Doctoral Dissertation (Shinshu University)

The Borel cohomology of the loop space of a homogeneous space

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Abstract

Let $B' \xrightarrow{f} B \xleftarrow{p} E$ be a diagram in which p is a fibration and the pair (f, p) of the maps is relatively formalizable. Then, we show that the rational cohomology algebra of the pullback of the diagram is isomorphic to the torsion product of algebras $H^*(B')$ and $H^*(E)$ over $H^*(B)$. Let M be a space which admits an action of a Lie group G. The isomorphism of algebras enables us to represent the cohomology of the Borel construction of the space of free (resp. based) loops on M in terms of the torsion product if M is equivariantly formal (resp. G-formal). Moreover, we compute explicitly the S^1-equivariant cohomology of the space of the based loops on the complex projective space $\mathbb{C}P^m$, where the S^1-action is induced by a linear action of S^1 on $\mathbb{C}P^m$.

Contents

Chapter 1

Introduction

1.1 Introduction

Let $f : B' \to B$ be a morphism between simply-connected spaces and $p : E \to B$ a fibration. Then we have a fibration $B' \times_B E \to B'$ which fits into the pullback diagram

$$\begin{tikzcd} B' \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"'] \\ B' \arrow[r, "f"] & B. \end{tikzcd}$$

Vigué-Poirrier [VP81, Proposition 4.4.5] has constructed the Eilenberg-Moore spectral sequence associated with the pullback diagram mentioned above by using a Sullivan representative for the map $f : B' \to B$. Moreover she proved that, as a graded vector space,

$$H^*(B' \times_B E; \mathbb{Q}) \cong \mathrm{Tor}_{H^*(B;\mathbb{Q})} \big(H^*(B'; \mathbb{Q}), H^*(E; \mathbb{Q})\big)$$

if $p : E \to B$ and $f : B' \to B$ are formalizable maps; see also [Tho82, Section V] and [FT88, Section V]. For an arbitrary underlying field, Anick constructed the Eilenberg-Moore spectral sequence with the Adams-Hilton model and exhibited existence of such an isomorphism; see [Ani85, Theorem 5.1].

One of the aims of this article is to establish an isomorphism of *algebras* between the cohomology $H^*(B' \times_B E; \mathbb{Q})$ and the torsion product mentioned above provided the given pair (p, f) of maps is *relatively formalizable*; see Definition 1.2.2 below.

Let M be a simply-connected space with an action of a connected Lie group G. Suppose that x is a base point of M which is fixed by the action of G. Then the space ΩM of loops based at x on M admits the action of G induced by that of G on M. By using the bar construction, Lillywhite has shown that there is an isomorphism,

$$H^* (EG \times_G \Omega M) \cong \mathrm{Tor}_{H^*(EG \times_G M)} (H^*(BG), H^*(BG))$$

if M is G-formal at x; see [Lil03, Proposition 6.1]. We can obtain such an isomorphism in our setting since the G-formality induces the relative formalizability of the pair of appropriate maps; see Theorem 3.2.2. Moreover we describe the Borel cohomology $H^* (EG \times_G LM)$ of the free loop space LM of an equivariantly formal space M in the sense of Goresky, Kottwitz and MacPherson [GKM98], in terms of the torsion functor; see Definition 3.2.3 and Theorem 3.2.6. This completes the program concerning the computation of the cohomology $H^* (EG \times_G LM)$, which is suggested in [Lil03, Remark 6.3]. In consequence, the torsion functor description allows us to compute explicitly the rational cohomology of the Borel construction of $\Omega \mathbb{C}P^m$ endowed with an S^1-action; see Theorems 1.2.4 and 1.2.5. We expect that our explicit computations of the Borel cohomology and our models for the Borel constructions of loop spaces advance the development of equivariant rational homotopy theory.

1.2 Results

In this section, we describe our results more precisely. In what follows, we assume that a differential graded module M is non-negative and connected, that is, $A^i = 0$ for $i < 0$ and $H^0(A) = \mathbb{Q}$. We write $H^*(X)$ for the cohomology $H^*(X;\mathbb{Q})$ of a space X with coefficients in the rational field.

We first recall the definitions of the torsion product and of a relatively formalizable pair of maps.

Definition 1.2.1. Let B, M and N be a differential graded algebra, a right B-algebra and a left B-algebra, respectively. The morphism $\varphi : B \to N$ defined by $\varphi(b) := b \cdot 1_N$ satisfies the condition that $H^0(\varphi)$ is the identity and $H^1(\varphi)$ is injective. Let $m : B \otimes \Lambda V \to N$ be a Sullivan model for φ; see Section 2.4. Then the *torsion product* $\mathrm{Tor}^*_B(M,N)$ *of* M *and* N *over* B is defined to be the homology of the derived tensor product $M \otimes^{\mathbb{L}}_B N$, namely

$$\mathrm{Tor}^*_B(M,N) := H^*\left(M \otimes^{\mathbb{L}}_B N\right).$$

Remark. Let $m : B \otimes \Lambda V \to N$ be a Sullivan model for φ. Then we see that

$$M \otimes^{\mathbb{L}}_B N = M \otimes_B (B \otimes \Lambda V).$$

Definition 1.2.2 (c.f.[Kur02, Definition 3.1])**.** Let $\alpha : X \to Z$ and $\beta : Y \to Z$ be maps with the same target. The pair (α, β) is a *relatively formalizable pair* if there exist Sullivan algebras ΛV_E, Λ_B and $\Lambda_{B'}$, quasi-isomorphisms m_E, m_B, $m_{B'}$, θ_E, θ_B and $\theta_{B'}$ and differential graded algebra morphisms φ and ψ which fit into the following homotopy commutative diagram

$$\begin{tikzcd}
A_{PL}(E) & \Lambda V_E \arrow[l, dotted, "m_E"', "\simeq"] \arrow[r, dotted, "\theta_E", "\simeq"'] & H^*(E) \\
A_{PL}(B) \arrow[u, "A_{PL}(\alpha)"] \arrow[d, "A_{PL}(\beta)"'] & \Lambda V_B \arrow[l, dotted, "m_B"', "\simeq"] \arrow[r, dotted, "\theta_B", "\simeq"'] \arrow[u, dotted, "\varphi"] \arrow[d, dotted, "\psi"'] & H^*(B) \arrow[u, "\alpha^*"'] \arrow[d, "\beta^*"] \\
A_{PL}(B') & \Lambda V_{B'} \arrow[l, dotted, "m_{B'}"', "\simeq"] \arrow[r, dotted, "\theta_{B'}", "\simeq"'] & H^*(B').
\end{tikzcd}$$

The relative formalizable pair (α, β) is nothing but to say that

$$A_{PL}(E) \xleftarrow{A_{PL}(\alpha)} A_{PL}(B) \xrightarrow{A_{PL}(\beta)} A_{PL}(B')$$

is quasi-isomorphic to the diagram

$$H^*(E) \xleftarrow{\alpha^*} H^*(B) \xrightarrow{\beta^*} H^*(B').$$

Indeed, the standard argument in the model, we have Lemma 3.1.1. Then we have the following proposition. One of our main results is described as follows. Let $p : E \to B$ be a fibration with fiber F over a simply-connected space and $f : B' \to B$ a map between simply-connected spaces. Suppose that one of $H_*(B)$, $H_*(F)$ has finite type and one of $H_*(B')$, $H_*(F)$ has finite type. The main theme of this article is concerned with the rational cohomology of the space $B' \times_B E$.

Proposition 1.2.3. *Under the same assumption as above, suppose further that (p, f) is a relatively formalizable pair. Then there exists a quasi-isomorphism $\varphi : A_{PL}(B' \times_B E) \underset{\simeq}{\to} H^*(B') \otimes^{\mathbb{L}}_{H^*(B)} H^*(E)$ of ΛV_B-algebras and ΛV_B is a minimal model for B. In particular, one have*

$$H(\varphi) : H^*(B' \times_B E) \underset{\cong}{\to} \mathrm{Tor}_{H^*(B)}\left(H^*(B'), H^*(E)\right)$$

is an isomorphism of $H^*(B)$-algebras. Here the cohomology is considered a differential graded algebra with the trivial differential.

We now discuss the cohomology of the Borel construction of the based loop space of the complex projective space $\mathbb{C}P^m$. We regard $\mathbb{C}P^m$ as a homogeneous space in the form $\frac{U(m+1)}{U(m)\times U(1)}$ whose base point is $\frac{U(m)\times U(1)}{U(m)\times U(1)}$. A homomorphism $\mu : S^1 \to U(m+1)$ induces an S^1-linear action of $\mathbb{C}P^m$. Then μ gives rise to the action on $\Omega\mathbb{C}P^m$. The Borel construction of $\Omega\mathbb{C}P^m$ associated with the action is denoted by $ES^1 \times^{\mu}_{S^1} \Omega\mathbb{C}P^m$.

Since $U(1) \times \cdots \times U(1)$ is a maximal torus of $U(m+1)$ and $\mu(S^1)$ is an abelian group, it follows that there exists an element $g \in U(m+1)$ such that $g\mu(S^1)g^{-1} \subset (U(1) \times \cdots \times U(1))$. Let $\overline{\mu} : S^1 \to U(m+1)$ be the map defined by $\overline{\mu}\left(e^{2\pi i\theta}\right) = g\mu\left(e^{2\pi i\theta}\right)g^{-1}$. Then there exist integers $\mu_1, \ldots, \mu_{m+1}$ such that $\overline{\mu}\left(e^{2\pi i\theta}\right) = \left(e^{2\pi i\theta\mu_1}, \ldots, e^{2\pi i\theta\mu_{m+1}}\right)$. We define a map $\varphi : ES^1 \times^{\mu}_{S^1} \Omega\mathbb{C}P^m \to ES^1 \times^{\overline{\mu}}_{S^1} \Omega\mathbb{C}P^m$ by $\varphi(x, m) = (x, gm)$. It is readily seen that φ is an isomorphism of the bundles over BS^1

$$\begin{tikzcd}
ES^1 \times^{\mu}_{S^1} \Omega\mathbb{C}P^m \arrow[rr, "\varphi", "\cong"'] \arrow[dr] & & ES^1 \times^{\overline{\mu}}_{S^1} \Omega\mathbb{C}P^m \arrow[dl] \\
& BS^1. &
\end{tikzcd}$$

Theorem 1.2.4. *The differential graded algebra*

$$(\mathbb{Q}[z] \otimes \Lambda(w_1) \otimes \mathbb{Q}[w_2],\ dw_2 = g(\overline{\mu})z^m w_1)$$

is a Sullivan model for $ES^1 \times^{\overline{\mu}}_{S^1} \Omega\mathbb{C}P^m$, where $|z| = 2$, $|w_1| = 1$, $|w_2| = 2m$ and $g(\overline{\mu}) = (\mu_{m+1} - \mu_1) \cdots (\mu_{m+1} - \mu_m)$. Moreover, this yields that

$$H^*\left(ES^1 \times^{\mu}_{S^1} \Omega\mathbb{C}P^m\right) \cong H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} \Omega\mathbb{C}P^m\right) \cong$$
$$\begin{cases} \mathbb{Q}[z,\ w_2] \otimes \Lambda(w_1) & (\mu_{m+1} \in \{\mu_1, \cdots, \mu_m\}) \\ \mathbb{Q}[z] \oplus \mathbb{Q}\{w_1 w_2^{l_1} z^{l_2} | l_1 \geq 0, 0 \leq l_2 \leq m-1\} & (\mu_{m+1} \notin \{\mu_1, \cdots, \mu_m\}), \end{cases}$$

as $H^*(BS^1)$-algebras.

Remark. If $m = 1$, μ_1 and μ_2 can be reordered. Indeed,

$$P(-)P^{-1} : \frac{U(2)}{U(1) \times U(1)} \longrightarrow \frac{U(2)}{U(1) \times U(1)}$$

is a morphism which preserves the base point, where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the case where $m \geq 2$, the cohomology $H^*\left(ES^1 \times^{\mu}_{S^1} \mathbb{C}P^m\right)$ does not characterize the integer μ_{m+1}, which appers in the representation $(\mu_1, \dots, \mu_{m+1})$ of the action $\overline{\mu}$; see Lemma 4.1.2. On the other hand Theorem 1.2.4 asserts that the cohomology $H^*\left(ES^1 \times^{\mu}_{S^1} \Omega\mathbb{C}P^m\right)$ characterizes μ_{m+1}.

We obtain a model for the Borel construction of the free loop space of the complex projective space $\mathbb{C}P^m$ with the S^1-action which is induced by the action on $U(m+1)$ mentioned above. In consequence, we establish the following theorem.

Theorem 1.2.5. *The differential graded algebra*

$$\left(\frac{\mathbb{Q}[c, z]}{(\rho)} \otimes \Lambda(\overline{c}) \otimes \mathbb{Q}[w], dw = \frac{\partial\rho}{\partial c}\overline{c}\right)$$

is a rational model for $ES^1 \times^{\overline{\mu}}_{S^1} L\mathbb{C}P^m$*, where* $|\overline{c}| = 1$*,* $|c| = |z| = 2$*,* $|w| = 2m$ *and* $\rho := (c - \mu_1 z) \cdots (c - \mu_{m+1} z)$*. Moreover, this yields that*

$$\begin{aligned} H^*\left(ES^1 \times^{\mu}_{S^1} L\mathbb{C}P^m\right) &\cong H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} L\mathbb{C}P^m\right) \\ &\cong H^*\left(\frac{\mathbb{Q}[c, z]}{(\rho)} \otimes \Lambda(\overline{c}) \otimes \mathbb{Q}[w], dw = \frac{\partial\rho}{\partial c}\overline{c}\right), \end{aligned}$$

as $H^*(BS^1)$*-algebras.*

The layout of the rest of this paper is as follows. In Section 3.1, we prove Proposition 1.2.3. In Section 3.2, we develop a general method for computing the Borel cohomology of loop spaces. Section 3.3 is devoted to investigating the Borel cohomology of the loop space of a homogeneous space. By relying on the results in Sections 3.2 and 3.3, we prove Theorems 1.2.4 and 1.2.5 in Sections 4.1 and 4.2.

Chapter 2

Rational homotopy theory

In this chapter, we recall briefly important facts in rational homotopy theory, which are used in this paper.

2.1 Sullivan algebras

Let $V = \bigoplus_{i=0}^{\infty} V^i$ be a graded module over $\mathbb{Q}$. The quotient graded algebra

$$\Lambda V := \frac{TV}{(x \cdot y - (-1)^{\deg x \ \deg y} y \cdot x)}$$

is called the *free commutative graded algebra* on V, where TV is the tensor algebra. If $\{v_i\}$ is a basis of V, we may write $\Lambda(\{v_i\})$ for ΛV.

A *differential graded algebra* is a graded algebra together with a linear map $d : R \to R$ of a degree 1 such that $d(xy) = d(x)y + (-1)^{\deg x} x d(y)$ and $d^2 = 0$.

Definition 2.1.1 (relative Sullivan algebra)**.** A *relative Sullivan algebra* is a commutative differential graded algebra of the form $(B \otimes \Lambda V, d)$ for which

- $(B, d) = (B \otimes 1, d)$ is a sub differential graded algebra, and $H^0(B) = \mathbb{Q}$,
- $V = \bigoplus_{p \geq 1} V^p$, (i.e. $V^0 = 0$)
- there exists an increasing sequence of graded modules $0 = V(-1) \subset V(0) \subset V(1) \subset \cdots \subset \bigcup_{k=0}^{\infty} V(k) = V$
  such that $d : V(k) \to B \otimes \Lambda V(k-1)$.

In particular, if $B = \mathbb{Q}$, we call $(\Lambda V, d)$ a *Sullivan algebra.*

2.2 The simplicial commutative cochain algebra A_{PL}

The first step is construction of the simplicial commutative cochain algebra, A_{PL}. To this end, we consider the free graded commutative algebra $\Lambda(t_0, \ldots, t_n, y_0, \ldots, y_n)$ in which the basis elements t_i have degree zero and the basis elements y_j have degree 1. Thus this algebra is the tensor product of the polynomial algebra in the variables t_i with the exterior algebra in the variables y_j. A unique derivation in this algebra is specified by $t_i \mapsto y_j$ and $y_j \mapsto 0$.

Now define $A_{PL} = \{(A_{PL})_n\}_{\geq 0}$ by:

- The cochain algebra $(A_{PL})_n$ is given by

$$(A_{PL})_n := \frac{\Lambda(t_0, \dots, t_n, y_0, \dots, y_n)}{\left(1 - \sum_{i=0}^{n} t_i,\ \sum_{j=0}^{n} y_j\right)},$$

where $dt_i = y_i$ and $dy_j = 0$.

- The face and degeneracy morphisms are the unique cochain algebra morphisms

$$\begin{cases} \partial_i : (A_{PL})_n \to (A_{PL})_{n-1} & (0 \leq i \leq n) \\ s_j : (A_{PL})_n \to (A_{PL})_{n+1} & (0 \leq j \leq n) \end{cases}$$

satisfying

$$\partial_i : t_k \mapsto \begin{cases} t_k & , k < i \\ 0 & , k = i \\ t_{k-1} & , k > i \end{cases} \text{ and } s_j : t_k \mapsto \begin{cases} t_k & , k < j \\ t_k + t_{k+1} & , k = j \\ t_{k+1} & , k > j. \end{cases}$$

The simplicial commutative cochain algebra $\{(A_{PL})_n\}_{n\geq 0}$ has differential d, face map ∂_i, and degeneracy map s_j fit into the following diagram,

$$\begin{tikzcd}
(A_{PL})^0 \arrow[r, "d"] & (A_{PL})^1 \arrow[r, "d"] & (A_{PL})^2 \arrow[r, "d"] & (A_{PL})^3 \arrow[r, "d"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\vdots \arrow[d, shift left] \arrow[d, shift left=2] & \vdots \arrow[d, shift left] \arrow[d, shift left=2] & \vdots \arrow[d, shift left] \arrow[d, shift left=2] & \vdots \arrow[d, shift left] \arrow[d, shift left=2] & & \vdots \arrow[d, shift left] \arrow[d, shift left=2] \\
(A_{PL})^0_2 \arrow[r, "d"] \arrow[u, shift left] \arrow[d, shift left, "\partial_i"] & (A_{PL})^1_2 \arrow[r, "d"] \arrow[u, shift left] \arrow[d, shift left, "\partial_i"] & (A_{PL})^2_2 \arrow[r, "d"] \arrow[u, shift left] \arrow[d, shift left, "\partial_i"] & (A_{PL})^3_2 \arrow[r] \arrow[u, shift left] \arrow[d, shift left, "\partial_i"] & \cdots & (A_{PL})_2 \arrow[u, shift left] \arrow[d, shift left, "\partial_i"] \\
(A_{PL})^0_1 \arrow[r, "d"] \arrow[u, shift left, "s_j"] \arrow[d, shift left, "\partial_i"] & (A_{PL})^1_1 \arrow[r, "d"] \arrow[u, shift left, "s_j"] \arrow[d, shift left, "\partial_i"] & (A_{PL})^2_1 \arrow[r, "d"] \arrow[u, shift left, "s_j"] \arrow[d, shift left, "\partial_i"] & (A_{PL})^3_1 \arrow[r, "d"] \arrow[u, shift left, "s_j"] \arrow[d, shift left, "\partial_i"] & \cdots & (A_{PL})_1 \arrow[u, shift left, "s_j"] \arrow[d, shift left, "\partial_i"] \\
(A_{PL})^0_0 \arrow[r, "d"] \arrow[u, shift left, "s_0"] & (A_{PL})^1_0 \arrow[r, "d"] \arrow[u, shift left, "s_0"] & (A_{PL})^2_0 \arrow[r, "d"] \arrow[u, shift left, "s_0"] & (A_{PL})^3_0 \arrow[r, "d"] \arrow[u, shift left, "s_0"] & \cdots & (A_{PL})_0 \arrow[u, shift left, "s_0"]
\end{tikzcd}$$

satisfying following formulas

$$\begin{aligned}
&\partial_i d = d\partial_i && \text{(for any } i\text{)}, \\
&s_j d = d s_j && \text{(for any } j\text{)}, \\
&\partial_i \partial_j = \partial_{j-1}\partial_i && \text{(for } i < j\text{)}, \\
&s_i s_j = s_{j+1} s_i && \text{(for } i \leq j\text{)}, \\
&\partial_i s_j = \begin{cases} s_{j-1}\partial_i & \text{(for } i < j\text{)}, \\ id_{(A_{PL})_n} & \text{(for } i = j,\ j+1\text{)}, \\ s_j \partial_{i-1} & \text{(for } i > j+1\text{)}. \end{cases}
\end{aligned}$$

Observe that $\left\{(A_{PL})^p_n,\ \{\partial_i\},\ \{s_j\}\right\}_{n\geq 0}$ is a simplicial set.

2.3 The commutative cochain algebra $A_{PL}(X)$

Let X be a topological space. Then we define a cochain algebra

$$A_{PL}(X) = \{(A_{PL})^p(X)\}_{p\geq 0}$$

by

$$A_{PL}{}^p(X) = \mathrm{Hom}_{\mathbf{Set}^{\mathbf{\Delta}^{\mathbf{op}}}}\big(S_*(X),\ A_{PL}{}^p\big);$$

that is, the set of morphisms of simplicial sets, where $S_*(X)$ is the singular simplicial set on a space X.

Proposition 2.3.1 ([FHT01, Corollary 10.10])**.** *For topological spaces X there are natural quasi-isomorphisms of cochain algebras*

$$C^*(X) \xrightarrow[\simeq]{} D(X) \xleftarrow[\simeq]{} A_{PL}(X),$$

where $D(X)$ is a third natural cochain algebra.

2.4 Sullivan models

Definition 2.4.1. 1. A *Sullivan model* for a commutative differential graded algebra (A, d) is a quasi-isomorphism

$$m : (\Lambda V, d) \xrightarrow[\simeq]{} (A, d)$$

from Sullivan algebra.

2. If X is a path-connected space, then a Sullivan model for $A_{PL}(X)$,

$$m : (\Lambda V, d) \xrightarrow[\simeq]{} A_{PL}(X)$$

is called a *Sullivan model* for X.

3. Let $\varphi : (B, d) \to (C, d)$ be a morphism between commutative differential graded algebras such that $H^0(B) = \mathbb{Q}$. A *Sullivan model* for φ is a quasi-isomorphism of the form

$$m : (B \otimes \Lambda V, d) \xrightarrow[\simeq]{} (C, d)$$

where $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra with base (B, d) and $m|_B = \varphi$.

4. If $f : X \to Y$ is a continuaus map then a Sullivan model for $A_{PL}(f)$ is called a *Sullivan model* for f.

Example 2.4.2. *The spheres, S^k.*

Let $[S^k]$ be the fundamental class of $H_k(S^k)$. This determines a unique class $\omega \in H^k\big(A_{PL}(S^k)\big)$ such that $<\omega,\ [S^k]>= 1$, where $<\cdot,\cdot>$, and $\{1,\ \omega\}$ is a basis for $H^*(S^k)$. Let Φ be a representing cocycle for ω.

Now *if k is odd* then a Sullivan model for S^k is given by

$$m : (\Lambda(e),\ 0) \underset{\simeq}{\longrightarrow} A_{PL}(S^k),$$

where deg $e = k$ and $me = \Phi$. Indeed, since k is odd, 1 and e are basis for the exterior algebra $\Lambda(e)$.

Suppose, on the other hand, *k is even.* We may still define $m : (\Lambda(e),\ 0) \underset{\simeq}{\longrightarrow} A_{PL}(S^k)$, where deg $e = k$ and $me = \Phi$. But now, deg e is even, $\Lambda(e)$ has as basis $\{1,\ e,\ e^2,\ e^3, \dots\}$ and this morphism is not a quasi-isomorphism. However, Φ^2 is certainly a coboundary. Write $\Phi^2 = d\Psi$ and extend m to

$$m : (\Lambda(e,\ e'),\ d) \underset{\simeq}{\longrightarrow} A_{PL}(S^k),$$

by setting deg $e' = 2k - 1$, $de' = e^2$ and $me' = \Psi$. This is a Sullivan model for S^k.

Lemma 2.4.3 ([FHT01, Propositions 12.1 and 14.3])**.** *1. Each commutative differential graded algebra (A, d) satisfying $H^0(A) = \mathbb{Q}$ has a Sullivan model*

$$m : (\Lambda V, d) \underset{\simeq}{\longrightarrow} (A, d).$$

2. A morphism $\varphi : (B, d) \to (C, d)$ of commutative differential graded algebras has a Sullivan model if $H^0(B) = H^0(C) = \mathbb{Q}$, $H^0(\varphi) = id_{\mathbb{Q}}$, and $H^1(\varphi)$ is injective.

$$\begin{tikzcd}
(B,d) \arrow[d, hook] \arrow[dr, "\varphi"] & \\
(B \otimes \Lambda V, d) \arrow[r, "m", "\simeq"'] & (C,d)
\end{tikzcd}$$

2.5 Models of fibrations

Let Y be a simply-connected space. Consider a Serre fibration of path connected spaces

$$p : X \to Y,$$

whose fibres are also path-connected. Let $j : F \to X$ be the inclusion of the fiber at $y_0 \in Y$. By applying the contravariant functor $A_{PL}(-)$ to the commutative diagram

$$\begin{tikzcd}
F \arrow[r, "j"] \arrow[d] & X \arrow[d, "p"] \\
\{y_0\} \arrow[r] & Y,
\end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} A_{PL}(F) & A_{PL}(X) \arrow[l, "A_{PL}(j)"'] \\ \mathbb{Q} \arrow[u] & A_{PL}(Y), \arrow[l, "\varepsilon"'] \arrow[u, "A_{PL}(p)"'] \end{tikzcd}$$

where ε is the augmentation corresponding to $\{y_0\}$.

Since Y is a simply-connected, it follows that $H^1(A_{PL}(p)) = 0$. By virtue of Lemma 2.4.3, we have a commutative diagram,

$$\begin{tikzcd} A_{PL}(Y) \arrow[r, "A_{PL}(p)"] & A_{PL}(X) \arrow[r, "A_{PL}(j)"] & A_{PL}(F) \\ (\Lambda V_Y, d) \arrow[u, "m_Y", "\simeq"'] \arrow[r, hook] & (\Lambda V_Y \otimes \Lambda V, d). \arrow[u, "m", "\simeq"'] & \end{tikzcd}$$

The augmentation $\varepsilon : \Lambda V_Y \to \mathbb{Q}$ defines a quotient Sullivan algebra

$$(\Lambda V, \overline{d}) := \mathbb{Q} \otimes_{(\Lambda V_Y, d)} (\Lambda V_Y \otimes \Lambda V, d),$$

Then we have a commutative diagram of differential graded algebras,

$$\begin{tikzcd} A_{PL}(Y) \arrow[r, "A_{PL}(p)"] & A_{PL}(X) \arrow[r, "A_{PL}(j)"] & A_{PL}(F) \\ (\Lambda V_Y, d) \arrow[u, "m_Y", "\simeq"'] \arrow[r, hook] & (\Lambda V_Y \otimes \Lambda V, d) \arrow[u, "m", "\simeq"'] \arrow[r, "\varepsilon \cdot id"] & (\Lambda V, \overline{d}). \arrow[u, "\overline{m}"] \end{tikzcd}$$

Proposition 2.5.1 ([FHT01, Proposition 15.5])**.** *Suppose one of the graded spaces $H_*(Y;\mathbb{Q})$ and $H_*(F;\mathbb{Q})$ are of finite type. Then*

$$\overline{m} : (\Lambda V, \overline{d}) \underset{\simeq}{\longrightarrow} A_{PL}(F)$$

is a quasi-isomorphism.

2.6 Models of pullbacks of fibrations

Consider the pullback diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ A \arrow[r, "f"] & Y \end{tikzcd}$$

in which p and q are Serre fibrations with fiber F, Z and X are path connected and A and Y are simply-connected. Choose basepoints a_0 and y_0 so that $f(a_0) = y_0$. Assume further that one of $H_*(F;\mathbb{Q})$ and $H_*(A;\mathbb{Q})$ has finite type and so is one of $H_*(F;\mathbb{Q})$ and $H_*(Y;\mathbb{Q})$.

Choose Sullivan models $m_Y : (\Lambda V_Y, d) \to A_{PL}(Y)$ and $n_A : (\Lambda W_A, d) \to A_{PL}(A)$. Let

$$\psi : (\Lambda V_Y, d) \to (\Lambda W_A, d)$$

a morphism of differential graded algebras satisfying $n_A\psi = A_{PL}(f)m_Y$. By applying Proposition 2.5.1, we have a commutative diagram

$$\begin{tikzcd}
A_{PL}(Y) \arrow[rr, "A_{PL}(p)"] & & A_{PL}(X) \arrow[rr] & & A_{PL}(F) & \\
& (\Lambda V_Y, d) \arrow[ul, "m_Y", "\simeq"'] \arrow[rr] & & (\Lambda V_Y \otimes \Lambda V, d) \arrow[ul, "m", "\simeq"'] \arrow[rr, "\varepsilon\cdot id"] & & (\Lambda V, \overline{d}), \arrow[ul, "\overline{m}", "\simeq"']
\end{tikzcd}$$

in which all the slanting arrows are Sullivan models.

Since $n_A\psi = A_{PL}(f)m_Y$, we have

$$\begin{tikzcd}
A_{PL}(Y) \arrow[rr, "A_{PL}(p)"] \arrow[dd, "A_{PL}(f)"'] & & A_{PL}(X) \arrow[rr] \arrow[dd, "A_{PL}(g)"'] & & A_{PL}(F) \arrow[dd, equal] & \\
& (\Lambda V_Y, d) \arrow[ul, "m_Y", "\simeq"'] \arrow[rr] \arrow[dd, "\psi"'] & & (\Lambda V_Y \otimes \Lambda V, d) \arrow[ul, "m", "\simeq"'] \arrow[rr, "\varepsilon\cdot id"] & & (\Lambda V, \overline{d}) \arrow[ul, "\overline{m}", "\simeq"'] \\
A_{PL}(A) \arrow[rr, "A_{PL}(q)"] & & A_{PL}(Z) \arrow[rr] & & A_{PL}(F) & \\
& (\Lambda W_A, d). \arrow[ul, "n_A", "\simeq"'] & & & &
\end{tikzcd}$$

By definition, we see that

$$(\Lambda W_A \otimes \Lambda V, d) := (\Lambda W_A, d) \otimes_{(\Lambda V_Y, d)} (\Lambda V_Y \otimes \Lambda V, d)$$

is a relative Sullivan algebra with base algebra $(\Lambda W_A, d)$. The pushout construction yields the morphism

$$\xi := A_{PL}(q)n_A \cdot A_{PL}(g)m : (\Lambda W_A, d) \otimes_{(\Lambda V_Y, d)} (\Lambda V_Y \otimes \Lambda V, d) \to A_{PL}(Z),$$

which fits into the commutative diagram

$$\begin{tikzcd}
A_{PL}(Y) \arrow[rr, "A_{PL}(p)"] \arrow[dd, "A_{PL}(f)"'] & & A_{PL}(X) \arrow[rr] \arrow[dd, "A_{PL}(g)"'] & & A_{PL}(F) \arrow[dd, equal] & \\
& (\Lambda V_Y, d) \arrow[ul, "m_Y", "\simeq"'] \arrow[rr] \arrow[dd, "\psi"'] & & (\Lambda V_Y \otimes \Lambda V, d) \arrow[ul, "m", "\simeq"'] \arrow[rr, "\varepsilon\cdot id"] \arrow[dd] & & (\Lambda V, \overline{d}) \arrow[ul, "\overline{m}", "\simeq"'] \arrow[dd, equal] \\
A_{PL}(A) \arrow[rr, "A_{PL}(q)"] & & A_{PL}(Z) \arrow[rr] & & A_{PL}(F) & \\
& (\Lambda W_A, d) \arrow[ul, "n_A", "\simeq"'] \arrow[rr] & & (\Lambda W_A \otimes \Lambda V, d) \arrow[ul, "\xi"] \arrow[rr, "\varepsilon\cdot id"] & & (\Lambda V, \overline{d}). \arrow[ul, "\overline{m}", "\simeq"']
\end{tikzcd}$$

Proposition 2.6.1 ([FHT01, Proposition 15.8]). *Under the same assumption as above, the morphism ξ is a Sullivan model for Z.*

Chapter 3

The Borel cohomology of loop spaces

3.1 The cohomology of the pullback with a relatively formalizable pair

In this short section, we prove Proposition 1.2.3.

Let $f : B' \to B$ be a map between simply-connected based spaces, $p : E \to B$ a fibration with fiber F, and $B' \times_B E$ the pullback

$$\begin{tikzcd} B' \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"'] \\ B' \arrow[r, "f"] & B. \end{tikzcd} \tag{3.1}$$

Assume that one of $H_*(B)$, $H_*(F)$ has finite type and one of $H_*(B')$, $H_*(F)$ has finite type. By using cofibrant replacements, we have the following lemma.

Lemma 3.1.1. *A relatively formalizable pair (p, f) induces a strictly commutative diagram*

$$\begin{tikzcd} A_{PL}(E) & \Lambda V_B \otimes \Lambda W_E \arrow[l, "n_E"', "\simeq"] \arrow[r, "\eta_E", "\simeq"'] & H^*(E) \\ A_{PL}(B) \arrow[u, "A_{PL}(p)"] \arrow[d, "A_{PL}(f)"'] & \Lambda V_B \arrow[l, "n_B"', "\simeq"] \arrow[r, "\eta_B", "\simeq"'] \arrow[u, hook, "i"] \arrow[d, hook, "j"'] & H^*(B) \arrow[u, "p^*"'] \arrow[d, "f^*"] \\ A_{PL}(B') & \Lambda V_B \otimes \Lambda W_{B'} \arrow[l, "n_{B'}"', "\simeq"] \arrow[r, "\eta_{B'}", "\simeq"'] & H^*(B'), \end{tikzcd}$$

in which $\Lambda V_B \otimes \Lambda W_E$, $\Lambda V_B \otimes \Lambda W_{B'}$ are relative Sullivan algebras with the base algebra ΛV_B, horizontal arrows are quasi-isomorphisms, i, j are the inclusions and $n_B^* = \eta_B^*$.

Proof. Recall the diagram mentioned in Definition 1.2.2. Put $n_B = m_B$ and $\eta_B = m_B^*(\theta_B^*)^{-1}\theta_B$. It is readily seen that $n_B^* = \eta_B^*$. We then have a diagram

$$\begin{tikzcd} A_{PL}(E) & (\Lambda V_E, d) \arrow[l, "m_E"', "\simeq"] \arrow[r, "\theta_E", "\simeq"'] & H^*(E) \arrow[r, "(\theta_E^*)^{-1}", "\cong"'] & H(\Lambda V_E, d) \arrow[r, "m_E^*", "\cong"'] & H^*(E) \\ A_{PL}(B) \arrow[u, "A_{PL}(p)"] \arrow[d, "A_{PL}(f)"'] & (\Lambda V_B, d) \arrow[l, "m_B"', "\simeq"] \arrow[r, "\theta_B", "\simeq"'] \arrow[u, "\varphi"'] \arrow[d, "\psi"] & H^*(B) \arrow[r, "(\theta_B^*)^{-1}", "\cong"'] \arrow[u, "p^*"'] \arrow[d, "f^*"] & H(\Lambda V_B, d) \arrow[r, "m_B^*", "\cong"'] \arrow[u, "H(\varphi)"'] \arrow[d, "H(\psi)"] & H^*(B) \arrow[u, "p^*"'] \arrow[d, "f^*"] \\ A_{PL}(B') & (\Lambda V_{B'}, d) \arrow[l, "m_{B'}"', "\simeq"] \arrow[r, "\theta_{B'}", "\simeq"'] & H^*(B') \arrow[r, "(\theta_{B'}^*)^{-1}", "\cong"'] & H(\Lambda V_{B'}, d) \arrow[r, "m_{B'}^*", "\cong"'] & H^*(B'), \end{tikzcd}$$

in which the left four squares are homotopy commutative and the right four squares are strictly commutative.

Consider the homotopy commutative squares consisting of solid arrows,

$$
\begin{tikzcd}
A_{PL}(E) & & (\Lambda V_E, d) \arrow[ll, "m_E", "\simeq"'] \arrow[rr, "m_E^*(\theta_E^*)^{-1}\theta_E", "\simeq"'] & & H^*(E) \\
& (\Lambda V_B \otimes \Lambda W'_E, d) \arrow[ul, dotted, "\alpha", "\simeq"'] \arrow[ur, dotted, "\beta", "\simeq"'] & & (\Lambda V_B \otimes \Lambda W_E, d) \arrow[ll, dotted, "\delta", "\simeq"'] \arrow[ur, dotted, "\gamma", "\simeq"'] & \\
A_{PL}(B) \arrow[uu, "A_{PL}(p)"] & & (\Lambda V_B, d) \arrow[ll, "n_B", "\simeq"'] \arrow[rr, "\eta_B", "\simeq"'] \arrow[uu, "\varphi"] \arrow[ul, hook, dotted, "i'"] \arrow[ur, hook', dotted, "i"'] & & H^*(B). \arrow[uu, "p^*"']
\end{tikzcd}
$$

Since B is simply-connected, it follows from [FHT01, Proposition 14.3], that there exists a Sullivan model α for $A_{PL}(p)n_B$. We see that $\alpha i' = A_{PL}(p)n_B \sim m_E\varphi$ (homotopic rel $(\Lambda V_B, d)$). By employing the Lifting Lemma [FHT01, Proposition 14.6] and [FHT95, Lemma 3.6], we deduce that there exists a morphism β such that $\beta i' = \varphi$ and $m_E\beta \sim \alpha$. We choose a Sullivan model γ for $p^*\eta_B$. The Lifting Lemma enables us to get a morphism δ. Put $n_E := \alpha\delta$ and $\eta_E := \gamma$. Then we see that $n_E i = A_{PL}(p)n_B$ and $\eta_E i = p^*\eta_B$. In the same way, we obtain quasi-isomorphisms $n_{B'}$ and $\eta_{B'}$ such that $n_{B'}j = A_{PL}(f)n_B$ and $\eta_{B'}j = f^*\eta_B$. $\square$

Proof of Proposition 1.2.3. By Lemma 3.1.1, we have the following commutative diagram

$$
\begin{tikzcd}
A_{PL}(E) & \Lambda V_B \otimes \Lambda W_E \arrow[l, "n_E", "\simeq"'] \arrow[r, "\eta_E", "\simeq"'] & H^*(E) \\
A_{PL}(B) \arrow[u, "A_{PL}(p)"] \arrow[d, "A_{PL}(f)"'] & \Lambda V_B \arrow[l, "n_B", "\simeq"'] \arrow[r, "\eta_B", "\simeq"'] \arrow[u, hook, "i"] \arrow[d, hook, "j"'] & H^*(B) \arrow[u, "p^*"'] \arrow[d, "f^*"] \\
A_{PL}(B') & \Lambda V_B \otimes \Lambda W_{B'} \arrow[l, "n_{B'}", "\simeq"'] \arrow[r, "\eta_{B'}", "\simeq"'] & H^*(B'),
\end{tikzcd}
$$

where $n_B^* = \eta_B^*$. By applying [FHT01, Proposition 15.8] to the pullback diagram (3.1), we have a quasi-isomorphism,

$$(n_{B'}j) \cdot (n_E i) : (\Lambda V_B \otimes \Lambda W_{B'}) \otimes_{\Lambda V_B} (\Lambda V_B \otimes \Lambda W_E) \to A_{PL}(B' \times_B E)$$

of differential graded ΛV_B-algebras. Consider the following pushout diagram

$$
\begin{tikzcd}
\Lambda V_B \arrow[r, hook, "i"] \arrow[d, "\eta_B", "\simeq"'] & \Lambda V_B \otimes \Lambda W_E \arrow[d, "\overline{\eta_B}", "\simeq"'] \arrow[ddr, bend left, "\eta_E", "\simeq"'] & \\
H^*(B) \arrow[r, hook, "j"] \arrow[drr, bend right, "p^*"] & H^*(B) \otimes \Lambda W_E \arrow[dr, dotted, "u", "\simeq"'] & \\
& & H^*(E).
\end{tikzcd}
$$

It follows from [FHT01, Lemma 14.2] that $\overline{\eta_B}$ is a quasi-isomorphism. By applying [FHT01, Theorem 6.10], we have a quasi-isomorphism $\eta_{B'} \otimes_{\eta_B} \overline{\eta_B}$ and a commutative diagram

$$\begin{tikzcd}
A_{PL}(B) \arrow[rr] & & A_{PL}(B' \times_B E) \\
\Lambda V_B \arrow[u, "n_B", "\simeq"'] \arrow[d, "\eta_B"', "\simeq"] \arrow[rr, hook] & & (\Lambda V_B \otimes \Lambda W_{B'}) \otimes_{\Lambda V_B} (\Lambda V_B \otimes \Lambda W_E) \arrow[u, "(n_{B'}j)\cdot(n_E i)", "\simeq"'] \arrow[d, "\eta_{B'}\otimes_{\eta_B}\overline{\eta_B}"', "\simeq"] \\
H^*(B) \arrow[rr, hook] & & H^*(B') \otimes_{H^*(B)} (H^*(B) \otimes \Lambda W_E).
\end{tikzcd}$$

This diagram yields a quasi-isomorphism $\varphi : A_{PL}(B' \times_B E) \underset{\simeq}{\to} H^*(B') \otimes^{\mathbb{L}}_{H^*(B)} H^*(E)$ of ΛV_B-algebras. Moreover,

$$\Big[H\big(\eta_{B'} \otimes_{\eta_B} \overline{\eta_B}\big)\Big] \circ \Big[H\big((n_{B'}j) \cdot (n_E i)\big)\Big]^{-1} :$$
$$H^*(B' \times_B E) \underset{\cong}{\to} H\left(H^*(B') \otimes^{\mathbb{L}}_{H^*(B)} H^*(E)\right)$$

is an isomorphism as an $H^*(B)$-algebras.

On the other hand, $H^*(B) \otimes \Lambda W_E$ is a free resolution of $H^*(E)$ as an $H^*(B)$-algebra because u is a quasi-isomorphism and $uj = p^*$. By definition, we have

$$\mathrm{Tor}_{H^*(B)}\big(H^*(B'), H^*(E)\big) = H^*\left(H^*(B') \otimes^{\mathbb{L}}_{H^*(B)} H^*(E)\right).$$

This completes the proof. □

3.2 The G-equivariant cohomology of loop spaces

Let G be a compact simply-connected Lie group, M a G-space and x an element of the fixed point set M^G. The based loop space ΩM and the free loop space LM are regarded as G-spaces with the actions induced by the action on M. Denote $m_{BG} : H^*(BG) \to A_{PL}(BG)$ by the quasi-isomporphism which is constructed in [FOT08, Example 2.42]. The maps $\xi_x : BG \to EG \times_G M$ and $\zeta : EG \times_G PM \to EG \times_G M$ are induced by the inclusion $\{x\} \hookrightarrow M$ and the natural surjection $PM \to M$, respectively. Let $\Delta : M \to M \times M$ be the diagonal map and $(e_0, e_1) : M^I \to M \times M$ the evaluation map. Then we discuss appropriate conditions that (ζ, ξ_x) and $(EG \times_G (e_0, e_1), EG \times_G \Delta)$ are relatively formalizable pairs. In consequence, we can describe the cohomologies of $EG \times_G \Omega M$ and $EG \times_G LM$ in terms of torsion products.

Definition 3.2.1 (c.f.[Lil03, Definition 3.2])**.** We call a G-space M *G-formal at x* if there are a relative Sullivan algebra $H^*(BG) \otimes \Lambda V$ with base $H^*(BG)$, a morphism φ and quasi-isomorphisms m, θ which fit into the following homotopy commutative diagram

$$\begin{tikzcd}
A_{PL}(EG \times_G M) \arrow[d, "A_{PL}(\xi_x)"'] & H^*(BG) \otimes \Lambda V \arrow[l, dotted, "m", "\simeq"'] \arrow[r, dotted, "\theta", "\simeq"'] \arrow[d, dotted, "\varphi"'] & H^*(EG \times_G M) \arrow[d, "\xi_x^*"'] \\
A_{PL}(BG) & H^*(BG) \arrow[l, "m_{BG}", "\simeq"'] \arrow[r, equal] & H^*(BG).
\end{tikzcd}$$

Remark. Strictly saying, Lillywhite describes the notion of G-formality in terms of the Cartan model for the Borel construction $EG \times_G M$.

Theorem 3.2.2 (c.f.[Lil03, Proposition 6.1])**.** *If M is G-formal at x then as an $H^*(BG)$-algebra*

$$H^*(EG \times_G \Omega M) \cong \mathrm{Tor}_{H^*(EG \times_G M)}\left(H^*(BG), H^*(BG)\right).$$

Remark. We stress that the G-formality of a G-space induces the relative formalizable pair (ζ, ξ_x). This fact plays a key role in our proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. Let $\chi : EG \times_G PM \to BG$ and $\omega : BG \to EG \times_G PM$ be the homotopy equivalences induced by the natural surjection $PM \to \{x\}$ and the inclusion $\{x\} \to PM$, respectively. Then we see that

$$A_{PL}(\omega)A_{PL}(\chi)m_{BG}\varphi = m_{BG}\varphi A_{PL}(\xi_x)m \sim A_{PL}(\omega)A_{PL}(\zeta)m_{BG}.$$

The result [FHT01, Proposition 12.9] enables us to obtain $A_{PL}(\chi)m_{BG}\varphi \sim A_{PL}(\zeta)m_{BG}$. Because $\omega^*\zeta^* = \xi_x^*$ and χ^* is the inverse of ω^*, then we see that $\chi^*\xi_x^* = \zeta^*$. Then we have a homotopy commutative diagram

$$\begin{tikzcd}
A_{PL}(EG \times_G PM) & A_{PL}(BG) \arrow[l, "A_{PL}(\chi)"', "\simeq"] & H^*(BG) \arrow[l, "m_{BG}"', "\simeq"] \arrow[r, "\chi^*", "\simeq"'] & H^*(EG \times_G PM) \\
A_{PL}(EG \times_G M) \arrow[u, "A_{PL}(\zeta)"] \arrow[d, "A_{PL}(\xi_x)"'] & & \Lambda V \arrow[ll, "m", "\simeq"'] \arrow[r, "\theta", "\simeq"'] \arrow[u, "\varphi"] \arrow[d, "\varphi"'] & H^*(EG \times_G M) \arrow[u, "\zeta^*"'] \arrow[d, "\xi_x^*"] \arrow[ul, "\xi_x^*"'] \\
A_{PL}(BG) & & H^*(BG) \arrow[ll, "m_{BG}", "\simeq"'] \arrow[r, equal] & H^*(BG).
\end{tikzcd}$$

Therefore $BG \xrightarrow{\xi_x} EG \times_G M \xleftarrow{\zeta} EG \times_G PM$ is a relatively formalizable pair. Since ζ is a fibration, we can apply Proposition 1.2.3 to the following diagram

$$\begin{tikzcd}
EG \times_G \Omega M \arrow[r] \arrow[d] & EG \times_G PM \arrow[d, "\zeta"'] \\
BG \arrow[r, "\xi_x"] & EG \times_G M.
\end{tikzcd}$$

This completes the proof. $\square$

Remark. According to the proof, if it says strictly, the isomorphism is as an $H^*(EG \times_G M)$-algebras.

Next we consider the G-equivariant cohomology of the free loop space LM.

Definition 3.2.3 ([GKM98, (1.2)])**.** We say that a G-space M is *equivariantly formal* if the spectral sequence

$$H^p(BG; H^q(M)) \Longrightarrow H^{p+q}(EG \times_G M)$$

for the fibration $EG \times_G M \to BG$ collapses at the E_2-term.

Definition 3.2.4 ([Lil03, Definition 3.2])**.** A G-space M is called *G-formal* if there are a relative Sullivan algebra ΛV with the base $H^*(BG)$ and a morphism φ and quasi-morphisms m, θ fit into the following commutative diagram

$$\begin{tikzcd} A_{PL}(EG \times_G M) & \Lambda V \arrow[l, dotted, "m"', "\simeq"] \arrow[r, dotted, "\theta", "\simeq"'] & H^*(EG \times_G M). \end{tikzcd}$$

Lemma 3.2.5 ([Lil03, Proposition 4.8])**.** *Let M be equivariantly formal. If M is G-formal, then so is $M \times M$.*

Proof. We have a pullback diagram of the form

$$\begin{tikzcd} EG \times_G (M \times M) \arrow[r, "EG\times_G pr_2"] \arrow[d, "EG\times_G pr_1"'] & EG \times_G M \arrow[d, "\pi"] \\ EG \times_G M \arrow[r, "\pi"] & BG, \end{tikzcd}$$

where $\pi : EG \times_G M \to BG$ is the Borel fibration and $pr_i : M \times M \to M$ denotes the projection on the ith factor. By proposition A.2.1, $EG \times_G pr_i : EG \times_G M \to BG$ is a Serre fibration. Apply [FHT01, Proposition 15.8] to the commutative diagram

$$\begin{tikzcd} A_{PL}(EG \times_G M) & A_{PL}(BG) \arrow[l, "A_{PL}(EG\times_G pr_1)"'] \arrow[r, "A_{PL}(EG\times_G pr_2)"] & A_{PL}(EG \times_G M) \\ \Lambda V \otimes H^*(BG) \arrow[u, "m", "\simeq"'] & H^*(BG) \arrow[l, hook'] \arrow[r, hook] \arrow[u, "m_{BG}", "\simeq"'] & H^*(BG) \otimes \Lambda V. \arrow[u, "m", "\simeq"'] \end{tikzcd}$$

We obtain a commutative diagram

$$\begin{tikzcd}
A_{PL}(EG \times_G (M \times M)) & & & A_{PL}(EG \times_G M) \arrow[lll, "A_{PL}(EG \times_G pr_2)"'] & & \\
& \Lambda W \arrow[ul, "m_W"', "\simeq"] \arrow[dr, "\theta\otimes\theta", "\simeq"'] & & & H^*(BG) \otimes \Lambda V \arrow[ul, "m"', "\simeq"] \arrow[lll, hook', "i_2"'] \arrow[dr, "\theta", "\simeq"'] & \\
& & P & & & H^*(EG \times_G M) \arrow[lll, hook'] \\
A_{PL}(EG \times_G M) \arrow[uuu, "A_{PL}(EG \times_G pr_1)"] & & & A_{PL}(BG) \arrow[lll, "A_{PL}(\pi)"'] \arrow[uuu, "A_{PL}(\pi)"'] & & \\
& \Lambda V \otimes H^*(BG) \arrow[ul, "m"', "\simeq"] \arrow[uuu, hook, "i_1"] \arrow[dr, "\theta", "\simeq"'] & & & H^*(BG) \arrow[ul, "m_{BG}"', "\simeq"] \arrow[lll, hook'] \arrow[uuu, hook] \arrow[dr, equal] & \\
& & H^*(EG \times_G M) \arrow[uuu, hook] & & & H^*(BG), \arrow[lll, "H(A_{PL}(\pi) m_{BG})"'] \arrow[uuu, "H(A_{PL}(\pi) m_{BG})"']
\end{tikzcd}$$

where m_W is the quasi-isomorphism $A_{PL}(EG \times_G pr_1) m \cdot A_{PL}(EG \times_G pr_2) m$, ΛW and $P = (P, 0)$ are the pushouts $(\Lambda V \otimes H^*(BG)) \otimes_{H^*(BG)} (H^*(BG) \otimes \Lambda V)$ and $H^*(EG \times_G M) \otimes_{H^*(BG)} H^*(EG \times_G M)$. Since M is equivariantly formal, we see that

$$H^*(BG) \otimes H^*(M) \cong H^*(EG \times_G M)$$

as a vector space. The $H^*(BG)$-module map $\varrho : H^*(BG) \otimes H^*(M) \to H^*(EG \times_G M)$ defined by $\varrho(x \otimes y) = \pi^*(x) \cdot \overline{y}$ is an epimorphism. Thus the map φ is an isomorphism of $H^*(BG)$-modules. So that $H^*(EG \times_G M)$ is a free $H^*(BG)$-module. Since θ is a quasi-isomorphism, it follows from [FHT01, Proposition 6.7(ii)] that $\theta \otimes \theta$ is a quasi-isomorphism.

We consider the Borel fibration $\pi' : EG \times_G (M \times M) \to BG$. Observe that

$$\pi' = \pi \circ (EG \times_G pr_1) = \pi \circ (EG \times_G pr_2)\,.$$

We have

$$A_{PL}(EG \times_G (M \times M)) \xleftarrow[\simeq]{m_W} \Lambda W \xrightarrow[\simeq]{\theta\otimes\theta} P.$$

In the cohomology, we have

$$H^*(EG \times_G (M \times M)) \xleftarrow[\cong]{H(m_W)} H(\Lambda W) \xrightarrow[\cong]{H(\theta\otimes\theta)} P.$$

The two diagrams above enable us to conclude that $M \times M$ is a G-formal space. In fact, we have

$$A_{PL}(EG \times_G (M \times M)) \xleftarrow[\simeq]{m_W} \Lambda W \xrightarrow[\simeq]{\theta_W} H^*(EG \times_G M),$$

where $\theta_W = m_W^*[(\theta \otimes \theta)^*]^{-1} \theta \otimes \theta$. $\square$

Theorem 3.2.6. *Let M be a G-formal space. Suppose further that M is equivariantly formal. Then as an $H^*(BG)$-algebra,*

$$H^*(EG \times_G LM) \cong \mathrm{Tor}_{H^*(EG\times_G(M\times M))}\left(H^*(EG \times_G M), H^*(EG \times_G M)\right).$$

Proof. The same argument as in the proof of Lemma 3.2.5 enables us to obtain a pullback diagram of fibrations

$$\begin{tikzcd} EG \times_G LM \arrow[r] \arrow[d] & EG \times_G M^I \arrow[d, "EG\times_G(e_0,e_1)"'] \\ EG \times_G M \arrow[r, "EG\times_G\Delta"] & EG \times_G (M \times M). \end{tikzcd}$$

It is enough to show that $(EG \times_G (e_0, e_1), EG \times_G \Delta)$ is a relatively formalizable pair. In fact, Proposition 1.2.3 deduces the result. We shall construct morphisms φ, ψ, m_I and θ_I which fit into the homotopy commutative diagram

$$\begin{tikzcd} A_{PL}\left(EG \times_G M^I\right) & \Lambda V \arrow[l, dotted, "m_I"', "\simeq"] \arrow[r, dotted, "\theta_I", "\simeq"'] & H^*\left(EG \times_G M^I\right) \\ A_{PL}\left(EG \times_G (M \times M)\right) \arrow[u, "A_{PL}(EG\times_G(e_0,e_1))"] \arrow[d, "A_{PL}(EG\times_G\Delta)"'] & \Lambda W \arrow[l, dotted, "m_W"', "\simeq"] \arrow[r, dotted, "\theta_W", "\simeq"'] \arrow[u, dotted, "\psi"] \arrow[d, dotted, "\varphi"] & H^*\left(EG \times_G (M \times M)\right) \arrow[u, "(EG\times_G(e_0,e_1))^*"'] \arrow[d, "(EG\times_G\Delta)^*"] \\ A_{PL}\left(EG \times_G M\right) & \Lambda V \arrow[l, "m"', "\simeq"] \arrow[r, "\theta", "\simeq"'] & H^*\left(EG \times_G M\right), \end{tikzcd}$$

where ΛW, m_W and θ_W are the same maps of differential graded algebras as in the proof of the Lemma 3.2.5.

Consider a pullback diagram of the fibrations

$$\begin{tikzcd} EG \times_G (M \times M) \arrow[r, "EG\times_G pr_2"] \arrow[d, "EG\times_G pr_1"'] & EG \times_G M \arrow[d, "\pi"] \\ EG \times_G M \arrow[r, "\pi"] & BG. \end{tikzcd}$$

where $pr_1,\ pr_2 : M \times M \to M$ are the projections.

Let $i_1,\ i_2 : \Lambda V \hookrightarrow \Lambda W$ be the inclusions and put $\varphi = id \cdot id$. Because $A_{PL}(EG \times_G pr_j)A_{PL}(EG \times_G \Delta) = id$ where $j = 1$ or 2, then we see that

$$m\varphi i_j = m = A_{PL}(EG \times_G pr_j)A_{PL}(EG \times_G \Delta)m.$$

It follows from the proof of Lemma 3.2.5 that $A_{PL}(EG \times_G pr_j)m = m_W i_j$. Then we see that $m\varphi i_j = A_{PL}(EG \times_G \Delta)m_W i_j$ and $m\varphi = A_{PL}(EG \times_G \Delta)m_W$. In the same way, we have $\theta\varphi = (EG \times_G^{\mu} \Delta)^*\theta_W$.

Homotopy equivalence $\rho : M^I \to M$ is defined by $\rho(\gamma) = \gamma(0)$ if $\gamma \in M^I$. We see that $\Delta\rho \sim (e_0, e_1)$. This enables us to construct a homotopy commutative diagram

$$\begin{tikzcd}
 & A_{PL}\left(EG \times_G M\right) \arrow[dl, "A_{PL}(EG\times_G\rho)"', "\simeq"] & & H^*\left(EG \times_G M\right) \arrow[dr, "(EG\times_G\rho)^*", "\cong"'] & \\
A_{PL}\left(EG \times_G M^I\right) & & \Lambda V \arrow[ul, "m"', "\simeq"] \arrow[ur, "\theta", "\simeq"'] & & H^*\left(EG \times_G M^I\right). \\
 & A_{PL}\left(EG \times_G (M\times M)\right) \arrow[ul, "A_{PL}(EG\times_G(e_0,e_1))"] \arrow[uu, "A_{PL}(EG\times_G\Delta)"'] & \Lambda W \arrow[l, "m_W"', "\simeq"] \arrow[u, "\varphi"] \arrow[r, "\theta_W", "\simeq"'] & H^*\left(EG \times_G (M\times M)\right) \arrow[uu, "(EG\times_G\Delta)^*"] \arrow[ur, "(EG\times_G(e_0,e_1))^*"'] &
\end{tikzcd}$$

Define $\psi = \varphi$, $m_I = A_{PL}\left(EG \times_G \rho\right) m$ and $\theta_I = \left(EG \times_G \rho\right)^* \theta$. It turns out that $(EG \times_G (e_0, e_1), EG \times_G \Delta)$ is a relatively formalizable pair with m_W and θ_W constructed in the proof of the Lemma 3.2.5. Then we can apply Proposition 1.2.3. $\square$

3.3 The Borel cohomology of the loop spaces of a homogeneous space

Let G and H be compact simply-connected Lie groups, K a closed subgroup of H and $\mu : G \to H$ a morphism of Lie groups. The homogeneous space H/K admits the action of G defined by $g \cdot hK = (\mu(g)h)\, K$. Let $EG \times^{\mu}_{G} H/K$ be the Borel construction defined by the action. We have the Borel fibration of the form $\pi : EG \times^{\mu}_{G} H/K \to BG$. Let $B\mu : BG \to BH$ and $B\nu : BK \to BH$ be the maps induced by μ and the inclusion $\nu : K \hookrightarrow H$, respectively. Put $\Lambda U = H^*(BG)$, $\Lambda V = H^*(BH)$ and $\Lambda W = H^*(BK)$. Let sV be the graded vector space defined by $(sV)^i = V^{i+1}$ for any i. Then we have a differential graded algebra of the from $\Lambda U \otimes \Lambda W \otimes \Lambda(sV)$, where d is defined by $dx = 0$ if $x \in U \oplus W$ and $d(sv) = (B\mu)^*(v) - (B\nu)^*(v)$ if $sv \in sV$.

Recall the differential graded algebra map $m_{BG} : H^*(BG) \to A_{PL}(BG)$ mentioned in Section 3.2.

Proposition 3.3.1. *With the same notation as above, the commutative differential graded algebra $\Lambda U \otimes \Lambda W \otimes \Lambda(sV)$ is a Sullivan model for the morphism $A_{PL}(\pi) m_{BG}$.*

Proof. Consider the following pullback diagram of the fibrations,

$$\begin{tikzcd}
EG \times^{\mu}_{G} H/K \arrow[r, "f"] \arrow[d, "\pi"'] & EH/K \arrow[d, "\pi'"] \\
BG \arrow[r, "B\mu"] & BH,
\end{tikzcd} \tag{3.2}$$

where EH admits the action of K which is induced by the inclusion $\nu : K \hookrightarrow H$. We construct a Sullivan model for $A_{PL}(\pi') m_{BH}$ as follows.

We define a differential d on $\Lambda V \otimes \Lambda W \otimes \Lambda(sV)$ by $dx = 0$ if $x \in V \oplus W$ and $d(sv) = (B\nu)^*(v) - v$ if $sv \in sV$.

In order to define a quasi-isomorphism

$$\big(\Lambda V \otimes \Lambda W \otimes \Lambda(sV),\ d(sv) = (B\nu)^*(v) - v\big) \to A_{PL}(EH/K),$$

we observe that, for a base $w \in W$, there exists a cycle w' in $A_{PL}(EH/K)$ such that $(E\nu/K)^*[w'] = w$. In fact, $E\nu/K : BK \to EH/K$ is a week homotopy equivalence. Moreover, $B\nu$ is regarded as the composite

$$BK \xrightarrow[\simeq]{E\nu/K} EH/K \xrightarrow{\pi'} BH.$$

It follows that $(\pi')^* = (B\nu)^* : H^*(BK) \to H^*(BH)$ and the element $m'(B\nu)^*(v) - A_{PL}(\pi')(v)$ is a boundary. Therefore, for a basis $v \in V$, there exists an element v' of $A_{PL}(EH/K)$ such that $dv' = m'(B\nu)^*(v) - A_{PL}(\pi')(v)$. Let

$$m' : \big(\Lambda V \otimes \Lambda W \otimes \Lambda(sV),\ d(sv) = (B\nu)^*(v) - v\big) \to A_{PL}(EH/K)$$

be a differential graded algebra map defined by $m'(v) = A_{PL}(\pi')m_{BH}(v)$ for $v \in V$, $m'(w) = w'$ for a base $w \in W$ and $m'(sv) = v'$ for a base $sv \in sV$.

We show that m' is a quasi-isomorphism. Let $\{v_i\}_{i=1}^m$ be a basis of V. Then $(B\nu)^*(v_1) - v_1, \ldots, (B\nu)^*(v_m) - v_m$ is a regular sequence. Therefore $H^*\left(\Lambda V \otimes \Lambda W \otimes \Lambda(sV)\right) \cong \Lambda W$ and $H(m')$ is the identity of ΛW. Then we obtain the following commutative diagram

$$\begin{tikzcd}
A_{PL}(BG) & \Lambda U \arrow[l, "m_{BG}"', "\simeq"] \\
A_{PL}(BH) \arrow[u, "A_{PL}(B\mu)"] \arrow[d, "A_{PL}(\pi')"'] & \Lambda V \arrow[l, "m_{BH}"', "\simeq"] \arrow[u, "(B\mu)^*"'] \arrow[d, hook] \\
A_{PL}(EH/K) & (\Lambda V \otimes \Lambda W \otimes \Lambda(sV), d(sv) = (B\nu)^*(v) - v) \arrow[l, "m'"', "\simeq"]
\end{tikzcd}$$

By [FHT01, Proposition 15.8], we have a Sullivan model $m = A_{PL}(\pi)m_{BG} \cdot A_{PL}(f)m'$ for $EG \times_G^\mu H/K$. □

We describe the G-equivariant cohomology of the loop spaces $\Omega(H/K)$ and $L(H/K)$ in terms of torsion products under the following hypothesis.

Hypothesis 3.3.2. Let $\{v_i\}_{i=1}^m$ be a basis of V. Assume that there exists an integer s such that $(B\nu)^*(v_1) - (B\mu)^*(v_1), \ldots, (B\nu)^*(v_s) - (B\mu)^*(v_s)$ is a regular sequence and $(B\nu)^*(v_{s+1}) - (B\mu)^*(v_{s+1}) = \cdots = (B\nu)^*(v_m) - (B\mu)^*(v_m) = 0$.

Proposition 3.3.3. *Under Hypothesis 3.3.2, H/K is a G-formal space.*

Proof. Consider the following diagram,

$$\begin{tikzcd} A_{PL}(EG \times^{\mu}_{G} H/K) & (\Lambda U \otimes \Lambda W \otimes \Lambda(sV), d) \arrow[l, "m"', "\simeq"] \arrow[r, dotted, "\theta", "\simeq"'] & H^*(EG \times^{\mu}_{G} H/K), \end{tikzcd}$$

where m is constructed in the proof of Proposition 3.3.1. Moreover the differential d on $\Lambda U \otimes \Lambda W \otimes \Lambda(sV)$ is defined by $d(sv) = (B\nu)^*(v) - v$.

Since $(B\nu)^*(v_1) - (B\mu)^*(v_1), \dots, (B\nu)^*(v_s) - (B\mu)^*(v_s)$ is a regular sequence, it follows that the natural surjection

$$p : (\Lambda U \otimes \Lambda W \otimes \Lambda(sV), d) \to \frac{\Lambda U \otimes \Lambda W}{((B\nu)^*(v_i) - (B\mu)^*(v_i))} \otimes \Lambda(sv_{s+1}, \dots, sv_m)$$

is a quasi-isomorphism. Then we define θ by the composite,

$$\begin{aligned} (\Lambda U \otimes \Lambda W \otimes \Lambda(sV), d) \xrightarrow[\simeq]{p} & \frac{\Lambda U \otimes \Lambda W}{((B\nu)^*(v_i) - (B\mu)^*(v_i))} \otimes \Lambda(sv_{s+1}, \dots, sv_m) \\ & \xrightarrow[\cong]{m^*(p^*)^{-1}} H^*(EG \times^{\mu}_{G} H/K). \end{aligned}$$

Then the right triangle in the diagram above is commutative. This completes the proof. $\square$

Theorem 3.3.4. *Under Hypothesis 3.3.2, as an $H^*(BG)$-algebra,*

$$H^*\left(EG \times^{\mu}_{G} \Omega(H/K)\right) \cong \mathrm{Tor}_{H^*(EG \times^{\mu}_{G} H/K)}\left(H^*(BG), H^*(BG)\right).$$

Proof. We recall the maps m and θ which is constructed in the proof of Proposition 3.3.3. By virtue of the Lifting Lemma, we see that there exists a map φ such that $m_{BG}\varphi \sim A_{PL}(\xi_{pt})m$; see the diagram below,

$$\begin{tikzcd} A_{PL}(EG \times_{G} H/K) \arrow[d, "A_{PL}(\xi_{pt})"'] & (\Lambda U \otimes \Lambda W \otimes \Lambda(sV), d) \arrow[l, "m"', "\simeq"] \arrow[r, "\theta", "\simeq"'] \arrow[d, dotted, "\varphi"] & H^*(EG \times_{G} H/K), \arrow[d, "\xi^*_{pt}"] \\ A_{PL}(BG) & H^*(BG) \arrow[l, "m_{BG}"', "\simeq"] \arrow[r, equal] & H^*(BG), \end{tikzcd}$$

where the differential d on $\Lambda U \otimes \Lambda W \otimes \Lambda(sV)$ is defined by $d(sv) = (B\nu)^*(v) - v$. If $x \in U \oplus W$, then $(\xi_{pt})^*\theta(x) = (\xi_{pt})^* m^*([x]) = [\varphi(x)] = \varphi(x)$. Since $H^*(BG)$ is a polynomial, it follows that $(\xi_{pt})^*\theta(sv_i) = 0 = \varphi(sv_i)$. By applying Theorem 3.2.2, we have the result. $\square$

Theorem 3.3.5. *Under Hypothesis 3.3.2, as an $H^*(BG)$-algebra,*

$$\begin{aligned} & H^*\left(EG \times^{\mu}_{G} L(H/K)\right) \\ & \quad \cong \mathrm{Tor}_{H^*\left(EG \times^{\mu}_{G} (H/K \times H/K)\right)}\left(H^*\left(EG \times^{\mu}_{G} H/K\right), H^*\left(EG \times^{\mu}_{G} H/K\right)\right) \end{aligned}$$

Proof. Since $H^*(H/K)$ and $H^*(BG)$ are generated by elements whose degree are even, we see that H/K is equivariantly formal. Then we can apply Theorem 3.2.6. $\square$

We conclude this section with describing the S^1-equivariant cohomology of loop spaces of the complex projective space $\mathbb{C}P^m$. By the following lemma, $\mathbb{C}P^m$ with the basepoint $[0,0,\ldots,0,1]$ is regarded as a homogeneous space $\frac{U(m+1)}{U(m)\times U(1)}$.

Lemma 3.3.6 ([Yok01, Proposition 27, Theorem 16]). *We define a subspace of $M(m+1;\mathbb{C})$ as follows,*

$$\mathbb{C}P(m) := \{X \in M(m+1;\mathbb{C}) | X^* = X,\ X^2 = X \text{ and } \mathrm{tr}(X) = 1\}.$$

1. *The morphism $f : \frac{U(m+1)}{U(m)\times U(1)} \to \mathbb{C}P(m)$ which is defined by $f(A) = AE_{m+1}A^*$ is a homeomorphism.*
2. *The morphism $g : \mathbb{C}P^m \to \mathbb{C}P(m)$ which is defined by $g\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \frac{1}{\sum_{K=0}^m |x_k|^2}\left(x_i\overline{x_j}\right)_{i,j}$ is a homeomorphism.*

A homomorphism $\mu : S^1 \to U(m+1)$ induces an S^1-linear action on $U(m+1)$. Then μ gives rise to the action on $\mathbb{C}P^m = \frac{U(m+1)}{U(m)\times U(1)}$. We denote by $ES^1 \times^{\mu}_{S^1} \mathbb{C}P^m$ the Borel construction of $\mathbb{C}P^m$. Since $U(1)\times\cdots\times U(1)$ is a maximal torus of $U(m+1)$ and $\mu(S^1)$ is an abelian group, it follows that there exists an element $g \in U(m+1)$ such that $g\mu(S^1)g^{-1} \subset U(1)\times\cdots\times U(1)$. We denote by $\overline{\mu}$ the composite $S^1 \xrightarrow{\mu} U(m+1) \xrightarrow[\cong]{g(-)g^{-1}} U(m+1)$. Note that there exist integers $\mu_1,\ldots,\mu_{m+1}$ such that

$$\overline{\mu}\left(e^{2\pi i\theta}\right) = \left(e^{2\pi i\theta\mu_1},\ldots,e^{2\pi i\theta\mu_{m+1}}\right).$$

We obtain the isomorphisms

$$H^*(BU(m+1)) \cong \mathbb{Q}[a_1,\ldots,a_{m+1}],\ H^*(BU(m)) \cong \mathbb{Q}[b_1,\ldots,b_m],$$
$$H^*(BU(1)) \cong \mathbb{Q}[c_1],\ H^*\left(BS^1\right) \cong \mathbb{Q}[z],$$
$$\text{and } H^*(B(U(1)\times\cdots\times U(1))) \cong \mathbb{Q}[t_1,\ldots,t_{m+1}]$$

where a_i, b_i and c_1 are the Chern classes and z and t_i are the first Chern classes. Since $(B\sigma)^*(a_i) = \sum_{1\le k_1<\cdots<k_i\le m+1} t_{k_1}\cdots t_{k_i}$ and $(B\tilde{\mu})^*(t_i) = \mu_i z$, it follows that $(B\overline{\mu})^*(a_i) = \lambda_i z^i$ where $\lambda_i = \sum_{1\le k_1<\cdots<k_i\le m+1} \mu_{k_1}\cdots\mu_{k_i}$, $\sigma : U(1)\times\cdots\times U(1) \to U(m+1)$ denotes the inclusion and $\tilde{\mu} : S^1 \to U(1)\times\cdots\times U(1)$ is the map defined by $g(-)g^{-1}\circ\mu$. Observe that $\overline{\mu} = \sigma\circ\tilde{\mu}$.

Let $\nu : U(m)\times U(1) \hookrightarrow U(m+1)$ be the canonical inclusion. Then we see that $(B\nu)^*(a_i) = b_i + b_{i-1}c_1$, where $b_0 = 1$ and $b_{m+1} = 0$. Therefore the sequence $(B\nu)^*(a_1) - (B\overline{\mu})^*(a_1),\ldots,(B\nu)^*(a_{m+1}) - (B\overline{\mu})^*(a_{m+1})$ is regular; see Lemma A.1.4 below.

By virtue of Proposition 1.2.3, we have

Proposition 3.3.7. *With the same notation as above, as an $H^*(BS^1)$-algebra*

$$H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} \Omega\mathbb{C}P^m\right) \cong \mathrm{Tor}_{H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} \mathbb{C}P^m\right)} \left(H^*(BS^1), H^*(BS^1)\right)$$

and

$$\begin{aligned} &H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} L\mathbb{C}P^m\right) \\ &\quad \cong \mathrm{Tor}_{H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} \mathbb{C}P^m \times \mathbb{C}P^m\right)} \left(H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} \mathbb{C}P^m\right), H^*\left(ES^1 \times^{\overline{\mu}}_{S^1} \mathbb{C}P^m\right)\right). \end{aligned}$$

Proof. Theorems 3.3.4 and 3.3.5 yield the results. □

Chapter 4

Proofs of main theorems

4.1 Proof of Theorem 1.2.4

In this section, we use the same notation as in Section 3.3. The purpose of this section is to compute the rational S^1-equivariant cohomology of the based loop space of the complex projective space by using Proposition 3.3.7.

Lemma 4.1.1. *Let $\pi : ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m \to BS^1$ be the Borel fibration. Then $(\mathbb{Q}[c_1, z] \otimes \Lambda(sa_{m+1}), d)$ is the minimal Sullivan model for $A_{PL}(\pi)m_{BS^1}$, where d is defined by $d(c_1) = d(z) = 0$ and $d(sa_{m+1}) = (c_1 - \mu_1 z) \cdots (c_1 - \mu_{m+1} z)$.*

Proof. Let $b_0 = 1$ and $b_{m+1} = 0$. Thanks to Proposition 3.3.1, we obtain a Sullivan model $(\mathbb{Q}[b_1, \ldots, b_m, c_1, z] \otimes \Lambda(sa_1, \ldots, sa_{m+1}), d)$ for $A_{PL}(\pi)m_{BS^1}$, where d is the differential defined by $d(b_i) = d(c_1) = d(z) = 0$ and $d(a_i) = b_i + b_{i-1}c_1 - \lambda_i z^i$.

We define differential graded algebra maps f and g over $\mathbb{Q}[z]$

$$\begin{array}{c} A_{PL}(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m) \\ \uparrow \simeq \\ \big(\mathbb{Q}[b_1, \ldots, b_m, c_1, z] \otimes \Lambda(sa_1, \ldots, sa_{m+1}),\ d(sa_i) = b_i + b_{i-1}c_1 - \lambda_i z^i\big) \\ f \uparrow \ \downarrow g \\ \big(\mathbb{Q}[c_1, z] \otimes \Lambda(sa_{m+1}),\ d(sa_{m+1}) = (c_1 - \mu_1 z) \cdots (c_1 - \mu_{m+1} z)\big) \end{array}$$

by

$$f(c_1) = c_1 \text{ and } f(sa_{m+1}) = \sum_{j=1}^{m+1} (-1)^{j-1} (c_1)^{m-j+1} \cdot (sa_j),$$

$$g(b_i) = \sum_{j=0}^{i} (-1)^{i+j} \lambda_j (c_1)^{i-j} z^j,\ g(c_1) = c_1 \text{ and } g(sac_i) = \begin{cases} 0 & (i \leq m) \\ sa_{m+1} & (i = m+1). \end{cases}$$

By Lemma A.1.4, we see that

$$H^*\left(\mathbb{Q}[b_1, \ldots, b_m, c_1, z] \otimes \Lambda(sa_1, \ldots, sa_{m+1})\right) \cong \frac{\mathbb{Q}[b_1, \ldots, b_m, c_1, z]}{(b_1 + b_0 c_1 - \lambda_1 z^1, \ldots, b_{m+1} + b_m c_1 - \lambda_{m+1} z^{m+1})}$$

$$H^*\left(\mathbb{Q}[c_1, z] \otimes \Lambda(sa_{m+1})\right) \cong \frac{\mathbb{Q}[c_1, z]}{((c_1 - \mu_1 z) \cdots (c_1 - \mu_{m+1} z))}.$$

By a straightforward computation, we see that g^* is the inverse of f^* and hence f is a quasi-isomorphism. $\square$

Lemma 4.1.1 enables us to establish the following lemmas.

Lemma 4.1.2. *As a $\mathbb{Q}[z]$-algebra,*

$$H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right) \cong \frac{\mathbb{Q}[c_1, z]}{\big((c_1 - \mu_1 z)\cdots(c_1 - \mu_{m+1} z)\big)}.$$

Lemma 4.1.3. *As a $\mathbb{Q}[z]$-algebra,*

$$H^*(BS^1) = H^*\left(ES^1 \times_{S^1} \frac{U(m)\times U(1)}{U(m)\times U(1)}\right) \cong \frac{\mathbb{Q}[c_1, z]}{(c_1 - \mu_{m+1} z)}\,(\cong \mathbb{Q}[z])\,.$$

Lemma 4.1.4. *Consider the isomorphisms in Lemmas 4.1.2 and 4.1.3, then we have $(\xi_{pt})^*(c_1) = c_1 = \mu_{m+1} z$, where $\xi_{pt} : ES^1 \times_{S^1} \frac{U(m)\times U(1)}{U(m)\times U(1)} \to ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m$ is induced by the inclusion $U(m)\times U(1) \hookrightarrow U(m+1)$.*

Proof of Lemma 4.1.4. See Appendix A.3. □

Proof of Theorem 1.2.4. Thanks to Proposition 3.3.7, it suffices to compute the torsion product $\mathrm{Tor}_{H^*\left(ES^1\times_{S^1}^{\overline{\mu}}\mathbb{C}P^m\right)}\left(H^*\left(BS^1\right),\ H^*\left(BS^1\right)\right)$. We first construct a free resolution $\mathscr{K}$ of $H^*\left(BS^1\right)$ as a $H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right)$-module.

We rewrite $H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right)$ as follows,

$$\begin{aligned}
&H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right)\\
&\quad\cong \frac{\mathbb{Q}[c_1, z]}{\big((c_1 - \mu_1 z)\cdots(c_1 - \mu_{m+1} z)\big)}\\
&\quad= \frac{\mathbb{Q}[c_1 - \mu_{m+1} z, z]}{\Big(\big((c_1 - \mu_{m+1}z + \mu_{m+1}z) - \mu_1 z\big)\cdots\big((c_1 - \mu_{m+1}z + \mu_{m+1}z) - \mu_m z\big)(c_1 - \mu_{m+1})\Big)}\\
&\quad\cong \frac{\mathbb{Q}[c, z]}{\Big(c\cdot\big(c - (\mu_1 - \mu_{m+1})z\big)\cdots\big(c - (\mu_m - \mu_{m+1})z\big)\Big)} \cong \frac{\mathbb{Q}[c,\ z]}{\big(c\cdot f(c,z)\big)},
\end{aligned}$$

where $f(c,z) := \big(c - (\mu_1 - \mu_{m+1})z\big)\cdots\big(c - (\mu_m - \mu_{m+1})z\big)$. By Lemma 4.1.4, we have $(\xi_{pt})^*(c) = (\xi_{pt})^*(c_1 - \mu_{m+1}z) = 0$. See the following diagram.

$$\begin{tikzcd}
 & \frac{\mathbb{Q}[c,\ z]}{\left(c\cdot f(c,z)\right)} \arrow[dl, hook'] \arrow[dr, "(\xi_{pt})^*"] \arrow[rr, "\cong"] & & \frac{\mathbb{Q}[c_1,z]}{\left((c_1-\mu_1 z)\cdots(c_1-\mu_{m+1}z)\right)} \arrow[d, "(\xi_{pt})^*"] \\
\mathscr{K} \arrow[rr, dotted, "\kappa", "\simeq"'] & & \mathbb{Q}[z] & \frac{\mathbb{Q}[c_1,z]}{(c_1-\mu_{m+1}z)} \arrow[l, "\cong"]
\end{tikzcd}$$

Now we construct the resolution of $\mathbb{Q}[z]$ over $\frac{\mathbb{Q}[c,\ z]}{\left(c\cdot f(c,z)\right)}$. We define a differential graded algebra $(\mathscr{K}, d)$ by $\mathscr{K} = \frac{\mathbb{Q}[c,z]}{(c\cdot f(c,z))} \otimes \Lambda(w_1) \otimes \mathbb{Q}[w_2]$, $d(c) = d(z) = 0$, $d(w_1) = c$ and $d(w_2) = f(c,z)w_1$, where $|w_1| = 1$, $|w_2| = 2m$. Moreover we define a morphism $\kappa : \mathscr{K} \to \mathbb{Q}[z]$ by $\kappa(c) = \kappa(w_1) = \kappa(w_2) = 0$ and $\kappa(z) = z$. We show that $\mathscr{K}$ is a free resolution of $H^*\left(BS^1\right)$ as an

$H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right)$-module. Since the triangle of the above diagram is commutative, it is enough to show that κ is a quasi-isomorphism.

We denote by $\mathscr{K}'$ a differential graded algebra $\mathbb{Q}[c, z, w_2] \otimes \Lambda(w_1, w_3)$, whose differential d is defined by $dw_1 = c$, $dw_2 = f(c, z)w_1 - w_3$ and $dw_3 = c \cdot f(c, z)$, where $|w_3| = 2m + 1$. Moreover we define a morphisms κ' : $\mathscr{K}' \to \mathbb{Q}[z]$ by $\kappa'(c) = \kappa'(w_1) = \kappa'(w_2) = \kappa'(w_3) = 0$ and $\kappa'(z) = z$, and $\epsilon : \mathscr{K}' \to \mathscr{K}$ by $\epsilon(c) = 0$, $\epsilon(z) = z$, $\epsilon(w_1) = w_1$, $\epsilon(w_2) = w_2$ and $\epsilon(w_3) = 0$. Then we see that the diagram

$$\begin{tikzcd} \mathscr{K}' \arrow[rr, "\epsilon"] \arrow[dr, "\kappa'"'] & & \mathscr{K} \arrow[dl, "\kappa"] \\ & \mathbb{Q}[z] & \end{tikzcd}$$

is commutative. Claims 4.1.5 and 4.1.6 yield that κ is a quasi-isomorphism.

Claim 4.1.5. The morphism $\epsilon : \mathscr{K}' \to \mathscr{K}$ is a quasi-isomorphism.

Claim 4.1.6. The morphism $\kappa' : \mathscr{K}' \to \mathbb{Q}[z]$ is a quasi-isomorphism.

We see that

$$\begin{aligned} H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \Omega\mathbb{C}P^m\right) &\cong \mathrm{Tor}_{H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right)}\left(H^*\left(BS^1\right), H^*\left(BS^1\right)\right) \\ &= H^*\left(\mathbb{Q}[z] \otimes_{\frac{\mathbb{Q}[c,z]}{(c\cdot f(c,z))}} \mathscr{K}\right) \\ &= H^*\left(\mathbb{Q}[z] \otimes \Lambda(w_1) \otimes \mathbb{Q}[w_2],\ dw_2 = g(\overline{\mu})z^m w_1\right), \end{aligned}$$

where $g(\overline{\mu}) = (\mu_{m+1}-\mu_1)\cdots(\mu_{m+1}-\mu_m)$. Therefore, we see that $g(\overline{\mu}) = 0$ if and only if μ_{m+1} is one of $\mu_1, \ldots, \mu_m$. Since $dz = dw_1 = 0$, a straightforward calculation deduces the result on the homology; see the figure (4.1) below. □

We give now proofs of the claims. To this end, we compare appropriate spectral sequences by making use of the technique in [KMN06] for computing the cohomology of a differential graded algebra.

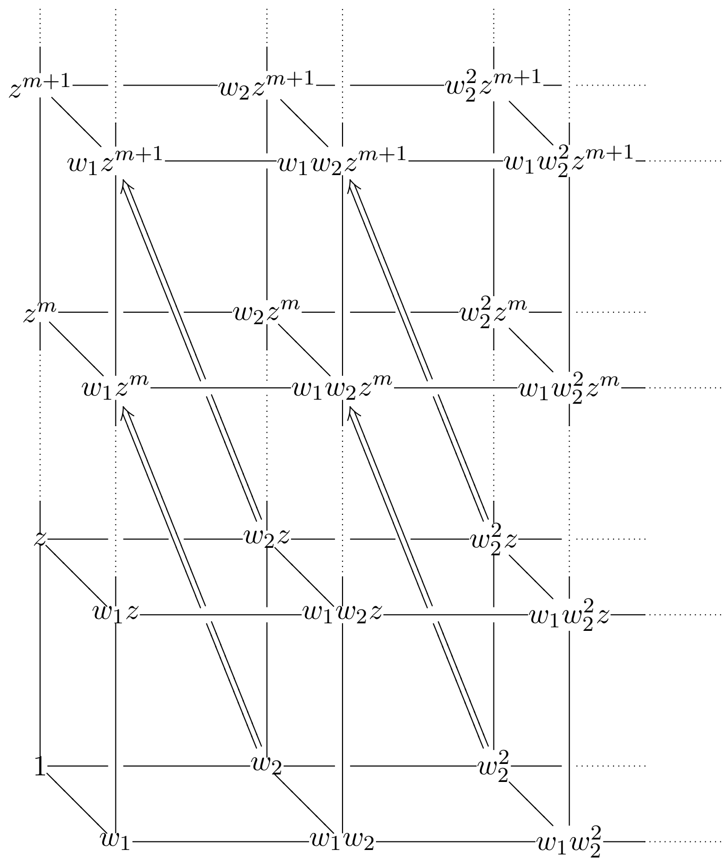
Proof of Claim 4.1.5. We assign the bidegree to each element in $\mathscr{K}'$ as follows: bideg c = bideg $z = (0, 2)$, bideg $w_1 = (-1, 2)$, bideg $w_2 = (-2, 2m+2)$ and bideg $w_3 = (0, 2m + 1)$. The bidegree of a monomial is defined as the sum of bidegree of each indecomposable element. Consider the filtration F^* of $\mathscr{K}'$ defined by

$F^i = \{x \in \mathscr{K}' |$the first component of bideg x is grater than or equal to $i\}$.

Then F^* induces a spectral sequence $\{^{\mathscr{K}'}E_r, d_r\}$ converging to $H(\mathscr{K}')$ as an algebra whose E_0-term is given by $^{\mathscr{K}'}E_0 = \sum F^i/F^{i+1}$. We see that, as a differential graded algebra,

$^{\mathscr{K}'}E_0 \cong \mathbb{Q}[c, z, w_2] \otimes \Lambda(w_1, w_3)$ and $d_0'(w_1) = d_0'(w_2) = 0,\ d_0'(w_3) = c \cdot f(c, z)$.

(4.1)

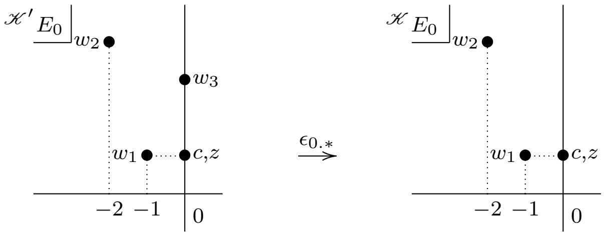


z^{m+1}
$w_2 z^{m+1}$
$w_2^2 z^{m+1}$
$w_1 z^{m+1}$
$w_1 w_2 z^{m+1}$
$w_1 w_2^2 z^{m+1}$
z^m
$w_2 z^m$
$w_2^2 z^m$
$w_1 z^m$
$w_1 w_2 z^m$
$w_1 w_2^2 z^m$
z
$w_2 z$
$w_2^2 z$
$w_1 z$
$w_1 w_2 z$
$w_1 w_2^2 z$
1
w_2
w_2^2
w_1
$w_1 w_2$
$w_1 w_2^2$

Moreover, we define the bidegree of each element in $\mathscr{K}$ by bideg $c =$ bideg $z = (0,2)$, bideg $w_1 = (-1,2)$ and bideg $w_2 = (-2, 2m+2)$. Then a spectral sequence $\{{}^{\mathscr{K}}E_r, d_r\}$ is constructed by using the same filtration of $\mathscr{K}$ as that of $\mathscr{K}'$. Then we see that, as a differential graded algebra,

$$^{\mathscr{K}}E_0 \cong \frac{\mathbb{Q}[c,z,w_2] \otimes \Lambda(w_1)}{(c \cdot f(c,z))} \text{ and } d_0(w_1) = d_0(w_2) = 0.$$

Since ϵ preserves the filtration, it follows that the map ϵ induces a morphism of spectral sequences $\{\epsilon_{r.*}\} : \{{}^{\mathscr{K}'}E_r, d_r\} \to \{{}^{\mathscr{K}}E_r, d_r\}$; see the figure below for the first step.



It is readily seen that $\epsilon_{1,*}$ is an isomorphism of algebras

$$^{\mathscr{K}'}E_1 \cong \frac{\mathbb{Q}[c,z,w_2] \otimes \Lambda(w_1)}{(c \cdot f(c,z))} \cong^{\mathscr{K}} E_1.$$

Thus we have the result. $\square$

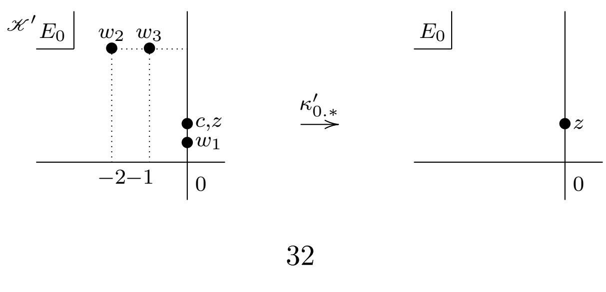
Proof of Claim 4.1.6. We define the bidegree of each element in $\mathscr{K}'$ by bideg $c =$ bideg $z = (0,2)$, bideg $w_1 = (0,1)$, bideg $w_2 = (-2, 2m+2)$ and bideg $w_3 = (-1, 2m+2)$. The filtration of $\mathscr{K}'$ defined by the first component as in the proof of Claim 4.1.5 constructs a spectral sequence $\{{}^{\mathscr{K}'}E_r, d_r\}$. Then we see that, as a differential graded algebra,

$$^{\mathscr{K}'}E_0 \cong \mathbb{Q}[c,z,w_2] \otimes \Lambda(w_1, w_3) \text{ and } d_0'(w_1) = c,\ d_0'(w_2) = d_0'(w_3) = 0.$$

Moreover, we define the bidegree to each element in $\mathbb{Q}[z]$ as follows: bideg $z = (0,2)$. We construct a spectral sequence $\{E_r, d_r\}$ by using the same filtration of $\mathbb{Q}[z]$ as that of $\mathscr{K}'$. We see that, as a differential graded algebra,

$$E_0 \cong \mathbb{Q}[z] \text{ and } d_0 \equiv 0.$$

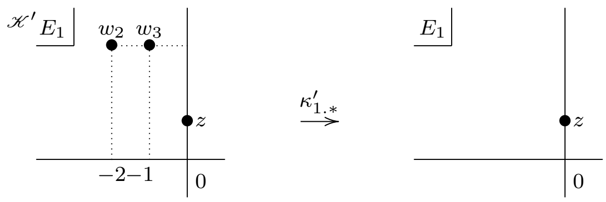
Since κ' preserves the filtration, it follows that the map κ' induces a morphism $\{\kappa'_{r.*}\}$ of spectral sequences; see the figure below.



A straightforward calculation yields that as an algebra,

$$ {}^{\mathscr{K}'}E_1 \cong H\left({}^{\mathscr{K}'}E_0, d_0'\right) \cong \mathbb{Q}[z, w_2] \otimes \Lambda(w_3) \text{ and } d_1'(w_2) = w_3,\ d_1'(w_3) = 0; $$

see the figure below.



It turns out that $\kappa'_{2.*}$ is an isomorphism of algebras

$$ {}^{\mathscr{K}'}E_2 \cong \mathbb{Q}[z] \cong E_2. $$

This completes the proof. $\square$

4.2 Proof of Theorem 1.2.5

The purpose of this section is to construct a model for a Borel construction associated with the free loop space $L\mathbb{C}P^m$.

Proof of Theorem 1.2.5. By Proposition 3.3.7, we see that as an $H^*(BS^1)$-algebra,

$$ H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} L\mathbb{C}P^m\right) \cong \mathrm{Tor}_{H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} (\mathbb{C}P^m \times \mathbb{C}P^m)\right)}\left(H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right), H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right)\right). $$

We put, respectively,

$$ A := H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} \mathbb{C}P^m\right) \cong \frac{\mathbb{Q}[c, z]}{(\rho)}, \quad A' := \mathbb{Q}[c, z], $$
$$ B := H^*\left(ES^1 \times_{S^1}^{\overline{\mu}} (\mathbb{C}P^m \times \mathbb{C}P^m)\right) \cong \frac{\mathbb{Q}[c] \otimes \mathbb{Q}[c] \otimes \mathbb{Q}[z]}{(\rho_1, \rho_2)}, $$
$$ B' := \mathbb{Q}[c] \otimes \mathbb{Q}[c] \otimes \mathbb{Q}[z]. $$

Here $\rho_1 := \sum_{i=0}^{m+1} \lambda_i\ c^{m-i+1} \otimes 1 \otimes z^i$ and $\rho_2 := \sum_{i=0}^{m+1} \lambda_i\ 1 \otimes c^{m-i+1} \otimes z^i$. We define elements $\zeta_i \in A'$ and $\zeta \in B'$ by

$$ \zeta_i := \begin{cases} 1 \otimes 1 & (i = 0) \\ (-1)^i c^i \otimes 1 + c^{i-1} \otimes c + \cdots + 1 \otimes c^i & (i = 1,\ 2, \cdots, m) \end{cases}, $$
$$ \zeta := \sum_{i=0}^{m} \lambda_{m-i} \zeta_i z^{m-i}. $$

Assume that $|w| = 2m$ and define a differential d on $B \otimes \Lambda(\bar{c}, w)$ by $d(b) = 0$ if $b \in B$, $d\bar{c} = c \otimes 1 - 1 \otimes c$ and $d(w) = \zeta\bar{c}$. We denote by $\mathscr{E}$ the differential graded algebra $(B \otimes \Lambda(\bar{c}, w), d)$. The same argument as in the proof of [Smi81, Proposition 3.5] shows that $\mathscr{E}$ is a free resolution of A as a B-module.

$$\begin{tikzcd} & B \arrow[dl, hook'] \arrow[dr, "\Delta^*"] & \\ \mathscr{E} \arrow[rr, "\epsilon", "\simeq"'] & & A \end{tikzcd} \qquad \begin{array}{c} \epsilon : \mathscr{E} \longrightarrow A \\ \epsilon(b) = \Delta^*(b),\ \epsilon(\bar{c}) = \epsilon(w) = 0 \end{array}$$

In fact, let $\mathscr{K} = B \otimes \Lambda(\bar{c})$ be a differential graded subalgebra of $\mathscr{E}$. We assign the bidegree to each element in $\mathscr{E}$ as follows: bideg $x = (0, \deg x)$ if $x \in \mathscr{K}$ and bideg $w = (-1, 2m+2)$. We construct a spectral sequence $\{{}^{\mathscr{E}}E_r, d_r\}$ by employing the filtration F^* of $\mathscr{E}$ defined by

$$F^i = \{x \in \mathscr{E} | \text{the first component of bideg } x \text{ is grater than or equal to } i\}.$$

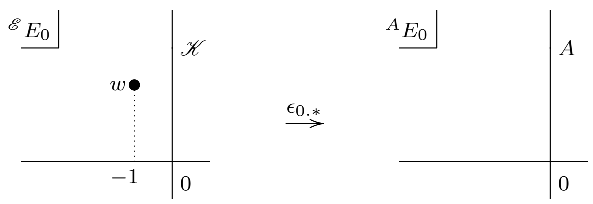
Then we see that, as a differential graded algebra,

$${}^{\mathscr{E}}E_0 \cong \mathscr{K} \otimes \Lambda(w) \text{ and } d_0^{\mathscr{E}}(w) = 0.$$

Moreover, we define the bidegree of element x of A by bideg $x = (0, \deg x)$. The same filtration of A as that of $\mathscr{E}$ defines a spectral sequence $\{{}^{A}E_r, d_r\}$. We see that, as a differential graded algebra,

$${}^{A}E_0 \cong A \text{ and } d_0 \equiv 0.$$

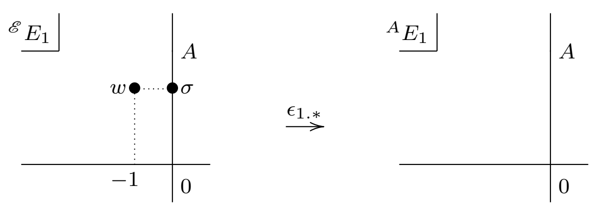
The map ϵ preserves the filtration so that we have the morphism of spectral sequences $\{\epsilon_{r.*}\} : \{{}^{\mathscr{E}}E_r, d_r\} \to \{{}^{A}E_r, d_r\}$, which is induced by ϵ.



Then Lemma 4.2.1 below enables us to conclude that

$${}^{\mathscr{E}}E_1 \cong A \otimes \Lambda(\sigma) \otimes \mathbb{Q}[w] \text{ and } d_1^{\mathscr{E}}(w) = \sigma.$$

Lemma 4.2.1. *The morphism $f : A \otimes \Lambda(\sigma) \to H^*(\mathscr{K})$ defined by $f(c) = c \otimes 1$, $f(z) = z$ and $f(\sigma) = \zeta\bar{c}$ is an isomorphism.*



It is readily seen that $\epsilon_{2,*}$ is an isomorphism of algebras

$$^{\mathscr{E}}E_2 \cong A \cong {}^{A}E_2.$$

We have the result. $\square$

Proof of Lemma 4.2.1. First we assume $|\alpha| = |\beta| = 2m+1$ and define a differential d' on $B' \otimes \Lambda(\bar{c}, \alpha, \beta)$ by $d'(\bar{c}) = c \otimes 1 - 1 \otimes c$, $d'(\alpha) = \rho_1$ and $d'(\beta) = \rho_2$. We denote by $\mathscr{K}'$ the differential graded algebra $(B' \otimes \Lambda(\bar{c}, \alpha, \beta), d')$. Let $\hat{f} : A \otimes \Lambda(\sigma) \to H^*(\mathscr{K}')$ be the morphism of algebras defined by $\hat{f}(a) := a \otimes 1$ and $\hat{f}(\sigma) = \zeta\bar{c} - \alpha + \beta$.

Claim 4.2.2. The morphism $\hat{f} : A \otimes \Lambda(\sigma) \to H(\mathscr{K}')$ is an isomorphism.

Next we define a differential d on $B \otimes \Lambda(\bar{c})$ by $d(b) = 0$ if $b \in B$ and $d\bar{c} = c \otimes 1 - 1 \otimes c$ and denote by $\mathscr{K}$ the differential graded algebra $(B \otimes \Lambda(\bar{c}), d)$. Consider the morphism $\pi : \mathscr{K}' \to \mathscr{K}$ defined by $\pi(b) = [b]$, $\pi(\bar{c}) = \bar{c}$ and $\pi(\alpha) = \pi(\beta) = 0$.

Claim 4.2.3. The morphism $\pi : \mathscr{K}' \to \mathscr{K}$ is a quasi-isomorphism.

The map f is nothing but the composite $A \otimes \Lambda(\sigma) \xrightarrow{\hat{f}} H^*(\mathscr{K}') \xrightarrow{(\pi)^*} H^*(\mathscr{K})$. Then we have the result. $\square$

Proof of Claim 4.2.2. Let $\mathscr{A}$ the differential graded subalgebra $B' \otimes \Lambda(\bar{c}) \otimes \Lambda(\alpha)$ of $\mathscr{K}'$. Then we see that,

$$H^*(\mathscr{A}) \cong \frac{\mathbb{Q}[c] \otimes \mathbb{Q}[c] \otimes \mathbb{Q}[z]}{(c \otimes 1 - 1 \otimes c,\ \rho_1)} \underset{\cong}{\xleftarrow{f_1}} \frac{\mathbb{Q}[c, z]}{(\rho)} = A,$$

where $f_1(c) = c \otimes 1$ and $f_1(z) = z$. Moreover we have a sequence of isomorphisms

$$H^*\left(\mathbb{Q} \otimes_{\mathscr{A}} \mathscr{K}'\right) \underset{\cong}{\xrightarrow{f_2}} H^*(\Lambda(\sigma),\ 0) \underset{\cong}{\xrightarrow{f_3}} \Lambda(\sigma),$$

where $f_2(1 \otimes \zeta\bar{c} - \alpha + \beta) = \sigma$ and $f_3(\sigma) = \sigma$. This enables us to obtain isomorphisms

$$A \otimes \Lambda(\sigma) \underset{\cong}{\xrightarrow{f_1 \otimes f_3 f_2}} H^*(\mathscr{A}) \otimes H^*\left(\mathbb{Q} \otimes_{\mathscr{A}} \mathscr{K}'\right) \underset{\cong}{\longrightarrow} H^*\left(\mathscr{A} \otimes \mathbb{Q} \otimes_{\mathscr{A}} \mathscr{K}'\right).$$

The natural quasi-isomorphism $f_4 : \mathscr{A} \otimes \mathbb{Q} \otimes_{\mathscr{A}} \mathscr{K}' \to \mathscr{K}'$ induces the following isomorphism

$$\hat{f} : A \otimes \Lambda(\sigma) \underset{\cong}{\longrightarrow} H^*\left(\mathscr{A} \otimes \mathbb{Q} \otimes_{\mathscr{A}} \mathscr{K}'\right) \underset{\cong}{\xrightarrow{(f_4)^*}} H^*\left(\mathscr{K}'\right),$$

which coincides with $\hat{f}$. $\square$

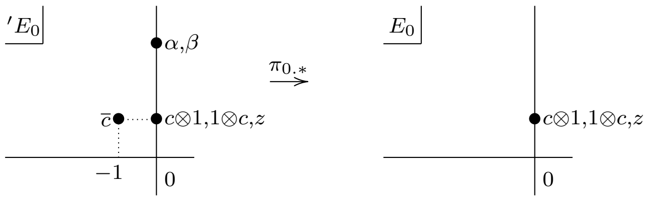
Proof of Claim 4.2.3. We define the bidegree of each element in $\mathscr{K}'$ by bideg $c \otimes 1$ = bideg $1 \otimes c$ = bideg $z = (0, 2)$, bideg $\overline{c} = (-1, 2)$ and bideg α = bideg $\beta = (0, 2m+1)$. The filtration associated with the bidegree constructs a spectral sequence $\{{}'E_r, d_r\}$. Then we see that, as a differential graded algebra,

$${}'E_0 \cong B' \otimes \Lambda(\overline{c}, \alpha, \beta) \text{ and } d_0'(\overline{c}) = 0,\ d_0'(\alpha) = \rho_1,\ d_0'(\beta) = \rho_2.$$

We define the bidegree to each element in $\mathscr{K}$ by bideg $c \otimes 1$ = bideg $1 \otimes c$ = bideg $z = (0, 2)$ and bideg $\overline{c} = (-1, 2)$. Then we have a spectral sequence $\{E_r, d_r\}$ converging to $H^*(\mathscr{K})$. We see that, as a differential graded algebra,

$$E_0 \cong B \otimes \Lambda(\overline{c}) \text{ and } d_0(w_1) = d_0(w_2) = 0.$$

Since π preserves the filtration, it follows that the map π induces a morphism $\{\pi_{r.*}\}$ of spectral sequences.



Then $\pi_{0.*}$ induces an isomorphism of algebras

$$\pi_{1.*} :\ {}'E_1 \cong B \otimes \Lambda(\overline{c}) \cong E_1.$$

This completes the proof. $\square$

Acknowledgement

I am deeply grateful to Katsuhiko Kuribayashi for his valuable comments and suggestions during the course of my study. I am also indebt to Dai Tamaki and Keiichi Sakai who have given me many comments and warm encouragement.

Appendix A

Appendix

A.1 Regular sequences

We describe a proposition on regular sequences.

Definition A.1.1. A sequence $a_1, \ldots, a_n$ of elements of differential graded algebra A and whose dimensions are natural number is *A-regular* when it satisfies the following conditions

1. a_1 is a non-zero-divisor on A,
2. a_i is a non-zero-divisor on $\dfrac{A}{(a_1, \ldots, a_{i-1})}$ for any $i = 2, \ldots, n$.

Definition A.1.2. Let $a_1, \ldots, a_n$ be a sequence of elements of differential graded algebra A and whose dimensions are even. Then we define *the Koszul complex of* $a_1, \ldots, a_n$ written by $K^*(a_1, \ldots, a_n; A)$ as follows

$$K^*(a_1, \ldots, a_n; A) := (\Lambda(b_1, \ldots, b_n) \otimes A,\ db_i = a_i, da = 0(a \in A)),$$

where $|b_1| = \cdots = |b_n| = -1$ and $|a| = 0$ for any $a \in A$.

Lemma A.1.3. *[BH93, Corollary 1.6.19] Suppose $a_1, \ldots, a_n$ is a sequence of elements of a differential graded algebra A and whose dimensions are even. Then the following are equivalence*

1. *$a_1, \ldots, a_n$ is A-regular,*
2. *$K^*(a_1, \ldots, a_n; A)$ is acyclic.*

Proposition A.1.4. *Let A and B be a differential graded algebra, $a_1, \ldots, a_m$ elements of A and $b_1, \ldots, b_{m+n}$ elements of B. Suppose that dimensions of $a_1, \ldots, a_m$ and $b_1, \ldots, b_{m+n}$ are even. If $a_1, \ldots, a_m$ is A-regular and $b_{m+1}, \ldots, b_{m+n}$ is B-regular, then $a_1 + b_1, \ldots, a_m + b_m, b_{m+1}, \ldots, b_{m+n}$ is $A \otimes B$-regular.*

Proof. By virtue of Lemma A.1.3, in order to prove Proposition A.1.4, it suffices to show that the Koszul complex

$$\begin{aligned} &K^*(a_1 + b_1, \ldots, a_m + b_m, b_{m+1}, \ldots, b_{m+n}; A \otimes B) \\ &\quad := (\Lambda(\alpha_1, \ldots, \alpha_m, \beta_{m+1}, \ldots, \beta_{m+n}) \otimes A \otimes B,\ d\alpha_i = a_i + b_i, d\beta_j = b_j) \end{aligned}$$

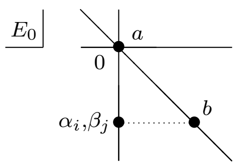
is acyclic. We assign the bidegree to each element in the Koszul complex as follows

$$\begin{array}{lll} \text{bideg } \alpha_i = (0, -1) & \text{bideg } a = (0, 0) & (a \in A), \\ \text{bideg } \beta_j = (0, -1) & \text{bideg } b = (1, -1) & (b \in B). \end{array}$$

We define a filtration degree by F^i consists the elements of the Koszul complex whose first component of the bidegree is gather then or equal to i. We construct a spectral sequence $\{E_r, d_r\}$ associated with the filtration F^* converging to $H(K^*(a_1+b_1, \ldots, a_m+b_m, b_{m+1}, \ldots, b_{m+n}; A \otimes B))$ as an algebra whose E_0-term is given by $E_0 = \sum F^i/F^{i+1}$. We see that, as a differential graded algebra,

$$E_0 \cong \Lambda(\alpha_1, \ldots, \alpha_m, \beta_{m+1}, \ldots, \beta_{m+n}) \otimes A \otimes B \text{ and } d_0(\alpha_i) = a_i + b_i, d_0(\beta_i) = 0,$$

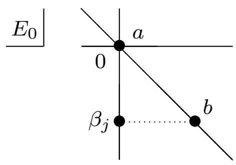
see the figure below.



$$(a \in A,\ b \in B)$$

The sequence $a_1, \ldots, a_n$ is A-regular. Then we have

$$E_1 = H^*(E_0) = \Lambda(\beta_{m+1}, \ldots, \beta_{m+n}) \otimes \frac{A}{(a_1, \ldots, a_m)} \otimes B \text{ and } d_1(\beta_j) = b_j.$$



$$(a \in \tfrac{A}{(a_1, \ldots, a_m)},\ b \in B)$$

It is readily seen that as an algebra,

$$E_2 \cong H^*(E_1) \cong \frac{A}{(a_1, \ldots, a_m)} \otimes \frac{B}{(b_{m+1}, \ldots, b_{m+n})} \text{ and } E_\infty \cong E_2.$$

Thus we see that

$$\mathrm{Tot}(E_\infty) \cong \mathrm{Tot}\left(\frac{A}{(a_1, \ldots, a_m)} \otimes \frac{B}{(b_{m+1}, \ldots, b_{m+n})}\right).$$

Since $\mathrm{Tot}(E_\infty)^i = 0$ $(i \neq 0)$, it follows that

$$H^i\left(K^*(a_1+b_1, \ldots, a_m+b_m, b_{m+1}, \ldots, b_{m+n}; A \otimes B)\right) = 0 \ (i \neq 0).$$

This implies that $K^*(a_1+b_1, \ldots, a_m+b_m, b_{m+1}, \ldots, b_{m+n}; A \otimes B)$ is acyclic. $\square$

A.2 The functor $EG \times_G -$

Proposition A.2.1. *Let E and B are G-spaces. If G-map $p : E \to B$ is a Serre fibration with fiber F, then $EG \times_G p : EG \times_G E \to EG \times_G B$ is a fibration with fiber F up to homotopy.*

Proof. By virtue of [Nei10, Proposition 3.2.2], we have the following commutative diagram

$$\begin{tikzcd}
EG\times_G E \arrow[dr, "\simeq"'] \arrow[drr, bend left, "\pi_E"] \arrow[ddr, bend right, "EG\times_G p"'] & & \\
& \widetilde{E} \arrow[r, "\widetilde{\pi_E}"] \arrow[d, "\widetilde{p}"'] & BG \arrow[d, equal, "id_{BG}"] \\
& EG\times_G B \arrow[r, "\pi_B"] & BG,
\end{tikzcd}$$

where the lower right side square is a totally fibred square. Because $EG\times_G E \underset{\simeq}{\to} \widetilde{E}$ is a weak homotopy equivalence, $\widetilde{\pi_E}$ is a fibration with fiber E up to homotopy. By [Nei10, Proposition 3.2.3], $F \to \widetilde{F} \to *$ is a fibration, see the following diagram;

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \widetilde{F} \arrow[r] \arrow[d] & * \arrow[d] \\
E \arrow[r] \arrow[d, "p"'] & \widetilde{E} \arrow[r, "\widetilde{\pi_E}"] \arrow[d, "\widetilde{p}"'] & BG \arrow[d, equal, "id_{BG}"] \\
B \arrow[r] & EG\times_G B \arrow[r, "\pi_B"] & BG.
\end{tikzcd}$$

Then we have the conclusion. □

A.3 Proof of Lemma 4.1.4

The following Lemma gives the proof of Lemma 4.1.4.

Lemma A.3.1. *Under Hypothesis 3.3.2, the same argument as in the proof of Proposition 3.3.3, enable us to obtain*

$$(\xi_{pt})^* : H^*\left(EG\times_G H/K\right) \to H^*(BG); (\xi_{pt})^*[w] = [1\otimes w]$$

where $\xi_{pt} : EG\times_G K/K \to EG\times_G H/K$ *is induced by the inclusion* $K \hookrightarrow H$.

Proof. Remember the construction of Sullivan model of $EG\times_G^{\overline{\mu}} H/K$. We use the following pullback diagram of fibrations,

$$\begin{tikzcd}
H/K \arrow[r, equal] \arrow[d] & H/K \arrow[d] \\
EG\times_G^{\overline{\mu}} H/K \arrow[r, "f"] \arrow[d, "\pi"'] & EH/K \arrow[d, "\pi'"] \\
BG \arrow[r, "B(\nu\circ\overline{\mu})"] & BH,
\end{tikzcd}$$

where $G \overset{\overline{\mu}}{\to} K \overset{\nu}{\hookrightarrow} H$ and ν is the inclusion. Moreover, their Sullivan models are the following,

$$\begin{tikzcd}
A_{PL}(BG) & A_{PL}(BH) \arrow[l, "A_{PL}(B(\nu\circ\overline{\mu}))"'] \arrow[r, "A_{PL}(\pi')"] & A_{PL}(EH/K) \\
\Lambda U \arrow[u, "m_{BG}", "\simeq"'] & \Lambda V \arrow[l, "\left(B(\nu\circ\overline{\mu})\right)^*"] \arrow[u, "m_{BH}", "\simeq"'] \arrow[r, hook] & \left(\Lambda V\otimes\Lambda W\otimes\Lambda(sV),\ d(sv) = (B\nu)^*(v) - v\right). \arrow[u, "m'", "\simeq"']
\end{tikzcd}$$

Especially, if $H = K$, the above diagram replaced the following diagram,

$$\begin{tikzcd}
A_{PL}(BG) & A_{PL}(BK) \arrow[l, "A_{PL}(B(\overline{\mu}))"'] \arrow[r, equal] & A_{PL}(BK) \\
\Lambda U \arrow[u, "m_{BG}", "\simeq"'] & \Lambda W \arrow[l, "\left(B(\overline{\mu})\right)^*"] \arrow[u, "m_{BK}", "\simeq"'] \arrow[r, hook] & \left(\Lambda W \otimes \Lambda W \otimes \Lambda(sW),\ d(sw) = 1 \otimes w - w \otimes 1\right). \arrow[u, "m''", "\simeq"']
\end{tikzcd}$$

Consider the following diagram,

$$\begin{tikzcd}[column sep=small]
H/K \arrow[rrr, equal] \arrow[dd] & & & H/K \arrow[dd] & \\
& K/K \arrow[ul] \arrow[rrr, equal] \arrow[dd] & & & K/K \arrow[ul] \arrow[dd] & = * \\
EG \times_G^{\overline{\mu}} H/K \arrow[rrr, "f"] \arrow[dd, "\pi"'] & & & EH/K \arrow[dd, "\pi'"'] & \\
& EG \times_G^{\overline{\mu}} K/K \arrow[ul, "\xi_{pt}"] \arrow[rrr] \arrow[dd] & & & EK/K \arrow[ul, "E\nu/K"'] \arrow[dd, "id_{BK}"] & = BK \\
BG \arrow[rrr, "B(\nu\circ\overline{\mu})"] \arrow[dr, equal] & & & BH & \\
& BG \arrow[rrr, "B\overline{\mu}"] & & & BK. \arrow[ul, "B\nu"']
\end{tikzcd}$$

Now we construct a model of right side square of above diagram. See the following diagram,

$$\begin{tikzcd}
A_{PL}(BH) \arrow[rr, "A_{PL}(\pi')"] \arrow[dd, "A_{PL}(B\nu)"'] & & A_{PL}(EH/K) \arrow[dd, "A_{PL}(E\nu/K)"'] & \\
& \Lambda V \arrow[ul, "m_{BH}", "\simeq"'] \arrow[rr, hook, "i_{\Lambda V}"] \arrow[dd, dotted, "\theta_1"] & & \left(\Lambda V \otimes \Lambda W \otimes \Lambda(sV),\ d(sv) = (B\nu)^*(v) - v\right) \arrow[ul, "m'", "\simeq"'] \arrow[dd, dotted, "\theta_2"] \\
A_{PL}(BK) \arrow[rr, equal] & & A_{PL}(BK) & \\
& \Lambda W \arrow[ul, "m_{BK}", "\simeq"'] \arrow[rr, hook, "i_{\Lambda W}"] & & \left(\Lambda W \otimes \Lambda W \otimes \Lambda(sW),\ d(sw) = 1 \otimes w - w \otimes 1\right). \arrow[ul, "m''", "\simeq"']
\end{tikzcd}$$

By employing the Lifting Lemma [FHT01, Proposition 14.6] and [FHT95, Lemma 3.6], we obtain θ_1 such that $m_{BK}\theta_1 \sim A_{PL}(B\nu)m_{BH}$ (homotopic). Moreover, there exists θ_2 such that $\theta_2 i_{\Lambda V} = i_{\Lambda W}\theta_1$ and $m''\theta_2 \sim A_{PL}(E\nu/K)m'$. On the other hand, because $H(\theta_1) = (B\nu)^*$, $\theta_1 = (B\nu)^*$. Since $\left((E\nu/K)^*(m')^*\right)[w] = w = (m'')^*[1 \otimes w]$, $H(\theta_2)[w] = [1 \otimes w]$ see the following diagram

$$\begin{tikzcd}
& H^*(EH/K) \arrow[d, "(E\nu/K)^*"', "\cong"] & H\left(\Lambda V \otimes \Lambda W \otimes \Lambda(sV),\ d(sv) = (B\nu)^*(v) - v\right) \arrow[l, "(m')^*"', "\cong"] \arrow[d, "H(\theta_2)"', "\cong"] \\
\Lambda W \arrow[r, equal] & H^*(BK) & H\left(\Lambda W \otimes \Lambda W \otimes \Lambda(sW),\ d(sw) = 1 \otimes w - w \otimes 1\right). \arrow[l, "(m'')^*"', "\cong"]
\end{tikzcd}$$

Therefore, we have

$$(\xi_{pt})^*[w] = H\big(id_{\Lambda U} \otimes_{(B\nu)^*} \theta_2\big)[w] = [1 \otimes w].$$

This complete the proof. $\square$

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