On the relative loop space homology and the Hochschild homology

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Abstract

The aim of this paper is to investigate relationships between relative free loop spaces and the Hochschild (co)homology and to give its application to relative string topology. In particular, we show that non-triviality of the Whitehead product of a mapping space implies non-commutativity of the loop product on the relative loop homology.

Let $f : X \to Y$ be a map from a $\mathbb{Q}$-Poincaré duality space X to a space Y and Y^I the space consisting of all paths on Y. We denote by $ev : L_fY \to X$ the evaluation fibration which is the pullback of the free path fibration $Y^I \to Y \times Y$ along the map $(f, f) : X \to Y \times Y$. Let $\mathrm{Sec}(ev)$ be the space of sections of the evaluation fibration ev with base point s which sends to x to $(x, c_{f(x)})$, where $c_{f(x)}$ denotes the constant path at $f(x)$. We show that there exists a natural injective map $\pi_*(\mathrm{Sec}(ev)) \otimes \mathbb{Q} \to H_{*+\dim X}(L_fY;\mathbb{Q})$ with degree of $\dim X$.

As an application of the result, we give a condition for the rational relative loop homology $\mathbb{H}_*(L_fY;\mathbb{Q})$ to be non-commutative provide X is a closed oriented manifold. Let $\mathrm{map}(X, Y; f)$ be the connected component of the mapping space $\mathrm{map}(X, Y)$ containing f. We prove that the non-trivial Whitehead product of $\mathrm{map}(X, Y; f)$ implies a non-commutativity of the loop homology $\mathbb{H}_*(L_fY;\mathbb{Q})$. This enables us to obtain an example of non-commutative algebra $\mathbb{H}_*(L_fY;\mathbb{Q})$ while the Chas-Sullivan loop homology $\mathbb{H}_*(LX;\mathbb{Q})$ is commutative in general.

Moreover, we describe the Whitehead products in the rational homotopy group of a mapping space in terms of the André-Quillen cohomology. As a consequence, an upper bound for the Whitehead length of a mapping space is given.

Contents

Chapter 1

Introduction

1.1 Backgrounds and Motivations

The main players in this paper are the free loop space and the Hochschild homology. Let X and Y be topological spaces and $\mathrm{map}(X, Y)$ the space of all continuous maps from X to Y with compact-open topology. The free loop space $LM = \mathrm{map}(S^1, M)$ is one of crucial and interesting objects in topology and geometry. For example, the unboundedness of the Betti number of the free loop space LM implies infiniteness of geodesics on a Riemannian manifold; see [19] and [46]. String topology initiated by Chas and Sullivan [9] is the study of algebraic structures on the homology $H_*(LM)$ of the free loop space. In particular, string operations on $H_*(LM)$ gives rise to 2-dimensional topological quantum field theory [8].

Relationships between free loop spaces and Hochschild homologies have been considered by several authors. One of ingredients for the study is a cosimplicial model for the free loop space. In [26], Jones has proved that the cohomology $H^*(LM; \mathbb{K})$ with coefficients in a field $\mathbb{K}$ is isomorphic as a vector space to the Hochschild homology of the simplicial cochain complex $S^*(\Omega^{\bullet}_{1,1})$ of a cosimplicial model $\Omega^{\bullet}_{1,1}$ for LM:

$$H^*(LM; \mathbb{K}) \cong HH_*(S^*(\Omega^{\bullet}_{1,1}; \mathbb{K})). \tag{1.1.1}$$

In the rational case, a commutative model for LM induces an isomorphism $H^*(LM) \cong HH_*(\Lambda V, \Lambda V)$, where ΛV is a minimal Sullivan model for M (see Chapter 2) and $HH_*(\Lambda V, \Lambda V)$ is the Hochschild homology of ΛV. Let M be a closed oriented manifold of dimension d, then the dual of the isomorphism and a Poincaré duality of M allow us to obtain the isomorphism $H_{*+d}(LM; \mathbb{Q}) \cong HH^{-*}(\Lambda V; \Lambda V)$. Félix and Thomas [17] have constructed an injective map, which is defined topologically, from the rational homotopy group of $\mathrm{aut}_1 M$ to the rational homology group of LM:

$$\Gamma : \pi_*(\Omega \mathrm{aut}_1 M) \otimes \mathbb{Q} \longrightarrow H_{*+d}(LM; \mathbb{Q}), \tag{1.1.2}$$

where $\mathrm{aut}_1 M$ denotes the path component of the monoid of the self-homotopy equivalences of M containing the identity map. In general, there is a direct sum decomposition of the Hochschild cohomology (called the Hodge decomposition or λ-decomposition)

$$HH^*(\Lambda V, \Lambda V) \cong \bigoplus_{i \geq 0} HH^*_{(i)}(\Lambda V, \Lambda V).$$

As for the summand $HH^*_{(1)}(\Lambda V, \Lambda V)$, Félix and Thomas [17] prove that the first piece is isomorphic to $\pi_*(\Omega \mathrm{aut}_1 M) \otimes \mathbb{Q}$ and fits into the commutative diagram

$$\begin{tikzcd} H_{n+d}(LM; \mathbb{Q}) \arrow[r, "\cong"] & HH^{-n}(\Lambda V, \Lambda V) \\ \pi_n(\Omega \mathrm{aut}_1(M)) \otimes \mathbb{Q} \arrow[u, "\Gamma"] \arrow[r, "\cong"'] & HH^{-n}_{(1)}(\Lambda V, \Lambda V). \arrow[u, "\text{inclusion}"'] \end{tikzcd} \tag{1.1.3}$$

This yields a topological description of the inclusion $HH^*_{(1)}(\Lambda V, \Lambda V) \subset HH^*(\Lambda V, \Lambda V)$. The results of Jones, Félix and Thomas mentioned above motivate us to generalize them to their relative versions.

1.2 Results

In what follows, we assume that a topological space has the homotopy type of a CW-complex whose homology with coefficients in a field $\mathbb{K}$ is of finite type.

Let X and Y be simply-connected spaces and $f, g : X \to Y$ continuous maps. Let Y^I denote the space consisting of all continuous maps form the closed unit interval $I = [0, 1]$ to the space Y. We denote by $ev_i : Y^I \to Y$ the evaluation map at i. Consider the pullback diagram:

$$\begin{tikzcd} P_{f,g} \arrow[r] \arrow[d, "ev"'] & Y^I \arrow[d, "(ev_0, ev_1)"] \\ X \arrow[r, "(f,g)"'] & Y \times Y. \end{tikzcd}$$

We observe that $P_{f,g}$ is homeomorphic to the space

$$\{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0),\ g(x) = \gamma(1)\}.$$

If $f = g$, then we write L_fY for $P_{f,f}$. For example, if X is the one point space and f, g are the constant map c_* to a base point of Y, then P_{c_*,c_*} is nothing but the based loop space ΩY. If $X = Y$ and both f, g are the identity map 1_Y, then $P_{1_Y,1_Y}$ is the free loop space LY.

Let $\Omega^\bullet_{f,g}$ be a cosimplicial model for $P_{f,g}$ and $C_*(S^*(\Omega^\bullet_{f,g}))$ a Hochschild chain complex of a simplicial cochain complex $S^*(\Omega^\bullet_{f,g}; \mathbb{K})$, that is a total complex of the double complex associated to a cosimplicial space $\Omega^\bullet_{f,g}$. Here $S^*(-;\mathbb{K})$ means the singular cochain functor over a field $\mathbb{K}$ (for proper definition, see Chapter 2). Our first result is described as follows.

Theorem 1.2.1. *Let X, Y be a simply-connected space and f, $g : X \to Y$ a continuous maps. Then, there is a quasi-isomorphism*

$$C_*(S^*(\Omega^\bullet_{f,g}; \mathbb{K})) \xrightarrow{\simeq} S^*(P_{f,g}; \mathbb{K}).$$

Theorem 1.2.1 is regarded as a generalization of Jones' work (1.1.1) mentioned in §1.1. Chen [10] proved Theorem 1.2.1 in the case where the underlying field $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers. We prove Theorem 1.2.1 relying on the ideal due to Chen.

Assume that X is a $\mathbb{Q}$-Poincaré duality space of formal dimension d, that is, the space X is equipped with a homology class $[X] \in H_d(X;\mathbb{Q})$ called an *orientation class* for which the cap product

$$- \cap [X] : H^*(X;\mathbb{Q}) \longrightarrow H_{d-*}(X;\mathbb{Q})$$

is an isomorphism. Let $\mathrm{Sec}(ev)$ be the space of all sections of the fibration $ev : L_fY \to X$ with a base point $s : X \to L_fY$ which sends x to $(x, c_{f(x)})$, where $c_{f(x)}$ is the constant path at $f(x)$. Then, for $n \geq 1$, we define Γ_1 by the composite

$$\Gamma_1 : \pi_n(\mathrm{Sec}(ev)) \otimes \mathbb{Q} \xrightarrow{H} H_n(\mathrm{Sec}(ev);\mathbb{Q}) \xrightarrow{\Gamma} H_{n+d}(L_fY;\mathbb{Q}).$$

Here, H is the Hurewicz map and Γ is the map defined by

$$\Gamma(a) = H_*(\mathrm{Ev})(a \otimes [X]),$$

$[X] \in S_d(X;\mathbb{Q})$ is the representative element of the orientation class of X and $\mathrm{Ev} : \mathrm{Sec}(ev) \times X \to L_fY$ is the evaluation map.

Theorem 1.2.2. *Let X be a simply connected $\mathbb{Q}$-Poincaré duality space with the homotopy type of a finite CW-complex and Y a simply-connected space. Let A be a $\mathbb{Q}$-Poincaré duality model for X in the sense of Lambrechts and Stanley [30] and $(\Lambda V, d)$ a minimal Sullivan model for Y. Then one sees that* *there exist isomorphisms* $H_{*+d}(L_fY;\mathbb{Q}) \cong HH^*(\Lambda V, A)$ *and* $\pi_*(\mathrm{Sec}(ev)) \otimes \mathbb{Q} \cong HH^*_{(1)}(\Lambda V, A)$ *such that the following diagram is commutative:*

$$\begin{tikzcd} H_{*+d}(L_fY;\mathbb{Q}) \arrow[r, "\cong"] & HH^*(\Lambda V, A) \\ \pi_*(\mathrm{Sec}(ev)) \otimes \mathbb{Q} \arrow[u, "\Gamma_1"] \arrow[r, "\cong"'] & HH^*_{(1)}(\Lambda V, A). \arrow[u, "\text{inclusion}"'] \end{tikzcd}$$

Let M be a closed oriented manifold of dimension d. In [9], Chas and Sullivan introduced a product on $H_*(LM)$ of degree d called the *loop product*. Moreover they have proved that the shifted homology $\mathbb{H}_*(LM) = H_{*+d}(LM)$ endowed with the loop product is a graded commutative algebra. Gruher and Salvatore [20] generalized the loop product to that on a relative loop space L_fY when X is a simply-connected d-dimensional closed oriented manifold. It turns out that $\mathbb{H}_*(L_fY)$ also has a graded algebra structure similar to the construction of loop products. We now recall the construction of loop product on $\mathbb{H}_*(L_fY)$. Consider the pullback diagram

$$\begin{tikzcd} L_fY \times_X L_fY \arrow[r] \arrow[d] & L_fY \times L_fY \arrow[d, "ev\times ev"] \\ X \arrow[r, "\Delta_X"'] & X \times X, \end{tikzcd}$$

where Δ_X is the diagonal map. Let $\mathrm{Comp} : L_fY \times_X L_fY \to L_fY$ be the concatenation of loops, that is, Comp is defined by $\mathrm{Comp}((x,\gamma_1),(x,\gamma_2)) = (x, \gamma_1 * \gamma_2)$ and

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & (0 \le t \le \frac{1}{2}) \\ \gamma_2(2t-1) & (\frac{1}{2} \le t \le 1) \end{cases}$$

for any $(x,\gamma_i) \in L_fY$ and $t \in [0,1]$. Gruher and Salvatore [20] constructed a homomorphism

$$\bar{\Delta}^!_X : H_*(L_fY \times L_fY) \longrightarrow H_{*-d}(L_fY \times_X L_fY)$$

with degree $-d$ by using the Thom construction. Then the loop product on $H_*(L_fY)$ is defined by the composite

$$\begin{tikzcd} H_*(L_fY) \otimes H_*(L_fY) \arrow[r] & H_*(L_fY \times L_fY) \arrow[d, "\bar{\Delta}^!_X"] & \\ & H_{*-d}(L_fY \times_X L_fY) \arrow[r, "\mathrm{Comp}_*"] & H_{*-d}(L_fY). \end{tikzcd}$$

The loop product on $\mathbb{H}_*(L_fY)$ is not necessarily graded commutative while the Chas-Sullivan loop homology $\mathbb{H}_*(LX)$ is graded commutative. Theorem 1.2.2 allows us to deduce a criterion for non-commutativity of the algebra $\mathbb{H}_*(L_fY;\mathbb{Q})$. Let $\mathrm{map}(X,Y;f)$ denote the connected component of the mapping space $\mathrm{map}(X,Y)$ containing f.

Proposition 1.2.3. *If the rational homotopy group $\pi_{\geq 2}(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ has a non-trivial Whitehead product, then $\mathbb{H}_*(L_fY;\mathbb{Q})$ is a non-commutative graded algebra.*

Examples of the spaces X and Y in which $\pi_{\geq 2}(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ has a non-trivial Whitehead product are described in Chapter 5 and Chapter 6.

The three assertions mentioned above, Theorems 1.2.1, 1.2.2 and Proposition 1.2.3 are stated in the article [38] by the author. However, there are some gaps in the proof of [38, Theorem 1.1, Theorem 1.2, Corollary 1.3]. Then modifying the proofs, we here refine these results and give proofs of Theorems 1.2.1 and 1.2.2 and Proposition 1.2.3.

Suppose that X is a finite CW-complex. Let ΛV be a minimal Sullivan model for Y, B a CDGA model for X and $\bar{f}$ a model for f. Denote by $H_{\mathrm{AQ}}^{-n}(\Lambda V,B;\bar{f})$ the homology of the complex of $\bar{f}$-derivations called the *André-Quillen cohomology* ([2]). The nth rational homotopy group of $\mathrm{map}(X,Y;f)$ is isomorphic to $H_{\mathrm{AQ}}^{-n}(\Lambda V,B;\bar{f})$ as abelian groups for $n\geq 2$. This fact has been proved by Block and Lazarev [2], Buijs and Murillo [7], Lupton and Smith [31]; see Chapter 5 for precise definitions and details. In order to study of the Whitehead product of the mapping space from rational homotopy theory point of view, Buijs and Murillo [7] defined a bracket in the André-Quillen cohomology $H_{\mathrm{AQ}}^{*}(\Lambda V,B;\bar{f})$ which coincides with the Whitehead product in $\pi_*(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ via the isomorphism mentioned above. We remark that the isomorphism due to Buijs and Murillo is constructed relying on the Sullivan model for $\mathrm{map}(X,Y;f)$ due to Haefliger [21] and Brown and Szczarba [4]. To this end, the finiteness of a model B for the source space X is assumed in the result [4, Theorem 1.3] and [21, §3].

On the other hand, the finiteness hypothesis on X assures that $\pi_n(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}$ is isomorphic to $\pi_n(\mathrm{map}(X,Y_{\mathbb{Q}};lf))$, where $l:Y\to Y_{\mathbb{Q}}$ the localization map; see [25, II. Theorem 3.11] and [42, Theorem 2.3]. Then the isomorphism constructed in [2] and [31] factors as follows:

$$\pi_n(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}\xrightarrow[\cong]{l_*}\pi_n(\mathrm{map}(X,Y_{\mathbb{Q}};lf))\xrightarrow[\cong]{\Theta}H_{\mathrm{AQ}}^{-n}(\Lambda V,B;\bar{f}).$$

For the detail of the map Θ, see Section 5.1. The proof of [31, Theorem 2.1] and the result [2, Theorem 3.8] show that the second map Θ is an isomorphism without the finiteness hypothesis on X. In this paper, we introduce a bracket in the André-Quillen cohomology which coincides with the Whitehead product in $\pi_*(\mathrm{map}(X, Y_{\mathbb{Q}}; f))$ up to the isomorphism Θ without assuming that X has a finite dimensional commutative model. Thus one might expect a generalization of the result [7, Theorem 2] due to Buijs and Murillo.

Let X be a simply-connected space with a CDGA model B and Y be a $\mathbb{Q}$-local, simply-connected space of finite type. Then we have a model $\bar{f} : \Lambda V \to B$ for a based map $f : X \to Y$. Now, we define a bracket in $H^*_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$.

$$[\ ,\ ] : H^n_{\mathrm{AQ}}(\Lambda V, B; \bar{f}) \otimes H^m_{\mathrm{AQ}}(\Lambda V, B; \bar{f}) \longrightarrow H^{n+m+1}_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$$

by

$$\begin{aligned}
&[\varphi, \psi](v) = (-1)^{n+m-1} \\
&\times \sum \Big(\sum_{i \neq j} (-1)^{\varepsilon_{ij}} \bar{f}(v_1 \cdots v_{i-1}) \varphi(v_i) \bar{f}(v_{i+1} \cdots v_{j-1}) \psi(v_j) \bar{f}(v_{j+1} \cdots v_s) \Big),
\end{aligned} \tag{1.2.1}$$

where v is a basis of V, $dv = \sum v_1 v_2 \cdots v_s$ and

$$\varepsilon_{ij} = \begin{cases} |\varphi|(\sum_{k=1}^{i-1} |v_k|) + |\psi|(\sum_{k=1}^{j-1} |v_k|) + |\varphi||\psi| & (i < j) \\ |\varphi|(\sum_{k=1}^{i-1} |v_k|) + |\psi|(\sum_{k=1}^{j-1} |v_k|) & (j < i). \end{cases}$$

The following is our main result on the Whitehead product in the rational homotopy group of a mapping space.

Theorem 1.2.4. *The isomorphism $\Theta : \pi_n(\mathrm{map}(X, Y; f)) \to H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$ is compatible with the Whitehead product in $\pi_n(\mathrm{map}(X, Y; f))$ and the bracket in $H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$ defined by the formula* (1.2.1).

It is important to remark that if X is finite, then the bracket in $H^*_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$ coincides with that due to Buijs and Murillo [7] up to sign. Let $\mathrm{map}_*(X, Y; f)$ be the path-component of the space of based maps from X to Y containing the based map $f : X \to Y$. We can apply the same argument as the case of the based mapping space $\mathrm{map}_*(X, Y; f)$; see Section 5.1 for details.

As an application of the result, we study the Whitehead length of a mapping space. The *Whitehead length* of a space Z, written $\mathrm{WL}(Z)$, is the length of non-zero iterated Whitehead products in $\pi_{\geq 2}(Z)$. By definition, $\mathrm{WL}(Z) = 1$ means that all Whitehead products vanish. In [32], Lupton and Smith give some results and examples related to a Whitehead length of $\mathrm{map}(X, Y; f)$ using a Quillen model for the mapping space. In this paper, we give another proof of their results using the bracket in the André-Quillen cohomology; see Proposition 5.2.1. In order to describe an upper bound for the Whitehead length of $\mathrm{map}_*(X, Y; f)$, we introduce a numerical invariant.

Definition 1.2.5 ([14, p315]). The *product length* of a connected graded algebra A, written $\mathrm{nil}A$, is the greatest integer n such that $A^+A^+\cdots A^+ \neq 0$ (n factors).

In [5], Buijs proved the following theorem.

Theorem 1.2.6 ([5, Theorem 0.3]). *Let X and Y be simply-connected spaces with finite type over $\mathbb{Q}$ and B a CDGA model for X. If $\mathrm{WL}(Y_{\mathbb{Q}}) = 1$, then*

$$\mathrm{WL}(\mathrm{map}_*(X, Y; f)_{\mathbb{Q}}) \leq \mathrm{nil}B - 1.$$

Using the bracket in the André-Quillen cohomology, we can prove the following proposition, which refines the above result; see Remark 5.2.4.

Proposition 1.2.7. *Let X and Y be simply-connected spaces with finite type over $\mathbb{Q}$, ΛV a minimal Sullivan model for Y and B a CDGA model for X. Assume further that Y is $\mathbb{Q}$-local and the differential of ΛV is non-trivial. If $\mathrm{WL}(Y) = 1$ and $\mathrm{nil}B \geq 2$, then*

$$\mathrm{WL}(\mathrm{map}_*(X, Y; f)) \leq \frac{1}{\omega - 1}(\mathrm{nil}B - 1) + 1,$$

where $\omega = \min\{n \geq 2 \mid d(V) \subset \Lambda^{\geq n}V\}$.

We here remark that the equation $\mathrm{WL}(Y) = 1$ implies that $\omega \geq 3$. Furthermore, ω is the largest number such that all Whitehead products of order less than ω vanish in Y [1, Proposition 6.4]. If Y has a minimal Sullivan model with a zero differential, it is readily seen that $\mathrm{WL}(\mathrm{map}_*(X, Y; f)) = 1$ by the bracket (1.2.1).

As computational examples, we compute the Whitehead length of $\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^m, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^n_{\mathbb{Q}}; f_1)$. Recall that $(\Lambda(x_2, x'_{2n+1}), dx_{2n+1} = x_2^{n+1})$ and $(\mathbb{Q}[z_2], 0)$ are the minimal Sullivan models for $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$, respectively. Let $f_1 : \mathbb{C}P^\infty \times \mathbb{C}P^m \to \mathbb{C}P^\infty \times \mathbb{C}P^n$ be the realization of the CDGA map

$$\bar{f}_1 : \mathbb{Q}[z_2] \otimes \Lambda(x_2, x'_{2n+1}) \longrightarrow \mathbb{Q}[w_2] \otimes \Lambda(y_2, y'_{2m+1})$$

defined by $\bar{f}_1(z_2) = q_1(w_2 \otimes 1)$, $\bar{f}_1(x_2) = q_2(w_2 \otimes 1) + q_3(1 \otimes y_2)$ and $\bar{f}_1(x'_{2n+1}) = 0$ for some $q_1, q_2, q_3 \in \mathbb{Q}$.

Proposition 1.2.8. *Let $m < n$. Then one has*

$$\mathrm{WL}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1)) = \begin{cases} 2 & (n-m=1,\ q_2=0,\ q_3 \neq 0) \\ 1 & (\textit{otherwize}) \end{cases}$$

By Proposition 1.2.3, we have an interesting example. Let X be a 3-dimensional sphere S^3 and Y be a space with a minimal Sullivan model $(\Lambda V, d) = (\Lambda(x_1, x_2, x_3, y), d)$ with $|x_1| = 2$, $|x_2| = |x_3| = 3$ and $|y| = 7$. The differential d is given by $dx_i = 0$ for any i and $dy = x_1x_2x_3$. Let $f_2 : S^3 \to Y$ be a map which is the realization of the CDGA map $\bar{f}_2 : \Lambda V \to M(S^3) = (\Lambda(e_3), 0)$ defined as

$$\bar{f}_2(x_1) = 0,\ \bar{f}_2(x_2) = \bar{f}_2(x_3) = e_3,\ \bar{f}_2(y) = 0.$$

In this setting, Lupton and Smith [32, Example 6.6] show that the Whitehead length of $\mathrm{map}(S^3, Y; f_2)$ is greater than 2 by using a Quillen model. We can give another proof of this result using the bracket (1.2.1) and have the following result.

Proposition 1.2.9. *One has*

$$\mathrm{WL}(\mathrm{map}(S^3, Y; f_2)) = 2$$

and hence $\mathbb{H}_*(L_{f_2}Y; \mathbb{Q})$ is non-commutative.

The organization of this paper is as follows. In Chapter 2, we will give preliminaries for our arguments in this paper. We will recall several fundamental definitions and results on rational homotopy theory. The precise definitions of the Hochschild (co)homology and the cubical singular chain complex are described in this chapter. Chapter 3 is devoted to proving Theorem 1.2.1. In Chapter 4, we will begin with an introduction of a commutative models for $\mathrm{Sec}(ev)$ and a Poincaré duality space. Theorem 1.2.2 and Proposition 1.2.3 are proved in this chapter. Chapter 6 will give a proof of Theorem 1.2.4. The Whitehead length of mapping spaces is also investigated. In the last section of this chapter, we will prove Propositions 1.2.8 and 1.2.9.

Chapter 2

Preliminaries

2.1 Rational homotopy theory

We refer the reader to the book [14] or [16] for the fundamental facts on rational homotopy theory.

Let V be a graded $\mathbb{Q}$-vector space of the form $V = \bigoplus_{i\geq 1} V^i$. Then a free commutative differential graded algebra (CDGA), $(\Lambda V, d)$ is called a *Sullivan algebra* if V has an increasing sequence of subspaces $V(0) \subset V(1) \subset \cdots$ which satisfies the conditions that $V = \bigcup_{i\geq 0} V(i)$, $d = 0$ in $V(0)$ and $d : V(i) \to \Lambda V(i-1)$ for any $i \geq 1$.

We recall a *minimal Sullivan model* for a simply-connected space X with finite type. It is a Sullivan algebra of the form $(\Lambda V, d)$ with $V = \bigoplus_{i\geq 2} V^i$, where each V^i is of finite dimension and d is decomposable; that is, $d(V) \subset \Lambda^{\geq 2} V$. Moreover, $(\Lambda V, d)$ is equipped with a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X)$ to the CDGA $A_{\mathrm{PL}}(X)$ of differential polynomial forms on X. Observe that, as algebras, $H^*(\Lambda V, d) \cong H^*(A_{\mathrm{PL}}(X)) \cong H^*(X; \mathbb{Q})$.

Example 2.1.1. 1. A minimal Sullivan model for the n-dimensional sphere S^n, $M(S^n)$, has the form $(\Lambda(e_n), 0)$ if n is odd and $(\Lambda(e_n, e_{2n-1}), de_{2n-1} = e_n^2)$ if n is even, where $|e_n| = n$ and $|e_{2n-1}| = 2n-1$.

2. A minimal Sullivan model for the complex projective space $\mathbb{C}P^n$ has the form $(\Lambda(x_2, y_{2n+1}), dx_2 = 0, dy_{2n+1} = x_2^{n+1})$, where $|x_2| = 2$ and $|y_{2n+1}| = 2n+1$.

A *CDGA model* for a space X is a connected CDGA (B, d) if there exists a quasi-isomorphism from a minimal Sullivan model for X to B. Let $\Lambda(t, dt)$ be the free CDGA with $|t| = 0$, $|dt| = 1$ and the differential d of $\Lambda(t, dt)$ sends t to dt. We defined the map $\varepsilon_i : \Lambda(t, dt) \to \mathbb{Q}$ by $\varepsilon_i(t) = i$. Two maps φ_0 and φ_1 of CDGA's from a Sullivan algebra ΛV to a CDGA A are *homotopic* if there exists a CDGA map $H : \Lambda V \to A \otimes \Lambda(t, dt)$ such that $(1 \cdot \varepsilon_i)H = \varphi_i$ for $i = 0, 1$. Denote by $[\Lambda V, A]$ the set of homotopy classes of CDGA maps from ΛV to A.

Let $f : X \to Y$ be a map between spaces of finite type. Then there exists a CDGA map $\widetilde{f}$ from a minimal Sullivan model $(\Lambda V_Y, d)$ for Y to a minimal Sullivan model $(\Lambda V_X, d)$ for X which makes the diagram

$$\begin{tikzcd} A_{\mathrm{PL}}(Y) \arrow[r, "A_{\mathrm{PL}}(f)"] & A_{\mathrm{PL}}(Y) \\ \Lambda V_Y \arrow[u, "\simeq"] \arrow[r, "\widetilde{f}"'] & \Lambda V_X \arrow[u, "\simeq"'] \end{tikzcd}$$

commutative up to homotopy. Let $\rho : \Lambda V_X \xrightarrow{\simeq} B$ a CDGA model for X, we call $\rho\widetilde{f}$ a *model* for f associated with models ΛV_Y and B and denote it by $\bar{f}$.

We now recall the following result.

Proposition 2.1.2 ([14, Proposition 12.9]). *Let A and C be CDGAs, ΛV a Sullivan algebra and $\pi : A \to C$ a quasi-isomorphism. Then the map*

$$\pi_* : [\Lambda V, A] \longrightarrow [\Lambda V, C]$$

induced by π is bijective.

In particular, we use the proposition when constructing a model for the Whitehead product of a mapping space in Chapter 5.

Remark 2.1.3. If π is a surjective quasi-isomorphism and ΛV is a minimal Sullivan model, then we can construct a CDGA map $\phi : \Lambda V \to A$ such that $\pi\phi = \psi$ for any CDGA map $\psi : \Lambda V \to C$ by induction on a degree of V [14, Lemma 12.4]. Let v be a basis of V and assume that ϕ is constructed in $\Lambda V^{<|v|}$. Then $\phi d(v)$ is defined. Since π is a surjective quasi-isomorphism and $\pi\phi d(v) = d\psi(v)$, we can find $a \in A$ such that $d(a) = \phi d(v)$ and $\pi(a) = \psi(v)$. Then, we extend ϕ with $\phi(v) = a$.

2.2 The Hochschild homology and cohomology

Let $E = \{(E_n, d)\}_{n \geq 0}$ be a simplicial cochain complex, that is, (E_n, d) is a cochain complex for any $n \geq 0$ together with cochain maps $\delta_i : E_{n-1} \to E_n$ and $\sigma_i : E_{n+1} \to E_n$ $(0 \leq i \leq n)$ satisfying simplicial identities. We then get a double complex $C_{p,q}(E)$ given by $C_{p,q}(E) = (E_p)^q$ with the vertical and horizontal differentials

$$d_1 : C_{p,q}(E) \longrightarrow C_{p,q+1}(E),\ d_1(x) = (-1)^p d(x),$$
$$d_2 : C_{p,q}(E) \longrightarrow C_{p-1,q}(E),\ d_2(x) = (-1)^{p-1} \sum_{i=0}^{p} (-1)^i \delta_i(x).$$

The total complex $(C_*(E), D)$ of the double complex $\{C_{p,q}(E)\}_{p,q}$ given by

$$C_n(E) = \bigoplus_{p-q=n} C_{p,q}(E),\ D = d_1 + d_2$$

is called the Hochschild complex of a simplicial cochain complex E and its homology, $HH_*(E)$, is called the Hochschild homology of E.

Let (A, d) be a differential graded algebra over a field $\mathbf{k}$ and M a differential graded A-A bimodule. Denote by sA the suspension of A, that is $(sA)^n = A^{n+1}$ and $T(sA)$ the tensor algebra on sA. The *Hochschild chain complex* of A with coefficient in M is the complex $C_*(A; M) = M \otimes T(sA)$ with the differential $D = D_1 + D_2$ defined by

$$\begin{aligned} D_1(m[a_1|a_2|\cdots|a_k]) =& d(m)[a_1|a_2|\cdots|a_k] \\ & - \sum_{i=1}^{k} (-1)^{\varepsilon_i} m[a_1|a_2|\cdots|d(a_i)|\cdots|a_k] \end{aligned}$$

$$\begin{aligned} D_2(m[a_1|a_2|\cdots|a_k]) =& (-1)^{|m|} m a_1[a_2|\cdots|a_k] \\ & + \sum_{i=2}^{k} (-1)^{\varepsilon_i} m[a_1|\cdots|a_{i-1}a_i|\cdots|a_k] \\ & - (-1)^{\varepsilon_k(|a_k|+1)} a_k m[a_1|a_2|\cdots|a_{k-1}]. \end{aligned}$$

Here $\varepsilon_i = |m| + \sum_{j<i} |sa_j|$ and $m[a_1|a_2|\cdots|a_k]$ the element $m \otimes (sa_1 \otimes sa_2 \otimes \cdots \otimes sa_k)$ of $M \otimes T(sA)$. The homology of the complex, $HH_*(A; M)$, is called the Hochschild homology. The *bar construction* of A is the complex

$$\mathrm{B}(A; A; A) = A \otimes T(sA) \otimes A$$

with the differential $d_{\mathrm{B}} = d_1 + d_2$ defined by

$$\begin{aligned} d_1(a[a_1|a_2|\cdots|a_k]b) =& d(a)[a_1|a_2|\cdots|a_k]b \\ & - \sum_{i=1}^{k} (-1)^{\varepsilon_i} a[a_1|a_2|\cdots|d(a_i)|\cdots|a_k]b \\ & + (-1)^{\varepsilon_{k+1}} a[a_1|a_2|\cdots|a_k]d(b), \end{aligned}$$

$$
\begin{aligned}
d_2(a[a_1|a_2|\cdots|a_k]b) =& (-1)^{|a|}aa_1[a_2|\cdots|a_k]b \\
&+\sum_{i=2}^{k}(-1)^{\varepsilon_i}a[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]b \\
&-(-1)^{\varepsilon_k}a[a_1|a_2|\cdots|a_{k-1}]a_kb.
\end{aligned}
$$

Let A be a differential graded algebra with an augmentation $A \to \mathbb{K}$ and $\bar{A}$ the augmentation ideal of A. The *normalized bar construction* $\overline{\mathrm{B}}(A, A, A)$ of A is the complex $A \otimes T(s\bar{A}) \otimes A$ with the differential defined as with the differential of $\mathrm{B}(A, A, A)$. We then see that the inclusion $\overline{\mathrm{B}}(A, A, A) \hookrightarrow \mathrm{B}(A, A, A)$ is a quasi-isomorphism.

Let A^{op} be the opposite graded algebra of A and A^e the tensor algebra $A \otimes A^{op}$. Recall that any A-bimodule can be considered a left (or right) A^e-module.

Lemma 2.2.1 ([13, Lemma 4.3]). *The left A^e-module map*

$$\varepsilon_A : \mathrm{B}(A; A; A) \to A$$

defined by $\varepsilon_A(a[\ ]b) = ab$ and $\varepsilon_A(a[a_1|a_2|\cdots|a_k]b) = 0$ for $k > 0$ is a semifree resolution of A as a left A^e-module.

We have an isomorphism

$$(C_*(A; M), D) \cong (M \otimes_{A^{op}} \mathrm{B}(A; A; A), d \otimes 1 + 1 \otimes d_{\mathrm{B}})$$

Therefore, the Hochschild homology is described by the torsion functor in the sense of Moore, see [12, Appendix],

$$HH_*(A, M) = \mathrm{Tor}^*_{A^e}(A, M).$$

Consider the complex

$$C^*(A; M) = (\mathrm{Hom}_{A^e}(\mathrm{B}(A; A; A), M), D'),$$

where $D'(\varphi) = d_M \circ \varphi - (-1)^{|\varphi|}\varphi \circ d_{\mathrm{B}}$ for $\varphi \in \mathrm{Hom}_{A^e}(\mathrm{B}(A; A; A), M)$. We call the complex *the Hochschild cochain complex* of A with coefficient in M. The *Hochschild cohomology* is the homology of the complex $C^*(A; M)$, $HH^*(A; M)$. We see that

$$HH^*(A, M) = \mathrm{Ext}^*_{A^e}(A, M).$$

2.3 The cubical singular chain complex

Let $I^n = [0,1]^n$ be the n times product of the closed unit interval $[0,1]$. An *n-cube* in a topological space Z is a continuous map $I^n \to Z$. An n-cube $\sigma : I^n \to Z$ is *degenerate* if there exist a integer i, $1 \le i \le n$, and an $(n-1)$-cube $\sigma' : I^{n-1} \to Z$ such that $\sigma(t_1, t_2, \cdots, t_n) = \sigma'(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$ for any $(t_1, t_2, \cdots, t_n) \in I^n$. Note that all 0-cube are non-degenerate. We denote by $C_n(Z;\mathbb{K})$ the free $\mathbb{K}$-module generated by the set of all non-degenerate n-cubes in Z. We define the map

$$\lambda_i^{\varepsilon} : I^{n-1} \longrightarrow I^n; (t_1, t_2, \cdots, t_{n-1}) \longmapsto (t_1, \cdots, t_{i-1}, \varepsilon, t_i, \cdots, t_{n-1})$$

for $\varepsilon = 0, 1$ and $1 \le i \le n$. Let $\partial = \sum_{i=1}^{n} (\lambda_i^{0*} - \lambda_i^{1*}) : C_n(Z;\mathbb{K}) \to C_{n-1}(Z;\mathbb{K})$. Then ∂ is a well-defined differential of $C_*(Z;\mathbb{K})$ ([33, p.13]) and the chain complex $(C_*(Z;\mathbb{K}), \partial)$ is called the *cubical singular chain complex* of Z. The *cubical singular cochain complex* of Z over $\mathbb{K}$ is the complex $C^n(Z;\mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}^{-n}(C_*(Z), \mathbb{K})$. The differential $d : C^{n-1}(Z;\mathbb{K}) \to C^n(Z;\mathbb{K})$ is defined by $d(\varphi) = \varphi\partial$ for $\varphi \in C^{n-1}(Z;\mathbb{K})$.

The Alexander-Whitney map and the Eilenberg-Zilber map are also defined in cubical singular chain complexes ([33, p.133, p137]). The Eilenberg-Zilber map

$$\mathrm{EZ} : C_n(Z_1;\mathbb{K}) \otimes C_m(Z_2;\mathbb{K}) \longrightarrow C_{n+m}(Z_1 \times Z_2;\mathbb{K})$$

is defined by $\mathrm{EZ}(\varphi \otimes \psi) = \varphi \times \psi$ where φ (resp. ψ) is an n (resp. m)-cube. The Alexander-Whitney map is defined as follows. Let J be any subset of $\{1, 2, \cdots, n+m\}$ and J^c the complementary subset of J. If $J = \{j_1, j_2, \cdots, j_l\}$, then denote $\lambda_J^{\varepsilon} = \lambda_{j_1}^{\varepsilon} \lambda_{j_2}^{\varepsilon} \cdots \lambda_{j_l}^{\varepsilon}$. For any $(n+m)$-cube $\sigma : I^{n+m} \to Z_1 \times Z_2$, we define a map $\mathrm{AW} : C_{n+m}(Z_1 \times Z_2;\mathbb{K}) \to (C_*(Z_1;\mathbb{K}) \otimes C_*(Z_2;\mathbb{K}))_{n+m}$ by

$$\mathrm{AW}(\sigma) = \sum_J (-1)^{\varepsilon(J)} (pr_1 \sigma \lambda_{J^c}^0) \otimes (pr_2 \sigma \lambda_J^1) \in (C_*(Z_1;\mathbb{K}) \otimes C_*(Z_2;\mathbb{K}))_{n+m}$$

where $pr_i : Z_1 \times Z_2 \to Z_i$ is the projection and $\varepsilon(J)$ is the cardinal number of the set $\{(i,j) \in J \times J^c \mid j < i\}$. We see that EZ and AW are chain maps; see [33, p.133, p138].

Remark 2.3.1. The cubical singular chain complex $C_*(Z;\mathbb{K})$ is quasi-isomorphic to the singular chain complex $S_*(Z;\mathbb{K})$. In fact, let

$$\Delta^n = \{(t_1, t_2 \cdots, t_n) \in \mathbb{R}^n \mid 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1\}$$

be the standard n-simplex and $\kappa_n : I^n \to \Delta^n$ be a non-degenerate cubical chain defined by

$$\kappa_n(t_1, t_2, \cdots, t_n) = (x_1, x_2, \cdots, x_n), \ x_i = 1 - t_1 t_2 \cdots t_i.$$

The method of acyclic models [40, Theorem 5.2.3'] allows us to conclude that the chain map $\kappa_* : S_*(Z;\mathbb{K}) \to C_*(Z;\mathbb{K})$ induced by κ is a quasi-isomorphism. Let $\mathrm{EZ} : S_n(Z_1) \otimes S_m(Z_1) \to S_{n+m}(Z_1 \times Z_2)$ and $\mathrm{AW} : S_{n+m}(Z_1 \times Z_2) \to (S_*(Z_1) \otimes S_*(Z_2))_{n+m}$ be the Eilenberg-Zilber map and the Alexander-Whitney map defined in singular chain complexes, see [39, §12] or [40, §5] for the definition. A straightforward computation shows that the diagram is strictly commutative:

$$\begin{tikzcd} S_*(Z_1 \times Z_2) \arrow[r, "\mathrm{AW}"] \arrow[d, "\kappa_*"'] & S_*(Z_1) \otimes S_*(Z_2) \arrow[d, "\kappa_* \otimes \kappa_*"] \\ C_*(Z_1 \times Z_2) \arrow[r, "\mathrm{AW}"'] & C_*(Z_1) \otimes C_*(Z_2). \end{tikzcd}$$

We see that the map $\kappa^* : C^*(Z) \to S^*(Z)$ induced by κ_n is an algebra map. On the other hand, the diagram

$$\begin{tikzcd} S_*(Z_1) \otimes S_*(Z_2) \arrow[r, "\mathrm{EZ}"] \arrow[d, "\kappa_* \otimes \kappa_*"'] & S_*(Z_1 \times Z_2) \arrow[d, "\kappa_*"] \\ C_*(Z_1) \otimes C_*(Z_2) \arrow[r, "\mathrm{EZ}"'] & C_*(Z_1 \times Z_2) \end{tikzcd}$$

does not commutes strictly, however commutative up to chain homotopy. This fact is shown by the method of acyclic models. We can choose the chain homotopy $h : S_*(Z_1) \otimes S_*(Z_2) \to C_*(Z_1 \times Z_2)$ so that the equation $(f_1 \times f_2)_* h = h(f_{1*} \otimes f_{2*})$ holds for any $f_i : Z_i \to W_i$ $(i = 1, 2)$.

In the rest of this section, we recall the *integration map* (the *slant product*). Let $\sigma \in C_q(Z_1;\mathbb{K})$, then define a map $\int_\sigma : C^{n+q}(Z_1 \times Z_2;\mathbb{K}) \to C^n(Z_2;\mathbb{K})$ by $(\int_\sigma x)(\varphi) = x(\sigma \times \varphi)$ for any $\varphi \in C_n(Z_2;\mathbb{K})$. It is easily seen the equality:

$$d\Big(\int_\sigma x\Big) = (-1)^q \Big(\int_\sigma dx - \int_{\partial\sigma} x\Big). \tag{2.3.1}$$

In fact, the following equations show the equality (2.3.1):

$$\begin{aligned} \Big(\int_\sigma dx\Big)(\varphi) &= dx(\sigma \times \varphi) \\ &= x(\partial\sigma \times \varphi) + (-1)^q x(\sigma \times \partial\varphi) \\ &= \Big(\int_{\partial\sigma} x\Big)(\varphi) + (-1)^q d\Big(\int_\sigma x\Big)(\varphi). \end{aligned}$$

We note that the equation (2.3.1) is a particular version of Stokes' theorem and the integration map is also defined in the singular cochain algebra of a space similarly.

Chapter 3

A cosimplicial model and the Hochschild homology

In this chapter, the ground field is an arbitrary field $\mathbb{K}$. For any space Z, we write $S^*(Z)$ (resp. $C^*(Z)$) for $S^*(Z;\mathbb{K})$ (resp. $C^*(Z;\mathbb{K})$).

3.1 Totalization of cosimplicial spaces

Let $Z^\bullet = \{Z^n\}$ be a cosimplicial space and $\Delta^\bullet$ be the cosimplicial space of the standard simplex, that is, $\Delta^\bullet$ is a family of the standard simplexes $\{\Delta^n\}_{n\geq 0}$ ($\Delta^0 = \{0\}$) together with coface operators $\delta^i : \Delta^{n-1} \to \Delta^n$ and codegeneracy operators $\sigma^i : \Delta^{n+1} \to \Delta^n$ for $1 \leq i \leq n$ given by

$$\delta^i(t_1, \cdots, t_{n-1}) = \begin{cases} (0, t_1, \cdots, t_{n-1}) & (i = 0) \\ (t_1, \cdots, t_i, t_i, \cdots, t_{n-1}) & (1 \leq i \leq n-1) \\ (t_1, \cdots, t_{n-1}, 1) & (i = n) \end{cases}$$

and

$$\sigma^i(t_1, \cdots, t_{n+1}) = (t_1, \cdots, t_i, t_{i+2}, \cdots, t_{n+1}).$$

The *totalization* (or *geometric realization*) of a cosimplicial space $Z^\bullet$ is the subspace

$$\mathrm{Tot}(Z^\bullet) = \{\Delta^\bullet \to Z^\bullet \mid \text{a map of cosimplicial space}\} \subset \prod_{n\geq 0} \mathrm{map}(\Delta^n, Z^n).$$

A *cosimplicial model* for a topological space W is a cosimplicial space $Z^\bullet$ such that the totalization $\mathrm{Tot}(Z^\bullet)$ is homeomorphic to W.

Example 3.1.1. Consider the cosimplicial space $\Omega^\bullet_{f,g}$ defined as $\Omega^n_{f,g} = X \times Y^{\times n}$. Coface operators $\delta^i : \Omega^{n-1}_{f,g} \to \Omega^n_{f,g}$ and Codegeneracy operators

$\sigma^i : \Omega_{f,g}^{n+1} \to \Omega_{f,g}^n$ are given by

$$\delta^i(x, y_1, \cdots, y_{n-1}) = \begin{cases} (x, f(x), y_1, \cdots, y_{n-1}) & (i = 0), \\ (x, y_1, \cdots, y_i, y_i, \cdots, y_{n-1}) & (1 \le i \le n-1), \\ (x, y_1, \cdots, y_{n-1}, g(x)) & (i = n), \end{cases}$$

and

$$\sigma^i(x, y_1, \cdots, y_{n+1}) = (x, y_1, \cdots, y_i, y_{i+2}, \cdots, y_{n+1}).$$

Then, we see that $\Omega_{f,g}^\bullet$ is a cosimplicial model for $P_{f,g}$.

Let $Z^\bullet$ be a cosimplicial space. Applying the singular cochain complex functor $S^*(-)$ over a field $\mathbb{K}$ to the cosimplicial space $Z^\bullet$, we get the simplicial cochain complex $S^*(Z^\bullet)$. This gives rise to the Hochschild complex $(C_*(S^*(Z^\bullet)), D)$, that is,

$$C_n(S^*(Z^\bullet)) = \bigoplus_{r-s=n} S^r(Z^s).$$

with the differential $D = D_1 + D_2$ defined by

$$D_1 : S^p(Z^q) \longrightarrow S^{p+1}(Z^q),\ D_1(x) = (-1)^q dx,$$

$$D_2 : S^p(Z^q) \longrightarrow S^p(Z^{q-1}),\ D_2(x) = (-1)^{q-1} \sum_{i=1}^{q} (-1)^i \delta^{i*}(x).$$

Let $\mathrm{ev}_n : \Delta^n \times \mathrm{Tot}(Z^\bullet) \to Z^n$ be the evaluation map and consider the composite map for any n:

$$\Phi : C_n(S^*(Z^\bullet)) \xrightarrow{\oplus \mathrm{ev}_s^*} \bigoplus_{r-s=n} S^r(\Delta^s \times \mathrm{Tot}(Z^\bullet)) \xrightarrow{\sum \int_{[\Delta^s]}} S^n(\mathrm{Tot}(Z^\bullet)).$$

Here $[\Delta^n] \in S_n(\Delta^n)$ denotes the identity map of Δ^n.

Lemma 3.1.2. *The map Φ is a chain map.*

Proof. The Equation (2.3.1) and commutativity of the diagram

$$\begin{tikzcd} \Delta^{s-1} \times \mathrm{Tot}(Z^\bullet) \arrow[r, "\mathrm{ev}_{s-1}"] \arrow[d, "\delta^i \times 1"'] & Z^{s-1} \arrow[d, "\delta^i"] \\ \Delta^s \times \mathrm{Tot}(Z^\bullet) \arrow[r, "\mathrm{ev}_s"'] & Z^s \end{tikzcd}$$

enables us to give

$$\begin{aligned} d\Phi &= \sum_{r-s=n} \Big((-1)^s \int_{[\Delta^s]} d\mathrm{ev}_s^* - (-1)^s \int_{\partial[\Delta^s]} \mathrm{ev}_s^* \Big) \\ &= \sum_{r-s=n} \Big(\int_{[\Delta^s]} \mathrm{ev}_s^* D_1 + \int_{[\Delta^{s-1}]} \mathrm{ev}_{s-1}^* D_2 \Big) = \Phi D. \end{aligned}$$

□

3.2 Proof of Theorem 1.2.1

Recall the chain map $\Phi_{f,g} = \Phi$ in the case that $Z^\bullet$ is the cosimplicial model $\Omega^\bullet_{f,g}$ for $P_{f,g}$. We show that the map $\Phi_{f,g}$ is a quasi-isomorphism for proving Theorem 1.2.1. Define a chain map $\Phi'_{f,g} : \mathrm{Tot}C^*(\Omega^\bullet_{f,g}) \to C^*(P_{f,g})$ as with the map $\Phi_{f,g}$, and consider the diagram:

$$\begin{tikzcd}
\mathrm{Tot}S^*(\Omega^\bullet_{f,g}) \arrow[r, "\Phi_{f,g}"] & S^*(P_{f,g}) \\
\mathrm{Tot}C^*(\Omega^\bullet_{f,g}) \arrow[r, "\Phi'_{f,g}"'] \arrow[u, "\mathrm{Tot}\kappa^*"] & C^*(P_{f,g}). \arrow[u, "\kappa^*"']
\end{tikzcd} \tag{3.2.1}$$

Then, the diagram is commutative up to chain homotopy. Indeed, given φ in $\mathrm{Tot}C^*(\Omega^\bullet_{f,g})^n$ ($\varphi \in C^r(\Omega^s_{f,g})$, $r - s = n$). Let $h : S_*(\Delta^s) \otimes S_*(P_{f,g}) \to C_*(\Delta^s \times P_{f,g})$ be a natural chain homotopy between $\kappa_*\mathrm{EZ}$ and $\mathrm{EZ}(\kappa_* \otimes \kappa_*)$ stated in Remark 2.3.1. Then, the map $\tilde{h} : \mathrm{Tot}C^*(\Omega^\bullet_{f,g}) \to S^*(P_{f,g})$ defined as $\tilde{h}(\varphi)(\sigma) = (-1)^n\varphi(\mathrm{ev}_{s*}h(1_{\Delta^s}\otimes\sigma))$ for $\sigma \in S_{n-1}(P_{f,g})$ is a chain homotopy from $\kappa^*\Phi'$ to $\Phi\mathrm{Tot}\kappa^*$.

We also see that the map $\mathrm{Tot}\kappa^*$ in (3.2.1) is a quasi-isomorphism by a spectral sequence argument. Define the decreasing filtration of $\mathrm{Tot}S^*(\Omega^\bullet_{f,g})$ by

$${}_1F^{-p} = \bigoplus_{\substack{r-s=n,\\ s\le p}} S^r(X \times Y^{\times s}) \subset \mathrm{Tot}S^*(\Omega^\bullet_{f,g})^n.$$

Let ${}_2F$ be the filtration of $\mathrm{Tot}C^*(\Omega^\bullet_{f,g})$ defined as with the filtration ${}_1F$. Is is easily seen that the map $\mathrm{Tot}\kappa^*$ preserves the filtration. Denote by $({}_iE_r, {}_id_r)$ for $i = 1, 2$ the second quadrant spectral sequences associated to the filtration ${}_iF$ and by $(\mathrm{Tot}\kappa_*)_r : {}_2E_r \to {}_1E_r$ the map induced by $\mathrm{Tot}\kappa_*$. Then, it follows that $(\mathrm{Tot}\kappa_*)_1$ is an isomorphism

$$\begin{tikzcd}
{}_2E_1^{-p,q} \arrow[r, "\cong"] \arrow[dd, "(\mathrm{Tot}\kappa_*)_1"'] & (H^*(X) \otimes H^*(Y)^{\otimes p})^q \arrow[d, "\cong"] \\
& (H^*(X) \otimes_{H^*(Y)^{\otimes 2}} \mathrm{B}^p(H^*(Y), H^*(Y), H^*(Y)))^{q-p} \\
{}_1E_1^{-p,q} \arrow[r, "\cong"] & (H^*(X) \otimes H^*(Y)^{\otimes p})^q \arrow[u, "\cong"']
\end{tikzcd}$$

and that the differential ${}_id_1$ on ${}_iE_1$ coincides with the differential $1 \otimes d_2$ on $H^*(X) \otimes_{H^*(Y)^{\otimes 2}} \mathrm{B}(H^*(Y), H^*(Y), H^*(Y))$, where d_2 is the horizontal differential on the bar construction $\mathrm{B}(H^*(Y), H^*(Y), H^*(Y))$. The bar construction $\mathrm{B}(H^*(Y), H^*(Y), H^*(Y))$ is quasi-isomorphic to the normalized bar construction $\overline{\mathrm{B}}(H^*(Y), H^*(Y), H^*(Y))$ and simply-connectivity of Y implies that $(\overline{H^*(Y)})^{\leq 1} = 0$. We also see that if $q < -2p$,

$$(H^*(X) \otimes_{H^*(Y)^{\otimes 2}} \overline{\mathrm{B}}(H^*(Y), H^*(Y), H^*(Y)))^{q-p} = 0$$

by degree reasons. This enables us to obtain that ${}_1E_2^{p,q} = 0$ and ${}_2E_2^{p,q} = 0$ if $q < -2p$ and using this fact implies that the spectral sequences converge strongly. Hence, by the comparison theorem [34, Theorem 3.9], the map $\mathrm{Tot}\kappa^*$ is a quasi-isomorphism.

By the commutative diagram (3.2.1), it is enough to prove that Φ' is a quasi-isomorphism for proving Theorem 1.2.1. We recall a Serre spectral sequence and construct a spectral sequence converging to $H^*(\mathrm{Tot}S^*(\Omega^\bullet_{f,g}))$ before proving Theorem 1.2.1.

We first introduce the Serre spectral sequence associated to the fibration $ev : P_{f,g} \to X$. For any non-degenerate p-cube $\sigma : I^p \to X$, a $(q+p)$-cube $\bar{\sigma} : I^q \times I^p \to P_{f,g}$ is *a fibered q-cube over σ* if the diagram

$$\begin{tikzcd}
I^q \times I^p \arrow[r, "\overline{\sigma}"] \arrow[d, "pr_2"'] & P_{f,g} \arrow[d, "ev"] \\
I^p \arrow[r, "\sigma"'] & X
\end{tikzcd}$$

is commutative. Denote by F_p the subcomplex of $C_*(P_{f,g})$ generated by non-degenerate cubes fibered by some $\sigma \in C_{\leq p}(X)$ and put

$$F^p = \{\varphi \in C^*(P_{f,g}) \mid \varphi|_{F_{p-1}} = 0\}. \tag{3.2.2}$$

Then, we get a spectral sequence, written by (E_r, d_r), associated to the filtration which is called the Serre spectral sequence.

Proposition 3.2.1 ([41, Chapter II 8 Proposition6]). *There is an isomorphism of $\mathbb{K}$-vector space*

$$E_2^{p,q} \cong H^p(X) \otimes H^q(\Omega Y).$$

Recall the correspondence of the proposition. Let $K_{p,q}$ be a $\mathbb{K}$-vector space generated by all pairs $(\bar{\sigma}, \sigma)$, where $\bar{\sigma}$ is $(p+q)$-cube on $P_{f,g}$ which is a fibered q-cube over $\sigma : I^p \to X$. Define two maps $\partial^h_{i,\varepsilon} : K_{p,q} \to K_{p-1.q}$ and $\partial^v_{j,\varepsilon} : K_{p,q} \to K_{p,q-1}$ as $\partial^h_{i,\varepsilon}(\bar{\sigma}, \sigma) = (\bar{\sigma}(1 \times \lambda^\varepsilon_i), \sigma\lambda^\varepsilon_i)$,

$$\begin{tikzcd}
I^q \times I^{p-1} \arrow[r, "1\times\lambda_i^\varepsilon"] \arrow[d, "pr_2"'] & I^q \times I^p \arrow[r, "\bar{\sigma}"] & P_{f,g} \arrow[d, "ev"] \\
I^{p-1} \arrow[r, "\lambda_i^\varepsilon"] & I^p \arrow[r, "\sigma"] & X
\end{tikzcd}$$

and $\partial^h_{j,\varepsilon}(\bar{\sigma},\sigma) = (\bar{\sigma}(\lambda^\varepsilon_j \times 1), \sigma)$,

$$\begin{CD}
I^{q-1}\times I^p @>{\lambda_j^\varepsilon\times 1}>> I^q\times I^p @>{\bar{\sigma}}>> P_{f,g}\\
@V{pr_2}VV @. @VV{ev}V\\
I^p @>{=}>> I^p @>{\sigma}>> X.
\end{CD}$$

for $\varepsilon = 0, 1$, $1 \le i \le p$ and $1 \le j \le q$. Put $d^h = \sum_{k=1}^{p}(-1)^k(\partial^h_{k,0} - \partial^h_{k,1}) : K_{p,q} \to K_{p-1,q}$ and $d^v = \sum_{k=1}^{q}(-1)^k(\partial^v_{k,0} - \partial^v_{k,1})$. It is readily seen the isomorphic

$$E_0^{p,q} = ((F^p/F^{p+1})^{p+q}, d_0) \cong (\mathrm{Hom}_{\mathbb{K}}(K_{p,q}, \mathbb{K}), d_v^*).$$

Let $0 = (0, 0, \cdots, 0) \in I^p$ be a base point. Then, we obtain the map

$$K_{p,q} \longrightarrow \bigoplus_{\sigma:I^p\to X:\text{nondegenerate}} C_q(F_{\sigma(0)}),\ (\bar{\sigma},\sigma) \longmapsto \bar{\sigma}|_{I^q\times\{0\}}$$

where $F_{\sigma(0)}$ is a fiber of $\sigma(0)$. By the definition of the differential d_v of $K_{p,q}$, the map induce a map

$$H_q(K_{p,q}, d_v) \longrightarrow \bigoplus_{\sigma:I^p\to X:\text{nondegenerate}} H_q(F_{\sigma(0)})$$

in homology. We see that the map is an isomorphism by the homotopy lifting property. For any σ, let γ_1, γ_2 be paths between $\sigma(0)$ and the base point $*$ of X, and $\bar{\gamma}_1$ and $\bar{\gamma}_2$ the induced map from $F_{\sigma(0)}$to $F_* \cong \Omega Y$. Simply connectivity of X shows that the equality $\bar{\gamma}_{1*} = \bar{\gamma}_{2*} : H_*(F_{\sigma(0)}) \to H_*(\Omega Y)$, and it implies that

$$H_q(K_{p,q}, d_v) \xrightarrow{\cong} \bigoplus_{\sigma:I^p\to X:\text{nondegenerate}} H_q(F_{\sigma(0)}) \xrightarrow{\cong} C_p(X)\otimes H_q(\Omega Y).$$

Apply the dual functor $\mathrm{Hom}_{\mathbb{K}}(-, \mathbb{K})$ to the isomorphism, we then obtain the isomorphism $C^p(X)\otimes H^q(\Omega Y) \xrightarrow{\cong} E_1^{p,q}$. The map compatible with the differentials d_1 on $E_1^{p,q}$ and $d\otimes 1$ on $C^p(X)\otimes H^q(\Omega Y)$, we hence see the assertion of Proposition 3.2.1.

In a similar fashion, we construct a spectral sequence converging to $HH_*(C^*(\Omega^\bullet_{f,g}))$. For each $s \ge 0$, the projection on the first factor $pr_1 : \Omega^s_{f,g} = X \times Y^{\times s} \to X$ is a fibration. Denote by $\tilde{F}_p$ the subcomplex of $C_*(\Omega^s_{f,g})$ generated by nondegenerate cubes fibered by some $C_{\le p}(\Omega^s_{f,g})$ and put

$$\tilde{F}^p = \{\varphi \in C_*(C^*(\Omega^\bullet_{f,g})) \mid \varphi|_{\tilde{F}_{p-1}} = 0\} \tag{3.2.3}$$

Proposition 3.2.2. *Let $(\tilde{E}_r, \tilde{d}_r)$ be the spectral sequence associated to the filtration $\tilde{F}^*$. Then,*

$$\tilde{E}_2^{p,q} \cong H^p(X) \otimes HH_q(C^*(\Omega^\bullet_{c_*,c_*})).$$

Proof. The assertion is shown by a similar argument of the proof Theorem 3.2.1. Let $\tilde{K}_{p,q}$ be a direct sum $\bigoplus_{r-s=p+q} \tilde{K}_{r,s}$, where each direct summand $\tilde{K}_{r,s}$ is a $\mathbb{K}$-vector space generated by pairs $(\bar{\sigma}, \sigma)$ for which the following diagram commutes:

$$\begin{tikzcd} I^r \arrow[r, "\bar{\sigma}"] \arrow[d, "pr_2"'] & X \times Y^{\times s} \arrow[d, "pr_1"] \\ I^p \arrow[r, "\sigma"'] & X. \end{tikzcd}$$

Differentials $\tilde{d}_h : \tilde{K}_{p,q} \to \tilde{K}_{p-1,q}$ and $\tilde{d}_v : \tilde{K}_{p,q} \to \tilde{K}_{p,q-1}$ are defined as with the definition of d_h and d_v. By the same argument above, we obtain a map

$$\tilde{K}_{p,q} \longrightarrow \bigoplus_{\sigma:I^p\to X:\text{nondegenerate}} \bigoplus_{r-s=q} C_r(\{\sigma(0)\} \times Y^{\times s}),\ (\bar{\sigma}, \sigma) \longmapsto \bar{\sigma}|_{I^{r-p}\times\{0\}}.$$

Moreover we see that the dual of the map induces the isomorphism of the assertion. $\square$

Proposition 3.2.3. *The map $\Phi'_{f,g}$ preserves the filtration defined in* (3.2.2) *and* (3.2.3). *Moreover, the morphism of spectral sequences induced by $\Phi'_{f,g}$ is of the form*

$$1 \otimes H^q(\Phi'_{c_*,c_*}) : \tilde{E}_2^{p,q} \longrightarrow E_2^{p,q}$$

at the E_2-term.

Proof. Given $\varphi \in C^s(X \times Y^{\times r})$ $(s - r = n)$. Assume that $\varphi \in \tilde{F}^p$. Then, for any $\bar{\sigma} : I^n \to P_{f,g}$ in F^{p-1}, there exists a nondegenerate m-cube σ $(m \le p-1 < n)$ such that the following square commutes:

$$\begin{tikzcd} I^{n-m} \times I^m \arrow[r, "\bar{\sigma}"] \arrow[d, "pr_2"'] & P_{f,g} \arrow[d, "ev"] \\ I^m \arrow[r, "\sigma"'] & X. \end{tikzcd}$$

Since $\mathrm{ev}_r(\kappa_r \times \bar{\sigma})$ is in $\tilde{F}_{p-1}$, $\Phi(\varphi)(\bar{\sigma}) = \varphi(\mathrm{ev}_r(\kappa_r \times \bar{\sigma})) = 0$. This finishes a proof of the first assertion. The second assertion is shown by the commutativity of the diagram:

$$\begin{tikzcd} K_{p,q} \arrow[r] \arrow[d] & \tilde{K}_{p,q} \arrow[d] \\ \displaystyle\bigoplus_{\sigma:I^p\to X:\text{nondegenerate}} C_q(F_{\sigma(0)}) \arrow[r] & \displaystyle\bigoplus_{\sigma:I^p\to X:\text{nondegenerate}} \bigoplus_{r-s=q} C_r(\{\sigma(0)\} \times Y^{\times s}). \end{tikzcd}$$

Here, the top horizontal map $K_{p,q} \to \tilde{K}_{p,q}$ sends $(\bar{\sigma}, \sigma)$ to $\sum_{s\geq 0}(\mathrm{ev}_s(\kappa_s \times \bar{\sigma}), \sigma)$. The bottom horizontal map is also defined as with the top map. $\square$

Before proving Theorem 1.2.1, we recall the following theorem.

Theorem 3.2.4 ([34, Theorem 3.26]). *Let E_r and $\tilde{E}_r$ be first quadrant spectral sequences of cohomological type over a field $\mathbb{K}$ and $\phi_r : E_r \to \tilde{E}_r$ a morphism of spectral sequences such that $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$, $\tilde{E}^{p,q} = \tilde{E}^{p,0} \otimes \tilde{E}^{0,q}$ and $\phi_2^{p,q} = \phi_2^{p,0} \otimes \phi_2^{0,q}$. Then any two of the following conditions imply the third:*

1. *$\phi_2^{p,0} : E_2^{p,0} \longrightarrow \tilde{E}_2^{p,0}$ is an isomorphism for all p.*
2. *$\phi_2^{0,q} : E_2^{0,q} \longrightarrow \tilde{E}_2^{0,q}$ is an isomorphism for all q.*
3. *$\phi_\infty^{p,q} : E_\infty^{p,q} \longrightarrow \tilde{E}_\infty^{p,q}$ is an isomorphism for all p, q.*

End of proof of Theorem 1.2.1. Since the both spectral sequences E_r and $\tilde{E}_r$ are strong convergent, it is only enough to show that $H(\Phi_{c_*,c_*})$ is an isomorphism to prove the proposition. We consider the following pull back diagram

$$\begin{array}{ccc} P_{1_Y,c_*} & \longrightarrow & Y^I \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle (ev_0,ev_1)} \\ Y & \xrightarrow[(1_Y,c_*)]{} & Y \times Y. \end{array}$$

The space P_{1_Y,c_*} is contractible, we see that $H^*(P_{1_Y}, c_*) \cong \mathbb{K}$. On the other hand, the homology $HH_*(C^*(\Omega^\bullet_{1_Y,c_*}))$ is isomorphic to $\mathbb{K}$. In effect, let $h_r : Y \times Y^{\times(r+1)} \to Y \times Y^{\times r}$ be the projection on the last $(r+1)$ factors, that is, $h_r(y, y_0, \cdots, y_r) = (y_0, \cdots, y_r)$. Then, for face operators $\delta^i : \Omega^r_{1_Y,c_*} \to \Omega^{r+1}_{1_Y,c_*}$, we see that

$$h_{r+1}\delta^0 = 1,\ h_{r+1}\delta^i = \delta^{i-1}h_r\ (1 \leq i \leq r+1). \tag{3.2.4}$$

The map h_q induces the map $\bar{h} : C_*(C^*(\Omega^\bullet_{1_Y,c_*})) \to C_*(C^*(\Omega^\bullet_{1_Y,c_*}))$ with degree -1 and the formula (3.2.4) enables us to give $D\bar{h} + \bar{h}D = 1$. It implies that $HH_*(C^*(\Omega^\bullet_{1_Y,c_*})) \cong \mathbb{K}$ and thus, by Theorem 3.2.4, the map $H(\Phi'_{c_*,c_*})$ is an isomorphism. $\square$

Chapter 4

Topological description of Hodge decomposition

4.1 A model for the space of sections

In this section, we introduce commutative models for the space of sections of a fibration and for its connected component given by Brown and Szczarba [4].

Let $p : E \to X$ be a nilpotent fibration with X a finite complex and $\mathrm{Sec}(p)$ the space of all sections of p. Let (B, d) denote a finite dimensional commutative model for X and $(B, d) \to (B \otimes \Lambda V, D)$ a relative Sullivan model for p. The dual space B^* of B is the complex $\mathrm{Hom}_{\mathbb{Q}}(B, \mathbb{Q})$ with the differential d^* which sends φ to $-(-1)^{|\varphi|}\varphi d$. We note that B^* is a $\mathbb{Z}$-grading vector space equipped with the coproduct

$$\delta : B^* \xrightarrow{\mu_B^*} (B \otimes B)^* \xleftarrow{\cong} B^* \otimes B^*,$$

where μ_B is the product of B. Now, consider the free commutative graded algebra $\Lambda(B \otimes \Lambda V \otimes B^*)$ with the differential $\bar{d}$ induced by D and d^*. Let I be the differential ideal of $\Lambda(B \otimes \Lambda V \otimes B^*)$ generated by $(1 \otimes 1) \otimes 1^* - 1$,

$$a_1 a_2 \otimes \beta - \sum (-1)^{|a_2||\beta'|}(a_1 \otimes \beta')(a_2 \otimes \beta'').$$

for a_1, $a_2 \in B \otimes \Lambda V$, $b \in B$, $\beta \in B^*$, where $\delta(\beta) = \sum \beta' \otimes \beta''$. We denote by J the differential ideal of $\Lambda(B \otimes \Lambda V \otimes B^*)$ generated by the ideal I and

$$b \otimes 1 \otimes \beta - (-1)^{|b|}\beta(b)$$

It follows from [4, Theorem 4.4] that the map

$$\rho : \Lambda(V \otimes B^*) \longrightarrow \Lambda(B \otimes \Lambda V \otimes B^*)/J$$

induced by the inclusion $V \otimes B^* \hookrightarrow B \otimes \Lambda V \otimes B^*$ is an isomorphism of graded algebra.

We now remark that the sign of the generator $b \otimes 1 \otimes \beta - (-1)^{|b|}\beta(b)$ in J is different from the generator introduced by [4, §4]. It is caused by the difference of the differential of the dual space B^*, that is, the differential of B^* in [4] sends φ to φd. We can show that the above map ρ is an isomorphism by the same argument as that in [4, §3] even if we change the differential of B^* in [4] into d^*. In consequence, we have the following theorem.

Theorem 4.1.1 ([4]). *The algebra* $(\Lambda(V \otimes B^*), \tilde{d} = \rho^{-1}\bar{d}\rho)$ *is a commutative model for* $\mathrm{Sec}(p)$.

Fix a section $\sigma : X \to E$ of p and denote by $\mathrm{Sec}_\sigma(p)$ the connected component of $\mathrm{Sec}(p)$ containing σ. Let $\phi : B \otimes \Lambda V \to B$ be a model for s and $\bar{\phi} : \Lambda(V \otimes B^*) \to \mathbb{Q}$ is the map defined by

$$\bar{\phi}(v \otimes \beta) = (-1)^{|v|}\beta(\phi(1 \otimes v)), \ \ v \otimes \beta \in V \otimes B^*.$$

Consider the differential ideal $K_{\bar{\phi}}$ of $\Lambda(V \otimes B^*)$ generated by

$$(V \otimes B^*)^{<0} \cup \tilde{d}(V \otimes B^*)^0 \cup \{w - \bar{\phi}(w) \mid w \in (V \otimes B^*)^0\}.$$

Theorem 4.1.2 ([4]). *The algebra* $(\Lambda(V \otimes B^*), \tilde{d})/K_{\bar{\phi}}$ *is a commutative model for* $\mathrm{Sec}_\sigma(p)$ *and the projection*

$$(\Lambda(V \otimes B^*), \tilde{d}) \longrightarrow (\Lambda(V \otimes B^*), \tilde{d})/K_{\bar{\phi}}$$

is a model for the inclusion $\mathrm{Sec}_\sigma(p) \hookrightarrow \mathrm{Sec}(p)$.

In the last of this section, we give a model for the evaluation map $\mathrm{Ev} : \mathrm{Sec}(p) \times X \to E$. Brown and Szczarba [4] have proved that the algebra $(\Lambda(B \otimes \Lambda V \otimes B^*), \bar{d})/I$ is the model for the mapping space $\mathrm{map}(X, E)$, where I is the ideal described above, and we see that the projection

$$\mathrm{proj} : \Lambda(B \otimes \Lambda V \otimes B^*), \bar{d})/I \longrightarrow \Lambda(B \otimes \Lambda V \otimes B^*), \bar{d})/J$$

is a commutative model for the inclusion $\mathrm{Sec}(p) \hookrightarrow \mathrm{map}(X, E)$ by the construction of commutative models. Buijs and Murillo [6] and Kuribayashi [28] have given a model for the evaluation map $\mathrm{map}(X, E) \times X \to E$. Let $\{b_i\}_{i=1}^N$ be a homogeneous basis of B and denote by $\{\beta_i\}_{i=1}^N$ a its dual basis of B^*, that is, $\{\beta_i\}_{i=1}^N$ is a basis of B^* satisfying $\beta_i(b_j) = \delta_{ij}$ where δ_{ij} is the Kronecker's delta.

Theorem 4.1.3 ([6], [28]). *The algebra map*

$$\omega : B \otimes \Lambda V \longrightarrow \Lambda(B \otimes \Lambda V \otimes B^*)/I \otimes B, \omega(a) = \sum_{i=1}^{N} (-1)^{|b_i|}(a \otimes \beta_i) \otimes b_i$$

for $a \in B \otimes \Lambda V$ *is a model for the evaluation map* $\mathrm{map}(X, E) \times X \to E$.

Theorem 4.1.3 yields that the composite

$$\begin{array}{ccc} B \otimes \Lambda V \xrightarrow{\omega} \Lambda(B \otimes \Lambda V \otimes B^*)/I \otimes B & & \\ \downarrow{\scriptstyle \mathrm{proj} \otimes 1} & & \\ \Lambda(B \otimes \Lambda V \otimes B^*)/J \otimes B & \xrightarrow[\cong]{\rho^{-1} \otimes 1} & \Lambda(V \otimes B^*) \otimes B \end{array} \tag{4.1.1}$$

is a model for the evaluation map $\mathrm{Sec}(p) \times X \to E$.

4.2 A Poincaré duality space and its commutative model

We recall the result [30] due to Lambrechts and Stanley in this section.

Definition 4.2.1. An oriented differential Poincaré duality algebra over $\mathbb{Q}$ of formal dimension d is a triple (A, d, ε_A) that satisfies the following properties

1. (A, d) is a connected commutative differential graded algebra,
2. (A, ε_A) is an oriented Poincaré duality algebra; that is, $\varepsilon_A : A^d \to \mathbb{Q}$ such that the induced bilinear forms $A^k \otimes A^{d-k} \to \mathbb{Q}$, $a \otimes b \mapsto \varepsilon_A(ab)$ are non-degenerate,
3. $\varepsilon_A(dA) = 0$.

The map ε_A is called an *orientation* of A. We see that the map $\theta_A : A \to A^*$ with degree $-d$ defined by $\theta_A(a)(b) = \varepsilon_A(ab)$ for $a, b \in A$ is a right A-linear isomorphism which commutes with differentials.

In [30], Lambrechts and Stanley have proved the following results.

Theorem 4.2.2 ([30]). *Let X be a simply-connected Poincaré duality space and ΛV a Sullivan model for X. Then, there are a simply-connected oriented differential Poincaré duality algebra A and a quasi-isomorphism*

$$\rho : \Lambda V \to A.$$

4.3 Proof of Theorem 1.2.2

Let $(\Lambda V, d)$ be a minimal Sullivan model for Y. Consider the commutative graded algebra $\Lambda V \otimes \Lambda V \otimes \Lambda(sV)$ with the differential D given by

$$D(v\otimes 1\otimes 1) = d(v)\otimes 1\otimes 1, D(1\otimes v\otimes 1) = 1\otimes d(v)\otimes 1,$$
$$D(1\otimes 1\otimes sv) = (-v\otimes 1 + 1\otimes v)\otimes 1 - \sum_{i=1}^{\infty}\frac{(sD)^i}{i!}(v\otimes 1\otimes 1),$$

where sV denotes the suspension of V and s is the unique derivation of the algebra $\Lambda V \otimes \Lambda V \otimes \Lambda(sV)$ defined by

$$s(v\otimes 1\otimes 1) = 1\otimes 1\otimes sv = s(1\otimes v\otimes 1),\ \ s(1\otimes 1\otimes sv) = 0.$$

Then, by [14, §15, Example1], $(\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D)$ is a Sullivan model for Y^I and

$$\bar{\varepsilon} = \mu\cdot\varepsilon : \Lambda V \otimes \Lambda V \otimes \Lambda(sV) \to \Lambda V$$

is a semifree resolution of ΛV as a $\Lambda V \otimes \Lambda V$-module. Here μ is the product of ΛV and $\varepsilon : \Lambda V \to \mathbb{Q}$ is the canonical augmentation. The map $\bar{\varepsilon}$ is a model for the map $c : Y \to Y^I$ which sends y in Y to the constant path c_y at y. Denote by $(\Lambda W, d)$ a minimal Sullivan model for X. Observe that ΛW is a ΛV-module via a map $\tilde{f} : (\Lambda V, d) \to (\Lambda W, d)$ which is a model for $f : X \to Y$. Then, the commutative differential graded algebra

$$(\Lambda W \otimes \Lambda(sV), \bar{d}) \cong (\Lambda W, d) \otimes_{\Lambda V\otimes\Lambda V} (\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D)$$

is a Sullivan model for the free loop space $L_f Y$. Here, the differential $\bar{d}$ is defined as

$$\bar{d}(w) = d(w),\ \ \bar{d}(sv) = -\sum\sum_{j=1}^{k} \pm\tilde{f}(v_1\cdots\hat{v_j}\cdots v_k)\otimes sv_j,$$

where $w \in V$, $v \in V$, $dv = \sum v_1 v_2 \cdots v_k$ and the sign $\pm$ denotes a Koszul sign convention.

Since X is a $\mathbb{Q}$-Poincaré duality space, by Theorem 4.2.2, it follows that there exists a quasi-isomorphism ρ from ΛW to a oriented Poincaré duality algebra (A, d). Since

$$\rho\otimes 1 : (\Lambda W \otimes \Lambda(sV), \bar{d}) \longrightarrow (A \otimes \Lambda(sV), \bar{d})$$

is a quasi-isomorphism by [14, Lemma 14.2], the inclusion $(A, d) \to (A \otimes \Lambda(sV), \bar{d})$ is a commutative model for $ev : L_f Y \to X$.

Let $\{e_i\}_{i=1}^N$ be a homogeneous basis of A with $e_N = \omega_A$ a basis of $A^d = \mathbb{Q}\omega_A$, and denote by $\{e_i^*\}_{i=1}^N$ a its dual basis of A^*. The composite in (4.1.1)

$$\overline{\mathrm{Ev}} : A \otimes \Lambda(sV) \to \Lambda(V \otimes A^*) \otimes A$$

is a model for the evaluation map $\mathrm{Ev} : \mathrm{Sec}(ev) \times X \to L_fY$. An explicit calculation shows that

$$\begin{aligned}
\overline{\mathrm{Ev}}(a \otimes sv) &= \sum_{i=1}^{N} (-1)^{|e_i|} \rho^{-1}(a \otimes sv \otimes e_i^*) \otimes e_i \\
&= \sum_{i=1}^{N} \sum_{j,k} (-1)^{|e_i|+|e_j|(|sv|+1)} \lambda_{j,k}^i e_j^*(a)(sv \otimes e_k^*) \otimes e_i,
\end{aligned}$$

where $\delta(e_i^*) = \sum_{j,k} \lambda_{j,k}^i e_j^* \otimes e_k^*$ for $\lambda_{j,k}^i \in \mathbb{Q}$. We now note that the differential $\bar{d}$ on $\Lambda(sV \otimes A^*)$ is defined by

$$\begin{aligned}
\bar{d}(sv \otimes e_k^*) = \sum \sum_{r=1}^{m} \sum_{i,j} (-1)^{\varepsilon_r + |e_i|(|sv_r|+1)} \lambda_{i,j}^k e_i^*(\rho \tilde{f}(v_1 \cdots \hat{v_r} \cdots v_m)) sv_r \otimes e_j^* \\
+ (-1)^{|sv|} sv \otimes d^* e_k \quad (4.3.1)
\end{aligned}$$

for $sv \otimes e_k^* \in sV \otimes A^*$, where $dv = \sum v_1 v_2 \cdots v_m$ and

$$\begin{aligned}
\varepsilon_r &= \sum_{l=1}^{r-1} |v_l| + |sv_r| \Big(\sum_{l=r+1}^{m} |v_l| \Big) + 1, \\
\delta(e_k^*) &= \sum_{i,j} \lambda_{i,j}^k e_i^* \otimes e_j^*.
\end{aligned}$$

Let $\pi : (A, d) \to (A/A^{<d}, 0) = (\mathbb{Q}\omega_A, 0)$ be the projection and $\varepsilon_A : A^d \to \mathbb{Q}$ the orientation of A. We then see that the dual of the composite

$$H^*(A \otimes \Lambda(sV)) \xrightarrow{H(\overline{\mathrm{Ev}})} H^*(\Lambda(sV \otimes A^*) \otimes A) \xrightarrow{H(1 \otimes \varepsilon_A \pi)} H^{*-d}(\Lambda(sV \otimes A^*))$$

coincides with the composite Γ stated in §1.2.

Let $s : X \to L_fY$ be a section defined by $s(x) = (x, c_{f(x)})$ for $x \in X$, where $c_{f(x)} : I \to Y$ is the constant path at $f(x)$. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "="] \arrow[dd, "s"'] & & Y \arrow[dr, "="] \arrow[dd, "c"] & \\
& X \arrow[rr, "f"] \arrow[dd, "="] & & Y \arrow[dd, "\Delta_Y"] \\
L_fY \arrow[rr] \arrow[dr, "ev"'] & & Y^I \arrow[dr, "(ev_0, ev_1)"] & \\
& X \arrow[rr, "(f,f)"'] & & Y \times Y,
\end{tikzcd}$$

where Δ_Y denotes the diagonal map. Since the top and bottom squares of the diagram are pullback diagram,

$$\bar{s} : (A \otimes \Lambda(sV), \bar{d}) \cong (A, d) \otimes_{(\Lambda V, d)^{\otimes 2}} (\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D) \xrightarrow{1 \otimes \mu \bar{\varepsilon}} (A, d)$$

is a model for s. It follows that the map $\bar{s}$ is the identity on A and send sV to 0. Hence, the differential ideal $K_{\bar{s}}$ of $\Lambda(sV \otimes A^*)$ described in §4.1 is generated by the set of the form

$$(sV \otimes A^*)^{<0} \cup \bar{d}(sV \otimes A^*)^0 \cup (sV \otimes A^*)^0.$$

The result [6, Proposition 4.2] implies that the ideal $K_{\bar{s}}$ coincides with the kernel of the surjective map

$$\varphi : (\Lambda(sV \otimes A^*), \bar{d}) \longrightarrow (\Lambda(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}), \bar{d})$$

given by

$$\varphi(x) = \begin{cases} 0 & (|x| \leq 0), \\ x & (|x| \geq 1) \end{cases}$$

for $x \in sV \otimes A^*$. Here, $\overline{(sV \otimes A^*)}^1$ is the quotient vector space $(sV \otimes A^*)^1/\bar{d}(sV \otimes A^*)^0$. Therefore, we see that $(\Lambda(sV \otimes A^*), \bar{d})/K_{\bar{s}}$ is isomorphic to $(\Lambda(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}), \bar{d})$.

Since a base point of $\mathrm{Sec}(ev)$ is the section s, it follows that the inclusion induces an isomorphism

$$\pi_n(\mathrm{Sec}_s(ev)) \otimes \mathbb{Q} \xrightarrow[\cong]{\mathrm{inc}_*} \pi_n(\mathrm{Sec}(ev)) \otimes \mathbb{Q}.$$

We now note that $\mathrm{Sec}_s(ev)$ is a homotopy associative and homotopy commutative H-space. Given s_1 and s_2 in $\mathrm{Sec}_s(ev)$ and, for any $x \in X$, we may write $s_i(x) = (x, \gamma_i)$ for some $\gamma_i \in Y^I$. Then, a multiplication μ of $\mathrm{Sec}_s(ev)$ is defined by $\mu(s_1, s_2)(x) = (x, \gamma_{1,2})$ and

$$\gamma_{1,2}(t) = \begin{cases} \gamma_1(2t) & (0 \leq t \leq \frac{1}{2}), \\ \gamma_2(2t-1) & (\frac{1}{2} \leq t \leq 1). \end{cases}$$

Hence the fundamental group $\pi_1(\mathrm{Sec}_s(ev))$ is abelian. Therefore the result [3, Lemma 11.8] deduces an isomorphism

$$\pi_n(\mathrm{Sec}_s(ev)) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(H^n(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}, \bar{d}_0), \mathbb{Q})$$

for $n \geq 1$, where $\bar{d}_0$ is the linear part of the differential $\bar{d}$ on $\Lambda(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2})$. Moreover, we see that the dual of the map $H^*(\rho_1)$ induced by the canonical projection

$$\rho_1 : (\Lambda(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}), \bar{d}) \to (\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}, \bar{d}_0)$$

on the indecomposable elements in homology coincides with the Hurewicz map $H : \pi_*(\mathrm{Sec}_s(ev)) \otimes \mathbb{Q} \to H_*(\mathrm{Sec}_s(ev); \mathbb{Q})$. It is readily seen that the homology $H^n(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}, \bar{d}_0)$ for $n \geq 1$ is isomorphic to the homology $H^n(sV \otimes A^*, \bar{d}_0)$ of the linear part of the commutative model $(\Lambda(sV \otimes A^*), \bar{d})$ for $\mathrm{Sec}(ev)$ via the map $H(\varphi)$. Observe that the diagram

$$\begin{tikzcd}
H^n(\Lambda(sV \otimes A^*), \bar{d}) \arrow[r, "H(\varphi)"] \arrow[d, "H(\rho_1)"'] & H^n(\Lambda(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}), \bar{d}) \arrow[d, "H(\rho_1)"] \\
H^n(sV \otimes A^*, \bar{d}_0) \arrow[r, "\cong", "H(\varphi)"'] & H^n(\overline{(sV \otimes A^*)}^1 \oplus (sV \otimes A^*)^{\geq 2}, \bar{d}_0)
\end{tikzcd}$$

is commutative. By combining the results described above, we have commutative diagrams for $n \geq 1$:

$$\begin{tikzcd}
H_{n+d}(L_fY; \mathbb{Q}) & \\
H_{n+d}(\mathrm{Sec}(ev); \mathbb{Q}) \arrow[u, "\Gamma"] & H_{n+d}(\mathrm{Sec}_s(ev); \mathbb{Q}) \arrow[l, "\mathrm{inc}_*"'] \arrow[ul, "\Gamma\mathrm{inc}_*"'] \\
\pi_n(\mathrm{Sec}(ev)) \otimes \mathbb{Q} \arrow[u, "H"] & \pi_n(\mathrm{Sec}_s(ev)) \otimes \mathbb{Q}, \arrow[l, "\mathrm{inc}_*", "\cong"'] \arrow[u, "H"']
\end{tikzcd} \tag{4.3.2}$$

$$\begin{tikzcd}
H_{n+d}(L_fY; \mathbb{Q}) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Q}}(H^{n+d}(A \otimes \Lambda(sV)), \mathbb{Q}) \\
H_n(\mathrm{Sec}_s(ev); \mathbb{Q}) \arrow[u, "\Gamma\mathrm{inc}_*"] & \mathrm{Hom}_{\mathbb{Q}}(H^n(\Lambda(sV \otimes A^*)), \mathbb{Q}) \arrow[u, "\mathrm{Hom}(H((1\otimes\varepsilon_A\pi)\overline{\mathrm{Ev}}),1)"'] \\
\pi_n(\mathrm{Sec}_s(ev)) \otimes \mathbb{Q} \arrow[u, "H"] \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Q}}(H^n(sV \otimes A^*), \mathbb{Q}). \arrow[u, "\mathrm{Hom}(H(\rho_1),1)"']
\end{tikzcd} \tag{4.3.3}$$

On the other hand, since $\bar{\varepsilon} : (\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D) \to \Lambda V$ is a semifree resolution of ΛV as a $\Lambda V^{\otimes 2}$-module, the Hochschild cohomology $HH^*(\Lambda V, A)$ is isomorphic to the homology of hom-complex $\mathrm{Hom}_{\Lambda V^{\otimes 2}}(\Lambda V \otimes \Lambda V \otimes \Lambda(sV), A)$.

Consider the canonical isomorphism

$$\zeta : \mathrm{Hom}_{\Lambda V^{\otimes 2}}(\Lambda V \otimes \Lambda V \otimes \Lambda(sV), A) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV), A)$$

and define $\bar{D} = \zeta D' \zeta^{-1}$, where D' is the differential of the hom-complex $\mathrm{Hom}_{\Lambda V^{\otimes 2}}(\Lambda V \otimes \Lambda V \otimes \Lambda(sV), A)$. Then, for $\psi \in \mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV), A)$ and

$sv_1 sv_2 \cdots sv_p \in \Lambda(sV)$, we have

$$
\begin{aligned}
&\bar{D}(\psi)(sv_1 sv_2 \cdots sv_p)\\
=&d\psi(sv_1 sv_2 \cdots sv_p)\\
+&(-1)^{|\psi|}\sum_{i=1}^{p}\sum_{v_i}\sum_{k=1}^{p}\\
&\qquad \Big(\pm\rho\tilde{f}(\omega_{i_1}\cdots\omega_{i_{k-1}}\omega_{i_{k+1}}\cdots\omega_{i_p})\psi(sv_1\cdots sv_{i-1}s\omega_{i_k}sv_{i+1}\cdots sv_p)\Big),
\end{aligned}
$$

where $dv_i = \sum_{v_i}\omega_{i_1}\omega_{i_2}\cdots\omega_{i_p}$ and the sign $\pm$ is the Koszul sign convention. In fact, for example $p=1$ and $v = v_1 \in V$ with $dv = \sum_v \omega_1 \cdots \omega_p$,

$$
\begin{aligned}
\bar{D}(\psi)(sv) &= d\zeta^{-1}(\psi)(1\otimes 1\otimes sv) - (-1)^{|\psi|}\zeta^{-1}(\psi)d(1\otimes 1\otimes sv)\\
&= d\psi(sv) + (-1)^{|\psi|}\zeta^{-1}(\psi)\Big(\sum_{i=1}^{\infty}\frac{(sd)^i}{i!}(v\otimes 1\otimes \bar{1})\Big)\\
&= d\psi(sv) + (-1)^{|\psi|}\sum_{v}\sum_{j=1}^{p}(-1)^{\varepsilon_j^{\psi}}\rho\tilde{f}(\omega_1\cdots\omega_{j-1}\omega_{j+1}\cdots\omega_p)\psi(s\omega_j)\\
&\qquad + \zeta^{-1}(\psi)\Big(\sum_{i=2}^{\infty}\frac{(sd)^i}{i!}(v\otimes 1\otimes \bar{1})\Big),
\end{aligned}
$$

where

$$\varepsilon_j^{\psi} = (|\psi|+1)(|\omega_1|+\cdots+|\omega_{j-1}|+|\omega_{j+1}|+\cdots+|\omega_p|)+|\omega_j|(|\omega_{j+1}|+\cdots+|\omega_p|).$$

An induction on the degree of v yields that $\zeta^{-1}(\psi)((sd)^2(v\otimes 1\otimes\bar{1})) = 0$. Therefore, we see that $\mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV), A)$ decomposes into a direct sum of complexes

$$(\mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV), A), \bar{D}) = \bigoplus_{p\geq 0}(\mathrm{Hom}_{\mathbb{Q}}(\Lambda^p(sV), A), \bar{D}). \tag{4.3.4}$$

This decomposition is the dual of the result [45] due to Vigué which asserts that the decomposition of the Hochschild complex coincides with the Hodge decomposition. Hence, the above decomposition of $\mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV), A)$ is precisely the Hodge decomposition of the Hochschild cochain complex and denote by $HH^*_{(1)}(\Lambda V, A)$ the homology of the direct summand $(\mathrm{Hom}_{\mathbb{Q}}(sV, A), \bar{D})$.

Consider the composite map

$$\Theta : \mathrm{Hom}_{\mathbb{Q}}(sV, A) \xrightarrow{\mathrm{Hom}(1,\nu_A)} \mathrm{Hom}_{\mathbb{Q}}(sV, (A^*)^*) \xrightarrow{\iota} \mathrm{Hom}_{\mathbb{Q}}(sV\otimes A^*, \mathbb{Q})$$

where $\nu_A : A \to (A^*)^*$ is the map given by $\nu_A(a)(\varphi) = (-1)^{|\varphi||a|}\varphi(a)$ for $a\in A$, $\varphi\in A^*$ and ι is the adjoint map, that is, $\iota(\psi)(sv\otimes\varphi') = \psi(sv)(\varphi')$. It is readily seen that ι is an isomorphism and that ε_A is also an isomorphism since A is finite dimensional.

Lemma 4.3.1. *The map Θ is compatible with the differentials $\bar{D}$ on $\mathrm{Hom}_{\mathbb{Q}}(sV, A)$ and $\mathrm{Hom}_1(\bar{d}_0, 1)$ on $\mathrm{Hom}_{\mathbb{Q}}(sV \otimes A^*, \mathbb{Q})$.*

Proof. The assertion is shown by a straightforward computation. We first note that if $e_ie_j = \sum_l \alpha^l_{i,j} e_l$ for some $\alpha^l_{i,j} \in \mathbb{Q}$, then the definition of δ shows that

$$\delta(e_k^*) = \sum_{i,j} (-1)^{|e_i||e_j|} \alpha^k_{i,j} e_i^* \otimes e_j^* \tag{4.3.5}$$

Given φ in $\mathrm{Hom}_{\mathbb{Q}}(sV, A)$ and $sv \otimes e_k^*$ in $sV \otimes A^*$. We have

$$\begin{aligned}
&\mathrm{Hom}_1(\bar{d}_0, 1)\Theta(\varphi)(sv \otimes e_k^*)\\
=&(-1)^{|\varphi|+1}\Theta(\varphi)\Big((-1)^{|sv|} sv \otimes d^*e_k^*\\
&+\sum_{r=1}^{m}\sum_{i,j=1}^{N}(-1)^{\varepsilon_r+|e_i^*|(|sv_r|+1)+|e_i||e_j|}\alpha^k_{i,j} e_i^*(\rho\tilde{f}(v_1\cdots\widehat{v_r}\cdots v_m))sv_r \otimes e_j^{\vee}\Big)\\
=&(-1)^{|\varphi|+1+|sv|+(|e_k|+1)(|\varphi|+|sv|)} d^*e_k^*(\varphi(sv))\\
&+\sum_{r=1}^{m}\sum_{i,j=1}^{N}(-1)^{|\varphi|+1+\varepsilon_r+|e_i|(|sv_r|+1)+|e_i||e_j|+|e_j|(|\varphi|+|sv_r|)}\\
&\qquad\times e_i^*(\rho\tilde{f}(v_1\cdots\widehat{v_r}\cdots v_m))e_j^*(\varphi(sv_r)).
\end{aligned}$$

We see that $e_k^* d\varphi(sv) = 0$ unless $|e_k| = |\varphi| + |sv| + 1$, $e_i^*(\rho\tilde{f}(v_1\cdots\widehat{v_r}\cdots v_m)) = 0$ unless $|e_i| = |v_1| + \cdots + \widehat{|v_r|} + \cdots + |v_m|$ and $e_j^*(\varphi(sv_r)) = 0$ unless $|e_j| = |\varphi| + |sv_r|$. Moreover, the equations $|e_i| = |v_1| + \cdots + \widehat{|v_r|} + \cdots + |v_m|$, $e_j^*(\varphi(sv_r)) = 0$ and $|e_k| = |e_i| + |e_j|$ imply that

$$\begin{aligned}
&|\varphi| + 1 + \varepsilon_r + |e_i|(|sv_r|+1) + |e_i||e_j| + |e_j|(|\varphi| + |sv_r|)\\
\equiv&|\varphi| + |e_k| + (|\varphi|+1)\Big(\sum_{\substack{l=1,\\ l\neq r}}^{r-1}|v_l|\Big) + |v_r|\Big(\sum_{l=r+1}^{m}|v_l|\Big) \mod 2\\
=&|\varphi| + |e_k| + \varepsilon_r^{\varphi}
\end{aligned}$$

and $|e_k| = |\varphi| + |v|$. Therefore, it turns out that

$$\begin{aligned}
&\mathrm{Hom}_1(\bar{d}_0, 1)\Theta(\varphi)(sv \otimes e_k^*)\\
=&-d^*e_k^*(\varphi(sv)) + \sum\sum_{r=1}^{m}(-1)^{|\varphi|+|e_k|+\varepsilon_r^{\varphi}} e_k^*(\rho\tilde{f}(v_1\cdots\widehat{v_r}\cdots v_m)\varphi(sv_r)).
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\Theta\bar{D}(\varphi)(sv\otimes e_k^*)\\
=&(-1)^{|e_k|(|\varphi|+1+|sv|)}e_k^*(\bar{D}(\varphi)(sv))\\
=&(-1)^{|e_k|}e_k^*\Big(d\varphi(sv)+(-1)^{|\varphi|}\sum\sum_{r=1}^{m}(-1)^{\varepsilon_r^\varphi}\rho\tilde{f}(v_1\cdots\widehat{v_r}\cdots v_m)\varphi(sv_r)\Big).
\end{aligned}
$$

This completes the proof. □

End of the proof of Theorem 1.2.2. By the commutative diagrams (4.3.2) and (4.3.3), the commutativity of the following diagram yields the assertion:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Q}}(A\otimes\Lambda(sV),\mathbb{Q}) \arrow[r, "\iota", "\cong"'] & \mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV),A^*) & \mathrm{Hom}_{\mathbb{Q}}(\Lambda(sV),A) \arrow[l, "\theta_A^*"', "\cong"] \\
\mathrm{Hom}_{\mathbb{Q}}(sV\otimes A^*,\mathbb{Q}) \arrow[u, "\mathrm{Hom}_1(\rho_1(1\otimes\varepsilon_A\pi)\overline{\mathrm{Ev}},1)"] & & \mathrm{Hom}_{\mathbb{Q}}(sV,A), \arrow[ll, "\cong", "\Theta"'] \arrow[u, "I_1"']
\end{tikzcd}
$$

where the map I_1 is the inclusion and ι is the adjoint map. It is readily seen that for φ in $\mathrm{Hom}_{\mathbb{Q}}(sV,A)$,

$$\theta_A^*I_1(\varphi)(sv_1\cdots sv_m)=0=\iota\mathrm{Hom}_1(\rho_1\varepsilon_A\pi\overline{\mathrm{Ev}},1)\Theta(\varphi)(sv_1\cdots sv_m),$$

which satisfies the condition $m=0$ or $m\geq 2$. We may write $e_ie_j=\sum_{k=1}^N\alpha_{i,j}^ke_k$ and $\varphi(sv)=\sum_{k=1}^N\beta_ke_k$ for any $sv\in sV$ and some $\alpha_{i,j}^k$, β_j in $\mathbb{Q}$. Then, for a generator e_k in A with $|e_k|=-|\varphi|-|sv|-d$, we see that

$$
\begin{aligned}
\theta_A^*I_1(\varphi)(sv)(e_k)&=\varepsilon_A(\varphi(sv)e_k)\\
&=(-1)^{(|\varphi|+|sv|)|e_k|}\varepsilon_A(e_k\varphi(sv))\\
&=(-1)^{(|\varphi|+|sv|)|e_k|}\sum_{|e_j|=d-|e_k|}\alpha_{k,j}^N\beta_j\\
&=(-1)^{d|e_k|+|e_k|}\sum_{|e_j|=d-|e_k|}\alpha_{k,j}^N\beta_j.
\end{aligned}
$$

On the other hand, the equation (4.3.5) shows that

$$
\begin{aligned}
&\iota \mathrm{Hom}_1(\rho_1(1 \otimes \varepsilon_A \pi)\overline{\mathrm{Ev}}, 1)\Theta(\varphi)(sv)(e_k) \\
=&(-1)^{|sv||e_k|+d|\varphi|}\Theta(\varphi)\Big(\rho_1(1 \otimes \varepsilon_A \pi)\overline{\mathrm{Ev}}(e_k \otimes sv)\Big) \\
=&(-1)^{|sv||e_k|+d|\varphi|} \\
&\times \Theta(\varphi)\Big(\sum_{j,l}(-1)^{|e_i|+|e_j|(|sv|+1)+|e_j||e_l|}\alpha^i_{j,l}\rho_1(1 \otimes \varepsilon_A \pi)e_j^*(e_k)(sv \otimes e_l^*) \otimes e_i\Big) \\
=&\Theta(\varphi)\Big(\sum_{|e_l|=d-|e_k|}(-1)^{|sv||e_k|+d|\varphi|+d+|e_k|(|sv|+1)+|e_k||e_l|+d(|sv|+|e_l|)}\alpha^N_{k,l}(sv \otimes e_l^*)\Big) \\
=&\sum_{|e_l|=d-|e_k|}(-1)^{|sv||e_k|+d|\varphi|+d+|e_k|(|sv|+1)+|e_k||e_l|+d(|sv|+|e_l|)+|e_l|(|\varphi|+|sv|)}\alpha^N_{k,l}\beta_l \\
=&\sum_{|e_l|=d-|e_k|}(-1)^{d|e_k|+|e_k|}\alpha^N_{k,l}\beta_l.
\end{aligned}
$$

The equations yield the assertion. □

4.4 Non-commutativity for $\mathbb{H}_*(L_fY;\mathbb{Q})$

We retain the notation described in the section above. Let X be a simply-connected d-dimensional closed oriented manifold, Y a simply-connected space of finite type and $f : X \to Y$ a based map. We see that the shifted homology $\mathbb{H}_*(L_fY)$ has a graded algebra structure by Gruher and Salvatore [20].

Proof of Proposition 1.2.3. We define a homeomorphism

$$\psi : \mathrm{Sec}(ev) \longrightarrow \Omega\mathrm{map}(X, Y; f),\ \psi(s)(t)(x) = s(x)(t)$$

by $\psi(s)(t)(x) = s(x)(t)$ for $s \in \mathrm{Sec}(ev)$, $t \in [0, 1]$ and $x \in X$. Note that the map ψ is a morphism of H-spaces. For $n \geq 2$, we have isomorphisms

$$\pi_n(\mathrm{map}(X, Y; f)) \cong \pi_{n-1}(\Omega\mathrm{map}(X, Y; f)) \cong \pi_{n-1}(\mathrm{Sec}(ev)).$$

By [44, Chapter X Theorem (7.10)], the rational homotopy group $\pi_{\geq 2}(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}$ has a non-trivial Whitehead product if and only if there is a non-trivial Samelson product on $\pi_{\geq 1}(\Omega\mathrm{Sec}(ev)) \otimes \mathbb{Q}$. We denote $\langle \beta_1, \beta_2 \rangle$ by the non-trivial Samelson product for some β_1 and β_2. Then, by [44, Chapter X Theorem (6.3)], we have the equality $H(\langle \beta_1, \beta_2 \rangle) = H(\beta_1)H(\beta_2) - (-1)^{|\beta_1||\beta_2|}H(\beta_2)H(\beta_1)$, where H denotes the Hurewicz map. We observe that a graded algebra structure on $H_*(\Omega\mathrm{map}(X, Y; f); \mathbb{Q})$ is determined by the H-space structure on $\Omega\mathrm{map}(X, Y; f)$. Since the map

$\mathrm{Ev} : \mathrm{Sec}(ev) \times X \to L_f Y$ is a morphism of fiberwise monoids from the projection $\mathrm{Sec}(ev) \times X \to X$ to the map $ev : L_f Y \to X$, it follows from [20, Theorem 4.1 (ii)] that the map $\Gamma : H_*(\mathrm{Sec}(ev); \mathbb{Q}) \to \mathbb{H}_*(L_f Y; \mathbb{Q})$ stated in Section 1 is an algebra map. Therefore, we see that

$$\Gamma_1(\langle \beta_1, \beta_2 \rangle) = \Gamma_1(\beta_1)\Gamma_1(\beta_2) - (-1)^{|\beta_1||\beta_2|}\Gamma_1(\beta_2)\Gamma_1(\beta_1)$$

and Theorem 1.2.2 shows that $\Gamma_1(\beta_1)\Gamma_1(\beta_2) \neq (-1)^{|\beta_1||\beta_2|}\Gamma_1(\beta_2)\Gamma_1(\beta_1)$. □

We give a example of a map $f : X \to Y$ for which the loop homology $\mathbb{H}_*(L_f Y; \mathbb{Q})$ is non-commutative.

Example 4.4.1. Let $\mathbb{C}P^n$ be the complex projective space and $i : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ the inclusion for $n \geq 2$. The existence of a non-zero Whitehead product in $\pi_*(\mathrm{map}(\mathbb{C}P^{n-1}, \mathbb{C}P^n; i)) \otimes \mathbb{Q}$ is showed by the results of Møller and Raussen [36, Example 3.4]. They proved that $\mathrm{map}(\mathbb{C}P^{n-1}, \mathbb{C}P^n; i)$ is of the rational homotopy type of $S^2 \times S^5 \times S^7 \times \cdots \times S^{2n+1}$ and the non-zero Whitehead product comes from the S^2 factor. Therefore Proposition 1.2.3 implies that $\mathbb{H}_*(L_i \mathbb{C}P^n; \mathbb{Q})$ is a non-commutative algebra.

Chapter 5

A model for the Whitehead product in rational mapping spaces

5.1 A model for the adjoint of the Whitehead product

We begin by recalling the definition of the Whitehead product. Let $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$ be elements represented by $a : S^n \to X$ and $b : S^m \to X$, respectively. Then the Whitehead product $[\alpha, \beta]_w$ is defined to be the homotopy class of composite

$$S^{n+m-1} \xrightarrow{\eta} S^n \vee S^m \xrightarrow{\nabla(a \vee b)} X$$

where η is the universal example and $\nabla : X \vee X \to X$ is the folding map. Recall that the differential d of ΛV can be written by $d = \sum_{i \geq 1} d_i$ with $d_i(V) \subset \Lambda^{i+1} V$. The map d_1 is called the *quadratic part* of d. We see that the quadratic part d_1 is related with the Whitehead products in $\pi_*(X)$. We denote by $Q(g)^n : V^n \to \mathbb{Q}e_n$ the linear part of a model $\overline{g}$ for g, where $\overline{g} : \Lambda V \to M(S^n)$. Define a paring and a trilinear map

$$\begin{aligned} \langle\ ;\ \rangle &: V \times \pi_*(X) \longrightarrow \mathbb{Q}, \\ \langle\ ;\ ,\ \rangle &: \Lambda^2 V \times \pi_*(X) \times \pi_*(X) \longrightarrow \mathbb{Q} \end{aligned}$$

by

$$\langle v; \alpha \rangle e_n = \begin{cases} Q(a)^n v & (|v| = n) \\ 0 & (|v| \neq n) \end{cases}$$

and

$$\langle vw; \alpha, \beta \rangle = \langle v; \alpha \rangle \langle w; \beta \rangle + (-1)^{|w||\alpha|} \langle w; \alpha \rangle \langle v; \beta \rangle,$$

respectively.

Proposition 5.1.1 ([14, Proposition 13.16]). *The following holds*

$$\langle d_1 v; \alpha, \beta\rangle = (-1)^{n+m-1}\langle v; [\alpha, \beta]_w\rangle,$$

where $v \in V$, $\alpha \in \pi_n(X)$, $\beta \in \pi_m(X)$.

We next recall the definition of the André-Quillen cohomology and the isomorphism Θ from $\pi_n(\mathrm{map}(X, Y_{\mathbb{Q}}; f))$ to $H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$ defined in [2] and [31]. We here recall the complex of $\bar{f}$-derivations from a Sullivan algebra $(\Lambda V, d)$ to a commutative differential graded algebra(B, d) which denoted by $\mathrm{Der}^*(\Lambda V, B; \bar{f})$. An element θ in $\mathrm{Der}^n(\Lambda V, B; \bar{f})$ is a $\mathbb{Q}$-linear map of degree n with $\theta(xy) = \theta(x)\bar{f}(y) + (1)^{n|x|}\bar{f}(x)\theta(y)$ for any $x, y \in \Lambda V$. The differential $\partial : \mathrm{Der}^n(\Lambda V, B; \overline{f}) \to \mathrm{Der}^{n+1}(\Lambda V, B; \bar{f})$ are defined by $\partial(\theta) = d\theta - (-1)^n\theta d$. The homology of $\mathrm{Der}^*(\Lambda V, B; \bar{f})$, $H^*_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$, is called the André-Quillen cohomology.

Let $\alpha \in \pi_n(\mathrm{map}(X, Y_{\mathbb{Q}}; f))$ and $g : S^n \times X \to Y_{\mathbb{Q}}$ the adjoint of α. Denote $\tilde{g} : \Lambda V_X \to M(S^n) \otimes \Lambda V_Y$ a Sullivan model for g. Since S^n is formal, there is a quasi-isomorphism $\phi : M(S^n) \to (H^*(S^n; \mathbb{Q}), 0)$ and, for any $v \in \Lambda V$, we may write

$$(\phi \otimes 1)\tilde{g}(v) = 1 \otimes \tilde{f}(v) + e_n \otimes v'.$$

Then we put $\Theta(\alpha)(v) = v'$.

Theorem 5.1.2 ([2] [31]). *The map*

$$\Theta : \pi_n(\mathrm{map}(X, Y_{\mathbb{Q}}; f)) \longrightarrow H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$$

is an isomorphism.

In order to consider the image of the Whitehead product in $\pi_*(\mathrm{map}(X, Y_{\mathbb{Q}}; f))$ by the isomorphism Θ, we construct an appropriate model for the adjoint of the Whitehead product. This is the key to proving Theorem 1.2.4. Let X be a simply-connected space, Y a $\mathbb{Q}$-local, simply-connected space of finite type and $f : X \to Y$ a based map. We denote by $(\Lambda V, d)$ and (B, d) a minimal Sullivan model for Y and a CDGA model for X, respectively. Let $\bar{f} : \Lambda V \to B$ be a model for f associated with such the models.

We prepare for proving Theorem 1.2.4. We see that a minimal Sullivan model for $S^n \vee S^m$ has the form

$$M(S^n \vee S^m) = (M(S^n) \otimes M(S^m) \otimes \Lambda(\iota_{n+m-1}, x_1, x_2, \cdots), d)$$

in which $d\iota_{n+m-1} = e_n e_m$ and $|\iota_{n+m-1}| = n + m - 1 < |x_i|$ for any $i \geq 1$; see [14, p177].

Lemma 5.1.3. *Let $g : S^n \times X \longrightarrow Y$ be a map with $g|_X = f$. Then there exists a model $\bar{g}$ for g such that the diagram is strictly commutative:*

$$\begin{tikzcd}
\Lambda V \arrow[rr, "\bar{g}"] \arrow[dr, "\bar{f}"'] & & M(S^n) \otimes B \arrow[dl, "\varepsilon\cdot 1"] \\
& B &
\end{tikzcd}$$

where $\varepsilon : M(S^n) \to \mathbb{Q}$ is the augmentation. Moreover, if g satisfy $g|_X = f$ and $g|_{S^n} = *$, where $* : S^n \to Y$ is the constant map to the base point, then there is a model $\bar{g}$ for g such that the following diagram commute strictly:

$$\begin{tikzcd}
& M(S^n) & \\
\Lambda V \arrow[ur, "u\varepsilon"] \arrow[rr, "\bar{g}"] \arrow[dr, "\bar{f}"'] & & M(S^n) \otimes B \arrow[ul, "1\cdot\varepsilon"'] \arrow[dl, "\varepsilon\cdot 1"] \\
& B, &
\end{tikzcd}$$

where $u : \mathbb{Q} \to M(S^n)$ is the unit map.

Proof. Let $\bar{g}'$ be a model for g. We define the map $\bar{g} : \Lambda V \to M(S_n) \otimes B$ by

$$\bar{g}(v) = 1 \otimes (\bar{f} - (\varepsilon \cdot 1)\bar{g}')(v) + \bar{g}'(v).$$

Then $\bar{g}$ and $\bar{g}'$ are homotopic. Indeed, $\bar{f}$ and $(\varepsilon \cdot 1) \circ \bar{g}'$ are homotopic and let $H : \Lambda V \longrightarrow B \otimes \Lambda(t, dt)$ be a its homotopy. Then, the map $\bar{H} : \Lambda V \longrightarrow M(S^n) \otimes B \otimes \Lambda(t, dt)$ defined by

$$\bar{H}(v) = 1 \otimes H(v) + \bar{g}'(v) \otimes 1 - 1 \otimes (\varepsilon \cdot 1)\bar{g}'(v) \otimes 1$$

is a homotopy from $\bar{g}'$ to $\bar{g}$. A similar argument shows the second assertion. $\square$

Given $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$ and $\beta \in \pi_m(\mathrm{map}(X, Y; f))$. Let $g : S^n \times X \to Y$ and $h : S^m \times X \to Y$ be the adjoint maps of α and β, respectively. In order to consider the image of $[\alpha, \beta]_w$ by Θ, we construct a model for the adjoint of $[\alpha, \beta]_w$

$$ad([\alpha, \beta]_w) : S^{n+m-1} \times X \xrightarrow{\eta\times 1} (S^n \vee S^m) \times X \xrightarrow{(g|h)} Y,$$

where $(g|h)$ is a map defined by $(g|h)(u_n, x) = g(u_n, x)$ and $(g|h)(u_m, x) = h(u_m, x)$ for any $u_n \in S^n$, $u_m \in S^m$ and $x \in X$. It is readily seen that the canonical map

$$\pi : M(S^n \vee S^m) \longrightarrow M(S^n) \times_{\mathbb{Q}} M(S^m)$$

is a surjective quasi-isomorphism, where $M(S^n) \times_{\mathbb{Q}} M(S^m)$ is the pull-back of the augmentations $M(S^n) \to \mathbb{Q}$ and $M(S^m) \to \mathbb{Q}$. By Proposition 2.1.2, we have the following homotopy commutative square

$$\begin{tikzcd} A_{\mathrm{PL}}(S^n \vee S^m) \arrow[r, "(A_{\mathrm{PL}}(i_1), A_{\mathrm{PL}}(i_2))"] & A_{\mathrm{PL}}(S^n) \times_{\mathbb{Q}} A_{\mathrm{PL}}(S^m) \\ M(S^n \vee S^m) \arrow[u, "\simeq"] \arrow[r, "\pi"'] & M(S^n) \times_{\mathbb{Q}} M(S^m), \arrow[u, "\simeq"'] \end{tikzcd}$$

where $i_1 : S^n \to S^n \vee S^m$ and $i_2 : S^m \to S^n \vee S^m$ are the inclusions. The commutative diagram

$$\begin{tikzcd} S^n \times X \arrow[r, "i_1 \times 1"] \arrow[dr, "g"'] & (S^n \vee S^m) \times X \arrow[d, "(g|h)"] & S^m \times X \arrow[l, "i_2 \times 1"'] \arrow[dl, "h"] \\ & Y & \end{tikzcd} \tag{5.1.1}$$

enables us to give the following homotopy commutative diagram:

$$\begin{tikzcd} & M(S^n \vee S^m) \otimes B \arrow[d, "\pi \otimes 1"] \\ \Lambda V \arrow[ur, "\overline{(g|h)}"] \arrow[r, "(\bar{g}, \bar{h})"'] & (M(S^n) \times_{\mathbb{Q}} M(S^m)) \otimes B, \end{tikzcd} \tag{5.1.2}$$

where $(\bar{g}, \bar{h})$ is the map defined by $(\bar{g}, \bar{h})(v) = -1 \otimes \bar{f}(v) + (j_1 \otimes 1)\bar{g}(v) + (j_2 \otimes 1)\bar{h}(v)$ for any $v \in V$ and $j_1 : M(S^n) \to M(S^n) \times_{\mathbb{Q}} M(S^m)$ and $j_2 : M(S^m) \to M(S^n) \times_{\mathbb{Q}} M(S^m)$ are the inclusions. Indeed, by the diagram (5.1.1), we see that the diagram

$$\begin{tikzcd} M(S^n) \otimes B & (M(S^n) \times_{\mathbb{Q}} M(S^m)) \otimes B \arrow[l, "p_1 \otimes 1"'] \arrow[r, "p_2 \otimes 1"] & M(S^m) \otimes B \\ & \Lambda V \arrow[ul, "\bar{g}"] \arrow[u, "(\pi \otimes 1)\overline{(g|h)}"] \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

is homotopy commutative, where p_1 and p_2 are the projection. Let H_1 and H_2 be homotopies from $(p_1\pi \otimes 1)\overline{(g|h)}$ to $\bar{g}$ and from $(p_2\pi \otimes 1)\overline{(g|h)}$ to $\bar{h}$, respectively. Then, a CDGA map $H : \Lambda V \to (M(S^n) \times_{\mathbb{Q}} M(S^m)) \otimes B \otimes \Lambda(t, dt)$ defined by

$$H(v) = -1 \otimes \bar{f}(v) \otimes 1 + (j_1 \otimes 1 \otimes 1)H_1(v) + (j_2 \otimes 1 \otimes 1)H_2(v)$$

for any $v \in V$ is a homotopy from $(\pi \otimes 1)\overline{(g|h)}$ to $(\bar{g}, \bar{h})$. If there is a map $\phi : \Lambda V \to M(S^n \vee S^m) \otimes B$ such that $(\pi \otimes 1)\phi = (\bar{g}, \bar{h})$, ϕ and $\overline{(g|h)}$ is homotopic by Proposition 2.1.2. Therefore, it is only necessary to construct of a lift ϕ of the diagram (5.1.2) for getting a model for $(g|h)$.

Lemma 5.1.4. *There is a model ϕ for $(g|h)$ such that for any $v \in V$, $\phi(v)$ has no term of the form $e_n e_m \otimes u$ for some $u \in B$ and the following diagram commutes strictly:*

$$\begin{tikzcd} \Lambda V \arrow[rr, "\phi"] \arrow[dr, "\bar{f}"'] & & M(S^n \vee S^m) \otimes B \arrow[dl, "\varepsilon\cdot 1"] \\ & B. & \end{tikzcd}$$

Proof. First, we recall the construction of a lift ϕ' in Remark 2.1.3. For any basis v of V, we can find $a \in M(S^n \vee S^m) \otimes B$ so that $da = \phi' dv$ and $(\pi \otimes 1)a = (\bar{g}, \bar{h})v$. We may write

$$a = 1 \otimes \bar{f}(a) + e_n \otimes a_2 + e_m \otimes a_3 + \iota_{n+m-1} \otimes a_4 + e_n e_m \otimes a_5 + \mathcal{O}_a,$$

where $a_i \in B$ and $\mathcal{O}_a$ denote other terms. We put

$$a' = 1 \otimes \bar{f}(a) + e_n \otimes a_2 + e_m \otimes a_3 + \iota_{n+m-1} \otimes (a_4 + da_5) + \mathcal{O}_a. \quad (5.1.3)$$

Then it follows that $d(a) = d(a')$ and $(\pi \otimes 1)(a) = (\pi \otimes 1)(a')$. Hence, the map ϕ defined by

$$\phi(v) = a'$$

satisfies the condition that $(\pi \otimes 1)\phi = (\bar{g}, \bar{h})$. Thus we see that ϕ is a model for $(g|h)$. The second assertion is shown using the equation (5.1.3). □

Combining these results we prove our main result.

Proof of Theorem 1.2.4. Given two elements $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$ and $\beta \in \pi_m(\mathrm{map}(X, Y; f))$. Let $g : S^n \times X \to Y$ and $h : S^m \times X \to Y$ be the adjoint maps of α and β, respectively. First, by the proof of Proposition 5.1.1, we see that a model $\overline{\eta}$ for the universal example η sends $\iota_{n+m-1} \in M(S^n \vee S^m)$ to $(-1)^{n+m-1} e_{n+m-1} \in M(S^{n+m-1})$. We choose a model ϕ for the map $(g|h)$ as in Lemma 5.1.4. We may write, modulo the ideal generated by elements of $M(S^n \vee S^m)$ of degree greater than $n + m - 1$ and generators e_{2n-1} and e_{2m-1} if there exists,

$$\begin{aligned} \phi(v) &\equiv 1 \otimes \bar{f}(v) + e_n \otimes u_2 + e_m \otimes u_3 + \iota_{n+m-1} \otimes u_4, \\ \phi(v_i) &\equiv 1 \otimes \bar{f}(v_i) + e_n \otimes u_{i2} + e_m \otimes u_{i3} + \iota_{n+m-1} \otimes u_{i4} \end{aligned}$$

for any $v \in V$ and $dv = \sum v_1 v_2 \cdots v_s$. Since, $(\overline{\eta} \otimes 1)\phi(v) = 1 \otimes \bar{f}(v) + (-1)^{n+m-1} e_{n+m-1} \otimes u_4$, it follows that $\Theta([\alpha, \beta]_w)(v) = (-1)^{n+m-1} u_4$. On the other hand, ϕ is a CDGA map and satisfies the condition of Lemma 5.1.4. We then have

$$e_n e_m \otimes u_4 =$$
$$e_n e_m \otimes \sum \Big(\sum_{i \neq j} (-1)^{\varepsilon_{ij}} \bar{f}(v_1 \cdots v_{i-1}) u_{i2} \bar{f}(v_{i+1} \cdots v_{j-1}) u_{j3} \bar{f}(v_{j+1} \cdots v_s) \Big).$$

By commutativity of the diagram (5.1.2) and the definition of Θ, we see that $u_{i2} = \Theta(\alpha)(v_i)$ and $u_{j3} = \Theta(\beta)(v_j)$. Therefore,

$$\Theta([\alpha, \beta]_w)(v) = (-1)^{n+m-1} u_4 = [\Theta(\alpha), \Theta(\beta)](v).$$

This completes the proof. □

In the rest of this section, we also consider the Whitehead product in a based mapping space $\mathrm{map}_*(X, Y; f)$. Given $\alpha \in \pi_n(\mathrm{map}_*(X, Y; f))$ and let $g : S^n \times X \to Y$ be the adjoint map of α. Since g satisfy $g|_X = f$ and $g|_{S^n} = *$, by Lemma 5.1.3, there exists a model for g, $\bar{g}$, such that $(\varepsilon \cdot 1)\bar{g} = \bar{f}$ and $(1 \cdot \varepsilon)\bar{g} = u\varepsilon$. The second equation shows that $\Theta(\alpha)$ is a $\bar{f}$-derivation of degree $-n$ from ΛV to the augmentation ideal B^+ of B. We then get the map of abelian groups

$$\Theta' : \pi_n(\mathrm{map}_*(X, Y; f)) \longrightarrow H^{-n}_{\mathrm{AQ}}(\Lambda V, B^+; \bar{f});\ \Theta'(\alpha) = \Theta(\alpha)$$

for $n \geq 2$ and a straight-forward modification of Theorem 5.1.2 deduces the following proposition:

Proposition 5.1.5. *The map $\Theta' : \pi_n(\mathrm{map}_*(X, Y; f)) \to H^{-n}_{\mathrm{AQ}}(\Lambda V, B^+; \bar{f})$ is an isomorphism for $n \geq 2$.*

This proposition also enables us to get the following corollary.

Corollary 5.1.6. *The restriction of the bracket defined by the formula (1.2.1) in $H^*_{\mathrm{AQ}}(\Lambda V, B; \bar{f})$ to $H^*_{\mathrm{AQ}}(\Lambda V, B^+; \bar{f})$ corresponds the Whitehead product in $\pi_*(\mathrm{map}_*(X, Y; f))$ via the isomorphism Θ' from $\pi_n(\mathrm{map}_*(X, Y; f))$ to $H^{-n}_{\mathrm{AQ}}(\Lambda V, B^+; \bar{f})$.*

Proof. Given $\alpha \in \pi_n(\mathrm{map}_*(X, Y; f))$ and $\beta \in \pi_m(\mathrm{map}_*(X, Y; f))$. Since $\varepsilon\Theta'(\alpha) = 0$ and $\varepsilon\Theta'(\beta) = 0$, it follows that $\varepsilon\Theta'([\alpha, \beta]_w) = 0$ by the formula (1.2.1). □

5.2 The Whitehead length of mapping spaces

In this section, we consider the Whitehead length of mapping spaces. We recall the definition of the Whitehead length; see Section 1. Now we consider a upper bound of $\mathrm{WL}(\mathrm{map}(X,Y;f))$. The following result is proved by Lupton and Smith.

Proposition 5.2.1 ([32, Theorem 6.4]). *Let X and Y be $\mathbb{Q}$-local, simply-connected spaces with finite type. If Y is coformal, that is, a minimal Sullivan model for Y of the form $(\Lambda V, d_1)$, then*

$$\mathrm{WL}(\mathrm{map}(X,Y;f)) \leq \mathrm{WL}(Y).$$

We give another proof of Proposition 5.2.1 using the bracket defined by Theorem 1.2.4. Before proving the proposition, we introduce a numerical invariant which is called the d_1-depth for a simply-connected space Z and recall the relationship between the Whitehead length and the d_1-depth.

Definition 5.2.2. Let $(\Lambda V, d)$ be a minimal Sullivan model for a simply-connected space Z and d_1 the quadratic part of d. The *d_1-depth* of Z, denoted by $d_1\text{-depth}(Z)$, is the greatest integer n such that V_{n-1} is a proper subspace of V_n with

$$V_{-1}=0,\ V_0=\{v\in V \mid d_1 v=0\} \text{ and } V_i=\{v\in V \mid d_1 v\in \Lambda V_{i-1}\}\ (i\geq 1).$$

Theorem 5.2.3 ([27, Theorem 4.15][29, Theorem 2.5]). *Let Y be a $\mathbb{Q}$-local, simply-connected space. Then $d_1\text{-depth}(Y)+1=\mathrm{WL}(Y)$.*

Proof of Proposition 5.2.1. Let ΛV be a minimal Sullivan model for Y and $m=d_1\text{-depth}(Y)$. For any $v\in V$, we may write $d_1(v)=\sum_{j=1}^{n} u_{j1}u_{j2}\cdots u_{jk_j}$ where u_{ji} are basis of V. Then, put

$$T'_{d_1}(v)=\{u_{j1}u_{j2}\cdots u_{jk_j} \mid j=1\ldots n\}$$

and

$$T_{d_1}(u_1u_2\cdots u_s)=\bigcup_{i=1\ldots s}\{u_1\cdots u_{i-1}u'u_{i+1}\cdots u_s \mid u'\in T'_{d_1}(u_i)\}.$$

We also set

$$T_{d_1}(U)=\bigcup_{u\in U} T_{d_1}(u)$$

where U is a set of terms of ΛV. By the definition of d_1-depth, $T_{d_1}^{(m+1)}(v)=\{0\}$ and it follows that

$$[\varphi_1,[\varphi_2,\cdots[\varphi_{m+1},\varphi_{m+2}]\cdots]](v)=0$$

] for any $\varphi_1, \varphi_2, \ldots, \varphi_{m+2} \in H_{\mathrm{AQ}}^{\leq -2}(\Lambda V, B; \bar{f})$. Hence, by Theorem 1.2.4 and Theorem 5.2.3, we have $\mathrm{WL}(\mathrm{map}(X, Y; f)) \leq m+1 = \mathrm{WL}(Y)$. $\square$

We next prove Proposition 1.2.7.

Proof of Proposition 1.2.7. Let $m = \mathrm{WL}(\mathrm{map}_*(X, Y; f))$. If $m = 1$, then the assertion is trivial and so we may assume that $m \geq 2$. By Corollary 5.1.6, there are elements $\varphi_1, \varphi_2, \cdots, \varphi_m$ in $H_{\mathrm{AQ}}^{\leq -2}(\Lambda V, B^+; \bar{f})$ such that

$$[\varphi_1, [\varphi_2, \cdots, [\varphi_{m-1}, \varphi_m] \cdots]](v) \neq 0 \tag{5.2.1}$$

for some $v \in V$. For any element $u_1 u_2 \cdots u_s \in T_{d_1}^m(v)$, the length s of $u_1 u_2 \cdots u_s$ is greater than or equal to $(m-2)(\omega - 1) + \omega$ by the definition of ω. Therefore, the equation (5.2.1) implies that

$$\mathrm{nil} B \geq (m-2)(\omega - 1) + \omega$$

and hence we have

$$m \leq \frac{1}{\omega - 1}(\mathrm{nil} B - 1) + 1.$$

$\square$

Remark 5.2.4. Suppose that $\mathrm{WL}(Y) = 1$ and $\mathrm{WL}(\mathrm{map}_*(X, Y; f)) > 1$. The proof of Proposition 1.2.7 enables us to conclude that $\mathrm{nil} B \geq \omega$ and that $\omega \geq 3$ since $V = \mathrm{Ker} d_1$. Moreover we have

$$\mathrm{WL}(\mathrm{map}_*(X, Y; f)) \leq \frac{1}{\omega - 1}(\mathrm{nil} B - 1) + 1 \leq \mathrm{nil} B - 1.$$

Thus our upper bound of the Whitehead length of the mapping space may be less than that described in Theorem 1.2.6.

5.3 Computational examples

We shall determine the Whitehead length of the mapping space from $\mathbb{C}P^\infty \times \mathbb{C}P^n$ to $\mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}$. For this, we first compute the rational homotopy group of the mapping space. Let $f_1 : \mathbb{C}P^\infty \times \mathbb{C}P^n \to \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}$ be the map stated in §1.2. Since $\mathbb{C}P^n$ is formal, that is the CDGA map ρ

$$(\Lambda(x_2, x'_{2n+1}),\ dx'_{2n+1} = x_2^{n+1}) \longrightarrow (\mathbb{Q}[x_2]/(x_2^{n+1}), 0) = H^*(\mathbb{C}P^n; \mathbb{Q})$$

defined by $\rho(x_2) = x_2$, $\rho(x'_{2n+1}) = 0$ is a quasi-isomorphism, the CDGA $(\mathbb{Q}[z_2] \otimes \mathbb{Q}[x_2]/(x_2^{n+1}), 0)$ is a CDGA model for $\mathbb{C}P^\infty \times \mathbb{C}P^n$.

Proposition 5.3.1. *Let $k \geq 2$ and $m < n$. Then*

$$\pi_k(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1))$$
$$= \begin{cases} \mathbb{Q} & (k=2 \text{ and } q_2 \neq 0) \\ \mathbb{Q} \oplus \mathbb{Q} & (k=2 \text{ and } q_2 = 0) \\ \displaystyle\bigoplus_{0 \leq i = n-m-l+1}^{n-l+1} \mathbb{Q} & (k = 2l-1,\ 2 \leq l \leq n+1) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. We put $\mathrm{Der}^n = \mathrm{Der}^n(\mathbb{Q}[z_2] \otimes \Lambda(x_2, x'_{2n+1}),\ \mathbb{Q}[w_2] \otimes \mathbb{Q}[y_2]/(y_2^{m+1}); \rho \bar{f}_1)$ for convenience. For any elements $\theta_{r,s} \in \mathrm{Der}^{-2}$, we may write

$$\theta_{r,s}(z_2) = r,\ \ \theta_{r,s}(x_2) = s \text{ and } \theta_{r,s}(x'_{2n+1}) = 0$$

for some $r, s \in \mathbb{Q}$. Then,

$$\partial\theta_{r,s}(z_2) = \partial\theta_{r,s}(x_2) = 0,\ \ \partial\theta_{r,s}(x'_{2n+1}) = -ns\Big(\sum_{i+j=n} q_2^i q_3^j w_2^i \otimes y_2^j\Big).$$

When $q_2 \neq 0$, we see that $\theta_{r,s}$ is a cycle if and only if $s = 0$, that is all cycles of Der^{-2} generated by $\theta_{1,0}$. When $q_2 = 0$, $\theta_{r,s}(x'_{2n+1}) = 0$ since $y_2^n = 0$. Hence, $\theta_{1,0}$ and $\theta_{0,1}$ are generators of all cycles of Der^{-2}. In general, $\mathrm{Der}^{-2l} = 0$ for $l \geq 2$ by degree reasons. It follows that

$$\pi_{2l}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1)) \cong H^{-2l}(\mathrm{Der}^*) = 0\ (l \geq 2).$$

For any $\theta \in \mathrm{Der}^{-2l+1}$, we may write

$$\theta(z_2) = 0,\ \ \theta(x_2) = 0 \text{ and } \theta(x'_{2n+1}) = \sum_{i=0}^{n-l+1} r_i w_2^i \otimes y_2^{n-l+1-i}.$$

Note that if $l > n+1$, $\mathrm{Der}^{-2l+1} = 0$ by degree reasons. It is easily seen that all elements of Der^{-2l+1} are cycles. Moreover, we see that $y_2^{n-l+1-i} = 0$ if and only if $0 \leq i \leq n-m-l$. Therefore, we have

$$\pi_2(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1))$$
$$\cong H^{-2}(\mathrm{Der}^*) \cong \begin{cases} \mathbb{Q} & (k=2 \text{ and } q_2 \neq 0) \\ \mathbb{Q} \oplus \mathbb{Q} & (k=2 \text{ and } q_2 = 0) \end{cases}$$

and

$$\pi_{2l-1}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1))$$
$$\cong H^{-2l+1}(\mathrm{Der}^*) \cong \bigoplus_{0 \leq i = n-m-l+1}^{n-l+1} \mathbb{Q} \quad (2 \leq l \leq n+1),$$

$$\pi_{2l-1}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1)) \cong H^{-2l+1}(\mathrm{Der}^*) = 0\ (l > n+1).$$

□

Proof of Proposition 1.2.8. By the definition of the bracket in $H^*(\mathrm{Der}^*)$, we see that if $\varphi, \psi \in H^{\leq -3}(\mathrm{Der}^*)$, then $[\varphi, \psi] = 0$ since $\varphi(x_2) = 0$ and $\psi(x_2) = 0$. That is $[\varphi', \psi'] \neq 0$ means $|\varphi'| = |\psi'| = -2$. It shows that

$$\mathrm{WL}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1)) \leq 2.$$

If $q_2 \neq 0$, by Proposition 5.3.1, $H^{-2}(\mathrm{Der}^*)$ is generated by $\theta_{1,0}$. The equality $[\theta_{1,0}, \theta_{1,0}] = 0$ shows that $\mathrm{WL}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f_1)) = 1$. On the other hand, if $q_2 = 0$, $\theta_{0,1}$ is a generator of $H^{-2}(\mathrm{Der}^*)$ and

$$[\theta_{0,1}, \theta_{0,1}](x'_{2n+1}) = q_3^{n-1} y_2^{n-1}.$$

This completes the proof. □

Proof of Proposition 1.2.9. Let θ_1 and θ_2 be $\bar{f}_2$-derivations defined by

$$\begin{aligned} &\theta_1(x_1) = 1,\ \theta_1(x_2) = \theta_1(x_3) = 0,\ \theta_1(y) = 0, \\ &\theta_2(x_1) = 0,\ \theta_1(x_2) = 1,\ \theta_1(x_3) = 0,\ \theta_1(y) = 0. \end{aligned}$$

It is readily seen that θ_1 and θ_2 are non-trivial homology class of $H^{-2}_{\mathrm{AQ}}(\Lambda V, \Lambda(e_3); \bar{f}_2)$ and $H^{-3}_{\mathrm{AQ}}(\Lambda V, \Lambda(e_3); \bar{f}_2)$, respectively. Then, we have

$$[\theta_1, \theta_2](x_i) = 0,\ [\theta_1, \theta_2](y) = e_3$$

for any i, and $[\theta_1, \theta_2]$ is a non-trivial homology class. Therefore, the Whitehead length of $\mathrm{map}(S^3, Y; f_2)$ is greater than 2 and also, by the definition of the differential d on ΛV and the bracket (1.2.1), we see that

$$\mathrm{WL}(\mathrm{map}(S^3, Y; f_2)) = 2.$$

This implies that the loop homology $\mathbb{H}_*(L_{f_2}Y; \mathbb{Q})$ is non-commutative. □

Bibliography

[1] P. Andrews, M. Arkowitz, Sullivan's minimal models and higher order Whitehead products. Canad. J. Math. 30 (1978), no. 5, 961-982.

[2] J. Block and A. Lazarev, André-Quillen cohomology and rational homotopy of function spaces, Adv. Math., 193 (2005), 18-39.

[3] A. K. Bousfield, V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type. Mem. Amer. Math. Soc. 8 (1976), no. 179

[4] E. H. Brown and R. H. Szczarba, On the rational homotopy type of function spaces, Trans. Amer. Math. Soc., 349 (1997), 4931-4951.

[5] U. Buijs, Upper bounds for the Whitehead-length of mapping spaces. Homotopy theory of function spaces and related topics, 43-53, Contemp. Math., 519, Amer. Math. Soc.

[6] U. Buijs, A. Murillo, Basic constructions in rational homotopy theory of function spaces. Ann. Inst. Fourier (Grenoble) 56 (2006), no. 3, 815-838.

[7] U. Buijs and A. Murillo, The rational homotopy Lie algebra of function spaces, Comment. Math. Helv., 83 (2008), 723-739.

[8] R. L. Cohen, V. Godin, A polarized view of string topology, Topology, geometry and quantum field theory, 127-154, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.

[9] M. Chas and D. Sullivan, String topology, Preprint, arXiv:math.GT/9911159, 1999.

[10] K. T. Chen, Pullback de Rham cohomology of the free path fibration, Trans. Amer. Math. Soc., 242 (1978), 307-318.

[11] S. Eilenberg, S. Mac Lane, On the groups of $H(\Pi, n)$. I. Ann. of Math. (2) 58, (1953). 55-106.

[12] Y. Félix, S. Halperin, J. -C. Thomas, Gorenstein spaces. Adv. in Math. 71 (1988), no. 1, 92-112.

[13] Y. Félix, S. Halperin and J. C. Thomas, Differential graded algebras in topology, in: Handbook of Algebraic Topology. Chapter 16, Elsevier, North-Holland-Amsterdam (1995), 829-865.

[14] Y. Félix, S. Halperin and J. C. Thomas, Rational homotopy theory. Graduate Texts in Mathematics, 205. Springer-Verlag, New York, 2001.

[15] Y. Félix, L. Menichi and J. C. Thomas, Gerstenhaber duality in Hochschild cohomology. J. Pure Appl. Algebra 199 (2005), no. 1-3, 43-59.

[16] Y. Félix, J. Oprea and D. Tanré, Algebraic models in geometry. Oxford Graduate Texts in Mathematics, 17. Oxford University Press, Oxford, 2008.

[17] Y. Félix and J. C. Thomas, Monoid of self-equivalences and free loop spaces. Proc. Amer. Math. Soc. 132 (2004), no. 1, 305-312

[18] Y. Félix, J. C. Thomas and M. Vigué-Poirrier, The Hochschild cohomology of a closed manifold, Publ. Math. Inst. Hautes Études Sci., 99 (2004), 235-252.

[19] D. Gromoll, W. Meyer, Periodic geodesics on compact riemannian manifolds. J. Differential Geometry 3 (1969) 493-510.

[20] K. Gruher and P. Salvatore, Generalized string topology operations. Proc. Lond. Math. Soc. (3) 96 (2008), no. 1, 78-106.

[21] A. Haefliger, Rational homotopy of the space of sections of a nilpotent bundle, Trans. Amer. Math. Soc., 273 (1982), no.2, 609-620.

[22] S. Halperin and J.-M. Lemaire, Notions of category in differential algebra. Algebraic topology-rational homotopy (Louvain-la-Neuve, 1986), 138-154, Lecture Notes in Math., 1318, Springer, Berlin, 1988.

[23] A. Hatcher, Algebraic topology. Cambridge University Press, Cambridge, 2002.

[24] K. Hess, P. -E. Parent and J. Scott, CoHochschild homology of chain coalgebras. J. Pure Appl. Algebra 213 (2009), no. 4, 536-556.

[25] P. Hilton, G. Mislin, J. Roitberg, Localization of nilpotent groups and spaces, North-Holland Mathematics Studies, No. 15. Notas de Matematica, No. 55. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975.

[26] J. D. S. Jones, Cyclic homology and equivariant homology, Invent. Math., 87 (1987) no. 2, 403-423.

[27] S. Kaji, On the nilpotency of rational H-spaces, J. Math. Soc. Japan, 57 (2005), 1153-1165.

[28] K. Kuribayashi, A rational model for the evaluation map. Georgian Math. J. 13 (2006), no. 1, 127-141.

[29] K. Kuribayashi and T. Yamaguchi, A rational splitting of based mapping space, Algebr. Geom. Topol., 6 (2006), 309-327.

[30] P. Lambrechts, D. Stanley, Poincaré duality and commutative differential graded algebras. Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 4, 495-509.

[31] G. Lupton and S. Smith, Rationalized evaluation subgroups of a map I : Sullivan models, derivations and G-sequences, Journal of Pure and Applied Algebra, 209 (2007), 159-171.

[32] G. Lupton and S. Smith, Whitehead products in function spaces: Quillen model formulae, J. Math. Soc. Japan 62 (2010), no. 1, 49-81.

[33] W. S. Massey, Singular Homology Theory, Graduate Texts in Mathematics, vol.70, Springer-Verlag, New York, 1980.

[34] J. McCleary, A User's Guide to Spectral Sequences, second ed., Cambridge Stud. Adv. Math., Cambridge University Press, Cambridge; 2001.

[35] J. M. Møller, Nilpotent spaces of sections. Trans. Amer. Math. Soc. 303 (1987), no. 2, 733-741.

[36] J. M. Møller and M. Raussen, Rational homotopy of spaces of maps into spheres and complex projective spaces. Trans. Amer. Math. Soc. 292 (1985), no. 2, 721-732.

[37] T. Naito, A model for the Whitehead product in rational mapping spaces, to appear in Mathematical Journal of Okayama University.

[38] T. Naito, On the mapping space homotopy groups and the free loop space homology groups, Algebr. Geom. Topol. 11 (2011), no. 4, 2369-2390.

[39] J. J. Rotman, An introduction to algebraic topology. Graduate Texts in Mathematics, 119. Springer-Verlag, New York, 1988.

[40] P. Selick, Introduction to Homotopy Theory, Fields Inst. Monogr., vol. 9, Amer. Math. Soc., Providence, RI, 1997.

[41] J. P. Serre, Homologie singuliére des espaces fibrés. Applications, (French) Ann. of Math. (2) 54, (1951). 425-505.

[42] S. Smith, Rational evaluation subgroups, Math. Z. 221 (1996), no. 3, 387-400.

[43] E. H. Spanier, Algebraic topology. Corrected reprint. Springer-Verlag, New York-Berlin, 1981.

[44] G. W. Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics, 61. Springer-Verlag, New York-Berlin, 1978.

[45] M. Vigué-Poirrier, Décompositions de l'homologie cyclique des algébres différentielles graduées commutatives. (French) [Decomposition of the cyclic homology of commutative graded differential algebras] K-Theory 4 (1991), no. 5, 399-410.

[46] M. Vigué-Poirrier, D. Sullivan, The homology theory of the closed geodesic problem. J. Differential Geometry 11 (1976), no. 4, 633-644.