Pointwise multipliers and isomorphisms between weighted Bergman spaces on the unit ball

A Doctoral Thesis
presented to
Graduate School of Science and Technology
Shinshu University

by
Toshinao Takeshima
2013.3

Abstract

This thesis contains two topics regarding weighted Bergman spaces on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. One is the complete determination of pointwise multipliers between two usual weighted Bergman spaces on $\mathbb{B}_n$. The other is the isomorphism theorem between two generalized weighted Bergman spaces on $\mathbb{B}_n$.

In 2004, R. Zhao [9] determined the pointwise multipliers between two usual weighted Bergman spaces on the unit disc $\mathbb{B}_1$. We prove that his result still holds for higher dimensional cases $n \geq 2$.

In 2008, R. Zhao and K. Zhu [10] proved that two generalized weighted Bergman spaces $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ and $\mathscr{A}^q_\beta(\mathbb{B}_n)$ are isomorphic when $p = q$. Their proof, however, includes some approximation theorem which is difficult for us to understand. We give a different proof of the isomorphism theorem by using our recent result [3] about the fractional integral operators $\mathscr{R}_{s,t}$.

This thesis consists of three parts. In Chapter 1, we investigate pointwise multipliers between two usual weighted Bergman spaces. In Chapter 2, we investigate the fractional differential and integral operators. In Chapter 3, we describe the precise definition of generalized weighted Bergman spaces and prove the isomorphism theorem mentioned above.

Contents

Introduction

Let n be a fixed positive integer. Let $\mathbb{B}_n$ and $\mathbb{S}_n = \partial\mathbb{B}_n$ denote the unit ball and the unit sphere of the n-dimensional complex Euclidian space $\mathbb{C}^n$, respectively. Let ν denote the normalized Lebesgue measure on $\mathbb{B}_n$, so that $\nu(\mathbb{B}_n) = 1$. Let σ denote the normalized surface measure on $\mathbb{S}_n$.

For any $\alpha \in \{t \in \mathbb{C} : \mathrm{Re}\, t > -1\} \cup (-\infty, -1]$, we define a weighted Lebesgue measure ν_α on $\mathbb{B}_n$ by

$$d\nu_\alpha(z) = c_\alpha \left(1 - |z|^2\right)^\alpha d\nu(z), \quad z \in \mathbb{B}_n,$$

where $c_\alpha = 1$ for $\alpha \leq -1$ and $c_\alpha = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)}$ if $\mathrm{Re}\,\alpha > -1$, which is a normalizing constant so that $\nu_\alpha(\mathbb{B}_n) = 1$. The space of all holomorphic functions in $\mathbb{B}_n$ denoted by $H(\mathbb{B}_n)$.

For $f \in H(\mathbb{B}_n)$, $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}_+$, we define

$$\|f\|_{A^p_\alpha(\mathbb{B}_n)} = \left(\int_{\mathbb{B}_n} |f|^p d\nu_\alpha\right)^{\frac{1}{p}} = \|f\|_{L^p(\nu_\alpha)},$$

where $\mathbb{R}_+$ is the set of all positive real numbers. The *weighted Bergman space* $A^p_\alpha(\mathbb{B}_n)$ is defined by

$$A^p_\alpha(\mathbb{B}_n) = \left\{f \in H(\mathbb{B}_n) : \|f\|_{A^p_\alpha(\mathbb{B}_n)} < \infty\right\}.$$

We note that the collection $\{A^p_\alpha(\mathbb{B}_n) : \alpha \in (-1, \infty), p \in \mathbb{R}_+\}$ are the *usual* weighted Bergman spaces on $\mathbb{B}_n$. When $\alpha = 0$ we simply write $A^p(\mathbb{B}_n)$ for $A^p_0(\mathbb{B}_n)$.

As usual, we define

$$\|f\|_{H^\infty(\mathbb{B}_n)} = \sup_{z \in \mathbb{B}_n} |f(z)| \qquad (f \in H(\mathbb{B}_n))$$

and

$$H^\infty(\mathbb{B}_n) = \left\{f \in H(\mathbb{B}_n) : \|f\|_{H^\infty(\mathbb{B}_n)} < \infty\right\}.$$

For $\alpha \in \mathbb{R}$ and $f \in H(\mathbb{B}_n)$, we define

$$\|f\|_{\mathscr{B}_\alpha(\mathbb{B}_n)} = |f(0)| + \sup_{z \in \mathbb{B}_n} \left(1 - |z|^2\right)^\alpha |(\nabla f)(z)|,$$

where ∇f is the holomorphic gradient of f. The $\alpha -$*Bloch space* $\mathscr{B}_\alpha(\mathbb{B}_n)$ is defined by

$$\mathscr{B}_\alpha(\mathbb{B}_n) = \left\{f \in H(\mathbb{B}_n) : \|f\|_{\mathscr{B}_\alpha(\mathbb{B}_n)} < \infty\right\}.$$

Suppose $\{\alpha,\beta\} \subset (-1,\infty)$ and $\{p,q\} \subset \mathbb{R}_+$. Then a function $g \in H(\mathbb{B}_n)$ is called a *pointwise multiplier* from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ if $\{fg : f \in A^p_\alpha(\mathbb{B}_n)\} \subset A^q_\beta(\mathbb{B}_n)$. The set of all pointwise multipliers from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ is denoted by $(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n))$. In 2004, R. Zhao [9] determined the pointwise multipliers between two usual weighted Bergman spaces on the unit disc $\mathbb{B}_1$. In Chapter 1, we exactly determine the set $(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n))$ that his result still holds for higher dimensional cases $n \geq 2$ as follows.

Theorem 1. *Let $\{\alpha,\beta\} \subset (-1,\infty)$ and $\{p,q\} \subset \mathbb{R}_+$. Put $\gamma = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}$.*

(i) *If $p \leq q$ and $\gamma > 0$, then $(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = \mathscr{B}_{1+\gamma}(\mathbb{B}_n)$.*

(ii) *If $p \leq q$ and $\gamma = 0$, then $(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$.*

(iii) *If $p \leq q$ and $\gamma < 0$, then $(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = \{0\}$.*

(iv) *If $p > q$, then $(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = A^s_\delta(\mathbb{B}_n)$, where $s = \frac{pq}{p-q}$ and $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$.*

According to custom we denote the set of all positive integers and that of all nonnegative integers by $\mathbb{N}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, respectively. For any two complex numbers s,t with $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$, we define two invertible linear operators

$$\mathscr{R}^{s,t} : H(\mathbb{B}_n) \longrightarrow H(\mathbb{B}_n)$$

and

$$\mathscr{R}_{s,t} : H(\mathbb{B}_n) \longrightarrow H(\mathbb{B}_n)$$

as follows. If $f = \sum_{k=0}^{\infty} f_k$ is the homogeneous expansion of $f \in H(\mathbb{B}_n)$ at the origin of $\mathbb{C}^n$, then

$$\mathscr{R}^{s,t} f = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s)\Gamma(n+1+k+s+t)}{\Gamma(n+1+s+t)\Gamma(n+1+k+s)} f_k$$

and

$$\mathscr{R}_{s,t} f = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s+t)\Gamma(n+1+k+s)}{\Gamma(n+1+s)\Gamma(n+1+k+s+t)} f_k.$$

For $f \not\equiv 0$ in $\mathbb{B}_n$, we put

$$\mu_f(0) = \min\{k \in \mathbb{Z}_+ : f_k \not\equiv 0\}.$$

We call $\mu_f(0)$ the *zero multiplicity* of f at the origin of $\mathbb{C}^n$.

It is evident that the inverse of $\mathscr{R}^{s,t}$ is $\mathscr{R}_{s,t}$ and $\mathscr{R}_{s,t} = \mathscr{R}^{s+t,-t}$. According to [12, Proposition 1.14], $\mathscr{R}^{s,t}$ and $\mathscr{R}_{s,t}$ are continuous operators of $H(\mathbb{B}_n)$ onto itself with respect to the topology of uniform convergence on compact subsets of $\mathbb{B}_n$.

For $f \in H(\mathbb{B}_n)$, $p \in \mathbb{R}_+$ and $r \in [0,1)$, we define

$$M_p(r,f) = \left\{ \int_{\mathbb{S}_n} |f(r\zeta)|^p \, d\sigma(\zeta) \right\}^{\frac{1}{p}}$$

and

$$M_\infty(r,f) = \sup_{\zeta \in \mathbb{S}_n} |f(r\zeta)|.$$

In 2009, Y. Matsugu and T. Yamada [2] proved the following two theorems ([2, Theorems 1 and 2]).

Theorem A . *Let $f \in H(\mathbb{B}_n)$, $p \in (0,\infty]$, $q \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$, $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Suppose that $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then*

$$\int_0^1 (1-r)^{\alpha+qt} M_p^q(r, \mathscr{R}^{s,t} f)\, dr \le C \int_0^1 (1-r)^\alpha M_p^q(r,f)\, dr,$$

where C is a positive constant depending only on p,q,α,s,t and n.

Theorem B . *Let $f \in H(\mathbb{B}_n)$, $p \in (0,\infty]$, $q \in \mathbb{R}_+$, $\alpha \in (-1,\infty)$, $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Suppose that $n+s > -1$. Then*

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t} f)\, dr \le C \int_0^1 (1-r)^{\alpha+qt} M_p^q(r,f)\, dr,$$

where C is a positive constant depending only on p,q,α,s,t and n.

The aim of Chapter 2 is to show that Theorem B still holds even if we weaken the assumption $n+s > -1$. Thus the main results in Chapter 2 are the following two theorems.

Theorem 2. *Let $f \in H(\mathbb{B}_n)$, $p \in (0,\infty]$, $q \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$, $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $\tilde{t} =$* Re$t \in \mathbb{R}_+$. *Suppose that $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then*

$$\int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r, \mathscr{R}^{s,t} f)\, dr \le C \int_0^1 (1-r)^\alpha M_p^q(r,f)\, dr,$$

where C is a positive constant depending only on p,q,α,s,t and n .

Theorem 3. *Let $f \in H(\mathbb{B}_n)$, $p \in (0,\infty]$, $q \in \mathbb{R}_+$, $\alpha \in (-1,\infty)$, $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $\tilde{t} = \mathrm{Re}\, t \in \mathbb{R}_+$. Suppose that $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then*

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t} f)\, dr \le C \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr,$$

where C is a positive constant depending only on p, q, α, s, t and n.

The proof of Theorem 2 is almost the same as that of Theorem A. To prove Theorem 3, we need to consider the zero multiplicity $\mu_f(0)$ of the holomorphic function f at the origin of $\mathbb{C}^n$. Take a nonnegative integer k_0 with $k_0 > -n - \mathrm{Re}\, s - 1$. Then we can decompose f such as $f = g + h$ where g is a polynomial of degree k_0 at most and h is a holomorphic function with $\mu_h(0) > k_0$. So it suffices to prove Theorm 3 for g and h, separately.

In Chapter 3, we investigate the generalized weighted Bergman spaces on $\mathbb{B}_n$. Let $f \in H(\mathbb{B}_n)$ and $f = \sum_{j=0}^{\infty} f_j$ be the homogeneous expansion of f at the origin of $\mathbb{C}^n$. We then define the *radial derivative* of f by

$$\mathscr{R} f = \sum_{j=1}^{\infty} j f_j .$$

Moreover, for each $k \in \mathbb{Z}_+$ we define an operator

$$\mathscr{R}^k : H(\mathbb{B}_n) \longrightarrow H(\mathbb{B}_n)$$

as follows:

$$\mathscr{R}^0 f = f = \sum_{j=0}^{\infty} f_j, \quad \mathscr{R}^1 f = \mathscr{R} f = \sum_{j=1}^{\infty} j f_j,$$

and for $k \ge 2$,

$$\mathscr{R}^k f = \mathscr{R}\left(\mathscr{R}^{k-1} f\right) = \sum_{j=1}^{\infty} j^k f_j.$$

Let $p \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$. We define the set $I_{\alpha,p}$ and the integer $\kappa_{\alpha,p}$ as follows:

$$I_{\alpha,p} = \{k \in \mathbb{Z}_+ : pk + \alpha > -1\} = \mathbb{Z}_+ \cap \left(-\frac{\alpha+1}{p}, \infty\right),$$

$$\kappa_{\alpha,p} = \min I_{\alpha,p}.$$

It is clear that the three conditions

$$I_{\alpha,p} \ni 0, \kappa_{\alpha,p} = 0 \text{ and } \alpha \in (-1,\infty)$$

are mutually equivalent. For each $k \in I_{\alpha,p}$ and $f \in H(\mathbb{B}_n)$, we define the quantity $\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$ as follows.

$$\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = \left\{ |f(0)|^p + \|\mathscr{R}^k f\|^p_{A^p_{pk+\alpha}(\mathbb{B}_n)} \right\}^{\frac{1}{p}} \quad \text{if } k \in \mathbb{N},$$

$$\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = \|f\|_{A^p_\alpha(\mathbb{B}_n)} \quad \text{if } k = 0.$$

For each $k \in I_{\alpha,p}$, we define the subspace $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ of $H(\mathbb{B}_n)$ as follows.

$$\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \mathscr{R}^k f \in A^p_{pk+\alpha}(\mathbb{B}_n) \right\}.$$

For any $f \in H(\mathbb{B}_n)$, we define

$$\|f\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)} = \|f\|_{\mathscr{A}^p_{\alpha,\kappa_{\alpha,p}}(\mathbb{B}_n)}.$$

The *generalized weighted Bergman space* $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ on $\mathbb{B}_n$ is defined by

$$\mathscr{A}^p_\alpha(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)} < \infty \right\}.$$

We can show that the following facts hold:

(i) For any $k \in I_{\alpha,p}$ and $f \in H(\mathbb{B}_n)$,

$$C_1 \|f\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)} \leq \|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq C_2 \|f\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)},$$

where C_1 and C_2 are positive constants depending only on n, p, α and k.

(ii) For any $k \in I_{\alpha,p}$,

$$\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n) = \mathscr{A}^p_\alpha(\mathbb{B}_n).$$

(iii) If $\alpha > -1$, then

$$\mathscr{A}^p_\alpha(\mathbb{B}_n) = A^p_\alpha(\mathbb{B}_n).$$

(iv) If $1 \leq p < \infty$, then $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)}$. If $0 < p < 1$, then $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ is an F-space with the metric $d_{\mathscr{A}^p_\alpha(\mathbb{B}_n)}(f,g) = \|f-g\|^p_{\mathscr{A}^p_\alpha(\mathbb{B}_n)}$ for $\{f,g\} \subset \mathscr{A}^p_\alpha(\mathbb{B}_n)$.

In 2008, R. Zhao and K. Zhu [10] proved that two generalized weighted Bergman spaces $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ and $\mathscr{A}^q_\beta(\mathbb{B}_n)$ are isomorphic when $p = q$. The goal of Chapter 3 is to give a new proof of the following isomorphism theorem, by using Theorems 2 and 3.

Theorem 4. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $p \in \mathbb{R}_+$. Put $t = \frac{\alpha-\beta}{p}$. Suppose $s \in \mathbb{C}$, $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,t}$ is an invertible bounded linear operator from $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ onto $\mathscr{A}^p_\beta(\mathbb{B}_n)$.*

Chapter 1 : Pointwise multipliers from one usual weighted Bergman space $A^p_\alpha(\mathbb{B}_n)$ into another $A^q_\beta(\mathbb{B}_n)$

1.1 Preliminaries

Proposition 1. *Suppose* $\{\alpha,\beta\}\subset(-1,\infty)$, $\{p,q\}\subset\mathbb{R}_+$ *and* $g\in H(\mathbb{B}_n)$. *Then the following two conditions are equivalent:*

(i) $g\in(\mathscr{PM})(A^p_\alpha(\mathbb{B}_n),A^q_\beta(\mathbb{B}_n))$.

(ii) $\sup\left\{\dfrac{\|fg\|_{A^q_\beta(\mathbb{B}_n)}}{\|f\|_{A^p_\alpha(\mathbb{B}_n)}}: f\in A^p_\alpha(\mathbb{B}_n)\setminus\{0\}\right\}<\infty.$

Proof. Since all the weighted Bergman spaces are F-spaces, by using the closed graph theorem (Theorem 2.15 of [5]) we can prove the proposition easily. □

Proposition 2. *Suppose* $s\in(n,\infty)$, $\alpha=s-(n+1)$ *and* $p\in\mathbb{R}_+$. *Then* $\alpha\in(-1,\infty)$ *and*

$$\int_{\mathbb{B}_n}\frac{|f(w)|^p\{1-|\varphi_z(w)|^2\}^s}{(1-|w|^2)^{n+1}}dv(w)=\frac{1}{c_\alpha}\|f\circ\varphi_z\|^p_{L^p(v_\alpha)}$$

for $f\in C(\mathbb{B}_n)$ *and* $z\in\mathbb{B}_n$, *where* φ_z *is the involutive biholomorphic map of* $\mathbb{B}_n$ *that exchanges 0 and z.*

Proof. It is clear that $\alpha\in(-1,\infty)$. By Lemma 1.2 and Proposition 1.13 of [12], for $f\in C(\mathbb{B}_n)$ and $z\in\mathbb{B}_n$, we have

$$\begin{aligned}
&\int_{\mathbb{B}_n}\frac{|f(w)|^p\{1-|\varphi_z(w)|^2\}^s}{(1-|w|^2)^{n+1}}dv(w)\\
&=\int_{\mathbb{B}_n}\frac{|f(w)|^p}{(1-|w|^2)^{n+1}}\left\{\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2}\right\}^s dv(w)\\
&=\frac{1}{c_\alpha}\int_{\mathbb{B}_n}|f(w)|^p\frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle w,z\rangle|^{2(n+1+\alpha)}}dv_\alpha(w)\\
&=\frac{1}{c_\alpha}\|f\circ\varphi_z\|^p_{L^p(v_\alpha)}.
\end{aligned}$$

□

The next lemma is Exercise 7.7 of [12]. The one dimensional case is the same proof as Proposition 7 of [11]. For the completeness, we prove it here.

Lemma 3. *For any $\alpha \in (1,\infty)$ it holds that*

$$\mathscr{B}_\alpha(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{\alpha-1}|f(z)| < \infty \right\}.$$

Proof. For $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$, we have

$$\begin{aligned}
&|f(z)-f(0)| = \left| \int_0^1 \langle (\nabla f)(tz), \bar{z} \rangle dt \right| \\
&\leq \sup_{w\in\mathbb{B}_n} \{(1-|w|^2)^\alpha |(\nabla f)(w)|\} \int_0^1 (1-|tz|^2)^{-\alpha}|z| dt \\
&\leq \frac{2^{\alpha-1}}{\alpha-1}(1-|z|^2)^{1-\alpha} \sup_{w\in\mathbb{B}_n} \{(1-|w|^2)^\alpha |(\nabla f)(w)|\}.
\end{aligned}$$

Hence we obtain

$$\sup_{z\in\mathbb{B}_n} \{(1-|z|^2)^{\alpha-1}|f(z)|\} \leq \left(1+\frac{2^{\alpha-1}}{\alpha-1}\right) \|f\|_{\mathscr{B}_\alpha(\mathbb{B}_n)}. \tag{1}$$

Conversely, suppose $f \in H(\mathbb{B}_n)$ and $\sup_{z\in\mathbb{B}_n} \{(1-|z|^2)^{\alpha-1}|f(z)|\} < \infty$. Then $f \in A^1_{\alpha-1}(\mathbb{B}_n)$. For any $z \in \mathbb{B}_n$, the Bergman integral formula (Theorem 2.2 of [12]) thus gives

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z,w\rangle)^{n+\alpha}} dv_{\alpha-1}(w).$$

Differentiating inside the integral sign, we have for $j \in \{1,\ldots,n\}$ and $z \in \mathbb{B}_n$

$$(D_j f)(z) = \int_{\mathbb{B}_n} \frac{(n+\alpha)\bar{w}_j f(w)}{(1-\langle z,w\rangle)^{n+\alpha+1}} dv_{\alpha-1}(w).$$

Hence we have

$$|(\nabla f)(z)| \leq n(n+\alpha)c_{\alpha-1} \sup_{w\in\mathbb{B}_n} \{|f(w)|(1-|w|^2)^{\alpha-1}\} \int_{\mathbb{B}_n} \frac{dv(w)}{|1-\langle z,w\rangle|^{n+\alpha+1}}. \tag{2}$$

By Proposition 1.4.10 of [6], we obtain

$$\int_{\mathbb{B}_n} \frac{dv(w)}{|1-\langle z,w\rangle|^{n+\alpha+1}} \leq \frac{C}{(1-|z|^2)^\alpha} \quad (z \in \mathbb{B}_n) \tag{3}$$

where C is a positive constant depending only on α and n. By (2) and (3), we obtain

$$\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^{\alpha}|(\nabla f)(z)|\}\leq n(n+\alpha)c_{\alpha-1}C\sup_{w\in\mathbb{B}_n}\{|f(w)|(1-|w|^2)^{\alpha-1}\}.$$

Hence we have

$$\|f\|_{\mathscr{B}_\alpha(\mathbb{B}_n)}\leq\{1+n(n+\alpha)c_{\alpha-1}C\}\sup_{w\in\mathbb{B}_n}\{|f(w)|(1-|w|^2)^{\alpha-1}\}. \tag{4}$$

Inqualities (1) and (4) show that

$$\mathscr{B}_\alpha(\mathbb{B}_n)=\left\{f\in H(\mathbb{B}_n):\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^{\alpha-1}|f(z)|\}<\infty\right\}.$$

□

1.2 Proof of Theorem 1 in case $p\leq q$

First of all, we consider some lemmas and propositons to prove Theorem 1 in case $p\leq q$. Let $M_+(\mathbb{B}_n)$ denote the set of all positive Borel measures on $\mathbb{B}_n$.

Lemma 4. *Let $\alpha\in(-1,\infty)$, $\{p,q\}\subset\mathbb{R}_+$ and $\mu\in M_+(\mathbb{B}_n)$. Suppose $p\leq q$. Then the following two conditions are equivalent:*

(i) $\sup\left\{\dfrac{\|f\|_{L^q(\mu)}}{\|f\|_{A^p_\alpha(\mathbb{B}_n)}}:f\in A^p_\alpha(\mathbb{B}_n)\backslash\{0\}\right\}<\infty.$

(ii) $\sup_{z\in\mathbb{B}_n}\left\{\displaystyle\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}d\mu(w)\right\}<\infty.$

Proof. See Theorem 50 of [10]. □

Proposition 5. *Let $\{\alpha,\beta\}\subset(-1,\infty)$ and $\{p,q\}\subset\mathbb{R}_+$. Put $\gamma=\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}$. Suppose $p\leq q$. Then for any $g\in H(\mathbb{B}_n)$ the following inequalities hold:*

$$\begin{aligned}C_1\sup_{z\in\mathbb{B}_n}\left\{(1-|z|^2)^{\gamma}|g(z)|\right\}^q&\leq\sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}\\&\leq C_2\sup_{z\in\mathbb{B}_n}\left\{(1-|z|^2)^{\gamma}|g(z)|\right\}^q,\end{aligned}$$

where C_1 and C_2 are positive constants depending only on α,β,p,q and n.

Proof. Put

$$s = \frac{q}{p}(n+1+\alpha) \text{ and } \alpha_0 = \beta - \gamma q.$$

Then, by the assumptions, we see that

$$s \in (n, \infty) \text{ and } \alpha_0 = s - (n+1).$$

Fix $g \in H(\mathbb{B}_n)$. For any $z \in \mathbb{B}_n$, put

$$G(z) = (1-|z|^2)^{\gamma}|g(z)|. \tag{1}$$

Then $G \in C(\mathbb{B}_n)$, and Proposition 2 imply that

$$\begin{aligned}
&\frac{1}{c_{\alpha_0}} \sup_{z\in\mathbb{B}_n} \|G\circ\varphi_z\|^q_{L^q(v_{\alpha_0})} \\
&= \sup_{z\in\mathbb{B}_n} \left\{ (1-|z|^2)^{\frac{q}{p}(n+1+\alpha)} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}} |g(w)|^q dv(w) \right\} \\
&= \frac{1}{c_\beta} \sup_{z\in\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}} |g(w)|^q dv_\beta(w) \right\}.
\end{aligned} \tag{2}$$

By (1) and Theorem 1.12 of [12], we obtain

$$\sup_{z\in\mathbb{B}_n} \|G\circ\varphi_z\|^q_{L^q(v_{\alpha_0})} \le c_{\alpha_0}C_1' \sup_{w\in\mathbb{B}_n} |G(w)|^q = c_{\alpha_0}C_1' \sup_{w\in\mathbb{B}_n} \{(1-|w|^2)^{\gamma}|g(w)|\}^q, \tag{3}$$

where C_1' is a positive constant depending only on α, β, p, q and n. By (2) and (3), we have

$$\sup_{z\in\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}} |g(w)|^q dv_\beta(w) \right\} \le c_\beta C_1' \sup_{z\in\mathbb{B}_n} \{(1-|z|^2)^{\gamma}|g(z)|\}^q. \tag{4}$$

Conversely, choose any $R \in \mathbb{R}_+$. By using Lemma 2.24 of [12], for any $z \in \mathbb{B}_n$, there is a positive constant C_2' depending only on β, R and n such that

$$|g(z)|^q \le \frac{C_2'}{(1-|z|^2)^{n+1+\beta}} \int_{D(z,R)} |g|^q dv_\beta, \tag{5}$$

where $D(z,R)$ is the Bergman metric ball with center at z and radius R (cf.[12]).

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}^q \le \sup_{z\in\mathbb{B}_n}\left\{\frac{C_2'}{(1-|z|^2)^{n+1+\beta-\gamma q}}\int_{D(z,R)}|g|^q dv_\beta\right\}\\
&=\sup_{z\in\mathbb{B}_n}\left\{\frac{C_2'}{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}\int_{D(z,R)}|g|^q dv_\beta\right\}\\
&=C_2'\sup_{z\in\mathbb{B}_n}\left\{\int_{D(z,R)}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{(1-|z|^2)^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}. \qquad (6)
\end{aligned}$$

By Lemma 2.20 of [12], for any $z\in\mathbb{B}_n$ and $w\in D(z,R)$, we see

$$\left(\frac{|1-\langle z,w\rangle|}{1-|z|^2}\right)^{\frac{2q}{p}(n+1+\alpha)} < C_3, \qquad (7)$$

where C_3 is a positive constant depending only on p,q,α,R and n. By (6) and (7), we have

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}^q\\
&\le C_1'C_2'C_3\sup_{z\in\mathbb{B}_n}\left\{\int_{D(z,R)}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}\\
&\le C_1'C_2'C_3\sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}. \qquad (8)
\end{aligned}$$

The assertion of the proposition follows from (4) and (8). □

Proposition 6. *Let* $\{\alpha,\beta\}\subset(-1,\infty)$, $\{p,q\}\subset\mathbb{R}_+$ *and* $g\in H(\mathbb{B}_n)$. *Suppose* $p\le q$. *Put* $\gamma=\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}$. *Then the following three conditions are equivalent:*

(i) $g\in(\mathscr{PM})(A^p_\alpha(\mathbb{B}_n),A^q_\beta(\mathbb{B}_n))$.

(ii) $\displaystyle\sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z,w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}<\infty.$

(iii) $\displaystyle\sup_{z\in\mathbb{B}_n}\left\{(1-|z|^2)^\gamma|g(z)|\right\}<\infty.$

Proof. Define $\mu_g \in M_+(\mathbb{B}_n)$ by $d\mu_g = |g|^q dv_\beta$. Then

$$\|f\|_{L^q(\mu_g)} = \|fg\|_{A^q_\beta(\mathbb{B}_n)} \text{ for all } f \in H(\mathbb{B}_n).$$

Hence, the present proposition follows from Proposition 1, Lemma 4 and Proposition 5. □

Proof of Theorem 1 in case $p \leq q$. When $\gamma > 0$, by Lemma 3,

$$\mathscr{B}_{1+\gamma}(\mathbb{B}_n) = \left\{ g \in H(\mathbb{B}_n) : \sup_{z \in \mathbb{B}_n} (1-|z|)^\gamma |g(z)| < \infty \right\}. \tag{1}$$

By Proposition 6 and (1), we obtain

$$(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = \mathscr{B}_{1+\gamma}(\mathbb{B}_n).$$

When $\gamma = 0$, by Proposition 6, we obtain

$$(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = \{g \in H(\mathbb{B}_n) : \sup_{z \in \mathbb{B}_n} |g(z)| < \infty\} = H^\infty(\mathbb{B}_n).$$

When $\gamma < 0$, it is easily shown that

$$\begin{aligned} &\{g \in H(\mathbb{B}_n) : \sup_{z \in \mathbb{B}_n} (1-|z|^2)^\gamma |g(z)| < \infty\} \\ &= \{g \in H(\mathbb{B}_n) : \lim_{|z| \to 1-0} |g(z)| = 0\} = \{0\}. \end{aligned} \tag{2}$$

By Proposition 6 and (2), we obtain

$$(\mathscr{P}\mathscr{M})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = \{0\}.$$

The proof of Theorem 1 in case $p \leq q$ is now complete. □

1.3 Proof of Theorem 1 in case $p > q$

For $\mu \in M_+(\mathbb{B}_n)$, $\alpha \in \mathbb{R}$ and $R \in \mathbb{R}_+$, we define the function $\hat{\mu}_{R,\alpha}$ on $\mathbb{B}_n$ by

$$\hat{\mu}_{R,\alpha}(z) = \frac{\mu(D(z,R))}{(1-|z|^2)^{n+1+\alpha}} \quad (z \in \mathbb{B}_n),$$

where $D(z,R)$ is the Bergman metric ball with center at z and radius R.

Lemma 7. *Let* $\alpha \in (-1,\infty), \{p,q\} \subset \mathbb{R}_+$ *and* $\mu \in M_+(\mathbb{B}_n)$. *Suppose* $p > q$. *Then the following two conditions are equivalent:*

(i) $\sup\left\{ \frac{\|f\|_{L^q(\mu)}}{\|f\|_{A^p_\alpha(\mathbb{B}_n)}} : f \in A^p_\alpha(\mathbb{B}_n)\backslash\{0\} \right\} < \infty.$

(ii) $\hat{\mu}_{R,\alpha} \in L^{\frac{p}{p-q}}(v_\alpha)$ for all $R \in \mathbb{R}_+$.

Proof. See Theorem 54 of [10]. □

Proposition 8. *Let* $\{\alpha,\beta\} \subset (-1,\infty)$, $\{p,q\} \subset \mathbb{R}_+$ *and* $g \in H(\mathbb{B}_n)$. *Define* $\mu_g \in M_+(\mathbb{B}_n)$ *by* $d\mu_g = |g|^q dv_\beta$. *Suppose* $p > q$. *Then the following two conditions are equivalent:*

(i) $g \in (\mathscr{PM})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n))$.

(ii) $(\hat{\mu_g})_{R,\alpha} \in L^{\frac{p}{p-q}}(v_\alpha)$ for all $R \in \mathbb{R}_+$.

Proof. The proposition thus follows from Proposition 1 and Lemma 7. □

Proposition 9. *Let* $\{\alpha,\beta\} \subset (-1,\infty)$ *and* $\{p,q\} \subset \mathbb{R}_+$. *Suppose* $p > q$. *Put* $s = \frac{pq}{p-q}$ *and* $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$. *Then for any pair* $\{f,g\} \subset H(\mathbb{B}_n)$,

$$\|fg\|_{A^q_\beta(\mathbb{B}_n)} \leq C\|f\|_{A^p_\alpha(\mathbb{B}_n)}\|g\|_{A^s_\delta(\mathbb{B}_n)},$$

where C *is a positive constant depending only on* α,β,p,q *and* n.

Proof. Put

$$p_0 = \frac{p}{p-q} \text{ and } q_0 = \frac{p_0}{p_0 - 1}.$$

Then

$$\{p_0, q_0\} \subset (1,\infty) \text{ and } \frac{1}{p_0} + \frac{1}{q_0} = 1.$$

By the assumptions, we have

$$\frac{sq}{s-q} = p \text{ and } \left(\beta - \frac{q\delta}{s}\right)\frac{s}{s-q} = \alpha.$$

Hence, we have

$$\begin{aligned}
\|fg\|^q_{A^q_\beta(\mathbb{B}_n)} &= \int_{\mathbb{B}_n} |fg|^q dv_\beta \\
&= c_\beta \int_{\mathbb{B}_n} |f(z)|^q (1-|z|^2)^{\beta-\frac{\delta}{p_0}} \cdot |g(z)|^q (1-|z|^2)^{\frac{\delta}{p_0}} dv(z) \\
&\leq c_\beta \left[\int_{\mathbb{B}_n} \left\{|f(z)|^q (1-|z|^2)^{\beta-\frac{\delta}{p_0}}\right\}^{q_0} dv(z)\right]^{\frac{1}{q_0}} \\
&\qquad \times \left[\int_{\mathbb{B}_n} \left\{|g(z)|^q (1-|z|^2)^{\frac{\delta}{p_0}}\right\}^{p_0} dv(z)\right]^{\frac{1}{p_0}} \\
&= c_\beta \left[\int_{\mathbb{B}_n} |f(z)|^{\frac{sq}{s-q}} (1-|z|^2)^{(\beta-\frac{q\delta}{s})\frac{s}{s-q}} dv(z)\right]^{\frac{s-q}{s}} \left[\int_{\mathbb{B}_n} |g(z)|^s (1-|z|^2)^{\delta} dv(z)\right]^{\frac{q}{s}} \\
&= c_\beta \left[\int_{\mathbb{B}_n} |f(z)|^p (1-|z|^2)^{\alpha} dv(z)\right]^{\frac{q}{p}} \left[\int_{\mathbb{B}_n} |g(z)|^s (1-|z|^2)^{\delta} dv(z)\right]^{\frac{q}{s}} \\
&= c_\beta c_\alpha^{-\frac{q}{p}} c_\delta^{-\frac{q}{s}} \|f\|^q_{A^p_\alpha(\mathbb{B}_n)} \|g\|^q_{A^s_\delta(\mathbb{B}_n)}.
\end{aligned}$$

This completes the proof. □

Proposition 10. *Let $\{\alpha,\beta\} \subset (-1,\infty)$ and $\{p,q,R\} \subset \mathbb{R}_+$. Put $s = \frac{pq}{p-q}$ and $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$. Suppose $p > q$. Then for any $g \in H(\mathbb{B}_n)$,*

$$\int_{\mathbb{B}_n} |g|^s dv_\delta \leq C \int_{\mathbb{B}_n} |(\hat{\mu_g})_{R,\alpha}|^{\frac{p}{p-q}} dv_\alpha,$$

where $d\mu_g = |g|^q dv_\beta$, and C is a positive constant depending only on α,β,p,q,R and n.

Proof. By the definition of $(\hat{\mu_g})_{R,\alpha}$, we obtain

$$\int_{\mathbb{B}_n} |(\hat{\mu_g})_{R,\alpha}|^{\frac{p}{p-q}} dv_\alpha = \int_{\mathbb{B}_n} \left\{\frac{1}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z.R)} |g|^q dv_\beta\right\}^{\frac{p}{p-q}} dv_\alpha(z). \quad (1)$$

By Lemma 2.24 of [12], for any $z \in \mathbb{B}_n$, we have

$$|g(z)|^q \leq \frac{C_1}{(1-|z|^2)^{n+1+\beta}} \int_{D(z,R)} |g|^q dv_\beta \quad (2)$$

where C_1 is a positive constant depending only on β, R and n. By (1) and (2), we have

$$\int_{\mathbb{B}_n} |(\hat{\mu_g})_{R,\alpha}|^{\frac{p}{p-q}} dv_\alpha \geq \int_{\mathbb{B}_n} \left\{ \frac{1}{(1-|z|^2)^{n+1+\alpha}} \frac{(1-|z|^2)^{n+1+\beta}}{C_1} |g(z)|^q \right\}^{\frac{p}{p-q}} dv_\alpha(z)$$
$$= c_\alpha C_1^{-\frac{p}{p-q}} \int_{\mathbb{B}_n} (1-|z|^2)^{\frac{p(\beta-\alpha)}{p-q}+\alpha} |g(z)|^{\frac{pq}{p-q}} dv_\alpha(z)$$
$$= c_\alpha C_1^{-\frac{p}{p-q}} \int_{\mathbb{B}_n} (1-|z|^2)^\delta |g(z)|^s dv(z) = c_\alpha C_1^{-\frac{p}{p-q}} c_\delta^{-1} \int_{\mathbb{B}_n} |g|^s dv_\delta.$$

This completes the proof. □

Proof of Theorem 1 in case $p > q$. By Proposition 9, we have

$$g \in A^s_\delta(\mathbb{B}_n) \implies g \in (\mathscr{PM})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)).$$

By Proposition 8 and Proposition 10, we obtain

$$g \in (\mathscr{PM})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) \implies g \in A^s_\delta(\mathbb{B}_n).$$

Hence, we have

$$(\mathscr{PM})(A^p_\alpha(\mathbb{B}_n), A^q_\beta(\mathbb{B}_n)) = A^s_\delta(\mathbb{B}_n).$$

The proof of Theorem 1 is now finished. □

Chapter 2 :The Stoll-Shi's theorem concerning the fractional derivatives and integrals of functions in $H(\mathbb{B}_n)$

2.1 Preliminaries

Proposition 11. (cf Proposition 3 of [2].) *Let* $s \in \mathbb{C}$, $t \in \mathbb{C}$ *and* $\tilde{t} = \operatorname{Re} t \in \mathbb{R}_+$. *Suppose* $n+s \notin (-\mathbb{N})$ *and* $n+s+t \notin (-\mathbb{N})$. *Then for any* $f \in H(\mathbb{B}_n)$ *with* $\mu_f(0) > -n - \operatorname{Re} s - 1$, *any* $r \in (0,1)$ *and any* $z \in \frac{1}{r}\mathbb{B}_n$,

$$(\mathscr{R}_{s,t} f)(rz) = \frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)} \int_0^r \rho^{n+s}(r-\rho)^{t-1} f(\rho z)\, d\rho.$$

Proof. Let $f = \sum_{k=0}^{\infty} f_k$ be the homogeneous expansion of f at the origin $0 \in \mathbb{C}^n$. Put $k_0 = \mu_f(0)$, $\tilde{s} = \operatorname{Re} s$ and $\tilde{t} = \operatorname{Re} t$. Since $n + \tilde{s} + k_0 > -1$ and $\tilde{t} > 0$, it follows that

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} |f_k(z)| \int_0^r \rho^{n+\tilde{s}+k}(r-\rho)^{\tilde{t}-1}\, d\rho \\
&= \sum_{k=k_0}^{\infty} |f_k(z)| r^{n+\tilde{s}+k+\tilde{t}} \int_0^1 \rho^{n+\tilde{s}+k}(1-\rho)^{\tilde{t}-1} d\rho \\
&= r^{n+\tilde{s}+\tilde{t}} \sum_{k=k_0}^{\infty} \frac{\Gamma(n+\tilde{s}+k+1)\Gamma(\tilde{t})}{\Gamma(n+\tilde{s}+k+1+\tilde{t})} |f_k(rz)| < \infty.
\end{aligned}$$

Hence the integral calculus according to the term gives

$$\begin{aligned}
&\int_0^r \rho^{n+s}(r-\rho)^{t-1} f(\rho z)\, d\rho = \int_0^r \rho^{n+s}(r-\rho)^{t-1} \sum_{k=k_0}^{\infty} f_k(\rho z)\, d\rho \\
&= \sum_{k=k_0}^{\infty} \int_0^r \rho^{n+s}(r-\rho)^{t-1} f_k(\rho z)\, d\rho = \sum_{k=k_0}^{\infty} r^{n+s+t} f_k(rz) \int_0^1 \rho^{n+s+k}(1-\rho)^{t-1}\, d\rho \\
&= r^{n+s+t} \sum_{k=k_0}^{\infty} \frac{\Gamma(n+s+k+1)\Gamma(t)}{\Gamma(n+s+k+1+t)} f_k(rz) \\
&= r^{n+s+t} \sum_{k=0}^{\infty} \frac{\Gamma(n+s+k+1)\Gamma(t)}{\Gamma(n+s+k+1+t)} f_k(rz).
\end{aligned}$$

Thus we have

$$\frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)}\int_0^r \rho^{n+s}(r-\rho)^{t-1}f(\rho z)\,d\rho$$
$$=\sum_{k=0}^{\infty}\frac{\Gamma(n+1+s+t)\Gamma(n+s+k+1)}{\Gamma(n+1+s)\Gamma(n+s+k+1+t)}f_k(rz)=(\mathscr{R}_{s,t}f)(rz).$$

□

Proposition 12. (cf Proposition 4 of [2].) *Let $s\in\mathbb{C}$, $t\in\mathbb{C}$ and $\tilde{t}=\mathrm{Re}\,t\in\mathbb{R}_+$. Suppose $n+s\notin(-\mathbb{N})$ and $n+s+t\notin(-\mathbb{N})$. Then for any $f\in H(\mathbb{B}_n)$ with $\mu_f(0)>-n-\mathrm{Re}\,s-1$, any $r\in(0,1)$ and any $p\in[1,\infty]$,*

$$M_p(r,\mathscr{R}_{s,t}f)\le\left|\frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)}\right|\int_0^r \rho^{n+\tilde{s}}(r-\rho)^{\tilde{t}-1}M_p(\rho,f)\,d\rho,$$

where $\tilde{s}=\mathrm{Re}\,s$.

Proof. By Proposition 11, for $\zeta\in\mathbb{S}_n$, we have

$$|(\mathscr{R}_{s,t}f)(r\zeta)|\le\left|\frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)}\right|\int_0^r \rho^{n+\tilde{s}}(r-\rho)^{\tilde{t}-1}|f(\rho\zeta)|\,d\rho.$$

In the case $p=\infty$, we get

$$M_p(r,\mathscr{R}_{s,t}f)=M_\infty(r,\mathscr{R}_{s,t}f)=\sup_{\zeta\in\mathbb{S}_n}\left|(\mathscr{R}_{s,t}f)(r\zeta)\right|$$
$$\le\left|\frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)}\right|\sup_{\zeta\in\mathbb{S}_n}\left\{\int_0^r \rho^{n+\tilde{s}}(r-\rho)^{\tilde{t}-1}|f(\rho\zeta)|\,d\rho\right\}$$
$$\le\left|\frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)}\right|\sup_{\zeta\in\mathbb{S}_n}\left\{\int_0^r \rho^{n+\tilde{s}}(r-\rho)^{\tilde{t}-1}M_\infty(\rho,f)\,d\rho\right\}$$
$$=\left|\frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)}\right|\int_0^r \rho^{n+\tilde{s}}(r-\rho)^{\tilde{t}-1}M_p(\rho,f)\,d\rho.$$

In the case $1 \le p < \infty$, the continuous form of Minkowski's inequality, we have

$$\begin{aligned}
M_p(r, \mathscr{R}_{s,t} f) &= \left\{ \int_{\mathbb{S}_n} |(\mathscr{R}_{s,t} f)(r\zeta)|^p \, d\sigma(\zeta) \right\}^{\frac{1}{p}} \\
&\le \left| \frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)} \right| \left[\int_{\mathbb{S}_n} \left\{ \int_0^r \rho^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} |f(\rho\zeta)| \, d\rho \right\}^p d\sigma(\zeta) \right]^{\frac{1}{p}} \\
&\le \left| \frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)} \right| \int_0^r \left[\int_{\mathbb{S}_n} \left\{ \rho^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} |f(\rho\zeta)| \right\}^p d\sigma(\zeta) \right]^{\frac{1}{p}} d\rho \\
&= \left| \frac{\Gamma(n+1+s+t)}{r^{n+s+t}\Gamma(n+1+s)\Gamma(t)} \right| \int_0^r \rho^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho, f) \, d\rho.
\end{aligned}$$

□

For $f \in H(\mathbb{B}_n)$ and $\zeta \in \mathbb{S}_n$, we define

$$(M_{\text{rad}} f)(\zeta) = \sup_{0 \le r < 1} |f(r\zeta)|.$$

For $p \in (0, \infty]$, we define

$$\|f\|_{H^p(\mathbb{B}_n)} = \sup_{0 \le r < 1} M_p(r, f).$$

The *Hardy space* on $\mathbb{B}_n$ is

$$H^p(\mathbb{B}_n) = \{ f \in H(\mathbb{B}_n) : \|f\|_{H^p(\mathbb{B}_n)} < \infty \}.$$

Lemma 13. *For any $f \in H(\mathbb{B}_n)$ and $p \in \mathbb{R}_+$,*

$$\int_{\mathbb{S}_n} (M_{\text{rad}} f)^p \, d\sigma \le C \|f\|^p_{H^p(\mathbb{B}_n)},$$

where C is a positive constant depending only on n.

Proof. See Theorem 5.6.5. of [6]. □

Let $\mathbb{Z}^n_+$ denote the set of all multi-index of nonnegative integers. For $f \in H(\mathbb{B}_n)$ and $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n_+$ we will employ the notation

$$D^m f = \frac{\partial^m f}{\partial z^m} = \frac{\partial^{|m|}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}},$$

where $|m| = m_1 + \cdots m_n$.

Lemma 14. *For any $f \in H(\mathbb{B}_n)$, $r \in (0,1)$, $m \in \mathbb{Z}_+^n$ and $p \in (0,\infty]$,*

$$|(D^m f)(0)| \leq \left(\frac{4}{3}\right)^{\frac{n}{p}} \frac{(n+|m|-1)!}{(n-1)!} \left(\frac{2}{r}\right)^{|m|} M_p(r,f).$$

Proof. By Theorem 3.2.4 of [6], for any $R \in (0,1)$ and $z \in \mathbb{B}_n$,

$$f(Rz) = f_R(z) = \int_{\mathbb{S}_n} \frac{f_R(\xi)}{(1-\langle z,\xi\rangle)^n} d\sigma(\xi) = \int_{\mathbb{S}_n} \frac{f(R\xi)}{(1-\langle z,\xi\rangle)^n} d\sigma(\xi).$$

Hence we have

$$R^{|m|}(D^m f)(Rz) = \int_{\mathbb{S}_n} \frac{n(n+1)\cdots(n+|m|-1)\bar{\xi}^m f(R\xi)}{(1-\langle z,\xi\rangle)^{n+|m|}} d\sigma(\xi).$$

Since $r \in (0,1)$, we have

$$|(D^m f)(0)| \leq \left(\frac{2}{r}\right)^{|m|} \frac{(n+|m|-1)!}{(n-1)!} \int_{\mathbb{S}_n} |f(\frac{r}{2}\xi)| d\sigma(\xi).$$

By Theorem 4.17 of [12], for $\xi \in \mathbb{S}_n$, we have

$$|f(\frac{r}{2}\xi)| = |f_r(\frac{1}{2}\xi)| \leq \frac{\|f_r\|_{H^p(\mathbb{B}_n)}}{(1-|\frac{1}{2}\xi|^2)^{\frac{n}{p}}} = \left(\frac{4}{3}\right)^{\frac{n}{p}} M_p(r,f).$$

Hence we obtain

$$|(D^m f)(0)| \leq \left(\frac{4}{3}\right)^{\frac{n}{p}} \frac{(n+|m|-1)!}{(n-1)!} \left(\frac{2}{r}\right)^{|m|} M_p(r,f).$$

□

Proposition 15. (cf Proposition 6 of [2].) *Let $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $\tilde{t} = \mathrm{Re}\, t \in \mathbb{R}_+$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then for any $f \in H(\mathbb{B}_n)$ with $\mu_f(0) > -n - \mathrm{Re}\, s - 1$, any $r \in (0,1)$ and any $p \in (0,1]$,*

$$M_p(r, \mathscr{R}_{s,t} f) \leq \frac{C}{r^{n+\tilde{s}+\tilde{t}}} \left\{ \int_0^r \rho^{p(n+\tilde{s})} (r-\rho)^{p\tilde{t}-1} M_p^p(\rho,f)\, d\rho \right\}^{\frac{1}{p}},$$

where $\tilde{s} = \mathrm{Re}\, s$ and C is a positive constant depending only on s,t,p and n.

Proof. By Proposition 11, for $z \in \frac{1}{r}\mathbb{B}_n$, we have

$$(\mathscr{R}_{s,t}f)(rz) = \frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)} \int_0^1 \rho^{n+s}(1-\rho)^{t-1} f(r\rho z)\, d\rho.$$

Fix $\zeta \in \mathbb{S}_n$. Hence we have

$$|(\mathscr{R}_{s,t}f)(r\zeta)| \le \left| \frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)} \right| \int_0^1 \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1} |f(r\rho\zeta)|\, d\rho. \tag{1}$$

Put

$$\rho_k = 1 - \frac{1}{2^k} \quad (k \in \mathbb{Z}_+). \tag{2}$$

Then we have

$$\rho_0 = 0 < \frac{1}{4n} < \frac{1}{2} = \rho_1 < \rho_2 < \dots, \qquad \lim_{k\to\infty} \rho_k = 1. \tag{3}$$

We put

$$I(\zeta) = \int_0^1 \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1} |f(r\rho\zeta)|\, d\rho, \tag{4}$$

$$I_1(\zeta) = \int_0^{\frac{1}{4n}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1} |f(r\rho\zeta)|\, d\rho, \tag{5}$$

$$I_2(\zeta) = \int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1} |f(r\rho\zeta)|\, d\rho \tag{6}$$

and

$$I_3(\zeta) = \sum_{k=1}^{\infty} \int_{\rho_k}^{\rho_{k+1}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1} |f(r\rho\zeta)|\, d\rho. \tag{7}$$

By (3)–(7),

$$I(\zeta) = \sum_{j=1}^{3} I_j(\zeta). \tag{8}$$

Put

$$k_0 = \mu_f(0).$$

By the assumption, we have

$$k_0 > -n - \tilde{s} - 1. \tag{9}$$

Let $f=\sum_{k=0}^{\infty} f_k$ be the homogeneos expansion of f at the origin of $\mathbb{C}^n$. For $\rho \in (0,\frac{1}{4n})$, by Lemma 14, we have

$$|f(r\rho\zeta)| = |\sum_{k=k_0}^{\infty} f_k(r\rho\zeta)| = \left|\sum_{k=k_0}^{\infty}\left\{\sum_{m\in\mathbb{Z}_+^n,|m|=k}\frac{(D^m f)(0)}{m!}(r\rho\zeta)^m\right\}\right|$$

$$\leq \sum_{k=k_0}^{\infty}\left\{\sum_{m\in\mathbb{Z}_+^n,|m|=k}\frac{|(D^m f)(0)|}{m!}(r\rho)^{|m|}\right\}$$

$$\leq \sum_{k=k_0}^{\infty}\left\{\sum_{m\in\mathbb{Z}_+^n,|m|=k}\frac{(r\rho)^{|m|}}{m!}\left(\frac{4}{3}\right)^{\frac{n}{p}}\left(\frac{5}{2r}\right)^{|m|}\frac{(n+|m|-1)!}{(n-1)!}M_p(\frac{4r}{5},f)\right\}$$

$$= \frac{(\frac{4}{3})^{\frac{n}{p}}M_p(\frac{4r}{5},f)}{(n-1)!}\sum_{k=k_0}^{\infty}\left(\frac{5n\rho}{2}\right)^k\frac{(n+k-1)!}{k!}. \tag{10}$$

By (5) and (10), we get

$$I_1(\zeta) \leq \frac{(\frac{4}{3})^{\frac{n}{p}}M_p(\frac{4r}{5},f)}{(n-1)!}\sum_{k=k_0}^{\infty}\left(\frac{5n}{2}\right)^k\frac{(n+k-1)!}{k!}\int_0^{\frac{1}{4n}}\rho^{n+\tilde{s}+k}(1-\rho)^{\tilde{t}-1}\,d\rho$$

$$\leq \frac{(\frac{4}{3})^{\frac{n}{p}}M_p(\frac{4r}{5},f)}{(n-1)!}\sum_{k=k_0}^{\infty}\left(\frac{5n}{2}\right)^k\frac{(n+k-1)!}{k!}\int_0^{\frac{1}{4n}}\rho^{n+\tilde{s}+k}\,\frac{4n}{3}\,d\rho$$

$$= \frac{n(\frac{4}{3})^{\frac{n}{p}+1}M_p(\frac{4r}{5},f)}{(4n)^{n+\tilde{s}+1}(n-1)!}\sum_{k=k_0}^{\infty}\frac{(n+k-1)!}{(n+\tilde{s}+k+1)k!}\left(\frac{5}{8}\right)^k,$$

and

$$C_1 = \frac{n(\frac{4}{3})^{\frac{n}{p}+1}}{(4n)^{n+\tilde{s}+1}(n-1)!}\sum_{k=k_0}^{\infty}\frac{(n+k-1)!}{(n+\tilde{s}+k+1)k!}\left(\frac{5}{8}\right)^k < \infty.$$

Hence we obtain

$$I_1(\zeta) \leq C_1 M_p(\frac{4r}{5},f). \tag{11}$$

By (6) and Theorem 4.17 of [12], we have

$$\begin{aligned} I_2(\zeta) &= \int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}|f(\frac{4r}{5}\frac{5\rho}{4}\zeta)|\,d\rho \\ &= \int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}|f_{\frac{4r}{5}}(\frac{5\rho}{4}\zeta)|\,d\rho \\ &\le \int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}\frac{\|f_{\frac{4r}{5}}\|_{H^p(\mathbb{B}_n)}}{(1-|\frac{5\rho}{4}\zeta|^2)^{\frac{n}{p}}}\,d\rho \\ &= \int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}\frac{M_p(\frac{4r}{5},f)}{\{1-(\frac{5\rho}{4})^2\}^{\frac{n}{p}}}\,d\rho \\ &\le \left(\frac{64}{39}\right)^{\frac{n}{p}} M_p(\frac{4r}{5},f)\int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}\,d\rho. \end{aligned} \tag{12}$$

Put

$$C_2 = \left(\frac{64}{39}\right)^{\frac{n}{p}} \int_{\frac{1}{4n}}^{\frac{1}{2}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}\,d\rho < \infty.$$

Hence we have

$$I_2(\zeta) \le C_2 M_p(\frac{4r}{5},f). \tag{13}$$

For $k \in \mathbb{Z}_+$, put

$$N_k(\zeta) = \sup_{\rho_k<\rho<\rho_{k+1}} |f(r\rho\zeta)|.$$

Hence, we have

$$I_3(\zeta) \le \sum_{k=1}^{\infty} N_k(\zeta)\int_{\rho_k}^{\rho_{k+1}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}\,d\rho. \tag{14}$$

For any $k \in \mathbb{N}$, put

$$J_k = \int_{\rho_k}^{\rho_{k+1}} \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1}\,d\rho.$$

Fix any $k \in \mathbb{N}$. By considering each of four cases

$n+\tilde{s} \ge 0, \tilde{t}-1 \ge 0$; $n+\tilde{s} \ge 0, \tilde{t}-1 < 0$; $n+\tilde{s} < 0, \tilde{t}-1 < 0$; $n+\tilde{s} < 0, \tilde{t}-1 \ge 0$, we now estimate J_k. In the first case that $n+\tilde{s} \geq 0$ and $\tilde{t}-1 \geq 0$, we have

$$0 < J_k \leq \int_{\rho_k}^{\rho_{k+1}} (\rho_{k+1})^{n+\tilde{s}}(1-\rho_k)^{\tilde{t}-1}\,d\rho = (\rho_{k+1})^{n+\tilde{s}}(1-\rho_k)^{\tilde{t}-1}(\rho_{k+1}-\rho_k)$$
$$= 2^{\tilde{t}-1}(\rho_{k+1})^{n+\tilde{s}}\left(\frac{1}{2^{k+1}}\right)^{\tilde{t}}. \tag{15}$$

In the second case that $n+\tilde{s} \geq 0$ and $\tilde{t}-1 < 0$, we get

$$0 < J_k \leq \int_{\rho_k}^{\rho_{k+1}} (\rho_{k+1})^{n+\tilde{s}}(1-\rho_{k+1})^{\tilde{t}-1}\,d\rho = (\rho_{k+1})^{n+\tilde{s}}\left(\frac{1}{2^{k+1}}\right)^{\tilde{t}}. \tag{16}$$

In the third case that $n+\tilde{s} < 0$ and $\tilde{t}-1 < 0$, we obtain

$$0 < J_k \leq \int_{\rho_k}^{\rho_{k+1}} (\rho_k)^{n+\tilde{s}}(1-\rho_{k+1})^{\tilde{t}-1}\,d\rho$$
$$= \left(\frac{1-\frac{1}{2^k}}{1-\frac{1}{2^{k+1}}}\right)^{n+\tilde{s}}(\rho_{k+1})^{n+\tilde{s}}\left(\frac{1}{2^{k+1}}\right)^{\tilde{t}}.$$

It follows that

$$0 < J_k \leq C_3(\rho_{k+1})^{n+\tilde{s}}\left(\frac{1}{2^{k+1}}\right)^{\tilde{t}} \tag{17}$$

where

$$C_3 = \sup_{l\in\mathbb{N}}\left\{\left(\frac{1-\frac{1}{2^l}}{1-\frac{1}{2^{l+1}}}\right)^{n+\tilde{s}}\right\} < \infty.$$

In the fourth case that $n+\tilde{s} < 0$ and $\tilde{t}-1 \geq 0$, we have

$$0 < J_k \leq \int_{\rho_k}^{\rho_{k+1}} (\rho_k)^{n+\tilde{s}}(1-\rho_k)^{\tilde{t}-1}\,d\rho = (\rho_k)^{n+\tilde{s}}(1-\rho_k)^{\tilde{t}-1}(\rho_{k+1}-\rho_k)$$
$$= (\rho_k)^{n+\tilde{s}}\left(\frac{1}{2^k}\right)^{\tilde{t}-1}\frac{1}{2^{k+1}} = \left(\frac{\rho_k}{\rho_{k+1}}\right)^{n+\tilde{s}}(\rho_{k+1})^{n+\tilde{s}}\left(\frac{2}{2^{k+1}}\right)^{\tilde{t}-1}\frac{1}{2^{k+1}}$$
$$\leq 2^{\tilde{t}-1}C_3(\rho_{k+1})^{n+\tilde{s}}\left(\frac{1}{2^{k+1}}\right)^{\tilde{t}}. \tag{18}$$

By (15)–(18), for $k \in \mathbb{N}$, we get

$$0 < J_k \leq C_4(\rho_{k+1})^{n+\tilde{s}}\frac{1}{2^{(k+1)\tilde{t}}}, \tag{19}$$

where

$$C_4 = \max\{2^{\tilde{t}-1}, 1, C_3, 2^{\tilde{t}-1}C_3\}.$$

Hence (14) and (19) imply that

$$I_3(\zeta) \le C_4 \sum_{k=1}^{\infty} N_k(\zeta)(\rho_{k+1})^{n+\tilde{s}} \frac{1}{2^{(k+1)\tilde{t}}}. \tag{20}$$

By (8), (11), (13) and (20), we obtain

$$I(\zeta) \le (C_1 + C_2) M_p(\frac{4r}{5}, f) + C_4 \sum_{k=1}^{\infty} N_k(\zeta)(\rho_{k+1})^{n+\tilde{s}} \frac{1}{2^{(k+1)\tilde{t}}}. \tag{21}$$

Since $0 < p \le 1$, (21) gives

$$\{I(\zeta)\}^p \le (C_1 + C_2)^p \{M_p(\frac{4r}{5}, f)\}^p + C_4^p \sum_{k=1}^{\infty} \{N_k(\zeta)\}^p (\rho_{k+1})^{p(n+\tilde{s})} \frac{1}{2^{(k+1)p\tilde{t}}}. \tag{22}$$

On the other hand, we have

$$\int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho \ge \int_{\frac{4r}{5}}^{\frac{5r}{6}} \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho$$

$$\ge M_p^p(\frac{4r}{5}, f) \int_{\frac{4r}{5}}^{\frac{5r}{6}} \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}\, d\rho. \tag{23}$$

Since $\frac{\rho}{5} < r - \rho < \frac{\rho}{4}$ for $\rho \in (\frac{4r}{5}, \frac{5r}{6})$,

$$(r-\rho)^{p\tilde{t}-1} \ge 5^{-|p\tilde{t}-1|} \rho^{p\tilde{t}-1} \quad \text{for any } \rho \in (\frac{4r}{5}, \frac{5r}{6}). \tag{24}$$

By (24), if $n + \tilde{s} + \tilde{t} \neq 0$, we get

$$\int_{\frac{4r}{5}}^{\frac{5r}{6}} \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}\, d\rho \ge 5^{-|p\tilde{t}-1|} \int_{\frac{4r}{5}}^{\frac{5r}{6}} \rho^{p(n+\tilde{s})+p\tilde{t}-1}\, d\rho$$

$$= 5^{-|p\tilde{t}-1|} \frac{(\frac{5r}{6})^{p(n+\tilde{s}+\tilde{t})} - (\frac{4r}{5})^{p(n+\tilde{s}+\tilde{t})}}{p(n+\tilde{s}+\tilde{t})}$$

$$= 5^{-|p\tilde{t}-1|} \frac{(\frac{5}{6})^{p(n+\tilde{s}+\tilde{t})} - (\frac{4}{5})^{p(n+\tilde{s}+\tilde{t})}}{p(n+\tilde{s}+\tilde{t})} r^{p(n+\tilde{s}+\tilde{t})}. \tag{25}$$

If $n+\tilde{s}+\tilde{t}=0$, (24) gives

$$\begin{aligned}
&\int_{\frac{4r}{5}}^{\frac{5r}{6}} \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}\,d\rho \geq 5^{-|p\tilde{t}-1|}\int_{\frac{4r}{5}}^{\frac{5r}{6}} \rho^{-1}\,d\rho \\
&= 5^{-|p\tilde{t}-1|}\left(\log\frac{5r}{6}-\log\frac{4r}{5}\right) = 5^{-|p\tilde{t}-1|}\log\frac{25}{24} \\
&= 5^{-|p\tilde{t}-1|}\log\frac{25}{24}\,r^{p(n+\tilde{s}+\tilde{t})}. \qquad (26)
\end{aligned}$$

By (23), (25) and (26), we have

$$M_p^p(\frac{4r}{5},f) \leq \frac{1}{C_5 r^{p(n+\tilde{s}+\tilde{t})}}\int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}M_p^p(\rho,f)\,d\rho, \qquad (27)$$

where C_5 is a positive constant depending only on s,t,p and n. Put

$$K(\zeta)=\int_{\frac{1}{2}}^{1} \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1}\{M_{\mathrm{rad}}(f_{r\rho})(\zeta)\}^p\,d\rho. \qquad (28)$$

By (3) and (28), we have

$$\begin{aligned}
K(\zeta) &= \sum_{k=1}^{\infty}\int_{\rho_k}^{\rho_{k+1}} \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1}\{M_{\mathrm{rad}}(f_{r\rho})(\zeta)\}^p\,d\rho \\
&\geq \sum_{k=1}^{\infty}\{M_{\mathrm{rad}}(f_{r\rho_k})(\zeta)\}^p\int_{\rho_k}^{\rho_{k+1}} \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1}\,d\rho. \qquad (29)
\end{aligned}$$

For $k\in\mathbb{Z}_+$, we have

$$N_k(\zeta) \leq \sup_{0\leq\rho<\rho_{k+1}}|f(r\rho\zeta)| = \sup_{0\leq\rho<1}|f_{r\rho_{k+1}}(\rho\zeta)| = \{M_{\mathrm{rad}}(f_{r\rho_{k+1}})\}(\zeta). \qquad (30)$$

For $k\in\mathbb{N}$, put

$$L_k=\int_{\rho_k}^{\rho_{k+1}} \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1}\,d\rho. \qquad (31)$$

By some calculation about J_k, we show that

$$L_k \geq C_7(\rho_k)^{p(n+\tilde{s})}\left(\frac{1}{2^k}\right)^{p\tilde{t}} \qquad (32)$$

where C_7 is a positive constant depending only on s,t,p and n. By (29)–(32), we have

$$K(\zeta) \geq \sum_{k=1}^{\infty} \{M_{\text{rad}}(f_{r\rho_k})(\zeta)\}^p L_k \geq \sum_{k=1}^{\infty} \{N_{k-1}(\zeta)\}^p L_k$$
$$\geq C_7 \sum_{k=1}^{\infty} \{N_{k-1}(\zeta)\}^p \rho_k^{p(n+\tilde{s})} \frac{1}{2^{kp\tilde{t}}} \geq C_7 \sum_{k=1}^{\infty} \{N_k(\zeta)\}^p (\rho_{k+1})^{p(n+\tilde{s})} \frac{1}{2^{(k+1)p\tilde{t}}}. \tag{33}$$

By (21) and (33), for $\zeta \in \mathbb{S}_n$, we obtain

$$\{I(\zeta)\}^p \leq (C_1+C_2)^p \{M_p(\frac{4r}{5},f)\}^p + \frac{C_4^p}{C_7} K(\zeta). \tag{34}$$

By (1), (4) and (34), we have

$$\begin{aligned}
&\{M_p(r,\mathscr{R}_{s,t}f)\}^p = \int_{\mathbb{S}_n} |(\mathscr{R}_{s,t}f)(r\zeta)|^p \, d\sigma(\zeta) \\
&\leq \left|\frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)}\right|^p \int_{\mathbb{S}_n} \left\{\int_0^1 \rho^{n+\tilde{s}}(1-\rho)^{\tilde{t}-1} |f(r\rho\zeta)| \, d\rho\right\}^p d\sigma(\zeta) \\
&= \left|\frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)}\right|^p \int_{\mathbb{S}_n} \{I(\zeta)\}^p \, d\sigma(\zeta) \\
&\leq C_8 M_p^p(\frac{4r}{5},f) + C_9 \int_{\mathbb{S}_n} K(\zeta) \, d\sigma(\zeta),
\end{aligned} \tag{35}$$

where

$$C_8 = (C_1+C_2)^p \left|\frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)}\right|^p \text{ and } C_9 = \frac{C_4^p}{C_7} \left|\frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)}\right|^p.$$

By (28), we have

$$\begin{aligned}
&\int_{\mathbb{S}_n} K(\zeta) \, d\sigma(\zeta) = \int_{\mathbb{S}_n} d\sigma(\zeta) \int_{\frac{1}{2}}^1 \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1} \{M_{\text{rad}}(f_{r\rho})(\zeta)\}^p \, d\rho \\
&= \int_{\frac{1}{2}}^1 \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1} \, d\rho \int_{\mathbb{S}_n} \{M_{\text{rad}}(f_{r\rho})(\zeta)\}^p \, d\sigma(\zeta).
\end{aligned} \tag{36}$$

By Lemma 13, for $g \in H(\mathbb{B}_n)$ and $q \in \mathbb{R}_+$, we obtain

$$\int_{\mathbb{S}_n} (M_{\text{rad}}\, g)^q \, d\sigma \leq C_{10} \|g\|^q_{H^q(\mathbb{B}_n)}, \tag{37}$$

where C_{10} is a positive constant depending only on n. By (36) and (37), we have

$$\begin{aligned}\int_{\mathbb{S}_n} K(\zeta)\,d\sigma(\zeta) &\le C_{10}\int_{\frac{1}{2}}^{1} \rho^{p(n+\tilde{s})}(1-\rho)^{p\tilde{t}-1}\|f_{r\rho}\|^p_{H^p(\mathbb{B}_n)}\,d\rho \\ &\le \frac{C_{10}}{r^{p(n+\tilde{s}+\tilde{t})}}\int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}\{M_p(\rho,f)\}^p\,d\rho. \qquad (38)\end{aligned}$$

By (35), (27) and (38), we get

$$\{M_p(r,\mathscr{R}_{s,t}f)\}^p \le \left(\frac{C_8}{C_5}+C_9C_{10}\right)\frac{1}{r^{p(n+\tilde{s}+\tilde{t})}} \times \int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}\{M_p(\rho,f)\}^p\,d\rho.$$

This means

$$M_p(r,\mathscr{R}_{s,t}f) \le \frac{C}{r^{n+\tilde{s}+\tilde{t}}}\left\{\int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1}M_p^p(\rho,f)\,d\rho\right\}^{\frac{1}{p}},$$

where

$$C=\left(\frac{C_8}{C_5}+C_9C_{10}\right)^{\frac{1}{p}}.$$

□

Lemma 16. *Suppose $s\in(-1,\infty), t\in\mathbb{R}_+$ and $\alpha\in(-1,\infty)$. Then for any $\rho\in[0,1)$,*

$$\int_0^1 \frac{x^\alpha(1-x)^s}{(1-\rho x)^{1+s+t}}\,dx \le \frac{C}{(1-\rho)^t},$$

where C is a positive constant depending only on s,t and α.

Proof. See Lemma 7 of [2]. □

Lemma 17. *Suppose $\alpha\in\mathbb{C}$ and $\tilde{\alpha}=\operatorname{Re}\alpha\in(-1,\infty)$. Then for any $f\in A^1_{\tilde{\alpha}}(B_n)$ and any $z\in\mathbb{B}_n$,*

$$f(z)=\int_{\mathbb{B}_n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}dv_\alpha(w).$$

Proof. The same proof as that of Theorem 2.2 of [12] (where the parameter α is real) also holds in this complex case. Here we make use of the Lemma 16 in place of Proposition 1.13 of [12]. □

Lemma 18. *Let $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $N \in \mathbb{N}$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then there exists a holomorphic polynomial h in $\mathbb{C}$ of degree N such that for any $w \in \mathbb{B}_n$*

$$\mathscr{R}^{s,t}\left(\frac{1}{(1-\langle\cdot,w\rangle)^{n+1+s+N}}\right)=\frac{h(\langle\cdot,w\rangle)}{(1-\langle\cdot,w\rangle)^{n+1+s+N+t}}.$$

Proof. The just same proof as that of Theorem 2.18 of [12] (where the parameters s and t are real) also holds in this complex case. □

Proposition 19. (cf Proposition 8 of [2].) *Let $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $\tilde{t} = \mathrm{Re}\,t \in \mathbb{R}_+$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+t \notin (-\mathbb{N})$. Then for any $f \in H(\mathbb{B}_n)$, any pair $\{r,\rho\} \subset (0,1)$ and any $p \in (0,\infty]$,*

$$M_p(r\rho,\mathscr{R}^{s,t}f) \le \frac{C}{(1-\rho^2)^{\tilde{t}}}\,M_p(r,f),$$

where C is a positive constant depending only on s,t,p and n.

Proof. Put $\tilde{s} = \mathrm{Re}\,s$. Choose an $N \in \mathbb{N}$ as follows.
In case $1 < p \le \infty$:

$$\frac{1}{p}-\tilde{s}-1 < N. \tag{1}$$

In case $0 < p \le 1$:

$$\max\left\{\frac{n}{p}-n-\tilde{s}-1,\ -\tilde{s}-1\right\} < N. \tag{2}$$

Put

$$\beta = s+N \ \text{ and } \ \tilde{\beta} = \mathrm{Re}\,\beta. \tag{3}$$

By (1)–(3),

$$-1 < \tilde{\beta} < \infty. \tag{4}$$

Since the dilation f_r is holomorphic in a neighborhood of $\overline{\mathbb{B}}_n$, by (4) and the Bergman integral formula (Lemma 17), for any $z \in \mathbb{B}_n$ we have

$$f_r(z) = \int_{\mathbb{B}_n} \frac{f_r(w)}{(1-\langle z,w\rangle)^{n+1+\beta}}\,dv_\beta(w). \tag{5}$$

Since $\mathscr{R}^{s,t}$ is a continuous operator on $H(\mathbb{B}_n)$ with respect to the topology of uniform convergence on compact subsets of $\mathbb{B}_n$, by (5), for $z \in \mathbb{B}_n$, we have

$$(\mathscr{R}^{s,t}f_r)(z) = \int_{\mathbb{B}_n} f_r(w)\left(\mathscr{R}^{s,t}\left(\frac{1}{(1-\langle\cdot,w\rangle)^{n+1+\beta}}\right)\right)(z)\,dv_\beta(w). \tag{6}$$

Since $N \in \mathbb{N}$, Lemma 18 gives for any $w \in \mathbb{B}_n$

$$\left(\mathscr{R}^{s,t}\left(\frac{1}{(1-\langle\cdot,w\rangle)^{n+1+s+N}}\right)\right)(z) = \frac{h(\langle z,w\rangle)}{(1-\langle z,w\rangle)^{n+1+s+N+t}} \quad (z \in \mathbb{B}_n) \qquad (7)$$

where h is a holomorphic polynomial in $\mathbb{C}$. By (3),(6) and (7), for $z \in \mathbb{B}_n$, we have

$$|(\mathscr{R}^{s,t} f_r)(z)| \leq C_1 \int_{\mathbb{B}_n} \frac{|f_r(w)|}{|1-\langle z,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} dv_{\tilde{\beta}}(w), \qquad (8)$$

where

$$C_1 = \sup\{|h(\lambda)| : \lambda \in \mathbb{C}, |\lambda| < 1\}.$$

By (1),(2) and (7), C_1 is a positive constant depending only on s,t,p and n. Since $0 < \rho < 1$, by (8), for $\zeta \in \mathbb{S}_n$,

$$|(\mathscr{R}^{s,t} f_r)(\rho\zeta)| \leq C_1 \int_{\mathbb{B}_n} \frac{|f_r(w)|}{|1-\langle \rho\zeta,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} dv_{\tilde{\beta}}(w). \qquad (9)$$

From now on, we consider the case $1 < p < \infty$. By Hölder's inequality, (9) gives

$$\begin{aligned} &|(\mathscr{R}^{s,t} f_r)(\rho\zeta)| \\ &\leq C_1 \left\{\int_{\mathbb{B}_n} \frac{|f_r(w)|^p}{|1-\langle \rho\zeta,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} dv_{\tilde{\beta}}(w)\right\}^{\frac{1}{p}} \cdot \left\{\int_{\mathbb{B}_n} \frac{dv_{\tilde{\beta}}(w)}{|1-\langle \rho\zeta,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}}\right\}^{\frac{1}{q}}, \end{aligned}$$

for $\zeta \in \mathbb{S}_n$ where

$$q = \frac{p}{p-1}.$$

Since $-1 < \tilde{\beta} < \infty$ and $\tilde{t} \in \mathbb{R}_+$, for $z \in \mathbb{B}_n$, Proposition 1.4.10 of [6] gives

$$\int_{\mathbb{B}_n} \frac{dv_{\tilde{\beta}}(w)}{|1-\langle z,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} \leq \frac{C_2}{(1-|z|^2)^{\tilde{t}}}, \qquad (10)$$

where C_2 is a positive constant depending only on s,t,p and n. For $\zeta \in \mathbb{S}_n$, we obtain

$$|(\mathscr{R}^{s,t} f_r)(\rho\zeta)|^p \leq \frac{C_3}{(1-\rho^2)^{\tilde{t}(p-1)}} \int_{\mathbb{B}_n} \frac{|f_r(w)|^p}{|1-\langle \rho\zeta,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} dv_{\tilde{\beta}}(w), \qquad (11)$$

where $C_3 = C_1^p C_2^{p-1}$ is a positive constant depending only on s,t,p and n. Hence we have

$$\begin{aligned}
\{M_p(r\rho,\mathscr{R}^{s,t}f)\}^p &= \int_{\mathbb{S}_n} |(\mathscr{R}^{s,t}f)(r\rho\zeta)|^p\,d\sigma(\zeta) = \int_{\mathbb{S}_n} |(\mathscr{R}^{s,t}f_r)(\rho\zeta)|^p\,d\sigma(\zeta) \\
&\leq \frac{C_3}{(1-\rho^2)^{\tilde{t}(p-1)}} \int_{\mathbb{S}_n} d\sigma(\zeta) \int_{\mathbb{B}_n} \frac{|f_r(w)|^p}{|1-\langle \rho\zeta, w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}}\,dv_{\tilde{\beta}}(w) \\
&= \frac{C_3}{(1-\rho^2)^{\tilde{t}(p-1)}} \int_{\mathbb{B}_n} |f_r(w)|^p\,dv_{\tilde{\beta}}(w) \int_{\mathbb{S}_n} \frac{d\sigma(\zeta)}{|1-\langle \rho w, \zeta\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} \qquad (12)
\end{aligned}$$

Since $0 < 1+\tilde{\beta}+\tilde{t} < \infty$, for $z \in \mathbb{B}_n$, Proposition 1.4.10 of [6] gives

$$\int_{\mathbb{S}_n} \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} \leq \frac{C_4}{(1-|z|^2)^{1+\tilde{\beta}+\tilde{t}}}, \qquad (13)$$

where C_4 is a positive constant depending only on s,t,p and n. By (12) and (13), we have

$$\begin{aligned}
\{M_p(r\rho,\mathscr{R}^{s,t}f)\}^p &\leq \frac{C_3C_4}{(1-\rho^2)^{\tilde{t}(p-1)}} \int_{\mathbb{B}_n} \frac{|f_r(w)|^p}{(1-|\rho w|^2)^{1+\tilde{\beta}+\tilde{t}}}\,dv_{\tilde{\beta}}(w) \\
&= \frac{C_3C_4c_{\tilde{\beta}}}{(1-\rho^2)^{\tilde{t}(p-1)}} \cdot 2n \int_0^1 \frac{u^{2n-1}(1-u^2)^{\tilde{\beta}}}{(1-\rho^2u^2)^{1+\tilde{\beta}+\tilde{t}}} \{M_p(ru,f)\}^p\,du \\
&\leq \frac{C_3C_4c_{\tilde{\beta}}}{(1-\rho^2)^{\tilde{t}(p-1)}} \cdot \{M_p(r,f)\}^p \cdot n \int_0^1 \frac{x^{n-1}(1-x)^{\tilde{\beta}}}{(1-\rho^2x)^{1+\tilde{\beta}+\tilde{t}}}\,dx. \qquad (14)
\end{aligned}$$

Since $-1 < \tilde{\beta} < \infty$ and $\tilde{t} \in \mathbb{R}_+$, Lemma 16 gives

$$n\int_0^1 \frac{x^{n-1}(1-x)^{\tilde{\beta}}}{(1-\rho^2x)^{1+\tilde{\beta}+\tilde{t}}}\,dx \leq \frac{C_5}{(1-\rho^2)^{\tilde{t}}} \qquad (15)$$

where C_5 is a positive constant depending only on s,t,p and n. By (14) and (15), we obtain

$$M_p(r\rho,\mathscr{R}^{s,t}f) \leq C\frac{M_p(r,f)}{(1-\rho^2)^{\tilde{t}}},$$

where $C = \{C_3C_4c_{\tilde{\beta}}C_5\}^{\frac{1}{p}}$ is a positive constant depending only on s,t,p and n.

Secondly, we consider case $0 < p \leq 1$. Put

$$\alpha = p(\beta+n+1)-(n+1) \text{ and } \tilde{\alpha} = \operatorname{Re}\alpha. \qquad (16)$$

By (2) and (16), we have

$$-1 < \tilde{\alpha} < \infty$$

and

$$\beta = \frac{n+1+\alpha}{p} - (n+1).$$

By (8), for $z \in \mathbb{B}_n$, we get

$$|(\mathscr{R}^{s,t} f_r)(z)| \leq C_1 c_{\tilde{\beta}} \int_{\mathbb{B}_n} \frac{|f_r(w)|(1-|w|^2)^{\frac{n+1+\tilde{\alpha}}{p}-(n+1)}}{|1-\langle z,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}} dv(w). \tag{17}$$

For $w \in \mathbb{B}_n$, we put

$$F_z(w) = \frac{f_r(w)}{(1-\langle w,z\rangle)^{n+1+\beta+t}}.$$

Then F_z is holomorphic in a neighborhood of $\overline{\mathbb{B}}_n$. Since $0 < p \leq 1$ and $-1 < \tilde{\alpha} < \infty$, the embedding lemma (Lemma 2.15 of [12]) gives

$$\int_{\mathbb{B}_n} |F_z(w)|(1-|w|^2)^{\frac{n+1+\tilde{\alpha}}{p}-(n+1)} dv(w) \leq \frac{1}{c_{\tilde{\alpha}}} \left(\int_{\mathbb{B}_n} |F_z|^p \, dv_{\tilde{\alpha}} \right)^{\frac{1}{p}}. \tag{18}$$

By (17) and (20), for $z \in \mathbb{B}_n$, we have

$$|(\mathscr{R}^{s,t} f_r)(z)|^p \leq C_6 \int_{\mathbb{B}_n} \frac{|f_r(w)|^p}{|1-\langle z,w\rangle|^{p(n+1+\tilde{\beta}+\tilde{t})}} dv_{\tilde{\alpha}}(w), \tag{19}$$

where $C_6 = \left(\frac{C_1 c_{\tilde{\beta}}}{c_{\tilde{\alpha}}}\right)^p$ is a positive constant depending only on s,t,p and n.Hence, we obtain

$$\begin{aligned}
\{M_p(r\rho,\mathscr{R}^{s,t} f)\}^p &= \int_{\mathbb{S}_n} |(\mathscr{R}^{s,t} f_r)(\rho\zeta)|^p \, d\sigma(\zeta) \\
&\leq C_6 \int_{\mathbb{S}_n} d\sigma(\zeta) \int_{\mathbb{B}_n} \frac{|f_r(w)|^p}{|1-\langle \rho\zeta,w\rangle|^{p(n+1+\tilde{\beta}+\tilde{t})}} dv_{\tilde{\alpha}}(w) \\
&= C_6 \int_{\mathbb{B}_n} |f_r(w)|^p \, dv_{\tilde{\alpha}}(w) \int_{\mathbb{S}_n} \frac{d\sigma(\zeta)}{|1-\langle \rho w,\zeta\rangle|^{p(n+1+\tilde{\beta}+\tilde{t})}}.
\end{aligned}$$

Hence, Proposition 1.4.10 of [6] gives

$$\int_{\mathbb{S}_n} \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{p(n+1+\tilde{\beta}+\tilde{t})}} \leq \frac{C_7}{(1-|z|^2)^{\tilde{\alpha}+1+p\tilde{t}}}, \tag{17}$$

where C_7 is a positive constant depending only on s,t,p and n. Hence we get

$$\begin{aligned}\{M_p(r\rho,\mathscr{R}^{s,t}f)\}^p &\le C_6C_7\int_{\mathbb{B}_n}\frac{|f_r(w)|^p}{(1-|\rho w|^2)^{\tilde{\alpha}+1+p\tilde{t}}}\,dv_{\tilde{\alpha}}(w)\\ &= C_6C_7\cdot 2n\int_0^1\frac{u^{2n-1}c_{\tilde{\alpha}}(1-u^2)^{\tilde{\alpha}}}{(1-\rho^2u^2)^{\tilde{\alpha}+1+p\tilde{t}}}\,du\int_{\mathbb{S}_n}|f_r(u\xi)|^p\,d\sigma(\xi)\\ &= C_6C_7c_{\tilde{\alpha}}\cdot 2n\int_0^1\frac{u^{2n-1}(1-u^2)^{\tilde{\alpha}}}{(1-\rho^2u^2)^{\tilde{\alpha}+1+p\tilde{t}}}\{M_p(ru,f)\}^p\,du\\ &\le C_6C_7c_{\tilde{\alpha}}\cdot\{M_p(r,f)\}^p\cdot n\int_0^1\frac{x^{n-1}(1-x)^{\tilde{\alpha}}}{(1-\rho^2x)^{\tilde{\alpha}+1+p\tilde{t}}}\,dx.\end{aligned}$$

Since $-1<\tilde{\alpha}<\infty$ and $p\tilde{t}\in\mathbb{R}_+$, Lemma 16 gives

$$n\int_0^1\frac{x^{n-1}(1-x)^{\tilde{\alpha}}}{(1-\rho^2x)^{\tilde{\alpha}+1+p\tilde{t}}}\,dx\le\frac{C_8}{(1-\rho^2)^{p\tilde{t}}},\tag{14}$$

where C_8 is a positive constant depending only on s,t,p and n. Hence we obtain

$$M_p(r\rho,\mathscr{R}^{s,t}f)\le C'\frac{M_p(r,f)}{(1-\rho^2)^{\tilde{t}}},$$

where $C'=\{C_6C_7c_{\tilde{\alpha}}C_8\}^{\frac{1}{p}}$ is a positive constant depending only on s,t,p and n.

In the third case $p=\infty$, we can derive the conclusion easily. Indeed, by (9) and (10), we get

$$\begin{aligned}M_p(r\rho,\mathscr{R}^{s,t}f) &= M_\infty(r\rho,\mathscr{R}^{s,t}f)=\sup_{\zeta\in\mathbb{S}_n}|(\mathscr{R}^{s,t}f_r)(\rho\zeta)|\\ &\le C_1\sup_{\zeta\in\mathbb{S}_n}\left\{\int_{\mathbb{B}_n}\frac{|f_r(w)|}{|1-\langle\rho\zeta,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}}\,dv_{\tilde{\beta}}(w)\right\}\\ &\le C_1M_\infty(r,f)\sup_{\zeta\in\mathbb{S}_n}\left\{\int_{\mathbb{B}_n}\frac{dv_{\tilde{\beta}}(w)}{|1-\langle\rho\zeta,w\rangle|^{n+1+\tilde{\beta}+\tilde{t}}}\right\}\\ &\le C_1M_\infty(r,f)\frac{C_2}{(1-\rho^2)^{\tilde{t}}}.\end{aligned}$$

By (31), we obtain

$$M_p(r\rho,\mathscr{R}^{s,t}f)\le C''\frac{M_p(r,f)}{(1-\rho^2)^{\tilde{t}}},$$

where $C''=C_1C_2$ is a positive constant depending only on s,t,p and n. □

Lemma 20. *Let $k \in [1,\infty), \mu \in \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$. Suppose $h : (0,1) \to [0,\infty]$ is measurable. Then*

$$\int_0^1 (1-r)^{k\mu-1} \left\{ \int_0^r (r-t)^{\delta-1} h(t)dt \right\}^k dr \le C \int_0^1 (1-r)^{k\mu-1+k\delta} h^k(r)dr$$

where C is a positive constant depending only on k, μ and δ.

Proof. See Lemma 1 of [7]. □

Lemma 21. *Let $\beta \in \mathbb{R}_+, p \in \mathbb{R}_+, q \in \mathbb{R}_+$ and $s \in (0,1)$. Suppose $p \le q$ and $h : (0,1) \to [0,\infty]$ is nondecreasing. Then*

$$\left\{ \int_0^1 (1-r)^{\beta q-1} h^q(rs)dr \right\}^{\frac{1}{q}} \le (\beta p)^{\frac{1}{p}-\frac{1}{q}} \left\{ \int_0^1 (1-r)^{\beta p-1} h^p(rs)dr \right\}^{\frac{1}{p}}.$$

Proof. See Lemma 5 of [7]. □

Proposition 22. *Let $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $\tilde{t} = \mathrm{Re}\, t \in \mathbb{R}_+$. Then for any $f \in H(\mathbb{B}_n)$ with $\mu_f(0) > -n - \mathrm{Re}\, s - 1$, $r \in (0,1)$ and any $p \in (0,\infty]$,*

$$\int_0^r \left(\frac{\rho}{r}\right)^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\, d\rho \le C \int_0^r (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\, d\rho,$$

where $\tilde{s} = \mathrm{Re}\, s$ and C is a positive constant depending only on s,t,p and n.

Proof. We define

$$I(r) = \int_0^r \left(\frac{\rho}{r}\right)^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\, d\rho,$$

$$I_1(r) = \int_0^{\frac{r}{4n}} \left(\frac{\rho}{r}\right)^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\, d\rho$$

and

$$I_2(r) = \int_{\frac{r}{4n}}^r \left(\frac{\rho}{r}\right)^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\, d\rho.$$

Hence, we have

$$I(r) = I_1(r) + I_2(r), \tag{1}$$

$$I_1(r) = r^{\tilde{t}} \int_0^{\frac{1}{4n}} \rho^{n+\tilde{s}} (1-\rho)^{\tilde{t}-1} M_p(r\rho,f)\, d\rho \tag{2}$$

and

$$I_2(r) \leq (4n)^{|n+\tilde{s}|} \int_{\frac{r}{4n}}^{r} (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho. \tag{3}$$

Similarly to (10) in the proof of Proposition 15, we have

$$|f(r\rho\zeta)| \leq \frac{(\frac{4}{3})^{\frac{n}{p}} M_p(\frac{4r}{5}, f)}{(n-1)!} \sum_{k=k_0}^{\infty} \left(\frac{5n\rho}{2}\right)^k \frac{(n+k-1)!}{k!} \tag{4}$$

for $\rho \in (0, \frac{1}{4n})$ and $\zeta \in \mathbb{S}_n$ where $k_0 = \mu_f(0)$. By (2) and (4), we obtain

$$\begin{aligned}
I_1(r) &\leq r^{\tilde{t}} \frac{(\frac{4}{3})^{\frac{n}{p}} M_p(\frac{4r}{5}, f)}{(n-1)!} \sum_{k=k_0}^{\infty} \left(\frac{5n}{2}\right)^k \frac{(n+k-1)!}{k!} \int_0^{\frac{1}{4n}} \rho^{n+\tilde{s}+k} (1-\rho)^{\tilde{t}-1}\, d\rho \\
&\leq r^{\tilde{t}} \frac{(\frac{4}{3})^{\frac{n}{p}} M_p(\frac{4r}{5}, f)}{(n-1)!} \left(\frac{4n}{4n-1}\right)^{|\tilde{t}-1|} \sum_{k=k_0}^{\infty} \left(\frac{5n}{2}\right)^k \frac{(n+k-1)!}{k!} \int_0^{\frac{1}{4n}} \rho^{n+\tilde{s}+k}\, d\rho \\
&= r^{\tilde{t}} \frac{(\frac{4}{3})^{\frac{n}{p}} M_p(\frac{4r}{5}, f)}{(n-1)!} \left(\frac{4n}{4n-1}\right)^{|\tilde{t}-1|} \\
&\times \sum_{k=k_0}^{\infty} \left(\frac{5}{8}\right)^k \left(\frac{1}{4n}\right)^{n+\tilde{s}+1} \frac{(n+k-1)!}{k!} \frac{1}{n+\tilde{s}+k+1}.
\end{aligned}$$

Hence, we have

$$I_1(r) \leq C_1 r^{\tilde{t}} M_p(\frac{4r}{5}, f), \tag{5}$$

where

$$C_1 = \frac{(\frac{4}{3})^{\frac{n}{p}}}{(n-1)!} \left(\frac{4n}{4n-1}\right)^{|\tilde{t}-1|} \left(\frac{1}{4n}\right)^{n+\tilde{s}+1} \sum_{k=k_0}^{\infty} \left(\frac{5}{8}\right)^k \frac{(n+k-1)!}{k!} \frac{1}{n+\tilde{s}+k+1}.$$

On the other hand, we get

$$\int_{\frac{4r}{5}}^{r} (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho \geq \int_{\frac{4r}{5}}^{r} (r-\rho)^{\tilde{t}-1} M_p(\frac{4r}{5}, f)\, d\rho = \frac{r^{\tilde{t}}}{5^{\tilde{t}}\tilde{t}} M_p(\frac{4r}{5}, f). \tag{6}$$

By (5) and (6), we have

$$I_1(r) \leq C_2 \int_{\frac{4r}{5}}^{r} (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho, \tag{7}$$

where

$$C_2 = C_1 5^{\tilde{t}}\tilde{t}.$$

By (1), (3) and (7), we have

$$I(r) \leq C\int_0^r (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\,d\rho,$$

where

$$C = (4n)^{|n+\tilde{s}|} + C_2.$$

This completes the proof. □

Proposition 23. *Suppose $s \in \mathbb{C}$, $t \in \mathbb{C}$, $\tilde{t} = \operatorname{Re} t \in \mathbb{R}_+$ and $p \in (0,1]$. Then for any $f \in H(\mathbb{B}_n)$ with $\mu_f(0) > -n - \operatorname{Re} s - 1$ and any $r \in (0,1)$,*

$$\int_0^r \left(\frac{\rho}{r}\right)^{p(n+\tilde{s})} (r-\rho)^{p\tilde{t}-1} M_p^p(\rho,f)\,d\rho \leq C\int_0^r (r-\rho)^{p\tilde{t}-1} M_p^p(\rho,f)\,d\rho,$$

where $\tilde{s} = \operatorname{Re} s$ and C is a positive constant depending only on s,t,p and n.

Proof. The proof is similar to that of Proposition 22. □

Proposition 24. *Let $f \in H(\mathbb{B}_n)$, $p \in (0,\infty]$, $q \in \mathbb{R}_+$, $\alpha \in (-1,\infty)$, $s \in \mathbb{C}$, $t \in \mathbb{C}$ and $\tilde{t} = \operatorname{Re} t \in \mathbb{R}_+$. Suppose that $n+s \notin (-\mathbb{N})$, $n+s+t \notin (-\mathbb{N})$ and $\mu_f(0) > -n - \operatorname{Re} s - 1$. Then*

$$\int_0^1 (1-r)^{\alpha} M_p^q(r,\mathscr{R}_{s,t}f)\,dr \leq C\int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\,dr,$$

where C is a positive constant depending only on p,q,α,s,t and n.

Proof. By Proposition 22, for any $r \in (0,1)$, we have

$$\int_0^r \left(\frac{\rho}{r}\right)^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\,d\rho \leq C_1\int_0^r (r-\rho)^{\tilde{t}-1} M_p(\rho,f)\,d\rho, \tag{1}$$

where $\tilde{s} = \operatorname{Re} s$ and C_1 is a positive constant depending only on s,t,p and n. Since $q\tilde{t} \in \mathbb{R}_+$, for any $x \in (0,1)$,

$$(q\tilde{t}+1)(1-x^{q\tilde{t}+1})^{\alpha} \leq C_2(1-x)^{\alpha}, \tag{2}$$

where C_2 is a positive constant depending only on t, q and α.
By a change of variables and (2),

$$\int_0^1 (1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t} f)\, dr = \int_0^1 (1-x^{q\tilde{t}+1})^\alpha M_p^q(x^{q\tilde{t}+1}, \mathscr{R}_{s,t} f) \cdot (q\tilde{t}+1) x^{q\tilde{t}}\, dx$$
$$\leq C_2 \int_0^1 x^{q\tilde{t}} (1-x)^\alpha M_p^q(x^{q\tilde{t}+1}, \mathscr{R}_{s,t} f)\, dx \leq C_2 \int_0^1 r^{q\tilde{t}} (1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t} f)\, dr. \tag{3}$$

Suppose

$$1 \leq p \leq \infty.$$

By Proposition 12, for $r \in (0,1)$, we get

$$M_p(r, \mathscr{R}_{s,t} f) \leq \frac{C_3}{r^{n+\tilde{s}+\tilde{t}}} \int_0^r \rho^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho \tag{4}$$

where

$$C_3 = \left| \frac{\Gamma(n+1+s+t)}{\Gamma(n+1+s)\Gamma(t)} \right|.$$

By (3) and (4), we have

$$\int_0^1 (1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t} f)\, dr$$
$$\leq C_2 \int_0^1 r^{q\tilde{t}} (1-r)^\alpha \left\{ \frac{C_3}{r^{n+\tilde{s}+\tilde{t}}} \int_0^r \rho^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho \right\}^q dr$$
$$= C_2 C_3^q \int_0^1 (1-r)^\alpha \left\{ \int_0^r \left(\frac{\rho}{r}\right)^{n+\tilde{s}} (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho \right\}^q dr. \tag{5}$$

By (1) and (5), we obtain

$$\int_0^1 (1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t} f)\, dr \leq C_4 \int_0^1 (1-r)^\alpha \left\{ \int_0^r (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho \right\}^q dr, \tag{6}$$

where

$$C_4 = C_1^q C_2 C_3^q.$$

In the first case $1 \leq p \leq \infty$ and $1 \leq q < \infty$, put

$$\mu = \frac{1+\alpha}{q}.$$

Since $1 \leq q < \infty$, $\mu \in \mathbb{R}_+$ and $\tilde{t} \in \mathbb{R}_+$, Lemma 20 shows that

$$\int_0^1 (1-r)^{q\mu-1} \left\{ \int_0^r (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho \right\}^q dr$$
$$\leq C_5 \int_0^1 (1-r)^{q\mu-1+q\tilde{t}} M_p^q(r,f)\, dr, \tag{7}$$

where C_5 is a positive constant depending only on t, q and α. By (6) and (7), we obtain

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t} f)\, dr \leq C \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr,$$

where $C = C_4 C_5$.

In the second case $1 \leq p \leq \infty$ and $0 < q \leq 1$, by Lemma 21, for $r \in (0,1)$, we have

$$\int_0^r (r-\rho)^{\tilde{t}-1} M_p(\rho, f)\, d\rho = r^{\tilde{t}} \cdot \int_0^1 (1-\rho)^{\tilde{t}-1} M_p(r\rho, f)\, d\rho$$
$$\leq r^{\tilde{t}} \cdot (\tilde{t}q)^{\frac{1}{q}-\frac{1}{1}} \left\{ \int_0^1 (1-\rho)^{q\tilde{t}-1} M_p^q(r\rho, f)\, d\rho \right\}^{\frac{1}{q}}$$
$$= (\tilde{t}q)^{\frac{1}{q}-1} \left\{ \int_0^r (r-\rho)^{q\tilde{t}-1} M_p^q(\rho, f)\, d\rho \right\}^{\frac{1}{q}}. \tag{8}$$

By (6) and (8), we obtain

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t} f)\, dr$$
$$\leq C_4 (\tilde{t}q)^{1-q} \int_0^1 (1-r)^{\alpha} \left\{ \int_0^r (r-\rho)^{q\tilde{t}-1} M_p^q(\rho, f)\, d\rho \right\} dr. \tag{9}$$

Put

$$\mu = 1 + \alpha.$$

Since $\mu \in \mathbb{R}_+$ and $q\tilde{t} \in \mathbb{R}_+$, Lemma 20 shows that

$$\int_0^1 (1-r)^{\mu-1} \left\{ \int_0^r (r-\rho)^{q\tilde{t}-1} M_p^q(\rho, f)\, d\rho \right\} dr$$
$$\leq C_6 \int_0^1 (1-r)^{\mu-1+q\tilde{t}} M_p^q(r,f)\, dr, \tag{10}$$

where C_6 is a positive constant depending only on t, q and α. By (9) and (10), we obtain

$$\int_0^1 (1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t}f)\, dr \leq C' \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr,$$

where $C' = C_4(\tilde{t}q)^{1-q}C_6$.

From now on, we suppose

$$0 < p < 1.$$

By Proposition 15, for $r \in (0,1)$, we have

$$M_p(r, \mathscr{R}_{s,t}f) \leq \frac{C_7}{r^{n+\tilde{s}+\tilde{t}}} \left\{ \int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho \right\}^{\frac{1}{p}}, \tag{11}$$

where C_7 is a positive constant depending only on s, t, p and n. Hence we obtain

$$\begin{aligned} &\int_0^1 r^{q\tilde{t}}(1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t}f)\, dr \\ &\leq C_7^q \int_0^1 r^{q\tilde{t}}(1-r)^\alpha \frac{1}{r^{q(n+\tilde{s}+\tilde{t})}} \left\{ \int_0^r \rho^{p(n+\tilde{s})}(r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho \right\}^{\frac{q}{p}} dr \\ &= C_7^q \int_0^1 (1-r)^\alpha \left\{ \int_0^r \left(\frac{\rho}{r}\right)^{p(n+\tilde{s})} (r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho \right\}^{\frac{q}{p}} dr. \end{aligned} \tag{12}$$

By Proposition 23 , for $r \in (0,1)$, we get

$$\int_0^r \left(\frac{\rho}{r}\right)^{p(n+\tilde{s})} (r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho \leq C_8 \int_0^r (r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho, \tag{13}$$

where C_8 is a positive constant depending only on s, t, p and n. Hence we have

$$\begin{aligned} &\int_0^1 (1-r)^\alpha M_p^q(r, \mathscr{R}_{s,t}f)\, dr \\ &\leq C_9 \int_0^1 (1-r)^\alpha \left\{ \int_0^r (r-\rho)^{p\tilde{t}-1} M_p^p(\rho, f)\, d\rho \right\}^{\frac{q}{p}} dr, \end{aligned} \tag{14}$$

where

$$C_9 = C_2 C_7^q C_8^{\frac{q}{p}}.$$

In the third case $0 < p < 1$ and $p \leq q$, put

$$\mu = \frac{p(1+\alpha)}{q}.$$

Since $1 \le \frac{q}{p} < \infty$, $\mu \in \mathbb{R}_+$ and $p\tilde{t} \in \mathbb{R}_+$, Lemma 20 shows that

$$\int_0^1 (1-r)^{\frac{q}{p}\mu-1} \left\{ \int_0^r (r-\rho)^{p\tilde{t}-1} M_p^p(\rho,f)\,d\rho \right\}^{\frac{q}{p}} dr$$
$$\le C_{10} \int_0^1 (1-r)^{\frac{q}{p}\mu-1+\frac{q}{p}\cdot p\tilde{t}} \left\{ M_p^p(r,f) \right\}^{\frac{q}{p}} dr, \tag{15}$$

where C_{10} is a positive constant depending only on t, p, q and α. By (14) and (15), we obtain

$$\int_0^1 (1-r)^{\alpha} M_p^q(r,\mathscr{R}_{s,t}f)\,dr \le C'' \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\,dr,$$

where $C'' = C_9 C_{10}$.
In the forth case $0<p<1$ and $p>q$, by (11) and (13), for $r \in (0,1)$, we get

$$M_p(r,\mathscr{R}_{s,t}f) \le \frac{C_7}{r^{\tilde{t}}} \left\{ \int_0^r \left(\frac{\rho}{r}\right)^{p(n+\tilde{s})} (r-\rho)^{p\tilde{t}-1} M_p^p(\rho,f)\,d\rho \right\}^{\frac{1}{p}}$$
$$\le \frac{C_7 C_8^{\frac{1}{p}}}{r^{\tilde{t}}} \left\{ \int_0^r (r-\rho)^{p\tilde{t}-1} M_p^p(\rho,f)\,d\rho \right\}^{\frac{1}{p}}$$
$$= C_7 C_8^{\frac{1}{p}} \left\{ \int_0^1 (1-\rho)^{p\tilde{t}-1} M_p^p(r\rho,f)\,d\rho \right\}^{\frac{1}{p}}. \tag{16}$$

Since $0<q<p<\infty$ and $\tilde{t} \in \mathbb{R}_+$, Lemma 21 shows that

$$\left\{ \int_0^1 (1-\rho)^{p\tilde{t}-1} M_p^p(r\rho,f)\,d\rho \right\}^{\frac{1}{p}} \le (\tilde{t}q)^{\frac{1}{q}-\frac{1}{p}} \left\{ \int_0^1 (1-\rho)^{q\tilde{t}-1} M_p^q(r\rho,f)\,d\rho \right\}^{\frac{1}{q}}. \tag{17}$$

By (14) and (15), for $r \in (0,1)$, we have

$$M_p^q(r,\mathscr{R}_{s,t}f) \le C_{11} \int_0^1 (1-\rho)^{q\tilde{t}-1} M_p^q(r\rho,f)\,d\rho = \frac{C_{11}}{r^{q\tilde{t}}} \int_0^r (r-\rho)^{q\tilde{t}-1} M_p^q(\rho,f)\,d\rho,$$

where

$$C_{11} = C_7^q C_8^{\frac{q}{p}} (\tilde{t}q)^{1-\frac{q}{p}}.$$

Hence we obtain

$$\int_0^1 r^{q\tilde{t}} (1-r)^{\alpha} M_p^q(r,\mathscr{R}_{s,t}f)\,dr$$
$$\le C_{11} \int_0^1 (1-r)^{\alpha} \left\{ \int_0^r (r-\rho)^{q\tilde{t}-1} M_p^q(\rho,f)\,d\rho \right\} dr. \tag{18}$$

By (3) and (18), we have

$$\int_0^1 (1-r)^\alpha M_p^q(r,\mathscr{R}_{s,t}f)\,dr \le C_2C_{11}\int_0^1 (1-r)^\alpha \left\{\int_0^r (r-\rho)^{q\tilde{t}-1}M_p^q(\rho,f)\,d\rho\right\}dr. \tag{19}$$

This is the same as the inequality (9) except the constant multiple of the integral in the right-hand side. By (19) and (10), we obtain

$$\int_0^1 (1-r)^\alpha M_p^q(r,\mathscr{R}_{s,t}f)\,dr \le C_{12}\int_0^1 (1-r)^{\alpha+q\tilde{t}}M_p^q(r,f)\,dr,$$

where $C_{12}=C_2C_{11}C_6$. This completes the proof. □

2.2 Proof of Theorem 2

By Proposition 19, for any $r\in(0,1)$,

$$M_p(r^2,\mathscr{R}^{s,t}f)\le \frac{C_1}{(1-r^2)^{\tilde{t}}}M_p(r,f),$$

where C_1 is a positive constant depending only on s,t,p and n. Hence, we have

$$\begin{aligned}
&\int_0^1 (1-r)^{\alpha+q\tilde{t}}M_p^q(r,\mathscr{R}^{s,t}f)\,dr \le \int_0^1 (1-r)^{\alpha+q\tilde{t}}\left\{\frac{C_1}{(1-r)^{\tilde{t}}}M_p(\sqrt{r},f)\right\}^q dr\\
&=C_1^q\int_0^1 (1-r)^\alpha M_p^q(\sqrt{r},f)\,dr = C_1^q\int_0^1 (1-r^2)^\alpha M_p^q(r,f)\cdot 2r\,dr\\
&\le 2^{1+|\alpha|}C_1^q\int_0^1 (1-r)^\alpha M_p^q(r,f)\,dr.
\end{aligned}$$

This completes the proof.

2.3 Proof of Theorem 3

Let be

$$f=\sum_{k=0}^\infty f_k$$

the homogeneous expansion of $f \in H(\mathbb{B}_n)$ at the origin of $\mathbb{C}^n$, then

$$\mathscr{R}_{s,t} f = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s+t)\Gamma(n+1+k+s)}{\Gamma(n+1+s)\Gamma(n+1+k+s+t)} f_k.$$

Put

$$k_0 = \min\{k \in \mathbb{Z}_+ : k > -n - \tilde{s} - 1\},$$

$$g = \sum_{k=0}^{k_0} f_k, \quad h = \sum_{k=k_0+1}^{\infty} f_k.$$

Then g is a holomorphic polynomial in $\mathbb{C}^n$ with $\deg(g) \leq k_0$ and

$$h \in H(\mathbb{B}_n), \quad \mu_h(0) > k_0 > -n - \tilde{s} - 1. \tag{1}$$

By Proposition 24, we have

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t} h)\, dr \leq C_1 \int_0^1 (1-r)^{\alpha + q\tilde{t}} M_p^q(r,h)\, dr, \tag{2}$$

where C_1 is a positive constant depending only on p, q, α, s, t and n. By Lemma 14, for $z \in \mathbb{B}_n$, we obtain

$$|g(z)| = \left| \sum_{m \in \mathbb{Z}_+^n, |m| \leq k_0} \frac{(D^m f)(0)}{m!} z^m \right| \leq \sum_{m \in \mathbb{Z}_+^n, |m| \leq k_0} \left| \frac{(D^m f)(0)}{m!} \right|$$

$$\leq \sum_{m \in \mathbb{Z}_+^n, |m| \leq k_0} \frac{1}{m!} \left(\frac{4}{3}\right)^{\frac{n}{p}} \frac{(n+|m|-1)!}{(n-1)!} 4^{|m|} M_p(\frac{1}{2}, f).$$

Hence, for $r \in (0,1)$, we have

$$M_p(r,g) \leq C_2 M_p(\frac{1}{2}, f), \tag{3}$$

where

$$C_2 = \left(\frac{4}{3}\right)^{\frac{n}{p}} \sum_{m \in \mathbb{Z}_+^n, |m| \leq k_0} \frac{4^{|m|}(n+|m|-1)!}{(n-1)!m!} = \left(\frac{4}{3}\right)^{\frac{n}{p}} \sum_{k=0}^{k_0} \frac{(4n)^k (n+k-1)!}{(n-1)!k!}.$$

Similarly to (3), for $r \in (0,1)$, we have

$$M_p(r, \mathscr{R}_{s,t} g) \leq C_3 M_p(\frac{1}{2}, f), \tag{4}$$

where

$$C_3 = \left(\frac{4}{3}\right)^{\frac{n}{p}} \sum_{k=0}^{k_0} \frac{(4n)^k (n+k-1)!}{(n-1)!k!} \left|\frac{\Gamma(n+1+s+t)\Gamma(n+1+k+s)}{\Gamma(n+1+s)\Gamma(n+1+k+s+t)}\right|.$$

Hence we get

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t}f)\, dr \\ \le C_4 \int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t}g)\, dr + C_4 \int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t}h)\, dr, \qquad (5)$$

where C_4 is a positive constant depending only on p and q. Similarly we have

$$\int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,h)\, dr = \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r, f-g)\, dr \\ \le C_4 \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr + C_4 \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,g)\, dr. \qquad (6)$$

Since $\alpha > -1$, (4) gives

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t}g)\, dr \le \int_0^1 (1-r)^{\alpha} \{C_3 M_p(\frac{1}{2}, f)\}^q\, dr \\ = \frac{C_3^q}{\alpha+1} M_p^q(\frac{1}{2}, f). \qquad (7)$$

Similarly, (3) gives

$$\int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,g)\, dr \le \int_0^1 (1-r)^{\alpha+q\tilde{t}} \{C_2 M_p(\frac{1}{2}, f)\}^q\, dr \\ = \frac{C_2^q}{\alpha+q\tilde{t}+1} M_p^q(\frac{1}{2}, f). \qquad (8)$$

By (2) and (5)–(8), we have

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t}f)\, dr \le C_5 M_p^q(\frac{1}{2}, f) + C_6 \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr, \qquad (9)$$

where $C_5 = \frac{C_3^q C_4}{\alpha+1} + \frac{C_1 C_2^q C_4^2}{\alpha+q\tilde{t}+1}$ and $C_6 = C_1 C_4^2$. On the other hand,

$$\int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr \ge \int_{\frac{1}{2}}^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r,f)\, dr \\ \ge \int_{\frac{1}{2}}^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(\frac{1}{2}, f)\, dr = \frac{\left(\frac{1}{2}\right)^{\alpha+q\tilde{t}+1}}{\alpha+q\tilde{t}+1} M_p^q(\frac{1}{2}, f). \qquad (10)$$

By (9) and (10), we have

$$\int_0^1 (1-r)^{\alpha} M_p^q(r, \mathscr{R}_{s,t} f)\, dr \leq C \int_0^1 (1-r)^{\alpha+q\tilde{t}} M_p^q(r, f)\, dr,$$

where $C = 2^{\alpha+q\tilde{t}+1}(\alpha + q\tilde{t} + 1)C_5 + C_6$. This completes the proof.

Chapter 3 : The isomorphisms between generalized weighted Bergman spaces

3.1 Preliminaries

Proposition 25. *Suppose $\alpha \in (-1,\infty)$ and $p \in \mathbb{R}_+$. Then for any $f \in H(\mathbb{B}_n)$*

$$\frac{2^{-|\alpha|}}{2nc_\alpha}\|f\|^p_{A^p_\alpha(\mathbb{B}_n)} \le \int_0^1 (1-r)^\alpha M_p^p(r,f)\,dr \le \frac{2^{2n+\alpha+|\alpha|}}{2nc_\alpha}\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}.$$

Proof. Using Lemma 1.8 of [12], we have

$$\|f\|^p_{A^p_\alpha(\mathbb{B}_n)} = \int_{\mathbb{B}_n} |f|^p\, dv_\alpha = 2nc_\alpha \int_0^1 r^{2n-1}(1-r^2)^\alpha dr \int_{\mathbb{S}_n} |f(r\zeta)|^p d\sigma(\zeta)$$
$$= 2nc_\alpha \int_0^1 r^{2n-1}(1-r^2)^\alpha M_p^p(r,f)dr \le 2^{1+|\alpha|}nc_\alpha \int_0^1 (1-r)^\alpha M_p^p(r,f)dr.$$

On the other hand,

$$\int_0^1 (1-r)^\alpha M_p^p(r,f)\,dr = \int_0^{\frac{1}{2}} (1-r)^\alpha M_p^p(r,f)\,dr + \int_{\frac{1}{2}}^1 (1-r)^\alpha M_p^p(r,f)\,dr$$
$$\le 2^{\alpha+1}\int_{\frac{1}{2}}^1 (1-r)^\alpha M_p^p(r,f)\,dr$$
$$\le \frac{2^{2n+\alpha+|\alpha|}}{2nc_\alpha}\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}.$$

Hence we have

$$\frac{1}{2^{1+|\alpha|}nc_\alpha}\|f\|^p_{A^p_\alpha(\mathbb{B}_n)} \le \int_0^1 (1-r)^\alpha M_p^p(r,f)\,dr \le \frac{2^{2n+\alpha+|\alpha|}}{2nc_\alpha}\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}.$$

□

For any $\alpha \in \mathbb{R}$ and any $p \in \mathbb{R}_+$, we define

$$I_{\alpha,p} = \{k \in \mathbb{Z}_+ : pk+\alpha > -1\}, \quad \kappa_{\alpha,p} = \min I_{\alpha,p}.$$

It is clear that the three conditions $I_{\alpha,p} \ni 0$, $\kappa_{\alpha,p} = 0$ and $\alpha \in (-1,\infty)$ are mutually equivalent.

For any $k \in I_{\alpha,p}$ and any $f \in H(\mathbb{B}_n)$, we define the quantity $\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$ as follows.

$$\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = \left\{ |f(0)|^p + \|\mathscr{R}^k f\|^p_{A^p_{pk+\alpha}(\mathbb{B}_n)} \right\}^{\frac{1}{p}} \quad \text{if} \quad k \in \mathbb{N},$$

$$\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = \|f\|_{A^p_\alpha(\mathbb{B}_n)} \quad \text{if} \quad k = 0.$$

For any $k \in I_{\alpha,p}$, we define the subspace $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ of $H(\mathbb{B}_n)$ as follows.

$$\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \mathscr{R}^k f \in A^p_{pk+\alpha}(\mathbb{B}_n) \right\}.$$

Proposition 26. *Suppose $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$, $k \in I_{\alpha,p}$, $f \in H(\mathbb{B}_n)$, $g \in H(\mathbb{B}_n)$ and $c \in \mathbb{C}$. Then the followings hold.*

(i) $0 \leq \|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq \infty$

(ii) $\|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = 0 \Longleftrightarrow f = 0$ *on* $\mathbb{B}_n$

(iii) $\|cf\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = |c| \, \|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$

(iv) $\|f+g\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq \|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} + \|g\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$ *if* $1 \leq p < \infty$

(v) $\|f+g\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq \|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} + \|g\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$ *if* $0 < p \leq 1$.

Proof. All of these can be easily shown. □

Proposition 27. *Suppose $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$ and $k \in I_{\alpha,p}$. Then the followings hold.*

(i) *$\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ is a linear subspace of $H(\mathbb{B}_n)$.*

(ii) *$\|\cdot\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$ is a norm on the linear space $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ if $1 \leq p < \infty$.*

(iii) *$d_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}$ is a translation invariant metric on the linear space $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ if $0 < p \leq 1$, where for any $f \in H(\mathbb{B}_n)$ and any $g \in H(\mathbb{B}_n)$*

$$d_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}(f,g) = \|f-g\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}.$$

Proof. This is an immeadiate consequence of Proposition 26. □

Lemma 28. *Let $\alpha \in \mathbb{R}_+$, $p \in \mathbb{R}_+$ and $s \in \mathbb{C}$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+\frac{\alpha}{p} \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. Since $\frac{\alpha}{p} \in \mathbb{R}_+$, it follows from Theorem 2.19 of [12] that for any $f \in H(\mathbb{B}_n)$

$$C_1 \|f\|^p_{A^p(\mathbb{B}_n)} \le \int_{\mathbb{B}_n} \left|(1-|z|^2)^{\frac{\alpha}{p}} (\mathscr{R}^{s,\frac{\alpha}{p}} f)(z)\right|^p dv(z) \le C_2 \|f\|^p_{A^p(\mathbb{B}_n)} \tag{1}$$

where C_1 and C_2 are both positive constants depending only on α, p, s and n. By (1), $\mathscr{R}^{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p(\mathbb{B}_n)$ onto $A^p_\alpha(\mathbb{B}_n)$. Since $(\mathscr{R}^{s,\frac{\alpha}{p}})^{-1} = \mathscr{R}_{s,\frac{\alpha}{p}}$ on $H(\mathbb{B}_n)$, it implies that $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$. □

Lemma 29. *Let $\alpha \in (-1,0)$, $p \in \mathbb{R}_+$ and $s \in \mathbb{C}$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+\frac{\alpha}{p} \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. By Lemma 1 of [10],

$$\mathscr{R}_{s,\frac{\alpha}{p}} = \mathscr{R}^{s+\frac{\alpha}{p},-\frac{\alpha}{p}} \text{ on } H(\mathbb{B}_n). \tag{1}$$

Since $\alpha \in (-1,0)$ and $-\frac{\alpha}{p} \in \mathbb{R}_+$, it follows from Theorem 2.19 of [12] that for any $f \in H(\mathbb{B}_n)$

$$C_1 \|f\|^p_{A^p_\alpha(\mathbb{B}_n)} \le \int_{\mathbb{B}_n} \left|(1-|z|^2)^{-\frac{\alpha}{p}} (\mathscr{R}^{s+\frac{\alpha}{p},-\frac{\alpha}{p}} f)(z)\right|^p dv_\alpha(z) \le C_2 \|f\|^p_{A^p_\alpha(\mathbb{B}_n)} \tag{2}$$

where C_1 and C_2 are both positive constants depending only on α, p, s and n. By (2), $\mathscr{R}^{s+\frac{\alpha}{p},-\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$. This fact and (1) imply that $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$. □

Proposition 30. *Let $\alpha \in (-1,\infty)$, $p \in \mathbb{R}_+$ and $s \in \mathbb{C}$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+\frac{\alpha}{p} \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. If $\alpha = 0$, then $\mathscr{R}_{s,\frac{\alpha}{p}} = \mathscr{R}_{s,0}$ is the identity operator on $H(\mathbb{B}_n)$ and $A^p_\alpha(\mathbb{B}_n) = A^p(\mathbb{B}_n)$. Hence $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.

If $\alpha \neq 0$, then $\alpha \in \mathbb{R}_+$ or $\alpha \in (-1,0)$. And so, it follows from Lemma 28 or Lemma 29 that $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $A^p_\alpha(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$. □

Lemma 31. *Suppose $f \in H(\mathbb{B}_n)$, $p \in \mathbb{R}_+$, $q \in \mathbb{R}_+$, $\alpha \in (-1,\infty)$ and $k \in \mathbb{N}$. Then*

$$\int_0^1 (1-r)^\alpha M_p^q(r,f)\,dr$$
$$\leq C\left\{ \sum_{m\in\mathbb{Z}^n_+,|m|<k} |(D^m f)(0)|^q + \sum_{m\in\mathbb{Z}^n_+,|m|=k} \int_0^1 (1-r)^{kq+\alpha} M_p^q(r,D^m f)\,dr \right\},$$

where C is a positive constant depending only on p,q,α,k and n.

Proof. See §3 "The proof of Theorem 2" of [8]. □

Lemma 32. *Suppose $\alpha \in (-1,\infty)$ and $p \in \mathbb{R}_+$. Then for any $f \in H(\mathbb{B}_n)$,*

$$\int_{\mathbb{B}_n} |f-f(0)|^p\,dv_\alpha \leq C \int_{\mathbb{B}_n} \{(1-|z|^2)|(\mathscr{R}f)(z)|\}^p\,dv_\alpha(z)$$

where C is a positive constant depending only on α, p and n.

Proof. Put

$$I = \int_{\mathbb{B}_n} \{(1-|z|^2)|(\mathscr{R}f)(z)|\}^p\,dv_\alpha(z)$$
$$= 2nc_\alpha \int_0^1 r^{2n-1}(1-r^2)^{p+\alpha} M_p^p(r,\mathscr{R}f)\,dr. \tag{1}$$

By §6.4.4(2) of [6], for any $\zeta \in \mathbb{S}_n$ and $\lambda \in \mathbb{B}_1$, we have

$$(\mathscr{R}f)(\lambda\zeta) = \lambda (f_\zeta)'(\lambda), \tag{2}$$

where $f_\zeta(\lambda) = f(\lambda\zeta)$ is a slice function of f. By Lemma 1.10 of [12] and the above (2), for $r \in (0,1)$, we have

$$M_p^p(r,\mathscr{R}f) = \int_{\mathbb{S}_n} |(\mathscr{R}f)(r\zeta)|^p d\sigma(\zeta) = \int_{\mathbb{S}_n} d\sigma(\zeta)\cdot\frac{1}{2\pi}\int_0^{2\pi} |(\mathscr{R}f)(re^{i\theta}\zeta)|^p d\theta$$
$$= \int_{\mathbb{S}_n} d\sigma(\zeta)\cdot\frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta}(f_\zeta)'(re^{i\theta})|^p d\theta$$
$$= r^p \int_{\mathbb{S}_n} d\sigma(\zeta)\cdot\frac{1}{2\pi}\int_0^{2\pi} |(f_\zeta)'(re^{i\theta})|^p d\theta = r^p \int_{\mathbb{S}_n} M_p^p(r,(f_\zeta)')d\sigma(\zeta). \tag{3}$$

By (1) and (3), we obtain

$$\begin{aligned}
I &= 2nc_\alpha \int_0^1 r^{2n-1+p}(1-r^2)^{p+\alpha}\,dr \int_{\mathbb{S}_n} M_p^p(r,(f_\zeta)')\,d\sigma(\zeta)\\
&\geq 2nc_\alpha \int_{\mathbb{S}_n} d\sigma(\zeta) \int_{\frac{1}{2}}^1 r^{2n-1+p}(1-r^2)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr\\
&\geq \frac{2nc_\alpha}{2^{2n+p}} \int_{\mathbb{S}_n} d\sigma(\zeta) \int_{\frac{1}{2}}^1 (1-r)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr. \qquad (4)
\end{aligned}$$

On the other hand, for $\zeta \in \mathbb{S}_n$, we get

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (1-r)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr \leq \int_0^{\frac{1}{2}} (1-r)^{p+\alpha} M_p^p(\frac{1}{2},(f_\zeta)')\,dr\\
&= M_p^p(\frac{1}{2},(f_\zeta)')\,\frac{1-(\frac{1}{2})^{p+\alpha+1}}{p+\alpha+1}\\
&= M_p^p(\frac{1}{2},(f_\zeta)')\,\frac{1-(\frac{1}{2})^{p+\alpha+1}}{p+\alpha+1}\,\frac{p+\alpha+1}{(\frac{1}{2})^{p+\alpha+1}} \int_{\frac{1}{2}}^1 (1-r)^{p+\alpha}\,dr\\
&\leq (2^{p+\alpha+1}-1) \int_{\frac{1}{2}}^1 (1-r)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr.
\end{aligned}$$

Hence we have

$$\int_0^1 (1-r)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr \leq 2^{p+\alpha+1} \int_{\frac{1}{2}}^1 (1-r)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr \qquad (5)$$

for $\zeta \in \mathbb{S}_n$. By (4) and (5), we get

$$I \geq \frac{2nc_\alpha}{2^{2n+2p+\alpha+1}} \int_{\mathbb{S}_n} d\sigma(\zeta) \int_0^1 (1-r)^{p+\alpha} M_p^p(r,(f_\zeta)')\,dr. \qquad (6)$$

By Lemma 31, for any $g \in H(\mathbb{B}_1)$

$$\int_0^1 (1-r)^\alpha M_p^p(r,g-g(0))\,dr \leq C_1 \int_0^1 (1-r)^{p+\alpha} M_p^p(r,g')\,dr \qquad (7)$$

where C_1 is a positive constant depending only on α and p. By (6) and (7), we have

$$I \geq C_2 \int_{\mathbb{S}_n} d\sigma(\zeta) \int_0^1 (1-r)^\alpha M_p^p(r,f_\zeta - f_\zeta(0))\,dr, \qquad (8)$$

where

$$C_2 = \frac{2nc_\alpha}{C_1 2^{2n+2p+\alpha+1}}.$$

On the other hand, for $\zeta \in \mathbb{S}_n$, we obtain

$$\begin{aligned}
&\int_0^1 (1-r)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr \geq \int_{\frac{1}{2}}^1 (1-r)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr \\
&\geq \int_{\frac{1}{2}}^1 r^{2n-1} 2^{-|\alpha|} (1-r^2)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr \\
&= 2^{-|\alpha|} \int_{\frac{1}{2}}^1 r^{2n-1} (1-r^2)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr. \qquad (9)
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
&\int_0^{\frac{1}{2}} r^{2n-1} (1-r^2)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr \\
&\leq \int_0^{\frac{1}{2}} r^{2n-1} (1-r^2)^\alpha M_p^p(\frac{1}{2}, f_\zeta - f_\zeta(0))\, dr \\
&= M_p^p(\frac{1}{2}, f_\zeta - f_\zeta(0)) \int_0^{\frac{1}{2}} r^{2n-1} (1-r^2)^\alpha\, dr \\
&= C_3 M_p^p(\frac{1}{2}, f_\zeta - f_\zeta(0)) \int_{\frac{1}{2}}^1 r^{2n-1} (1-r^2)^\alpha\, dr \\
&\leq C_3 \int_{\frac{1}{2}}^1 r^{2n-1} (1-r^2)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr
\end{aligned}$$

where

$$C_3 = \frac{\int_0^{\frac{1}{2}} r^{2n-1} (1-r^2)^\alpha\, dr}{\int_{\frac{1}{2}}^1 r^{2n-1} (1-r^2)^\alpha\, dr} \in \mathbb{R}_+.$$

Hence we have

$$\begin{aligned}
&\int_0^1 r^{2n-1} (1-r^2)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr \\
&\leq (1+C_3) \int_{\frac{1}{2}}^1 r^{2n-1} (1-r^2)^\alpha M_p^p(r, f_\zeta - f_\zeta(0))\, dr \qquad (10)
\end{aligned}$$

for $\zeta \in \mathbb{S}_n$. By (8)–(10), we obtain

$$\begin{aligned} I &\geq \frac{C_2 2^{-|\alpha|}}{1+C_3}\int_{\mathbb{S}_n} d\sigma(\zeta)\int_0^1 r^{2n-1}(1-r^2)^{\alpha} M_p^p(r, f_\zeta - f_\zeta(0))\,dr \\ &= \frac{C_2 2^{-|\alpha|}}{1+C_3}\int_{\mathbb{S}_n} d\sigma(\zeta)\int_0^1 r^{2n-1}(1-r^2)^{\alpha}\,dr\cdot\frac{1}{2\pi}\int_0^{2\pi}|f_\zeta(re^{i\theta})-f_\zeta(0)|^p d\theta \\ &= \frac{C_2 2^{-|\alpha|}}{2nc_\alpha(1+C_3)}\int_{\mathbb{B}_n}|f-f(0)|^p dv_\alpha. \end{aligned} \tag{11}$$

By (11) and (1), we have

$$\int_{\mathbb{B}_n}|f-f(0)|^p dv_\alpha \leq \frac{2^{1+|\alpha|}nc_\alpha(1+C_3)}{C_2}\int_{\mathbb{B}_n}\{(1-|z|^2)|(\mathscr{R}f)(z)|\}^p\,dv_\alpha(z).$$

This completes the proof. □

Lemma 33. *Suppose $\alpha \in \mathbb{C}$ and $\tilde{\alpha} = \operatorname{Re}\alpha \in (-1,\infty)$. Then for any $f \in L^1(\mathbb{B}_n, dv_{\tilde{\alpha}})$*

$$\int_{\mathbb{B}_n} f\circ\varphi\,dv_\alpha = \int_{\mathbb{B}_n} f(z)\frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}}dv_\alpha(z)$$

where φ is any automorphism of $\mathbb{B}_n$ and $a = \varphi(0)$.

Proof. The same proof as that of Proposition 1.13 of [12] (where the parameter α is real) also holds in this complex case. □

For $f \in H(\mathbb{B}_n)$, we write $\tilde{\nabla}f(z) = \nabla(f\circ\varphi_z)(0)$, where φ_z is the biholomorphic mapping of $\mathbb{B}_n$ that interchanges 0 and z.

Lemma 34. *Suppose $\alpha \in (-1,\infty)$ and $p \in \mathbb{R}_+$. Then for any $f \in H(\mathbb{B}_n)$,*

$$\int_{\mathbb{B}_n}|\tilde{\nabla}f|^p\,dv_\alpha \leq C\int_{\mathbb{B}_n}|f|^p\,dv_\alpha,$$

where C is a positive constant depending only on α, p and n.

Proof. (cf Theorem 2.16 of [12].)
Fix $f \in H(\mathbb{B}_n)$. Pick $\beta \in (\alpha,\infty)$. By Proposition 2.4 of [12], for any $g \in H(\mathbb{B}_n)$ we have

$$|(\nabla g)(0)| \leq C_1\|g\|_{A_\beta^p(\mathbb{B}_n)}, \tag{1}$$

where C_1 is a positive constant depending only on β, p and n. For any $z \in \mathbb{B}_n$, (1) and Lemma 33 give

$$|(\tilde{\nabla} f)(z)|^p = |\nabla(f \circ \varphi_z)(0)|^p \leq C_1^p \|f \circ \varphi_z\|^p_{A^p_\beta(\mathbb{B}_n)} = C_1^p \int_{\mathbb{B}_n} |f \circ \varphi_z|^p dv_\beta$$
$$= C_1^p \int_{\mathbb{B}_n} |f(w)|^p \frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle w,z\rangle|^{2(n+1+\beta)}} dv_\beta(w).$$

Hence we have

$$\int_{\mathbb{B}_n} |(\tilde{\nabla} f)(z)|^p dv_\alpha(z) \leq C_1^p \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} |f(w)|^p \frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle w,z\rangle|^{2(n+1+\beta)}} dv_\beta(w)$$
$$= C_1^p \int_{\mathbb{B}_n} |f(w)|^p dv_\beta(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle w,z\rangle|^{2(n+1+\beta)}} dv_\alpha(z).$$

By Proposition 1.4.10 of [6], for any $w \in \mathbb{B}_n$, we get

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\beta}}{|1-\langle w,z\rangle|^{2(n+1+\beta)}} dv_\alpha(z) \leq \frac{c_\alpha C_2}{(1-|w|^2)^{\beta-\alpha}},$$

where C_2 is a positive constant depending only on α, β and n. Hence we obtain

$$\int_{\mathbb{B}_n} |\tilde{\nabla} f|^p dv_\alpha \leq C_1^p C_2 c_\beta \int_{\mathbb{B}_n} |f|^p dv_\alpha.$$

This completes the proof. □

Lemma 35. *Suppose $\alpha \in (-1,\infty)$ and $p \in \mathbb{R}_+$. Then for any $f \in H(\mathbb{B}_n)$*

$$C_1 \int_{\mathbb{B}_n} |f - f(0)|^p dv_\alpha \leq \int_{\mathbb{B}_n} \{(1-|z|^2)|(\mathscr{R} f)(z)|\}^p dv_\alpha(z)$$
$$\leq \int_{\mathbb{B}_n} \{(1-|z|^2)|(\nabla f)(z)|\}^p dv_\alpha(z) \leq \int_{\mathbb{B}_n} |\tilde{\nabla} f|^p dv_\alpha \leq C_2 \int_{\mathbb{B}_n} |f - f(0)|^p dv_\alpha,$$

where C_1 and C_2 are both positive constants depending only on α, p and n.

Proof. (cf Theorem 2.16 of [12].)
By Proposition 2.14 of [12], for any $f \in H(\mathbb{B}_n)$ and any $z \in \mathbb{B}_n$,

$$(1-|z|^2)|(\mathscr{R} f)(z)| \leq (1-|z|^2)|(\nabla f)(z)| \leq |(\tilde{\nabla} f)(z)|.$$

Hence the present lemma follows from Lemma 32 and Lemma 34. □

Lemma 36. *Suppose $\alpha \in (-1, \infty)$, $p \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Then for any $f \in H(\mathbb{B}_n)$*

$$C_1 \int_{\mathbb{B}_n} |f - f(0)|^p \, dv_\alpha \le \int_{\mathbb{B}_n} \{(1-|z|^2)^k |(\mathscr{R}^k f)(z)|\}^p \, dv_\alpha(z)$$
$$\le C_2 \int_{\mathbb{B}_n} |f - f(0)|^p \, dv_\alpha,$$

where C_1 and C_2 are both positive constants depending only on α, p, k and n.

Proof. (cf Theorem 2.16 of [12].)
When $k = 1$, the lemma is true by Lemma 35. Assume that $k \ge 2$ and the assertion is true for $k - 1$. Then for any $f \in H(\mathbb{B}_n)$,

$$C_1 \int_{\mathbb{B}_n} |f - f(0)|^p \, dv_{p+\alpha} \le \int_{\mathbb{B}_n} \{(1-|z|^2)^{k-1} |(\mathscr{R}^{k-1} f)(z)|\}^p \, dv_{p+\alpha}(z)$$
$$\le C_2 \int_{\mathbb{B}_n} |f - f(0)|^p \, dv_{p+\alpha}, \tag{1}$$

where C_1 and C_2 are both positive constants depending only on α, p, k and n. Moreover, Lemma 35 gives for any $f \in H(\mathbb{B}_n)$,

$$C_3 \int_{\mathbb{B}_n} |f - f(0)|^p \, dv_\alpha \le \int_{\mathbb{B}_n} \{(1-|z|^2) |(\mathscr{R} f)(z)|\}^p \, dv_\alpha(z)$$
$$\le C_4 \int_{\mathbb{B}_n} |f - f(0)|^p \, dv_\alpha, \tag{2}$$

where C_3 and C_4 are both positive constants depending only on α, p and n.

Let $f \in H(\mathbb{B}_n)$. Then $\mathscr{R} f \in H(\mathbb{B}_n)$ and $(\mathscr{R} f)(0) = 0$. It follows from (1) that

$$C_1 \int_{\mathbb{B}_n} \{(1-|z|^2) |(\mathscr{R} f)(z)|\}^p \, dv_\alpha(z) = \frac{C_1 c_\alpha}{c_{p+\alpha}} \int_{\mathbb{B}_n} |\mathscr{R} f - (\mathscr{R} f)(0)|^p \, dv_{p+\alpha}$$
$$\le \frac{c_\alpha}{c_{p+\alpha}} \int_{\mathbb{B}_n} \{(1-|z|^2)^{k-1} |(\mathscr{R}^{k-1} \mathscr{R} f)(z)|\}^p \, dv_{p+\alpha}(z)$$
$$\le \frac{C_2 c_\alpha}{c_{p+\alpha}} \int_{\mathbb{B}_n} |\mathscr{R} f - (\mathscr{R} f)(0)|^p \, dv_{p+\alpha}$$
$$= C_2 \int_{\mathbb{B}_n} \{(1-|z|^2) |(\mathscr{R} f)(z)|\}^p \, dv_\alpha(z). \tag{3}$$

Since

$$\frac{c_\alpha}{c_{p+\alpha}} \int_{\mathbb{B}_n} \{(1-|z|^2)^{k-1} |(\mathscr{R}^{k-1} \mathscr{R} f)(z)|\}^p \, dv_{p+\alpha}(z)$$
$$= \int_{\mathbb{B}_n} \{(1-|z|^2)^k |(\mathscr{R}^k f)(z)|\}^p \, dv_\alpha(z),$$

by (2) and (3) we obtain

$$C_1C_3\int_{\mathbb{B}_n}|f-f(0)|^p\,dv_\alpha \le \int_{\mathbb{B}_n}\{(1-|z|^2)^k|(\mathscr{R}^k f)(z)|\}^p\,dv_\alpha(z)$$
$$\le C_2C_4\int_{\mathbb{B}_n}|f-f(0)|^p\,dv_\alpha.$$

This completes the proof. □

Lemma 37. *Let $\alpha\in(-1,\infty)$, $p\in\mathbb{R}_+$, $k\in I_{\alpha,p}\cap\mathbb{N}$ and $s\in\mathbb{C}$. Suppose $n+s\notin(-\mathbb{N})$ and $n+s+\frac{\alpha}{p}\notin(-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. By Proposition 30, for any $f\in H(\mathbb{B}_n)$, we have

$$C_1\|f\|^p_{A^p_\alpha(\mathbb{B}_n)} \le \|\mathscr{R}_{s,\frac{\alpha}{p}}f\|^p_{A^p(\mathbb{B}_n)} \le C_2\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}, \tag{1}$$

where C_1 and C_2 are both positive constants depending only on α,p,s and n. Moreover, Lemma 36 gives for $f\in H(\mathbb{B}_n)$, we obtain

$$C_3\int_{\mathbb{B}_n}|f-f(0)|^p\,dv_\alpha \le \int_{\mathbb{B}_n}\{(1-|z|^2)^k|(\mathscr{R}^k f)(z)|\}^p\,dv_\alpha(z)$$
$$\le C_4\int_{\mathbb{B}_n}|f-f(0)|^p\,dv_\alpha,$$

where C_3 and C_4 are both positive constants depending only on α,p,k and n. Hence we get

$$\|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = |f(0)|^p+\|\mathscr{R}^k f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)}$$
$$\ge |f(0)|^p+\frac{C_3c_{kp+\alpha}}{c_\alpha}\|f-f(0)\|^p_{A^p_\alpha(\mathbb{B}_n)} \ge \frac{C_5}{\gamma_p}\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}, \tag{2}$$

where $C_5=\min\{1,\frac{C_3c_{kp+\alpha}}{c_\alpha}\}$ and $\gamma_p=\max\{1,2^{p-1}\}$. On the other hand, for $f\in H(\mathbb{B}_n)$, we get

$$\|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = |f(0)|^p+\|\mathscr{R}^k f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)}$$
$$\le \left(1+2\frac{\gamma_pC_4c_{kp+\alpha}}{c_\alpha}\right)\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}. \tag{3}$$

By (1)–(3), we have

$$C_6 \|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq \|\mathscr{R}_{s,\frac{\alpha}{p}} f\|^p_{A^p(\mathbb{B}_n)} \leq C_7 \|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)}, \tag{4}$$

where $C_6 = \dfrac{C_1 c_\alpha}{c_\alpha + 2\gamma_p C_4 c_{kp+\alpha}}$ and $C_7 = \dfrac{\gamma_p C_2}{C_5}$. Hence $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$. □

Lemma 38. *Let $\alpha \in (-\infty, 0)$, $p \in \mathbb{R}_+$, $k \in I_{\alpha,p} \cap \mathbb{N}$ and $s \in \mathbb{C}$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+\frac{\alpha}{p} \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. By Lemma 1 of [10], we get

$$\mathscr{R}_{s,\frac{\alpha}{p}} = \mathscr{R}^{s+\frac{\alpha}{p},-\frac{\alpha}{p}} \text{ on } H(\mathbb{B}_n).$$

Since $kp+\alpha \in (-1,\infty)$ and $-\frac{\alpha}{p} \in \mathbb{R}_+$, it follows from Theorem 2.19 of [12] that for any $f \in H(\mathbb{B}_n)$

$$\begin{aligned} C_1 \|f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} &\leq \int_{\mathbb{B}_n} \left|(1-|z|^2)^{-\frac{\alpha}{p}} (\mathscr{R}^{s+\frac{\alpha}{p},-\frac{\alpha}{p}} f)(z)\right|^p dv_{kp+\alpha}(z) \\ &\leq C_2 \|f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)}, \end{aligned}$$

where C_1 and C_2 are both positive constants depending only on α, p, s and n. Hence we have

$$\begin{aligned} C_1 \|f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} &\leq c_{kp+\alpha} \int_{\mathbb{B}_n} \left|(1-|z|^2)^k (\mathscr{R}^{s+\frac{\alpha}{p},-\frac{\alpha}{p}} f)(z)\right|^p dv(z) \\ &\leq C_2 \|f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)}. \end{aligned}$$

Moreover, Lemma 36 gives for $f \in H(\mathbb{B}_n)$,

$$\begin{aligned} C_3 \int_{\mathbb{B}_n} |f-f(0)|^p dv &\leq \int_{\mathbb{B}_n} \{(1-|z|^2)^k |(\mathscr{R}^k f)(z)|\}^p dv(z) \\ &\leq C_4 \int_{\mathbb{B}_n} |f-f(0)|^p dv, \end{aligned}$$

where C_3 and C_4 are both positive constants depending only on p,k and n. It follows that for $f \in H(\mathbb{B}_n)$,

$$\|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = |f(0)|^p + \|\mathscr{R}^k f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} \geq \frac{C_5}{\gamma_p} \|\mathscr{R}_{s,\frac{\alpha}{p}} f\|^p_{A^p(\mathbb{B}_n)},$$

where $C_5 = \min\{1, \frac{C_3 c_{kp+\alpha}}{C_2}\}$ and $\gamma_p = \max\{1, 2^{p-1}\}$. On the other hand, for $f \in H(\mathbb{B}_n)$, we have

$$\begin{aligned}
\|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} &= |f(0)|^p + \|\mathscr{R}^k f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} \\
&\leq \left(1 + \frac{\gamma_p C_4 c_{kp+\alpha}}{C_1}\right) |(\mathscr{R}_{s,\frac{\alpha}{p}} f)(0)|^p + \frac{\gamma_p C_4 c_{kp+\alpha}}{C_1} \|\mathscr{R}_{s,\frac{\alpha}{p}} f\|^p_{A^p(\mathbb{B}_n)} \\
&\leq \left(1 + \frac{2\gamma_p C_4 c_{kp+\alpha}}{C_1}\right) \|\mathscr{R}_{s,\frac{\alpha}{p}} f\|^p_{A^p(\mathbb{B}_n)}.
\end{aligned}$$

Hence we obtain

$$C_6 \|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq \|\mathscr{R}_{s,\frac{\alpha}{p}} f\|^p_{A^p(\mathbb{B}_n)} \leq C_7 \|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)},$$

where $C_6 = \frac{C_1}{C_1 + 2\gamma_p C_4 c_{kp+\alpha}}$ and $C_7 = \frac{\gamma_p}{C_5}$. This implies that $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$. □

Proposition 39. *Let $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$, $k \in I_{\alpha,p}$ and $s \in \mathbb{C}$. Suppose $n + s \notin (-\mathbb{N})$ and $n + s + \frac{\alpha}{p} \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. If $k = 0$, then $\alpha \in (-1, \infty)$ and $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n) = A^p_\alpha(\mathbb{B}_n)$. Hence the assertion follows from Proposition 30. In the case $k \in \mathbb{N}$, the assertion follows from Lemma 37 and Lemma 38. □

Corollary 40. *Suppose $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$, $k \in I_{\alpha,p}$ and $k' \in I_{\alpha,p}$. Then $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n) = \mathscr{A}^p_{\alpha,k'}(\mathbb{B}_n)$ and for any $f \in H(\mathbb{B}_n)$*

$$C_1 \|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} \leq \|f\|_{\mathscr{A}^p_{\alpha,k'}(\mathbb{B}_n)} \leq C_2 \|f\|_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)},$$

where C_1 and C_2 are both positive constants depending only on α, p, k, k' and n.

Proof. It follows from Proposition 39. □

From Corollary 40, we understand that $\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)$ does not depend on how to get k as space for any $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$, $k \in I_{\alpha,p}$. We can define the generalised weighted Bergman space on $\mathbb{B}_n$. By Corollary 40 and the definition of $\mathscr{A}_\alpha^p(\mathbb{B}_n)$, for $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$ and $k \in I_{\alpha,p}$, we have

$$\mathscr{A}_{\alpha,k}^p(\mathbb{B}_n) = \mathscr{A}_\alpha^p(\mathbb{B}_n)$$

and for $f \in H(\mathbb{B}_n)$

$$C_1\|f\|_{\mathscr{A}_\alpha^p(\mathbb{B}_n)} \leq \|f\|_{\mathscr{A}_{\alpha,k}^p(\mathbb{B}_n)} \leq C_2\|f\|_{\mathscr{A}_\alpha^p(\mathbb{B}_n)},$$

where C_1 and C_2 are both positive constants depending only on α, p, k and n.

Proposition 41. (cf Theorem 10 of [10].) *Let $\alpha \in \mathbb{R}$, $p \in \mathbb{R}_+$ and $s \in \mathbb{C}$. Suppose $n+s \notin (-\mathbb{N})$ and $n+s+\frac{\alpha}{p} \notin (-\mathbb{N})$. Then $\mathscr{R}_{s,\frac{\alpha}{p}}$ is an invertible bounded linear operator from $\mathscr{A}_\alpha^p(\mathbb{B}_n)$ onto $A^p(\mathbb{B}_n)$.*

Proof. By the above definition of the space $\mathscr{A}_\alpha^p(\mathbb{B}_n)$, this proposition is just only a restatement of Proposition 39. □

Corollary 42. *Let $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}_+$. Then the followings hold.*

(i) $\mathscr{A}_\alpha^p(\mathbb{B}_n) = (\mathscr{A}_\alpha^p(\mathbb{B}_n), \|\cdot\|_{\mathscr{A}_\alpha^p(\mathbb{B}_n)})$ *is a Banach space if* $1 \leq p < \infty$.

(ii) $\mathscr{A}_\alpha^p(\mathbb{B}_n) = (\mathscr{A}_\alpha^p(\mathbb{B}_n), d_{\mathscr{A}_\alpha^p(\mathbb{B}_n)})$ *is an F-space if* $0 < p \leq 1$.

Proof. This is clear from Proposition 27 and Proposition 41. □

3.2 Proof of Theorem 4

Proof of Theorem 4. If $t = 0$, then $\mathscr{R}_{s,t} = \mathscr{R}_{s,0}$ is the identity operator on $H(\mathbb{B}_n)$ and $\mathscr{A}_\alpha^p(\mathbb{B}_n) = \mathscr{A}_\beta^p(\mathbb{B}_n)$. Hence $\mathscr{R}_{s,t}$ is an invertible bounded linear operator from $\mathscr{A}_\alpha^p(\mathbb{B}_n)$ onto $\mathscr{A}_\beta^p(\mathbb{B}_n)$.

Choose a $k \in \mathbb{N} \cap I_{\alpha,p} \cap I_{\beta,p}$. Then $\mathscr{A}_\alpha^p(\mathbb{B}_n) = \mathscr{A}_{\alpha,k}^p(\mathbb{B}_n)$, $\mathscr{A}_\beta^p(\mathbb{B}_n) = \mathscr{A}_{\beta,k}^p(\mathbb{B}_n)$ and for any $f \in H(\mathbb{B}_n)$

$$C_1\|f\|_{\mathscr{A}_\alpha^p(\mathbb{B}_n)} \leq \|f\|_{\mathscr{A}_{\alpha,k}^p(\mathbb{B}_n)} \leq C_2\|f\|_{\mathscr{A}_\alpha^p(\mathbb{B}_n)}, \tag{1}$$

$$C_3\|f\|_{\mathscr{A}_\beta^p(\mathbb{B}_n)} \leq \|f\|_{\mathscr{A}_{\beta,k}^p(\mathbb{B}_n)} \leq C_4\|f\|_{\mathscr{A}_\beta^p(\mathbb{B}_n)}, \tag{2}$$

where C_1 and C_2 are both positive constants depending only on α, p, k and n. Similarly, C_3 and C_4 are both positive constants depending only on β, p, k and n.

We now consider the case $t > 0$. Since $kp+\beta > -1$ and $kp+\beta+tp = kp+\alpha$, Theorem 2 and Theorem 3 give for $f \in H(\mathbb{B}_n)$

$$\int_0^1 (1-r)^{kp+\alpha} M_p^p(r, \mathscr{R}^{s,t} f) dr \le C_5 \int_0^1 (1-r)^{kp+\beta} M_p^p(r,f) dr, \tag{3}$$

$$\int_0^1 (1-r)^{kp+\beta} M_p^p(r, \mathscr{R}_{s,t} f) dr \le C_6 \int_0^1 (1-r)^{kp+\alpha} M_p^p(r,f) dr, \tag{4}$$

where C_5 and C_6 are both positive constants depending only on α, β, p, s and n. Moreover, Proposition 25 gives for $f \in H(\mathbb{B}_n)$

$$\begin{aligned} \frac{2^{-|kp+\alpha|}}{2nc_{kp+\alpha}} \|f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} &\le \int_0^1 (1-r)^{kp+\alpha} M_p^p(r,f) dr \\ &\le \frac{2^{2n+|kp+\alpha|+kp+\alpha}}{2nc_{kp+\alpha}} \|f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{2^{-|kp+\beta|}}{2nc_{kp+\beta}} \|f\|^p_{A^p_{kp+\beta}(\mathbb{B}_n)} &\le \int_0^1 (1-r)^{kp+\beta} M_p^p(r,f) dr \\ &\le \frac{2^{2n+|kp+\beta|+kp+\beta}}{2nc_{kp+\beta}} \|f\|^p_{A^p_{kp+\beta}(\mathbb{B}_n)}. \end{aligned} \tag{6}$$

It follows from (5),(4) and (6) that for $f \in H(\mathbb{B}_n)$

$$\begin{aligned} &\|f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} = |f(0)|^p + \|\mathscr{R}^k f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} \\ &\ge |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{2n+|kp+\alpha|+kp+\alpha}} \int_0^1 (1-r)^{kp+\alpha} M_p^p(r, \mathscr{R}^k f) dr \\ &\ge |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{2n+|kp+\alpha|+kp+\alpha}} \cdot \frac{1}{C_6} \int_0^1 (1-r)^{kp+\beta} M_p^p(r, \mathscr{R}_{s,t} \mathscr{R}^k f) dr \\ &= |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{2n+|kp+\alpha|+kp+\alpha}} \cdot \frac{1}{C_6} \int_0^1 (1-r)^{kp+\beta} M_p^p(r, \mathscr{R}^k \mathscr{R}_{s,t} f) dr \\ &\ge |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{2n+|kp+\alpha|+kp+\alpha}} \cdot \frac{1}{C_6} \cdot \frac{2^{-|kp+\beta|}}{2nc_{kp+\beta}} \|\mathscr{R}^k \mathscr{R}_{s,t} f\|^p_{A^p_{kp+\beta}(\mathbb{B}_n)} \\ &= |(\mathscr{R}_{s,t} f)(0)|^p + \frac{2nc_{kp+\alpha}}{2^{2n+|kp+\alpha|+kp+\alpha}} \cdot \frac{1}{C_6} \cdot \frac{2^{-|kp+\beta|}}{2nc_{kp+\beta}} \|\mathscr{R}^k \mathscr{R}_{s,t} f\|^p_{A^p_{kp+\beta}(\mathbb{B}_n)} \\ &\ge C_7 \|\mathscr{R}_{s,t} f\|^p_{\mathscr{A}^p_{\beta,k}(\mathbb{B}_n)}, \end{aligned} \tag{7}$$

where

$$C_7 = \min\left\{1, \frac{2nc_{kp+\alpha}}{2^{2n+|kp+\alpha|+kp+\alpha}} \cdot \frac{1}{C_6} \cdot \frac{2^{-|kp+\beta|}}{2nc_{kp+\beta}}\right\}.$$

On the other hand, it follows from (5),(3) and (6) that for $f \in H(\mathbb{B}_n)$

$$\begin{aligned}
\|\mathscr{R}^{s,t}f\|^p_{\mathscr{A}^p_{\alpha,k}(\mathbb{B}_n)} &= |(\mathscr{R}^{s,t}f)(0)|^p + \|\mathscr{R}^k\mathscr{R}^{s,t}f\|^p_{A^p_{kp+\alpha}(\mathbb{B}_n)} \\
&\leq |(\mathscr{R}^{s,t}f)(0)|^p + \frac{2nc_{kp+\alpha}}{2^{-|kp+\alpha|}}\int_0^1 (1-r)^{kp+\alpha}M_p^p(r, \mathscr{R}^k\mathscr{R}^{s,t}f)dr \\
&= |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{-|kp+\alpha|}}\int_0^1 (1-r)^{kp+\alpha}M_p^p(r, \mathscr{R}^{s,t}\mathscr{R}^k f)dr \\
&\leq |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{-|kp+\alpha|}}C_5\int_0^1 (1-r)^{kp+\beta}M_p^p(r, \mathscr{R}^k f)dr \\
&\leq |f(0)|^p + \frac{2nc_{kp+\alpha}}{2^{-|kp+\alpha|}}C_5\frac{2^{2n+|kp+\beta|+kp+\beta}}{2nc_{kp+\beta}}\|\mathscr{R}^k f\|_{A^p_{kp+\beta}(\mathbb{B}_n)} \\
&\leq C_8\|f\|^p_{\mathscr{A}^p_{\beta,k}(\mathbb{B}_n)},
\end{aligned} \tag{8}$$

where

$$C_8 = \max\left\{1, \frac{C_5 c_{kp+\alpha}2^{2n+|kp+\beta|+kp+\beta+|kp+\alpha|}}{c_{kp+\beta}}\right\}.$$

By (2),(7) and (1), for $f \in H(\mathbb{B}_n)$, we have

$$\|\mathscr{R}_{s,t}f\|_{\mathscr{A}^p_\beta(\mathbb{B}_n)} \leq \frac{C_2}{C_3}\left(\frac{1}{C_7}\right)^{\frac{1}{p}}\|f\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)}. \tag{9}$$

By (1),(8) and (2), for any $f \in H(\mathbb{B}_n)$, we have

$$\|\mathscr{R}^{s,t}f\|_{\mathscr{A}^p_\alpha(\mathbb{B}_n)} \leq \frac{C_4}{C_1}(C_8)^{\frac{1}{p}}\|f\|_{\mathscr{A}^p_\beta(\mathbb{B}_n)}. \tag{10}$$

(9) implies that $\mathscr{R}_{s,t}$ is a bounded linear operator from $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ into $\mathscr{A}^p_\beta(\mathbb{B}_n)$. By (10), $\mathscr{R}^{s,t}$ is a bounded linear operator from $\mathscr{A}^p_\beta(\mathbb{B}_n)$ into $\mathscr{A}^p_\alpha(\mathbb{B}_n)$. Since $(\mathscr{R}^{s,t})^{-1} = \mathscr{R}_{s,t}$ on $H(\mathbb{B}_n)$, these two facts show that $\mathscr{R}_{s,t}$ is an invertible bounded linear operator from $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ onto $\mathscr{A}^p_\beta(\mathbb{B}_n)$.

Next we consider the case $t < 0$. Since $\frac{\beta-\alpha}{p} = -t > 0$, the previous case gives that $\mathscr{R}_{s+t,-t}$ is an invertible bounded linear operator from $\mathscr{A}^p_\beta(\mathbb{B}_n)$ onto $\mathscr{A}^p_\alpha(\mathbb{B}_n)$.

By Lemma 1 of [10], it holds that $\mathscr{R}_{s+t,-t} = \mathscr{R}^{(s+t)+(-t),-(-t)} = \mathscr{R}^{s,t} = (\mathscr{R}_{s,t})^{-1}$ on $H(\mathbb{B}_n)$. Hence $\mathscr{R}_{s,t}$ is an invertible bounded linear operator from $\mathscr{A}^p_\alpha(\mathbb{B}_n)$ onto $\mathscr{A}^p_\beta(\mathbb{B}_n)$. □

References

[1] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl., **38** (1972), 746–765.

[2] Y. Matsugu and T. Yamada, On Stoll-Shi's theorem concerning the fractional derivatives of holomorphic functions, Far East J. Math. Sci., **32** (2009), 359–386.

[3] Y. Matsugu and T. Takeshima, A complete form of the Stoll-Shi's theorem concerning the fractional integrals of holomorphic functions, Far East J. Math. Sci. **44** (2010), 197–218.

[4] Y. Matsugu and T. Takeshima, Pointwise multipliers from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$, Far East J. Math. Sci. **58** (2011), 157–172.

[5] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[6] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[7] J. Shi, On the rate of growth of the means M_p of holomorphic and pluriharmonic functions on bounded symmetric domains of $\mathbb{C}^n$, J. Math. Anal. Appl., **126** (1987), 161–175.

[8] J. Shi, Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of $\mathbb{C}^n$, Trans. Amer. Math. Soc., **328** (1991), 619–632.

[9] R. Zhao, Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces, Ann. Acad. Sci. Fenn. Math., **29** (2004), 139–150.

[10] R. Zhao and K. Zhu, *Theory of Bergman Spaces in the the Unit Ball of* $\mathbb{C}^n$, Mémoires de la SMF 115, 2008.

[11] K. Zhu, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math. 23(1993), 1143–1177.

[12] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.