 0 & f_1 & 0 \\
p_2 & f_{10} & f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} & f_{17} & f_{16} & f_{15} & f_{14} & f_{13} & f_{12} & f_{11} & f_{10} & p_2 & 0 & 0 \\
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_9 & p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & 0
\end{array}\right)$$

with the $p_i \geq 1$ and the $f_j \geq 0$, where

$$f_1 = p_1 - p_2, f_2 = p_1 - p_3, f_3 = p_1 - p_4, f_4 = p_1 - p_5, f_5 = p_1 - p_6,$$
$$f_6 = p_1 - p_7, f_7 = p_1 - p_8, f_8 = p_1 - p_9, f_9 = p_1 - p_{10},$$
$$f_{10} = -p_1 + p_2 + p_3, f_{11} = -p_1 + p_3 + p_4, f_{12} = -p_1 + p_4 + p_5, f_{13} = -p_1 + p_5 + p_6,$$
$$f_{14} = -p_1 + p_6 + p_7, f_{15} = -p_1 + p_7 + p_8, f_{16} = -p_1 + p_8 + p_9, f_{17} = -p_1 + p_9 + p_{10},$$
$$f_{18} = p_1 + p_2 - p_3 - p_4, f_{19} = p_1 + p_2 - p_4 - p_5, f_{20} = p_1 + p_2 - p_5 - p_6,$$
$$f_{21} = p_1 + p_2 - p_6 - p_7, f_{22} = p_1 + p_2 - p_7 - p_8, f_{23} = p_1 + p_2 - p_8 - p_9,$$
$$f_{24} = p_1 + p_2 - p_9 - p_{10},$$
$$f_{25} = -p_1 - p_2 + p_3 + p_4 + p_5, f_{26} = -p_1 - p_2 + p_4 + p_5 + p_6,$$
$$f_{27} = -p_1 - p_2 + p_5 + p_6 + p_7, f_{28} = -p_1 - p_2 + p_6 + p_7 + p_8,$$
$$f_{29} = -p_1 - p_2 + p_7 + p_8 + p_9, f_{30} = -p_1 - p_2 + p_8 + p_9 + p_{10},$$
$$f_{31} = -p_1 - p_2 + p_9 + p_9 + p_{10},$$
$$f_{32} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6, f_{33} = p_1 + p_2 + p_3 - p_5 - p_6 - p_7,$$
$$f_{34} = p_1 + p_2 + p_3 - p_6 - p_7 - p_8, f_{35} = p_1 + p_2 + p_3 - p_7 - p_8 - p_9,$$
$$f_{36} = p_1 + p_2 + p_3 - p_8 - p_9 - p_{10}, f_{37} = p_1 + p_2 + p_3 - p_9 - p_9 - p_{10},$$
$$f_{38} = -p_1 - p_2 - p_3 + p_4 + p_5 + p_6 + p_7, f_{39} = -p_1 - p_2 - p_3 + p_5 + p_6 + p_7 + p_8,$$
$$f_{40} = -p_1 - p_2 - p_3 + p_6 + p_7 + p_8 + p_9, f_{41} = -p_1 - p_2 - p_3 + p_7 + p_8 + p_9 + p_{10},$$
$$f_{42} = -p_1 - p_2 - p_3 + p_8 + p_9 + p_9 + p_{10},$$
$$f_{43} = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8,$$
$$f_{44} = p_1 + p_2 + p_3 + p_4 - p_6 - p_7 - p_8 - p_9,$$
$$f_{45} = p_1 + p_2 + p_3 + p_4 - p_7 - p_8 - p_9 - p_{10},$$
$$f_{46} = p_1 + p_2 + p_3 + p_4 - p_8 - p_9 - p_9 - p_{10},$$
$$f_{47} = -p_1 - p_2 - p_3 - p_4 + p_5 + p_6 + p_7 + p_8 + p_9,$$
$$f_{48} = -p_1 - p_2 - p_3 - p_4 + p_6 + p_7 + p_8 + p_9 + p_{10},$$
$$f_{49} = -p_1 - p_2 - p_3 - p_4 + p_7 + p_8 + p_9 + p_9 + p_{10},$$
$$f_{50} = -p_1 - p_2 - p_3 - p_4 + p_8 + p_8 + p_9 + p_9 + p_{10},$$
$$f_{51} = p_1 + p_2 + p_3 + p_4 + p_5 - p_6 - p_7 - p_8 - p_9 - p_{10},$$
$$f_{52} = p_1 + p_2 + p_3 + p_4 + p_5 - p_7 - p_8 - p_9 - p_9 - p_{10},$$
$$f_{53} = p_1 + p_2 + p_3 + p_4 + p_5 - p_8 - p_8 - p_9 - p_9 - p_{10},$$
$$f_{54} = -p_1 - p_2 - p_3 - p_4 - p_5 + p_6 + p_7 + p_8 + p_9 + p_9 + p_{10},$$
$$f_{55} = -p_1 - p_2 - p_3 - p_4 - p_5 + p_7 + p_8 + p_8 + p_9 + p_9 + p_{10},$$
$$f_{56} = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - p_7 - p_8 - p_8 - p_9 - p_9 - p_{10},$$
$$f_{57} = -p_1 - p_2 - p_3 - p_4 - p_5 - p_6 + p_7 + p_7 + p_8 + p_8 + p_9 + p_9 + p_{10},$$

and the following cases are possible:

1. $f_i = 0$ for $i$ =6, 10, 11, 13, 16, 25, 26, 28, 30, 31, 33, 35, 45, 50, 54, 56 and

$f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (4p, 3p, p, 3p, 3p, p, 4p, 2p, 2p, 3p)$$

and

$$A \cong R[X,Y]/(X^6 - Y^7, X^2Y - c^p)$$

2. $f_i = 0$ for $i =$10, 12, 14, 16, 17, 18, 20, 22, 31, 38, 40, 42, 43, 49, 54, 55 and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (5p, p, 4p, 2p, 3p, 3p, 2p, 4p, p, 4p)$$

and

$$A \cong R[X,Y]/(X^8 - Y^6, X^2Y - c^p)$$

3. $f_i = 0$ for $i =$9, 10, 12, 14, 16, 18, 20, 22, 24, 36, 38, 40, 43, 45, 51, 53 and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (5p, p, 4p, 2p, 3p, 3p, 2p, 4p, p, 5p)$$

and

$$A \cong R[X,Y]/(X^{10} - Y^5, X^2Y - c^p)$$

4. $f_i = 0$ for $i =$9, 10, 12, 14, 16, 18, 20, 22, 31, 36, 38, 40, 43, 49, 51, 53, 57 and $f_i \geq 1$ otherwise: Setting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (9p, 2p, 7p, 4p, 5p, 6p, 3p, 8p, p, 9p)$$

and

$$A \cong R[X,Y]/(X^9 - Y^{11}, XY - c^p)$$

5. $f_i = 0$ for $i =$1, 2, 5, 6, 8, 12, 21, 25, 26, 30, 38, 39, 41, 42, 44, 50, 51, 52 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, p, p, q, p - q, p, p, q, p, p - q)$$

and

$$A \cong R[X,Y]/(X^5 - c^q, Y^4 - c^{p-q})$$

6. $f_i = 0$ for $i =$1, 2, 4, 5, 6, 8, 9, 20, 21, 24, 33, 37, 43, 44, 45, 46, 51, 52 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, p, p, q, p, p, p, q, p, p)$$

and

$$A \cong R[X,Y]/(X^5 - c^q, Y^4 - Xc^{p-q})$$

7. $f_i = 0$ for $i$ =1, 2, 3, 5, 6, 7, 8, 18, 21, 22, 23, 34, 35, 44, 51, 52, 53, 56 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, p, p, p, q, p, p, p, p, q)$$

and
$$A \cong R[X, Y]/(X^4 - c^q, Y^5 - Xc^{p-q})$$

8. $f_i = 0$ for $i$ =9, 10, 11, 13, 14, 16, 25, 26, 27, 28, 29, 31, 36, 40, 45, 47, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (3p, 2p, p, 2p, 2p, p, 2p, 2p, p, 3p)$$

and
$$A \cong R[X, Y]/(X^5 - Y^5, X^3Y - c^p)$$

9. $f_i = 0$ for $i$ =9, 11, 12, 15, 16, 24, 25, 26, 28, 29, 37, 38, 39, 40, 45, 46, 51, 52 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, q, p - q, q, q, q, p - q, q, p)$$

and
$$A \cong R[X, Y]/(X^5 - c^{p-q}, Y^4 - X^2c^{2q-p})$$

10. $f_i = 0$ for $i$ =6, 12, 13, 17, 21, 22, 25, 26, 30, 31, 34, 35, 42, 44, 50, 51, 52, 56 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, q, q, p - q, q, p, q, q, p - q)$$

and
$$A \cong R[X, Y]/(X^4 - c^{p-q}, Y^5 - X^2c^{2q-p})$$

11. $f_i = 0$ for $i$ =2, 5, 7, 12, 17, 18, 21, 22, 23, 31, 34, 38, 41, 42, 44, 49, 51, 53 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, p, q, p - q, p, q, p, q, p - q)$$

and
$$A \cong R[X, Y]/(X^4 - c^{p-q}, Y^5 - X^3c^{2q-p})$$

12. $f_i = 0$ for $i$ =1, 4, 6, 9, 11, 16, 25, 26, 28, 29, 30, 33, 40, 42, 43, 45, 50, 55 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, p, q, p - q, p, q, p, p - q, q, p)$$

and
$$A \cong R[X, Y]/(X^5 - c^{p-q}, Y^4 - X^4c^{2q-p})$$

13. $f_i = 0$ for $i =$3, 6, 9, 10, 13, 16, 19, 28, 32, 33, 35, 36, 37, 45, 47, 50, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (4p, 3p, p, 4p, 3p, p, 4p, 2p, 2p, 4p)$$

and
$$A \cong R[X, Y]/(X^{12} - Y^4, X^2Y - c^p)$$

14. $f_i = 0$ for $i =$3, 6, 9, 10, 13, 16, 25, 28, 31, 32, 33, 35, 36, 45, 47, 50, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_3$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (7p, 6p, p, 7p, 5p, 2p, 7p, 4p, 3p, 7p)$$

and
$$A \cong R[X, Y]/(X^{13} - Y^7, XY - c^p)$$

15. $f_i = 0$ for $i =$1, 3, 4, 7, 8, 14, 19, 23, 27, 28, 32, 35, 36, 37, 46, 48, 53, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, 2p, p, 2p, 2p, p, p, 2p, 2p, p)$$

and
$$A \cong R[X, Y]/(X^7 - Y^3, X^2Y^2 - c^p)$$

16. $f_i = 0$ for $i =$9, 10, 11, 12, 14, 15, 16, 20, 24, 29, 36, 38, 40, 43, 45, 46, 51, 53, 57 and $f_i \geq 1$ otherwise: Setting $p = p_4$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (3p, p, 2p, p, 2p, 2p, p, 2p, p, 3p)$$

and
$$A \cong R[X, Y]/(X^8 - Y^3, X^4Y - c^p)$$

17. $f_i = 0$ for $i =$3, 6, 9, 10, 13, 16, 19, 22, 31, 32, 33, 34, 35, 36, 45, 47, 50, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_9$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (3p, 2p, p, 3p, 2p, p, 3p, 2p, p, 3p)$$

and
$$A \cong R[X, Y]/(X^{11} - Y^3, X^3Y - c^p)$$

18. $f_i = 0$ for $i =$2, 4, 7, 9, 14, 18, 19, 20, 22, 23, 24, 36, 38, 40, 43, 45, 46, 53, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, 2p, p, 2p, p, p, 2p, p, 2p)$$

and
$$A \cong R[X, Y]/(X^4 - Y^4, X^2Y^3 - c^p)$$

19. $f_i = 0$ for $i =$2, 4, 6, 8, 18, 19, 20, 21, 22, 23, 24, 33, 35, 37, 43, 44, 45, 46, 52, 56 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, p, q, p, q, p, q, p, q)$$

and

$$A \cong R[X, Y]/(X^{10} - c^q, Y^2 - Xc^{p-q})$$

20. $f_i = 0$ for $i =$1, 4, 6, 9, 11, 16, 25, 26, 28, 29, 30, 31, 33, 40, 42, 43, 45, 50, 54, 55 and $f_i \geq 1$ otherwise: Setting $p = p_8$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, 2p, p, p, 2p, p, 2p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^4 - Y^4, X^4Y - c^p)$$

21. $f_i = 0$ for $i =$10, 11, 12, 13, 14, 15, 16, 17, 26, 28, 30, 38, 39, 40, 41, 42, 48, 50, 54, 55 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, p-q, q, p-q, q, p-q, q, p-q, q)$$

and

$$A \cong R[X, Y]/(X^{10} - c^q, Y^2 - X^3c^{p-2q})$$

22. $f_i = 0$ for $i =$6, 10, 11, 12, 13, 16, 17, 21, 22, 26, 30, 33, 35, 42, 43, 44, 45, 50, 52, 56 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, p-q, q, p-q, q, p, q, p-q, q)$$

and

$$A \cong R[X, Y]/(X^{10} - c^q, Y^2 - X^5c^{p-2q})$$

23. $f_i = 0$ for $i =$2, 4, 8, 14, 15, 18, 19, 20, 23, 24, 28, 37, 38, 39, 40, 41, 46, 48, 54, 55 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, q, p, q, p, q, p-q, q, p, q)$$

and

$$A \cong R[X, Y]/(X^{10} - c^q, Y^2 - X^9c^{p-2q})$$

24. $f_i = 0$ for $i =$1, 2, 3, 4, 5, 6, 7, 8, 18, 19, 20, 21, 22, 23, 32, 33, 34, 35, 43, 44 and $f_i \geq 1$ otherwise: Setting $p = p_1$ and $q = p_{10}$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (p, p, p, p, p, p, p, p, p, q)$$

and

$$A \cong R[X, Y]/(X^2 - c^q, Y^{10} - Xc^{p-q})$$

25. $f_i = 0$ for $i$ =2, 4, 8, 14, 15, 18, 19, 20, 23, 24, 28, 37, 38, 39, 40, 41, 46, 48, 54, 55, 57 and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, 2p, p, 2p, p, p, p, 2p, p)$$

and

$$A \cong R[X, Y]/(X^2 - Y^9, X^2Y - c^p)$$

26. $f_i = 0$ for $i$ =2, 5, 7, 12, 17, 18, 20, 21, 22, 23, 31, 34, 38, 41, 42, 43, 44, 49, 51, 53, 57 and $f_i \geq 1$ otherwise: Setting $p = p_5$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, 2p, p, p, 2p, p, 2p, p, p)$$

and

$$A \cong R[X, Y]/(X^5 - Y^3, X^5Y - c^p)$$

27. $f_i = 0$ for $i$ =3, 6, 9, 10, 13, 16, 18, 19, 21, 22, 24, 32, 33, 34, 35, 36, 37, 45, 47, 50, 51, 52, 56 and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, p, 2p, p, p, 2p, p, p, 2p)$$

and

$$A \cong R[X, Y]/(X^8 - Y^2, X^6Y - c^p)$$

28. $f_i = 0$ for $i$ =6, 10, 11, 12, 13, 16, 17, 21, 22, 25, 26, 30, 31, 33, 34, 35, 42, 43, 44, 45, 50, 51, 52 ,56 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, p, p, p, p, 2p, p, p, p)$$

and

$$A \cong R[X, Y]/(X^5 - Y^2, X^5Y^2 - c^p)$$

29. $f_i = 0$ for $i$ =1, 2, 3, 4, 7, 8, 9, 14, 18, 19, 23, 24, 27, 28, 36, 37, 38, 39, 40, 46, 47, 48, 53, 54, 56 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, 2p, 2p, 2p, 2p, p, p, 2p, 2p, 2p)$$

and

$$A \cong R[X, Y]/(X^{16} - Y^2, X^2Y - c^p)$$

30. $f_i = 0$ for $i$ =1, 2, 3, 4, 7, 8, 9, 14, 18, 19, 23, 24, 27, 28, 36, 37, 38, 39, 40, 46, 47, 48, 53, 54, 57 and $f_i \geq 1$ otherwise: Setting $p = p_7$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (3p, 3p, 3p, 3p, 3p, 2p, p, 3p, 3p, 3p)$$

and

$$A \cong R[X, Y]/(X^3 - Y^{17}, XY - c^p)$$

31. $f_i = 0$ for $i$ =9, 10, 11, 12, 13, 14, 15, 16, 24, 25, 26, 27, 28, 29, 36, 37, 38, 39, 40, 45, 46, 47, 51, 52, 53, 56 and $f_i \geq 1$ otherwise: Setting $p = p_6$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, p, p, p, p, p, p, p, 2p)$$

and
$$A \cong R[X, Y]/(X^4 - Y^2, X^8 Y - c^p)$$

32. $f_i = 0$ for $i$ =10, 11, 12, 13, 14, 15, 16, 17, 25, 26, 27, 28, 29, 30, 31, 38, 39, 40, 41, 42, 47, 48, 49, 50, 54, 55, 57 and $f_i \geq 1$ otherwise: Setting $p = p_2$, we have

$$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = (2p, p, p, p, p, p, p, p, p, p)$$

and
$$A \cong R[X, Y]/(X^3 - Y^2, X^6 Y - c^p)$$

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