The 1-type Model Structure on the Category of Small Categories Related to Coverings

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Abstract

We construct a model structure on the category of small categories which is closely related to coverings and fundamental groups of small categories. It has morphisms inducing isomorphisms on fundamental groups as weak equivalences and categories fibered and cofibered in groupoids as fibrations. The class of fibrant objects in this model category is the class of groupoids, and coverings are characterized as fibrations whose fibers are all discrete. This is Quillen equivalent to the category of simplicial sets and spaces with the 1-type model structure, and the category of groupoids with the Anderson model structure. We also prove that the model structure is equipped with a factorization of morphisms, which induces universal covers.

Contents

Introduction

The category **Cat** of small categories has a couple of interesting model structures. One of them is introduced by Joyal and Tierney in [JT91] which has equivalences of categories as weak equivalences. There also exists the restricted model structure of the above on the category **Grd** of groupoids [And78]. On the other hand, Thomason found another model structure [Tho80] on **Cat** which is Quillen equivalent to the category $\mathbf{Set}^{\Delta^{op}}$ of simplicial sets with the Kan model structure, and the category **Top** of topological spaces with the Quillen model structure [Qui67]. These model categories are related to each other by the following adjunctions

$$\mathbf{Grd} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathbf{Cat} \underset{c}{\overset{N}{\rightleftarrows}} \mathbf{Set}^{\Delta^{op}} \underset{S_*}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}$$

where ι, π, N, c, $|-|$ and S_* are the inclusion, groupoidification, nerve, categorization, geometric realization and the singular simplicial set functor, respectively. In [Qui68], Quillen shows that Serre fibrations in **Top** are related to Kan fibrations in $\mathbf{Set}^{\Delta^{op}}$ by $|-|$ and S_*. Similarly, Gabriel and Zisman define coverings in $\mathbf{Set}^{\Delta^{op}}$ corresponding to coverings in **Top** through $|-|$ and S_* [GZ67]. They also give the notion of coverings of groupoids, and prove that the fundamental groupoid functor **Top** $\longrightarrow$ **Grd** preserves coverings.

In this paper, we consider coverings in **Cat** related to coverings in $\mathbf{Set}^{\Delta^{op}}$, **Top** and **Grd** by the above functors. Our aim is to deal with coverings in **Cat** in terms of model categories.

Main theorem 1 (Theorem 4.2.3). *The category of small categories admits a model structure by the following choices of morphisms:*

- *A morphism f is a weak equivalence if it is a weak 1-equivalence.*

- *A morphism f is a cofibration if it is injective on the set of objects.*

- *A morphism f is a fibration if it is a category fibered and cofibered in groupoids.*

We call the above model structure the "1-type model structure" and denote the category of small categories equipped with the model structure by $\mathbf{Cat}_1$. This model category has the following properties.

Main theorem 2. *The model category $\mathbf{Cat}_1$ has the following properties:*

1. *This is a left Bousfield localization [Hir03] of the Joyal-Tierney model category [Definition 4.1.1].*

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2. *The class of fibrant objects is the class of groupoids [Corollary 4.1.7].*

3. *A covering of small categories coincides with a fibration whose fibers are all discrete [Proposition 4.2.8].*

4. *There exists a factorization of morphisms in $\mathbf{Cat}_1$ which induces universal covers and groupoidification [Corollary 4.3.2 and 4.3.3].*

Elvira-Donazar and Hernandez-Paricio already discovered the 1-type model structure on $\mathbf{Set}^{\Delta^{op}}$ and $\mathbf{Top}$ in [DP95]. We prove that $\mathbf{Cat}_1$ is Quillen equivalent to the two model categories. Categories (co)fibered in groupoids appearing in the definition of fibrations in $\mathbf{Cat}_1$ are used for theory of stacks. Hollander gave a model structure on the category of categories fibered in groupoids in order to characterize stacks in [Hol07]. We compare the overcategory $\mathbf{Cat} \downarrow C$ with the category $\mathcal{F}(C)$ of categories fibered in groupoids on C as model categories for a site (C, J).

This paper is organized as follows. In chapter 1, we recall the basic background in category theory and simplicial method. We prepare some notations and terminologies for categories and functors used in the main text.

Chapter 2 provides the notion of fundamental groups and coverings in $\mathbf{Cat}$. There exists a Galois-type correspondence between them, namely, subgroups of the fundamental group of C are classified by coverings over C.

Chapter 3 describes the definition of model categories and how to obtain a new model structure from already known model structures. We introduce two techniques called the Bousfield localization and the transfer principle.

The 1-type model structure on $\mathbf{Cat}$ is defined as the left Bousfield localization of the Joyal-Tierney model structure in Chapter 4. After that, we verify that the model structure coincides with the one described in Main theorem 1. A covering in $\mathbf{Cat}$ is a spacial case of fibrations in $\mathbf{Cat}_1$, and universal covers are induced from a factorization of morphisms in $\mathbf{Cat}_1$. Finally, we compare $\mathbf{Cat}_1$ with other model categories.

Acknowledgments

Chapter 1

Preliminaries and Terminologies

We begin with an introduction to basic category theory and simplicial method.

1.1 Categories and functors

This section is a review of basic concepts in category theory. We present some notations and terminologies for categories and functors used through this paper.

Definition 1.1.1 (Category). A *category* C consists of the following data;

1. a class C_0 of objects,

2. a set $C(X, Y)$ of morphisms from X to Y for every pair of objects $X, Y \in C_0$. A morphism $f \in C(X, Y)$ is also denoted by $f : X \longrightarrow Y$ or $X \xrightarrow{f} Y$.

3. a map
$$\circ : C(Y, Z) \times C(X, Y) \longrightarrow C(X, Z); (g, f) \mapsto g \circ f$$
called *composition* for every three objects $X, Y, Z \in C_0$. These data have to satisfy the following axioms:

   (a) Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ for every composable triple of morphisms f, g and h.

   (b) Identity: There exists an *identity morphism* $1_X \in C(X, X)$ for every object X such that $1_Y \circ f = f \circ 1_X = f$ whenever $f \in C(X, Y)$.

Remark 1.1.2. Note that the identity morphism $1_X \in C(X, X)$ exists uniquely. If $1'_X \in C(X, X)$ satisfies the condition of identity morphisms, then

$$1_X = 1_X \circ 1'_X = 1'_X.$$

Definition 1.1.3. A morphism $f : X \longrightarrow Y$ of C is called an *isomorphism* if there exists an inverse morphism $g : Y \longrightarrow X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. We denote $X \cong Y$ if there exists an isomorphism between them.

Example 1.1.4. Here are some examples of categories.

- Set is the category of sets and maps.

- **Top** is the category of topological spaces and continuous maps.

- **Set**$^{\Delta^{\mathrm{op}}}$ is the category of simplicial sets and simplicial maps (see Section 1.3).

- **Cat** is the category of small categories and functors (see Definition 1.1.8).

- **Grd** is the category of groupoids as a fullsubcategory of **Cat** (see Definition 1.1.8).

- Let C be a category and let X be an object of C. The *overcategory* $C \downarrow X$ has the set of objects consisting of morphisms $f : Y \longrightarrow X$ of C, and the set of morphisms from $f : Y \longrightarrow X$ to $g : Z \longrightarrow X$ consisting of morphisms $h : Y \longrightarrow Z$ of C such that $g \circ h = f$.

  Dually, the *undercategory* $X \downarrow C$ has the set of objects consisting of morphisms $f : X \longrightarrow Y$ of C, and the set of morphisms from $f : X \longrightarrow Y$ to $g : X \longrightarrow Z$ consisting of morphisms $h : Y \longrightarrow Z$ of C such that $g = h \circ f$.

- Given a category C, the *opposite category* C^{op} consists of $C_0^{\mathrm{op}} = C_0$ and $C^{\mathrm{op}}(X, Y) = C(Y, X)$.

Definition 1.1.5. Let C be a category and let X be an object of C.

1. The object X is called an *initial object* if $C(X, Y)$ consists of a single morphism for any $Y \in C_0$.

2. The object Y is called a *terminal object* if $C(X, Y)$ consists of a single morphism for any $X \in C_0$.

Note that if there exists an initial object or a terminal object in a category, it is determined uniquely up to isomorphism.

Example 1.1.6. The empty set ϕ is an initial object, and a single point set $*$ is a terminal object in **Set**.

Example 1.1.7. The undercategory $X \downarrow C$ given in Example 1.1.4 has 1_X as an initial object and the overcategory $C \downarrow X$ has 1_X as a terminal object.

Definition 1.1.8. A small category is a category C whose class of objects C_0 is a set. A small category C is called a *groupoid* when every morphism of C is an isomorphism.

Example 1.1.9. Here are some examples of small categories and groupoids.

- A monoid is a small category with a single object, furthermore, a group is a groupoid with a single object.

- A poset (partially ordered set) $(P, \leqq)$ is a small category whose set of objects is P and set of morphisms from x to y is a single point whenever $x \leqq y$, or empty otherwise.

- For two small categories C and D, the product $C \times D$ is given by $(C \times D)_0 = C_0 \times D_0$ and $(C \times D)((c_1, d_1), (c_2, d_2)) = C(c_1, c_2) \times D(d_1, d_2)$.

- We can regard a set S as a small category which has S as the set of objects and only identity morphisms. We call such small categories consisting of only identity morphisms *discrete*.

Definition 1.1.10 (Functor). Given categories C and D, a (covariant) *functor* $F : C \longrightarrow D$ consists of;

1. a map $F : C_0 \longrightarrow D_0$,

2. a map $F = F_{X,Y} : C(X,Y) \longrightarrow D(FX, FY)$ for each pair of objects $X, Y \in C_0$ preserving composition and identities, that is,

    (a) $F(g \circ f) = Fg \circ Ff$ for any composable morphisms f and g of C; and
    (b) $F(1_X) = 1_{FX}$ for any object X of C.

Given functors $F : C \longrightarrow D$ and $G : D \longrightarrow E$, then the composition functor $G \circ F : C \longrightarrow E$ is given by composing maps on the set of objects and morphisms respectively. For a small category C, the identity functor $1_C : C \longrightarrow C$ is given by the identity maps on the set of objects and morphisms of C.

Example 1.1.11. Let X be an object of a category D. For any category C, the *constant functor* on X

$$\Delta(X) : C \longrightarrow D$$

is defined by $\Delta(X)(Y) = X$ and $\Delta(f) = 1_X$ for any object $Y \in C_0$ and morphism f of C.

Notation 1.1.12. We use the following notations for small categories:

- ϕ is the empty category.

- $*$ is the category with a unique object $*$ with the only identity morphism.

- $[n]$ is the poset $0 < 1 < 2 < \cdots < n$ regarded as a category

$$0 \to 1 \to 2 \to \cdots \to n.$$

- S^0 is the category with two objects $\{0, 1\}$ and the only identity morphisms.

- S^1 is the category with two objects and having two parallel morphisms between them $0 \rightrightarrows 1$.

- I_n is the category $0 \to 1 \leftarrow 2 \to \cdots \leftarrow n$ (case n even).

- CS^1 is the category with three objects $0 \longrightarrow 1 \rightrightarrows 2$ where 0 is the initial object,

- S^∞ is the groupoid with two objects and two non-trivial morphisms $0 \leftrightarrows 1$.

Also we use the next notations for sets of functors:

- $K = \{k : * \longrightarrow S^\infty\}$ where $k(*) = 0$.

- $I = \{\phi \longrightarrow *, \ i : S^0 \longrightarrow [1], \ i' : S^1 \longrightarrow [1]\}$ where both i and i' are the identity maps on the set of objects.

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- $J_1 = \{j_1 : * \longrightarrow [1] \ , \ j_1^{\mathrm{op}} : * \longrightarrow [1]^{\mathrm{op}}\}$ where $j_1(*) = 0$.

- $J_2 = \{j_2 : I_2 \longrightarrow [2] \ , \ j_2^{\mathrm{op}} : I_2^{\mathrm{op}} \longrightarrow [2]^{\mathrm{op}}\}$ where $j_2(0) = 0$, $j_2(1) = 2$ and $j_2(2) = 1$.

- $J_3 = \{j_3 : CS^1 \longrightarrow [2] \ , \ j_3^{\mathrm{op}} : (CS^1)^{\mathrm{op}} \longrightarrow [2]^{\mathrm{op}}\}$ where j_3 is the identity map on the set of objects.

- $J = J_1 \cup J_2 \cup J_3$.

Definition 1.1.13 (Natural transformation). Given two categories C, D and two functors $F, G : C \longrightarrow D$, a *natural transformation* $t : F \Longrightarrow G$ assigns a morphism $t_X : FX \longrightarrow GX$ of D to each object $X \in C_0$ making the following diagram in D commutative

$$
\begin{array}{ccc}
FX & \xrightarrow{\ t_X\ } & GX \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\
FY & \xrightarrow[\ t_Y\ ]{} & GY
\end{array}
$$

for a morphism $f : X \longrightarrow Y$ of C. If t_X is an isomorphism in D for any X, then t is said to be a *natural isomorphism*. We denote $F \cong D$ if there exists a natural isomorphism between them.

Definition 1.1.14. For a small category C and a category D, the *functor category* D^C consists of functors from C to D as objects and natural transformations as morphisms. An object of D^C is called a diagram in D indexed by C. In particular, $D^{[1]}$ is denoted by $\mathrm{Mor}(D)$ and called the *morphism category* of D.

The following lemma mentions a relation between general categories and functor categories. It allows the embedding of any small category into a functor category valued in **Set**.

Lemma 1.1.15 (Yoneda lemma). *Let C be a category and let X be an object of C. Define a functor $h^X : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$ by $h^X(Y) = C(X, Y)$. If $\mathrm{Nat}(h^X, F)$ denotes the set of natural transformations from h^X to a functor $F : C^{\mathrm{op}} \longrightarrow \mathbf{Set}$, then there exists a bijection $\mathrm{Nat}(h^X, F) \cong F(X)$.*

Proof. For a natural transformation $t : h^X \Longrightarrow F$, we obtain a map

$$ t_X : h^X(X) = C(X, X) \longrightarrow F(X). $$

The map $\mathrm{Nat}(h^X, F) \longrightarrow F(X)$ sending t to $t_X(1_X)$ is a bijection. $\qquad\square$

Corollary 1.1.16. *The functor $h^- : C \longrightarrow \mathbf{Set}^{C^{\mathrm{op}}}$ is an embedding, i.e, it satisfies $X \cong Y$ if $h^X \cong h^Y$ and $h^- : C(X, Y) \cong \mathbf{Set}^{C^{\mathrm{op}}}(h^X, h^Y)$ for any $X, Y \in C_0$.*

Definition 1.1.17. A functor $F : C \longrightarrow D$ is an *equivalence of categories* if there exists an inverse functor $G : D \longrightarrow C$ such that $G \circ F \cong 1_C$ and $F \circ G \cong 1_D$.

Definition 1.1.18. Let $F : C \longrightarrow D$ and $G : D \longrightarrow C$ be functors. We say that the functor F is *left adjoint* to G or G is *right adjoint* to F if there exists a natural isomorphism $C(X, GY) \cong D(FX, Y)$ for any $X \in C_0$ and $Y \in D_0$. In that case, we write $F : C \Longleftrightarrow D : G$ and call (F, G) to be a pair of adjoint functors.

Definition 1.1.19. Let $F : C \Longleftrightarrow D : G$ be a pair of adjoint functors. For an object $X \in C_0$, the *counit map* $\eta : X \longrightarrow GFX$ is a morphism corresponding to the identity morphism 1_{FX} under $C(X, GFX) \cong D(FX, FX)$.

Dually, for an object $Y \in D_0$, the *unit map* $\varepsilon : FGY \longrightarrow Y$ is a morphism corresponding to the identity morphism 1_{GY} under $C(GY, GY) \cong D(FGY, Y)$.

Proposition 1.1.20. *If a functor $F : C \longrightarrow D$ is an equivalence of categories, then F is left and right adjoint to the inverse functor G of F.*

Proof. For $X \in C_0$ and $Y \in D_0$, the natural isomorphism $FG \cong 1$ induces the following isomorphism

$$C(X, GY) \overset{F}{\cong} D(FX, FGY) \cong D(FX, Y).$$

Therefore, F is left adjoint to G. Similarly, we can show that F is right adjoint to G. $\qquad\qquad\square$

Proposition 1.1.21. *A functor $f : C \longrightarrow D$ between small categories is an equivalence of categories if and only if the following two conditions hold:*

1. *Essentially surjective: For any $y \in D_0$, there exists $x \in C_0$ such that $fx \cong y$.*

2. *Fully faithful: $f : C(x, y) \longrightarrow D(fx, fy)$ is bijective for any pair of objects $x, y \in C_0$.*

1.2 Limits and Colimits

In this section, we study the notion of limits and colimits in a general category. A model category requires to have limits and colimits for diagrams indexed by every small categories.

Definition 1.2.1 (limit and colimit). Let $F : C \longrightarrow D$ be a functor from a small category C to a category D. A *limit* $\lim(F)$ is an object of D together with a natural transformation $p : \Delta(\lim(F)) \longrightarrow F$ satisfying the following universality. For any object $X \in D_0$ and a natural transformation $t : \Delta(X) \longrightarrow F$, there exists a unique morphism $f : X \longrightarrow \lim(F)$ of D such that $p_c \circ f = t_c$ for every $c \in C_0$.

Dually, a *colimit* $\mathrm{colim}(F)$ is an object of D together with a natural transformation $i : F \longrightarrow \Delta(\mathrm{colim}(F))$ satisfying the following universality. For any object $X \in D_0$ and a natural transformation $t : F \longrightarrow \Delta(X)$, there exists a unique morphism $f : \mathrm{colim}(F) \longrightarrow X$ of D such that $f \circ i_c = t_c$ for every $c \in C_0$.

If there exist (co)limits of diagrams in D indexed by every small categories, then we say that D is *closed under (co)limits*.

Example 1.2.2. Many of our familiar examples of categories have limits and colimits. All of the categories **Set**, **Top**, **Set**$^{\Delta^{\mathrm{op}}}$, **Cat** and **Grd** introduced in Example 1.1.4 are closed under limits and colimits.

Definition 1.2.3. Let C be a category closed under limits and colimits.

- Let S be a set and regard it as a discrete small category. A (co)limit of a diagram in C indexed by S is called a *direct (co)product* in C.

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- A (co)limit of a diagram in C indexed by S^1 in Notation 1.1.12 is called an *(co)equalizer* in C.

- A limit of a diagram in C indexed by I_2 in Notation 1.1.12 is called a *pullback* in C.

- A colimit of a diagram in C indexed by I_2^{op} is called a *pushout* in C.

Proposition 1.2.4 ([Mac98]). *Let C be a category.*

1. *If C has all equalizers and direct products, then C is closed under limits.*

2. *If C has all coequalizers and direct coproducts, then C is closed under colimits.*

A natural transformation $F \Longrightarrow F'$ induces morphisms $\lim(F) \longrightarrow \lim(F')$ and $\mathrm{colim}(F) \longrightarrow \mathrm{colim}(F')$ by the universality. Hence, limits and colimits give the following functors

$$\lim, \mathrm{colim} : D^C \longrightarrow D.$$

Proposition 1.2.5. *Let C be a small category and let D be a category closed under limits and colimits. The functor $\lim : D^C \longrightarrow D$ is right adjoint to the functor $\Delta : D \longrightarrow D^C$ sending an object X of D to the constant functor $\Delta(X)$. Dually, the functor $\mathrm{colim} : D^C \longrightarrow D$ is left adjoint to Δ.*

Proof. For $F \in (D^C)_0$ and $X \in D_0$, a natural transformation $t : \Delta(X) \Longrightarrow F$ induces a unique morphism $f : X \longrightarrow \lim(F)$ by the definition of limits. It gives an isomorphism $D^C(\Delta(X), F) \cong D(X, \lim(F))$. Similarly, we obtain an isomorphism $D^C(F, \Delta(X)) \cong D(\mathrm{colim}(F), X)$ by the definition of colimits. $\qquad\square$

Proposition 1.2.6. *If C is a category closed under limits and colimits, then C has an initial object and a terminal object.*

Proof. Let ϕ be an empty category, then the colimit of $\phi \longrightarrow C$ is an initial object and the limit is a terminal object. $\qquad\square$

Definition 1.2.7. Let C and D be categories closed under limits and colimits. Let $F : C \longrightarrow D$ be a functor. Given a diagram X in C indexed by a small category, then we have canonical morphisms $\varphi_X : F(\lim X) \longrightarrow \lim(FX)$ and $\psi_X : \mathrm{colim}(FX) \longrightarrow F(\mathrm{colim} X)$ of D by the universality. The functor F is said to *preserve limits* if φ_X is an isomorphism for any diagram X, and said to *preserve colimits* if ψ_X is an isomorphism for any diagram X.

Proposition 1.2.8. *Let C and D be categories closed under limits and colimits. Suppose that there exists a pair of adjoint functors $F : C \Longleftrightarrow D : G$, then F preserves colimits and G preserves limits.*

Proof. By Proposition 1.2.5, $\mathrm{colim} : C^I :\longrightarrow C$ is left adjoint to Δ for a small category I. Hence, $F \circ \mathrm{colim} : C^I \longrightarrow D$ is left adjoint to $\Delta \circ G$. On the other hand, F induces the functor $F_* : C^I \longrightarrow D^I$ given by $F_*(X) = F \circ X$. Also the functor $\mathrm{colim} \circ F_* : C^I \longrightarrow D$ is left adjoint to $\Delta \circ G$, therefore $F \circ \mathrm{colim} \cong \mathrm{colim} \circ F_*$. Similarly, we have $F \circ \lim \cong \lim \circ F_*$. $\qquad\square$

1.3 Simplicial sets

The notion of simplicial sets is a generalized idea of simplicial complexes. It consists of a set of n-simplices for each $n \geq 0$, and face and degeneracy maps satisfying the simplicial identities.

Definition 1.3.1. A *simplicial set K* is a graded set indexed by non-negative integers with maps $d_j : K_n \longrightarrow K_{n-1}$ called face maps and $s_j : K_n \longrightarrow K_{n+1}$ called degeneracy maps for $0 \leq j \leq n$, which satisfy the following simplicial identities:

1. $d_i d_j = d_{j-1} d_i$ if $i < j$,

2. $s_i s_j = s_{j+1} s_i$ if $i \leq j$,

3. $d_i s_j = s_{j-1} d_i$ if $i < j$,

4. $d_j s_j = 1 = d_{j+1} s_j$,

5. $d_i s_j = s_j d_{i-1}$ if $i > j+1$.

Sometimes a simplicial set K is called a complex, and an element of K_n is called an n-simplex of K.

Remark 1.3.2. Let Δ be the small category consisting of posets $[n]$ in Notation 1.1.12 for $n \geq 0$ as objects and poset maps $[n] \longrightarrow [m]$ as morphisms from $[n]$ to $[m]$. A simplicial set can be regarded as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

Example 1.3.3. Here are some basic examples of simplicial sets.

1. For $n \geq 0$, the *standard n-complex $\Delta[n] : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$* is defined by $\Delta(-, [n])$. Denote the identity morphism in $\Delta[n]_n = \Delta([n], [n])$ by Δ_n.

2. For $n \geq 0$, $\partial \Delta[n]$ is the subcomplex of $\Delta[n]$ generated by $\{d_j \Delta_n \mid 0 \leq j \leq n\}$.

3. For $n \geq 0$ and $0 \leq k \leq n$, the *k-th horn Λ_k^n* is the subcomplex $\Delta[n]$ generated by $\{d_j \Delta_n \mid j \neq k\}$.

4. For a space X, the *singular simplicial set $S_* X$* is defined by $S_n X = \mathrm{Top}(\Delta^n, X)$.

5. For two complexes K and L, the product $K \times L$ is given by $(K \times L)_n = K_n \times L_n$.

Definition 1.3.4. A *simplicial map $f : K \longrightarrow L$* is a map of degree zero of graded sets commuting with the face and degeneracy maps. Denote the category of simplicial sets and simplicial maps by $\mathrm{Set}^{\Delta^{\mathrm{op}}}$.

Lemma 1.3.5. *Let K be a simplicial set, then $\mathrm{Set}^{\Delta^{\mathrm{op}}}(\Delta[n], K) \cong K_n$.*

Proof. By Lemma 1.1.15 (Yoneda lemma). $\qquad\square$

Definition 1.3.6 (Geometric realization). Let K be a simplicial set. The *geometric realization* of K is defined as the following space

$$|K| = \left(\coprod_{n \geq 0} \Delta^n \times K_n \right) \Big/ \sim$$

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where $(u, \varphi^*(x)) \sim (\varphi_*(u), x)$ for $u \in \Delta^n$, $x \in K_m$ and $\varphi : [n] \longrightarrow [m]$. A simplicial map $f : K \longrightarrow L$ induces a continuous map $|f| : |K| \longrightarrow |L|$ given by $|f|(u, x) = (u, fx)$. It gives a functor $|-| : \mathrm{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathrm{Top}$.

Proposition 1.3.7. *The geometric realization functor* $|-|$ *is left adjoint to the singular simplicial set functor* S_*.

Proof. See [May92]. □

Proposition 1.3.8. *For two simplicial sets K and L, there exists an isomorphism* $|K \times L| \cong |K| \times |L|$.

Proof. The projections $p_1 : K \times L \longrightarrow K$ and $p_2 : K \times L \longrightarrow L$ induce $|p_1| : |K \times L| \longrightarrow |K|$ and $|p_2| : |K \times L| \longrightarrow |L|$. The product of these maps

$$(|p_1|, |p_2|) : |K \times L| \longrightarrow |K| \times |L|$$

is an isomorphism. □

A simplicial set which has an extension condition is a useful object for homotopy theory of simplicial sets.

Definition 1.3.9 (Kan complex). A simplicial set K is called a *Kan complex* if for every collection of $n + 1$ n-simplices $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1}$ satisfying the compatibility condition $d_i x_j = d_{j-1} x_i$, $i < j$, $i \neq k$, $j \neq k$, there exists an $(n+1)$-simplex x such that $d_j x = x_j$ for $j \neq k$.

Remark 1.3.10. A simplicial set K is a Kan complex if and only if for every simplicial map $\Lambda_k^n \longrightarrow K$ can be extended to $\Delta[n] \longrightarrow K$ for $n \geq 0$ and $0 \leq k \leq n$.

Example 1.3.11. The singular simplicial set $S_* X$ is a Kan complex for any space X.

Example 1.3.12. A simplicial group is a Kan complex.

Definition 1.3.13. A simplicial map $p : E \longrightarrow B$ is said to be a *Kan fibration* if for a commutative diagram

$$
\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{\ f\ } & E \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
\Delta[n] & \xrightarrow{\ g\ } & B,
\end{array}
$$

there exists a simplicial map $h : \Delta[n] \longrightarrow E$ such that $p \circ h = g$ and $h \circ i = f$ for $n \geq 0$ and $0 \leq k \leq n$. If b denotes the complex generated by a vertex b of B, the subcomplex $F = p^{-1}(b)$ of E is called the *fiber* of p over b. Note that b gives a simplicial map $* \longrightarrow B$ from the terminal object $*$ in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ generated by a point.

Remark 1.3.14. A simplicial set K is a Kan complex if and only if $K \longrightarrow *$ is a Kan fibration.

Lemma 1.3.15. *Let $p : E \longrightarrow B$ be a Kan fibration.*

1. The fiber F is a Kan complex.

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2. If B is a Kan complex, then E is a Kan complex.

3. If E is a Kan complex and p is onto, then B is a Kan complex.

Proof. See [May92]. □

Proposition 1.3.16 ([Qui68]). *If $p : E \longrightarrow B$ is a Kan fibration, then $|p| : |E| \longrightarrow |B|$ is a Serre fibration.*

We can define homotopy groups for Kan complexes. See [May92] for details.

Definition 1.3.17. Let K be a simplicial set. For $x, y \in K_n$, define a relation $x \simeq y$ by $d_i x = d_i y$ for any $0 \leq i \leq n$ and there exists $z \in K_{n+1}$ such that $d_n z = x, d_{n+1} z = y$ and $d_i z = s_{n-1} d_i x = s_{n-1} d_i y$ for $0 \leq i < n$. The above simplex z is called a homotopy from x to y.

Proposition 1.3.18. *If K is a Kan complex, then $\simeq$ is an equivalence relation on K_n.*

Definition 1.3.19. Let K be a Kan complex and choose a vertex $* \in K_0$. Define a set $(K, *)_n$ as the subset $\{x \in K_n \mid d_i x = *, 0 \leq i \leq n\}$ of K_n. The *n-dimensional homotopy group* $\pi_n(K, *)$ of a pair $(K, *)$ is defined by $(K, *)_n / \simeq$. For $n \geq 1$, the group structure of $\pi_n(K, *)$ is given by the following. For $[x], [\tilde{y}] \in \pi_n(K, *)$, the collection

$$*, \cdots, *, x, -, y \in (K, *)_n \subset K_n$$

satisfies the compatibility condition in Definition 1.3.9, hence there exists $z \in K_{n+1}$ such that $d_{n-1} z = x$, $d_{n+1} z = y$ and $d_j z = *$ for $0 \leq i \leq n - 2$. The multiplication $[x][y]$ is defined by $[d_n z]$ and this is commutative for $n \geq 2$. A simplicial map $f : K \longrightarrow L$ between Kan complexes induces a map $f_* : \pi_n(K, *) \longrightarrow \pi_n(L, f(*))$ given by $f_*[x] = [f(x)]$. This is a homomorphism of groups for $n \geq 1$.

Proposition 1.3.20. *The homotopy groups of spaces and simplicial sets are related by S_* and $|-|$ as follows.*

1. There is a natural isomorphism $\pi_n(X, *) \cong \pi_n(S_* X, *)$ for any space X.

2. There is a natural isomorphism $\pi_n(K, *) \cong \pi_n(|K|, *)$ for any Kan complex K.

Theorem 1.3.21. *Let $p : E \longrightarrow B$ be a Kan fibration between Kan complexes. Choose vertices $b \in B_0$ and $e \in p^{-1}(b)_0 = F_0$, then there exists the following long exact sequence*

$$\cdots \longrightarrow \pi_n(F, e) \longrightarrow \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \longrightarrow \pi_{n-1}(F, e) \longrightarrow \cdots.$$

Chapter 2

Fundamental groups and coverings of small categories

We define the notion of fundamental groups and coverings of small categories, and study relations between them. The fundamental group of a small category C is defined as the endomorphism group of the groupoidification $\pi(C)$, and a covering is defined as a functor satisfying the unique right lifting property with respect to J_1 given in Notation 1.1.12.

2.1 Fundamental groups of small categories

Minian defined the fundamental group $\pi_1(C, x)$ for a pointed category (C, x) as a colimit of the set of strong homotopy classes of functors from I_n to C for $n \geqq 0$ [Min02]. It is the endomorphism group of the groupoidification of C. The groupoidification is an operation to add formal inverses to all morphisms of a small category. We give a concrete construction in the following definition.

Definition 2.1.1. For a small category C, let $(C^I; a, b)$ be the set of functors

$$(C^I; a, b) = \{\alpha : I_n \longrightarrow C \mid \alpha(0) = a, \alpha(n) = b, n \geqq 0\}$$

for $a, b \in C_0$. An element of $(C^I; a, b)$ forms a zigzag sequence of morphisms from a to b. Let $\sim$ be the equivalence relation on $(C^I; a, b)$ generated by

1. $(c \xrightarrow{f} d \xleftarrow{f} c) \sim c \sim (c \xleftarrow{f} d \xrightarrow{f} c)$,

2. $(c \xrightarrow{f} d \xlongequal{=} d \xrightarrow{g} e) \sim (c \xrightarrow{g \circ f} e)$, $(c \xleftarrow{f} d \xrightarrow{=} d \xleftarrow{g} e) \sim (c \xleftarrow{f \circ g} e)$,

3. $(c \xrightarrow{f} b \xlongequal{=} b) \sim (c \xrightarrow{f} b)$.

Define a small category $\pi(C)$ by $\pi(C)_0 = C_0$ and $\pi(C)(a, b) = (C^I; a, b)/\sim$ with concatenation of sequences of morphisms as the composition. Then, $\pi(C)$ is a groupoid since all of the morphisms are invertible, and it gives a functor $\pi : \mathbf{Cat} \longrightarrow \mathbf{Grd}$.

Definition 2.1.2. For a category C, $\pi_0(C)$ denotes the set of connected components $C_0/\sim_0$, where $x \sim_0 y$ is defined by $\pi(C)(x, y) \neq \phi$. We say that C is connected if $\pi_0(C) = *$.

A pointed category (C, x) is a pair of small category C and an object x of C. Define the fundamental group $\pi_1(C, x)$ as the endomorphism group $\pi(C)(x, x)$ on x of $\pi(C)$. It is easy to show that the relation $\sim$ on $(C^I; a, b)$ is equal to the one defined by strong homotopy in [Min02], hence the fundamental group coincides with Minian's. We say that C is simply connected if it is connected and $\pi_1(C, x)$ is trivial for any $x \in C_0$.

If C is connected, it is obvious that $\pi_1(C, x) \cong \pi_1(C, y)$ for all $x, y \in C_0$, in which case we write $\pi_1(C, x)$ simply as $\pi_1(C)$.

The groupoid $\pi(C)$ is called the groupoidification of C. It is the minimal groupoid containing C as a subcategory.

Proposition 2.1.3. *The groupoidification functor π is left adjoint to the inclusion $\iota : \mathbf{Grd} \longrightarrow \mathbf{Cat}$.*

Proof. Let C be a small category and G be a groupoid. For a functor $f : C \longrightarrow G$,

$$\tilde{f} : \pi(C) \longrightarrow G$$

is defined by $\tilde{f}(x) = f(x)$ for an object $x \in \pi(C)_0 = C_0$ and $\tilde{f}(\alpha) = f(\alpha)$, $\tilde{f}(\alpha^{-1}) = f(\alpha)^{-1}$ for a morphism α in C. The map $\tilde{f}$ is the unique functor satisfying $\tilde{f} \circ \ell = f$ where $\ell : C \longrightarrow \pi(C)$ is the canonical inclusion. It gives a natural isomorphism $\mathbf{Cat}(C, G) \longrightarrow \mathbf{Grd}(\pi(C), G)$. $\square$

Theorem 2.1.4 ([Min02]). *Let (C, x) be a pointed category, then there exists a natural isomorphism*

$$\pi_1(C, x) \cong \pi_1(BC, x)$$

where BC is the classifying space of C (see Definition 2.2.17).

By the above theorem, $\pi_1(C, *)$ can be studied by using homotopy theoretic properties of BC. However, we can describe $\pi_1(C, *)$ in terms of morphisms in C in certain cases.

Proposition 2.1.5. *If the base point $*$ of C is an initial or a terminal object, then $\pi_1(C, *)$ is trivial.*

Proof. Let $*$ be an initial object and consider a sequence γ of the following form

$$(* = c_0 \xrightarrow{f_1} c_1 \xleftarrow{f_2} c_2 \xrightarrow{f_3} \cdots \longleftarrow c_n = *),$$

then there exists a unique morphism $\alpha_2 : * = c_0 \longrightarrow c_2$. On the other hand, the set $C(c_0, c_1)$ consists of the single morphism $f_2 \circ \alpha_2 = f_1$. Therefore, γ is

$$(* = c_0 \xrightarrow{\alpha_2} c_2 \xleftarrow{=} c_2 \xrightarrow{f_2} c_1 \xleftarrow{f_2} c_2 \xrightarrow{f_3} \cdots \longleftarrow c_n = *)$$
$$= (* = c_0 \xrightarrow{f_3 \circ \alpha_2} c_3 \longleftarrow \cdots \longleftarrow c_n = *).$$

By iterating this operation, the above sequence can be shown to be equivalent to $* \xrightarrow{=} *$, thus $\pi_1(C, *)$ is trivial. Similarly, we can prove that $\pi_1(C, *)$ is trivial if $*$ is a terminal object. $\square$

Example 2.1.6. Recall the category S^1 given in Notation 1.1.12. It consists of two objects $0, 1$ and two parallel morphisms f, g and identity morphisms

$$0 \underset{g}{\overset{f}{\rightrightarrows}} 1.$$

The fundamental group $\pi_1(S^1)$ is generated by $0 \overset{f}{\longrightarrow} 1 \overset{g}{\longleftarrow} 0$, thus $\pi_1(S^1) \cong \mathbb{Z}$.

Example 2.1.7. Let G be a group regarded as a groupoid with a single object. An element of $\pi_1(G)$ can be written as a sequence $(g_1, g_2, \cdots, g_n)$ of elements of G. The relations in Definition 2.1.1 imply that

$$(g_1, g_2, \cdots, g_n) = g_1 g_2^{-1} \cdots g_n^{(-1)^{n-1}}$$

in $\pi_1(G)$. It follows that $\pi_1(G) \cong G$.

2.2 Coverings of small categories

The notion of coverings is already defined in the category of spaces, simplicial sets [GZ67] and groupoids [GZ67], [May99]. We define coverings in the category of small categories as functors which have the unique lifting property with respect to J_1 in Notations 1.1.12.

Definition 2.2.1. Let M be a category and let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be morphisms of M. We say that p has the *right lifting property* with respect to i or i has the *left lifting property* with respect to p if for every commutative diagram in M of the following form

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & X \\ i \downarrow & & \downarrow p \\ B & \underset{g}{\longrightarrow} & Y, \end{array}$$

there exists a morphism $h : B \longrightarrow X$ such that $h \circ i = f$ and $p \circ h = g$. If such h exists uniquely, then we say that p has the unique right lifting property with respect to i or i has the unique left lifting property with respect to p.

Let S be a class of morphisms in M. A morphism is called an *S-injection* if it has the right lifting property with respect to every morphisms in S. Denote the class of S-injections by S-inj.

Dually, a morphism is called an *S-projection* if it has the left lifting property with respect to every morphisms in S, and denote the class of S-projections by S-proj.

Definition 2.2.2. A functor $p : E \longrightarrow B$ is called a *covering* if it has the unique right lifting property with respect to J_1 given in Notation 1.1.12. A covering $p : E \longrightarrow B$ is called a *universal cover* if E is simply connected and B is connected.

A covering is also defined as a functor which induces bijections on stars [Hig05], [CRS12].

Remark 2.2.3. A functor $p : E \longrightarrow B$ is a covering if and only if p is surjective on objects, and

$$p : \left(\coprod_{e \in E_0} E(e, x) \right) \coprod \left(\coprod_{e \in E_0} E(x, e) \right) \longrightarrow \left(\coprod_{b \in B_0} B(b, y) \right) \coprod \left(\coprod_{b \in B_0} B(y, b) \right)$$

is bijective for any $y \in B_0$ and $x \in p^{-1}(y)$.

Lemma 2.2.4. *A functor $p : E \longrightarrow B$ is a covering if and only if p has the unique right lifting property with respect to the inclusions $* \longrightarrow [n]$ for all $n \geqq 0$.*

Proof. Since $[n] = 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n$, we can repeat taking lifts of $i \longrightarrow i + 1$ starting at the point of the image $* \longrightarrow [n]$. $\qquad\square$

Example 2.2.5. Recall the category S^1 given in Notation 1.1.12

$$0 \underset{g}{\overset{f}{\rightrightarrows}} 1.$$

Let I_∞ be the poset $\mathbb{Z}$ with the partial order given $2n - 1 > 2n < 2n + 1$ for any $n \in \mathbb{Z}$

$$\cdots \longleftarrow (-2) \longrightarrow (-1) \longleftarrow 0 \longrightarrow 1 \longleftarrow 2 \longrightarrow \cdots .$$

Define a functor $p : I_\infty \longrightarrow S^1$ by $p(2n) = 0$, $p(2n+1) = 1$ and $p(2n \rightarrow (2n+1)) = f$, $p(2n \rightarrow (2n-1)) = g$ for $n \in \mathbb{Z}$, then p is a covering. Furthermore, this is a universal cover since $B(I_\infty) \cong \mathbb{R}$ is contractible.

Next, we construct a universal cover over any connected category using the Grothendieck construction [Tho79].

Definition 2.2.6. Let I be a small category. The *Grothendieck construction* of a functor $F : I \longrightarrow$ Set is a small category $\mathrm{Gr}(F)$ defined as follows. The set of objects of $\mathrm{Gr}(F)$ consists of pairs (i, x) of an object $i \in I_0$ and an element $x \in F(i)$. A morphism $(i, x) \longrightarrow (j, y)$ in $\mathrm{Gr}(F)$ is a morphism $f : i \longrightarrow j$ in I such that $F(f)(x) = y$. It is equipped with the canonical projection $\mathrm{Gr}(F) \longrightarrow I$ given by $(i, x) \mapsto i$.

Definition 2.2.7. Let $(C, *)$ be a pointed and connected category, then the category $\widehat{C}$ is defined by the Grothendieck construction of

$$\pi(C)(*, -) : C \longrightarrow \mathrm{Set}.$$

The canonical projection $T : \widehat{C} \longrightarrow C$ carries an object of $\widehat{C}$ formed

$$(* \longrightarrow c_1 \longleftarrow c_2 \longrightarrow \cdots \longleftarrow c_n)$$

to the last object c_n.

Lemma 2.2.8. *The canonical projection $T : \widehat{C} \longrightarrow C$ is a covering.*

Proof. Suppose we have the following commutative diagram

$$
\begin{array}{ccc}
* & \xrightarrow{\;x\;} & \widehat{C} \\
{\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle T} \\
[1] & \xrightarrow{\;g\;} & C.
\end{array}
$$

The above x gives a class of zigzag sequence of morphisms in C and g gives a morphism

$$g(0 \longrightarrow 1) : g(0) = x_n \longrightarrow g(1)$$

in C where x_n is the last object of x. Define $h : [1] \longrightarrow \widehat{C}$ by $h(0) = x$, $h(1) = g(0 \longrightarrow 1) \circ x$ and $h(0 \longrightarrow 1) = g(0 \longrightarrow 1)$. It makes the above diagram commutative and exists uniquely. Similarly, T has the unique lifting property with respect to $j_1^{op} : * \longrightarrow [1]^{op}$. $\qquad\square$

Proposition 2.2.9. *The category $\widehat{C}$ is simply connected.*

Proof. For an object

$$(*) = (c_*^0) \xrightarrow{f_1} (c_*^1) \xleftarrow{f_2} (c_*^2) \longrightarrow \cdots \xleftarrow{f_n} (c_*^n) = (*)$$

in $\pi_1(\widehat{C})$, it suffices to show that

$$(* \xrightarrow{f_1} T(c_*^1) \xleftarrow{f_2} T(c_*^2) \longrightarrow \cdots \xleftarrow{f_n} *) = 1$$

in $\pi_1(C)$. By iterating the following process, we obtain

$$
\begin{aligned}
& (* \xrightarrow{f_1} T(c_*^1) \xleftarrow{f_2} T(c_*^2) \longrightarrow \cdots \xleftarrow{f_n} *) \\
=& (* \xrightarrow{c_*^1} T(c_*^1) \xleftarrow{f_2} T(c_*^2) \longrightarrow \cdots \xleftarrow{f_n} *) \\
=& (* \xrightarrow{c_*^2} T(c_*^2) \longrightarrow \cdots \xleftarrow{f_n} *) = \cdots = 1.
\end{aligned}
$$

$\qquad\square$

Corollary 2.2.10. *The canonical projection $T : \widehat{C} \longrightarrow C$ is a universal cover.*

Proof. By Proposition 2.2.9 and Lemma 2.2.8. $\qquad\square$

We recall the definition of coverings of simplicial sets and groupoids [GZ67], [May99].

Definition 2.2.11. A simplicial map $p : E \longrightarrow B$ is called a covering if it has the unique right lifting property with respect to the inclusions $\Delta[0] \longrightarrow \Delta[n]$, $n \geqq 0$.

Definition 2.2.12. A functor $p : E \longrightarrow B$ in **Grd** is called a covering if it has the unique right lifting property with respect to K in Notation 1.1.12.

Proposition 2.2.13 ([GZ67]). *Both $S_* : \mathbf{Top} \longrightarrow \mathbf{Set}^{\Delta^{op}}$ and $|-| : \mathbf{Set}^{\Delta^{op}} \longrightarrow \mathbf{Top}$ preserve coverings.*

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Proposition 2.2.14. *Both* $\iota : \mathbf{Grd} \longrightarrow \mathbf{Cat}$ *and* $\pi : \mathbf{Cat} \longrightarrow \mathbf{Grd}$ *preserve coverings.*

Proof. Since ι is right adjoint to π, it preserves the unique right lifting property and coverings. Conversely, assume that $p : E \longrightarrow B$ is a covering in $\mathbf{Cat}$. Consider the following commutative diagram in $\mathbf{Grd}$

$$
\begin{array}{ccc}
* & \xrightarrow{\ e\ } & \pi(E) \\
\downarrow & & \downarrow{\scriptstyle \pi(p)} \\
S^\infty & \xrightarrow[\ f\ ]{} & \pi(B).
\end{array}
$$

Let s be the image of morphism $0 \longrightarrow 1$ in S^∞ by f. It is a zigzag sequence of morphisms of B starting at $p \circ e(*)$. Since p is a covering, we can find lifts of morphisms appearing in s, uniquely. It gives a functor $S^\infty \longrightarrow \pi(E)$ making the diagram commutative, therefore $\pi(p)$ is a covering. $\qquad\square$

The category of simplicial sets and the category of small categories are related with each other by the nerve functor and the categorization functor [GZ67].

Definition 2.2.15. The nerve NC of a small category C is defined by

$$N_n C = \mathbf{Cat}([n], C)$$

and

$$d_i(f_1, \cdots, f_n) = (f_1, \cdots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \cdots, f_n)$$

and

$$s_j(f_1, \cdots, f_n) = (f_1, \cdots, f_j, 1, f_{j+1}, \cdots, f_n).$$

It gives a functor $N : \mathbf{Cat} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}}$.

The categorization cK of a simplicial set K is defined as follows. The set of objects cK_0 is K_0 and the morphisms in cK are freely generated by the set K_1 subject to relations given by elements of K_2, namely, $x_1 = x_2 x_0$ in cK if there exists a 2-simplex x such that $d_2 x = x_2$, $d_0 x = x_0$ and $d_1 x = x_1$. It gives a functor $c : \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Cat}$

Proposition 2.2.16 ([GZ67]). *The pair of functors*

$$c : \mathbf{Set}^{\Delta^{\mathrm{op}}} \Longleftrightarrow \mathbf{Cat} : N$$

is a pair of adjoint functors, and $cN \cong 1_{\mathbf{Cat}}$.

Definition 2.2.17. For a small category C, the classifying space BC of C is given by $|NC|$.

Proposition 2.2.18. *A functor p is a covering in* $\mathbf{Cat}$ *if and only if $N(p)$ is a covering in* $\mathbf{Set}^{\Delta^{\mathrm{op}}}$.

Proof. Since N is right adjoint to c, N preserves the unique right lifting property. Therefore, Lemma 2.2.4 implies that N preserves coverings. Conversely, let $N(p)$ be a covering, then $N(p)$ has the unique right lifting property for

$$d_0^*, d_1^* : \Delta[0] \longrightarrow \Delta[1].$$

Since $cN \cong 1_{\mathbf{Cat}}$, the functor p has the unique right lifting property with respect to J_1. $\square$

Before the end of this section, let us define a Galois-type correspondence between subgroups of $\pi_1(C)$ and coverings over C for a connected category C. In the case of groupoids, Peter May proved the following [May99].

Theorem 2.2.19 ([May99]). *For a connected groupoid G, let $\mathrm{Cov}_{\mathbf{Grd}}(G)$ be the category of coverings over G in $\mathbf{Grd}$ and let $\mathrm{O}(\pi_1(G))$ be the category of subgroups of $\pi_1(G)$ as objects and subconjugacy relations as morphisms. Then, there exists an equivalence of categories between $\mathrm{Cov}_{\mathbf{Grd}}(G)$ and $\mathrm{O}(\pi_1(G))$.*

Proposition 2.2.20. *For a connected category C, let $\mathrm{Cov}_{\mathbf{Cat}}(C)$ be the category of coverings over C in $\mathbf{Cat}$. Then, there is an equivalence of categories between $\mathrm{Cov}_{\mathbf{Cat}}(C)$ and $\mathrm{Cov}_{\mathbf{Grd}}(\pi(C))$.*

Proof. The groupoidification functor induces $\pi_* : \mathrm{Cov}_{\mathbf{Cat}}(C) \longrightarrow \mathrm{Cov}_{\mathbf{Grd}}(\pi(C))$ by Proposition 2.2.14. On the other hand, let q be a covering in $\mathbf{Grd}$ over $\pi(C)$, the pullback of q along the canonical inclusion functor $C \longrightarrow \pi(C)$ induces a covering over C in $\mathbf{Cat}$. This correspondence gives an inverse functor $\mathrm{Cov}_{\mathbf{Grd}}(\pi(C)) \longrightarrow \mathrm{Cov}_{\mathbf{Cat}}(C)$ of π_*. $\square$

Corollary 2.2.21. *For a connected category C, there is an equivalence of categories between $\mathrm{Cov}_{\mathbf{Cat}}(C)$ and $\mathrm{O}(\pi_1(C))$.*

Proof. By Theorem 2.2.19 and Proposition 2.2.20. $\square$

Chapter 3

Model categories

Model categories introduced by Quillen in [Qui67] form the foundation of homotopy theory. This is a framework to do homotopy theory in general categories.

3.1 Model categories

Model category is a category equipped with three distinguished classes of morphisms called weak equivalences, cofibrations and fibrations. We can do homotopy theory in a general category using such three sorts of morphisms.

Definition 3.1.1. A *model structure* on a category M consists of three distinguished subcategories, *weak equivalences* W, *cofibrations* C and *fibrations* F satisfying the following properties.

1. (2-out-of-3): If f and g are morphisms of M such that $g \circ f$ is defined and two of f, g and $g \circ f$ are weak equivalences, then so is the third.

2. (Retract) : Each W, C and F is closed under retracts.

3. (Lifting): Every morphism in $W \cap C$ has the right lifting property with respect to F, and every morphism in C has the right lifting property with respect to $W \cap F$.

4. (Factorization): Every morphisms f can be written as $p \circ i$ for $i \in C$ and $p \in W \cap F$, moreover, f can be also written as $q \circ j$ for $j \in W \cap C$ and $q \in F$.

A morphism in $W \cap C$ is called a *trivial cofibration*, and a morphism in $W \cap F$ is called a *trivial fibration*, respectively.

A *model category* is a category M closed under small limits and colimits together with a model structure on M.

Cofibrations and fibrations are characterized as morphisms having lifting property.

Proposition 3.1.2 ([Hov99]). *Let M be a model category*

1. *$(W \cap C)\text{-}inj = F$*

2. *$C\text{-}inj = W \cap F$*

3. $W \cap C = F\text{-}proj$

4. $C = (W \cap F)\text{-}proj$

By the above fact, the class of cofibrations C is determined from W and F, and the class of fibrations F is determined from W and C. Moreover, by the 2-out-of-3 and factorization axioms of model categories, a weak equivalence can be factored as composition of a trivial cofibration and a trivial fibration. Since $W \cap C$ is determined by F, and $W \cap F$ is determined by C, the class W is also determined by F and C. After all, two classes of W, C, F determine the third.

Example 3.1.3. The following are some basic examples of model categories.

- **Top**$_Q$ is **Top** with the Quillen model structure [Qui67] given as follows:

  1. A morphism is a weak equivalence if it is a weak homotopy equivalence.

  2. A morphism is a fibration if it is a Serre fibration.

- **Top**$_S$ is **Top** with the Strøm model structure [Str72] given as follows:

  1. A morphism is a weak equivalence if it is a homotopy equivalence.

  2. A morphism is a fibration if it is a Hurewicz fibration.

- **Set**$_K^{\Delta^{op}}$ is **Set**$^{\Delta^{op}}$ with the Kan model structure [Hov99] given as follows:

  1. A morphism is a weak equivalence if the geometric realization is a weak homotopy equivalence in **Top**.

  2. A morphism is a fibration if it is a Kan fibration.

- **Cat**$_{JT}$ is **Cat** with the Joyal-Tierney model structure [JT91], [Rez00] given as follows:

  1. A morphism is a weak equivalence if it is an equivalence of categories.

  2. A morphism is a cofibration if it is injective on the set of objects.

- **Cat**$_T$ is **Cat** with the Thomason model structure [Tho80] given as follows:

  1. A morphism is a weak equivalence if it induces a weak homotopy equivalence between classifying spaces in **Top**.

  2. A morphism is a cofibration if it is a pseudo Dwyer morphism [Rap10].

- **Grd**$_A$ is **Grd** with the Anderson model structure [And78] given as follows:

  1. A morphism is a weak equivalence if it is an equivalence of categories.

  2. A morphism is a cofibration if it is injective on the set of objects.

- **DGM**$_I$ is the category **DGM** of bounded below differential graded modules over a commutative ring R with the injective model structure [Hov99] given as follows:

  1. A morphism is a weak equivalence if it is a quasi isomorphism.

  2. A morphism is a fibration if it is a surjection with injective kernel.

- DGM_P is DGM with the projective model structure [Hov99] given as follows:

  1. A morphism is a weak equivalence if it is a quasi isomorphism.

  2. A morphism is a cofibration if it is a injection with projective cokernel.

As we saw in Section 2.3, in the case of simplicial sets, Kan complexes play an important role in homotopy theory. A model category provides two classes of such good objects called cofibrant, and fibrant. Indeed, Kan complexes are fibrant in $Set_K^{\Delta^{op}}$.

Definition 3.1.4. Let M be a model category.

1. An object X is called *cofibrant* if the unique morphism from the initial object $\phi \to X$ is a cofibration.

2. An object Y is called *fibrant* if the unique morphism to the terminal object $Y \to *$ is a fibration.

Example 3.1.5. The followings are examples of cofibrant and fibrant objects of the model categories in Example 3.1.3.

- In Top_Q, every objects are fibrant and CW-complexes are cofibrant.

- In Top_S, every objects are fibrant and cofibrant.

- In $Set_K^{\Delta^{op}}$, Kan complexes are fibrant and every objects are cofibrant.

- In Cat_{JT}, every objects are fibrant and cofibrant.

- In Cat_T, posets are cofibrant.

- In Grd_A, every objects are fibrant and cofibrant.

- In DGM_I, injective complexes are fibrant and every objects are cofibrant.

- In DGM_P, every objects are fibrant and projective complexes are cofibrant.

In order to compare model categories, we introduce functors between them.

Definition 3.1.6. Let M and N be model categories and let

$$F : M \Longleftrightarrow N : G$$

be a pair of adjoint functors. We say that (F, G) is a *Quillen adjunction* if F preserves cofibrations and G preserves fibrations. Furthermore, a Quillen adjunction (F, G) is called a *Quillen equivalence* if for a cofibrant object X in M, a fibrant object Y in N and a morphism $f : X \longrightarrow GY$ in M, the morphism f is a weak equivalence in M if and only if the morphism $f^\sharp : FX \longrightarrow Y$ is a weak equivalence in N.

Proposition 3.1.7 ([Hir03]). *Let M and N be model categories and let*

$$F : M \Longleftrightarrow N : G$$

be a pair of adjoint functors. Then the following are equivalent:

1. (F, G) is a Quillen adjunction.

2. F preserves both cofibrations and trivial cofibrations.

3. G preserves both fibrations and trivial fibrations.

Example 3.1.8. The pair of identity functors $1 : \mathrm{Top}_Q \Longleftrightarrow \mathrm{Top}_S : 1$ is a Quillen adjunction.

Example 3.1.9. The pair of functors $|-| : \mathrm{Set}_K^{\Delta^{op}} \Longleftrightarrow \mathrm{Top}_Q : S_*$ is a Quillen equivalence.

3.2 Simplicial model categories

In this section, we introduce the notion of simplicial model categories. A simplicial model category M is a model category enriched over $\mathrm{Set}^{\Delta^{op}}$, i.e, it is given a simplicial set $\mathrm{Map}(X, Y)$ called the function complex such that $\mathrm{Map}(X, Y)_0 = M(X, Y)$ for every pair of objects X and Y.

Definition 3.2.1. A *simplicial category M* is a category together with

1. a simplicial set $\mathrm{Map}(X, Y)$ called the *function complex* for every two objects X and Y,

2. a morphism of simplicial sets called the composition of function complexes
$$\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(X, Z)$$
   for every three objects X, Y and Z satisfying the associativity condition,

3. a morphism of simplicial sets $* \longrightarrow \mathrm{Map}(X, X)$ for every object X satisfying the unit condition,

4. an isomorphism $\mathrm{Map}(X, Y)_0 \cong M(X, Y)$ commuting with the composition for every two objects X and Y.

Example 3.2.2. Let X and Y be simplicial sets. Let $\mathrm{Map}(X, Y)$ be the simplicial set given by $\mathrm{Map}(X, Y)_n = \mathrm{Set}^{\Delta^{op}}(X \times \Delta[n], Y)$. This gives a simplicial category structure on $\mathrm{Set}^{\Delta^{op}}$.

Definition 3.2.3. A *simplicial model category* is a model category M that is also a simplicial category satisfying the following two axioms:

1. For every two objects X and Y of M and every simplicial set K, there exist objects $X \otimes K$ and Y^K of M such that there are natural isomorphisms of simplicial sets
$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(X, Y^K).$$

2. If $i : A \longrightarrow B$ is a cofibration and $p : X \longrightarrow Y$ is a fibration in M, then the map of simplicial sets induced from the pullback
$$\mathrm{Map}(B, X) \longrightarrow \mathrm{Map}(A, X) \times_{\mathrm{Map}(A, Y)} \mathrm{Map}(B, Y)$$
   is a fibration in $\mathrm{Set}_K^{\Delta^{op}}$. Moreover, it is a trivial fibration when either i or p is a weak equivalence.

Example 3.2.4. The Kan model category $\mathrm{Set}_K^{\Delta^{op}}$ is a simplicial model category with $X \otimes K = X \times K$ and $X^K = \mathrm{Map}(K, X)$ for any simplicial sets X and K.

3.3 Homotopy limits and colimits in a simplicial model category

Let M be a model category and let I be a small category. The (co)limit functor $M^I \longrightarrow M$ does not send an objectwise weak equivalence to a weak equivalence of M in general. The homotopy (co)limit is an attempt to solve this problem of the ordinary (co)limit.

Definition 3.3.1 (Homotopy limit and colimit). Let $F : I \longrightarrow M$ be a diagram in a simplicial model category M indexed by a small category I.

1. The *homotopy colimt* hocolim$F$ of F is defined to be the following coequalizer

$$\coprod_{f \in I(x,y)} F(x) \otimes N(y \downarrow I)^{\mathrm{op}} \rightrightarrows \coprod_{x \in I_0} F(x) \otimes N(x \downarrow I)^{\mathrm{op}},$$

   where the above parallel maps are given by

   $$f_* \otimes 1 : F(x) \otimes N(y \downarrow I)^{\mathrm{op}} \longrightarrow F(y) \otimes N(y \downarrow I)^{\mathrm{op}}$$

   and

   $$1 \otimes f^* : F(x) \otimes N(y \downarrow I)^{\mathrm{op}} \longrightarrow F(x) \otimes N(x \downarrow I)^{\mathrm{op}}.$$

2. The *homotopy limit* holim$F$ of F is defined to be the following equalizer

$$\prod_{x \in I_0} F(x)^{N(I \downarrow x)} \rightrightarrows \prod_{f \in I(x,y)} F(y)^{N(I \downarrow x)},$$

   where the above parallel maps are given by

   $$f_*^1 : F(x)^{N(I \downarrow x)} \longrightarrow F(y)^{N(I \downarrow x)}$$

   and

   $$1^{f_*} : F(y)^{N(I \downarrow y)} \longrightarrow F(y)^{N(I \downarrow x)}.$$

Homotopy (co)limits have the following homotopy invariance.

Theorem 3.3.2. *[Hir03] Let M be a simplicial model category and let I be a small category.*

1. *If $f : F \longrightarrow G$ is a morphism of diagrams in M by indexed by I that is an objectwise weak equivalence between cofibrant objects, the induced morphism of homotopy colimits*

   $$f_* : \mathrm{hocolim}F \longrightarrow \mathrm{hocolim}G$$

   is a weak equivalence in M.

2. *If $f : F \longrightarrow G$ is a morphism of diagrams in M by indexed by I that is an objectwise weak equivalence between fibrant objects, the induced morphism of homotopy limits*

   $$f_* : \mathrm{holim}F \longrightarrow \mathrm{holim}G$$

   is a weak equivalence in M.

Theorem 3.3.3. *[Hir03] Let M be a simplicial model category and let I be a small category. If F is a diagram in M indexed by I and X is an object of M, then there exists a natural isomorphism of simplicial sets*

$$\mathrm{Map}(\mathrm{hocolim}F, X) \cong \mathrm{holimMap}(F, X)$$

3.4 The Bousfield localization of model categories

It tends to be difficult to prove that a category admits a model structure. The axioms of model structure are always hard to check. Hence, we introduce two techniques to construct a new model structure from another good model structure. These are called the Bousfield localization and the transfer principle.

Definition 3.4.1. Let M be a model category and let S be a class of morphisms.

1. An object W is *S-local* if it is fibrant and for any morphism $f : A \longrightarrow B$ in S, the induced map on *homotopy function complexes* (see [Hir03]) $f^* :$ $\mathrm{map}(B, W) \longrightarrow \mathrm{map}(A, W)$ is a weak equivalence in $\mathrm{Set}_K^{\Delta^{\mathrm{op}}}$. If S consists of the single map $f : A \longrightarrow B$, then an S-local object is called f-local.

2. A morphism $g : X \longrightarrow Y$ is an *S-local equivalence* if the induced map on homotopy function complexes $g^* : \mathrm{map}(Y, W) \longrightarrow \mathrm{map}(X, W)$ is a weak equivalence in $\mathrm{Set}_K^{\Delta^{\mathrm{op}}}$ for any S-local object W. If S consists of the single map $f : X \longrightarrow Y$, then an S-local equivalence is called an f-local equivalence.

Note that the above homotopy function complex $\mathrm{map}(X, Y)$ is not equal to the function complex $\mathrm{Map}(X, Y)$ in Definition 3.2.1 of simplicial model categories. It is defined in a general model category, however if X is cofibrant and Y is fibrant in a simplicial model category, then $\mathrm{map}(X, Y)$ and $\mathrm{Map}(X, Y)$ are weakly equivalent as simplicial sets.

Remark 3.4.2. Let M be a simplicial model category whose objects are cofibrant and fibrant and let S be a class of morphisms.

1. An object W is S-local if and only if for any morphism $f : A \longrightarrow B$ in S, the induced map on function complexes $f^* : \mathrm{Map}(B, W) \longrightarrow \mathrm{Map}(A, W)$ is a weak equivalence in $\mathrm{Set}_K^{\Delta^{\mathrm{op}}}$.

2. A morphism $g : X \longrightarrow Y$ is an S-local equivalence if and only if the induced map on function complexes $g^* : \mathrm{Map}(Y, W) \longrightarrow \mathrm{Map}(X, W)$ is a weak equivalence in $\mathrm{Set}_K^{\Delta^{\mathrm{op}}}$ for any S-local object W.

Definition 3.4.3. Let M be a model category and let S be a class of morphisms. The *left Bousfield localization* of M with respect to S is a model structure $L_S M$ on the underlying category M such that

1. the class of weak equivalences of $L_S M$ equals the class of S-local equivalences,

2. the class of cofibrations of $L_S M$ equals the class of cofibrations of M.

Proposition 3.4.4 ([Hir03]). *Let $L_S M$ be the left Bousfield localization of M with respect to S.*

1. Every weak equivalence of M is a weak equivalence of $L_S M$.

2. The class of trivial fibrations of $L_S M$ coincides with the class of trivial fibrations of M.

3. Every fibration of $L_S M$ is a fibration of M.

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4. Every trivial cofibration of M is a trivial cofibration of $L_S M$.

5. The class of fibrant objects of $L_S M$ coincides with the class of S-local objects.

Proposition 3.4.5. *The identity functors $1_M : M \Longleftrightarrow L_S M : 1_M$ is a Quillen adjunction.*

Proof. By Proposition 3.4.4. $\square$

Although a model structure consists of three classes of morphisms, the classes of cofibrations and trivial cofibrations determine the model structure. Indeed, the class of fibrations is determined by the class of trivial cofibrations by the lifting axiom of model categories. Moreover, a weak equivalence can be written as composition of a trivial cofibration and trivial fibration by the factorization axiom. Hence, the class of weak equivalences is also determined by the classes of cofibrations and trivial cofibrations. The following is a model category which has generators of cofibrations and trivial cofibrations.

Definition 3.4.6. We say that a model category M is *cofibrantly generated* if there exist sets A and B of morphisms such that

   1. both A and B permit the small object argument (see [Hir03]),

   2. $W \cap F = A$-inj and $F = B$-inj.

The above set A is called a *generating cofibrations*, and B is called a *generating trivial cofibrations*. Moreover, we say that M is *combinatorial* if it is cofibrantly generated and *locally presentable* [KL01].

One of useful techniques to define a model structure on a category is transferring another model structure to the category through a pair of adjoint functors.

Theorem 3.4.7 (Transfer principle). *Let M be a cofibrantly generated model category with generating cofibrations A and generating trivial cofibrations B. Let N be a category closed under limits and colimits, and let $F : M \Longleftrightarrow N : G$ be a pair of adjoint functors. If $FA = \{Fa \mid a \in A\}$ and $FB = \{Fb \mid b \in B\}$ and if*

   1. both of the set FA and FB permit the small object argument,

   2. G takes relative FB-cell complexes (see [Hir03], [Hov99]) to weak equivalences,

then there exists a cofibrantly generated model structure on N in which FA is a set of generating cofibrations, FB is a set of generating trivial cofibrations, and the weak equivalences are the morphisms that G takes into a weak equivalence in M. Furthermore, with respect to this model structure, (F, G) is a Quillen adjunction.

A model category is called proper if weak equivalences are preserved pushing them out along cofibrations and pulling them back along fibrations.

Definition 3.4.8. Let M be a model category.

   1. It is called *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence.

2. It is called *right proper* if every pullback of a weak equivalence along a fibration is a weak equivalence.

3. It is called *proper* if it is both left and right proper.

Proposition 3.4.9. *Let M be a model category.*

1. *If every object is cofibrant, then M is left proper.*

2. *If every object is fibrant, then M is right proper.*

Most of the examples in Example 3.1.3 are combinatorial, simplicial and proper model categories. We focus on the Joyal-Tierney model category $\mathbf{Cat}_{JT}$ here.

Example 3.4.10. The model category $\mathbf{Cat}_{JT}$ has the generating cofibrations I and the trivial cofibrations K in Notation 1.1.12. Since $\mathbf{Set}$ is locally presentable, so is $\mathbf{Cat}$ [KL01]. For a small category C, let $\rho(C)$ be the maximal groupoid contained in C, i.e, $\rho : \mathbf{Cat} \longrightarrow \mathbf{Grd}$ is right adjoint to the inclusion. The function complex $\mathrm{Map}(C, D) = N\rho(D^C)$ gives rise to a simplicial enrichment on $\mathbf{Cat}$. Let $C \otimes K = C \times \pi c(K)$ and $C^K = C^{\pi c(K)}$, then $\mathbf{Cat}_{JT}$ is a simplicial model category [Rez00]. Since every object in $\mathbf{Cat}_{JT}$ is fibrant and cofibrant, $\mathbf{Cat}_{JT}$ is proper. Thus, $\mathbf{Cat}_{JT}$ is a combinatorial and simplicial proper model category.

The following theorem guarantees the existence of the Bousfield localization for nice model categories.

Theorem 3.4.11 ([Lur09]). *If M is a combinatorial and simplicial left proper model category and S is a set of morphisms. Then the left Bousfield localization of M with respect to S does exist as a combinatorial and simplicial left proper model category.*

Chapter 4

The 1-type model structure on Cat

This chapter provides a model structure on the category of small categories, which is closely related to the notion of coverings, fundamental groups and groupoidification.

4.1 The 1-type model structure on the category of small categories

By Theorem 3.4.11 and Example 3.4.10, we obtain a new model structure on Cat by the Bousfield localization of Joyal-Tierney model structure if we give a set of functors.

Definition 4.1.1. Define the *1-type model category* on Cat as the left Bousfield localization of the Cat_{JT} with respect to the inclusion $\varphi : [1] \longrightarrow S^{\infty}$. Denote the category Cat with the 1-type model structure by Cat_1

It has φ-local equivalences as weak equivalences and cofibrations of Cat_{JT} as cofibrations. We introduce the notion of weak 1-equivalences in Cat, after that, we show that weak 1-equivalence coincides with φ-local equivalence.

Definition 4.1.2. A functor $f : C \longrightarrow D$ is called a *weak 1-equivalence* if the both induced maps $\pi_0(C) \longrightarrow \pi_0(D)$ and $\pi_1(C, x) \longrightarrow \pi_1(D, f(x))$ are isomorphisms for all $x \in C_0$.

Lemma 4.1.3. *Let G be a groupoid, then the canonical inclusion $G \longrightarrow \pi(G)$ is an isomorphism of categories.*

Proof. The inverse functor $\pi(G) \longrightarrow G$ is given by the identity map on the set of objects, and

$$(\cdot \xrightarrow{f_1} \cdot \xleftarrow{f_2} \cdot \xrightarrow{f_3} \cdots \xleftarrow{f_n} \cdot) \mapsto f_n^{-1} \circ \cdots \circ f_3 \circ f_2^{-1} \circ f_1$$

on the set of morphisms. $\qquad\square$

Proposition 4.1.4. *The canonical inclusion $\ell : C \longrightarrow \pi(C)$ is a weak 1-equivalence for any small category C.*

Proof. The map ℓ on the set of objects is the identity map. By Lemma 4.1.3, the functor $\pi(\ell) : \pi(C) \longrightarrow \pi(\pi(C))$ is an isomorphism of small categories. Thus $\ell_* : \pi_1(C, *) \longrightarrow \pi_1(\pi(C), *)$ is an isomorphism for any base point $*$. $\qquad\square$

Lemma 4.1.5. *A functor* $f : C \longrightarrow D$ *is a weak 1-equivalence if and only if the functor* $\pi(f) : \pi(C) \longrightarrow \pi(D)$ *is an equivalence of categories.*

Proof. Assume that $\pi(f)$ is an equivalence of categories, then it is obvious that f is a weak 1-equivalence by Proposition 4.1.4 and the following diagram is commutative

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & D \\
\downarrow & & \downarrow \\
\pi(C) & \xrightarrow[\pi(f)]{} & \pi(D).
\end{array}
$$

Conversely, assume that $f : C \longrightarrow D$ is a weak 1-equivalence. Since $\pi_n(BG, *) = 0$ for any pointed groupoid $(G, *)$ and $n \geq 2$, the induced map $B\pi(f)_* : \pi_n(B\pi(C), *) \longrightarrow \pi_n(B\pi(D), *)$ is an isomorphism for all $n \geq 0$. This is a weak equivalence in **Cat** with the Thomason model structure in Example 3.1.3. A functor between groupoids is a weak equivalence in the Thomason model structure if and only if it is an equivalence of categories [CGT04]. Thus $\pi(f)$ is an equivalence of categories. $\qquad\square$

Lemma 4.1.6. *A small category is φ-local if and only if it is a groupoid.*

Proof. If G is a groupoid,

$$
\mathrm{Map}([1], G) = N\rho(G^{[1]}) \cong NG^{[1]} \cong NG^{S^\infty} = \mathrm{Map}(S^\infty, G).
$$

Therefore $\varphi^* : \mathrm{Map}(S^\infty, G) \longrightarrow \mathrm{Map}([1], G)$ is a weak equivalence in $\mathbf{Set}_K^{\Delta^{op}}$. Conversely, assume that G is φ-local. Since φ is a cofibration in $\mathbf{Cat}_{JT}$,

$$
\varphi^* : \mathrm{Map}(S^\infty, G) \longrightarrow \mathrm{Map}([1], G)
$$

is a trivial fibration in $\mathbf{Set}_K^{\Delta^{op}}$. Therefore, the map $\varphi^* : G_0^{S^\infty} \longrightarrow G_0^{[1]}$ is surjective and G is a groupoid. $\qquad\square$

Corollary 4.1.7. *A small category is fibrant in* $\mathbf{Cat}_1$ *if and only if it is a groupoid.*

Proof. By Lemma 4.1.6 and Proposition 3.4.4. $\qquad\square$

Proposition 4.1.8. *A functor* $f : X \longrightarrow Y$ *is a φ-local equivalence if and only if it is a weak 1-equivalence.*

Proof. Let f be a φ-local equivalence, the induced map

$$
f^* : N(G^Y) \longrightarrow N(G^X)
$$

is a weak equivalence in $\mathbf{Set}_K^{\Delta^{op}}$ for any groupoid G by Lemma 4.1.6. Since the both G^X and G^Y are groupoids, $\pi(f)^* : G^{\pi(Y)} \longrightarrow G^{\pi(X)}$ is an equivalence of categories. Take $G = \pi(X)$, there exists an inverse functor of $\pi(f)$. Thus, f is a weak 1-equivalence. Conversely, we can prove that f^* is a weak equivalence if f is a weak 1-equivalence using the reverse procedure above. $\qquad\square$

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The notions of weak 1-equivalences in Top and $\text{Set}^{\Delta^{op}}$ are already known.

Definition 4.1.9. A morphism $f : X \longrightarrow Y$ in Top is called a *weak 1-equivalence* if the both induced maps $\pi_0(X) \longrightarrow \pi_0(Y)$ and $\pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ are isomorphisms for all $x \in X$. On the other hand, a morphism f in $\text{Set}^{\Delta^{op}}$ is called a weak 1-equivalence if the geometric realization $|f|$ is a weak 1-equivalence in Top. By Theorem 2.1.4, f is a weak 1-equivalence in Cat if and only if the nerve Nf is a weak 1-equivalence in $\text{Set}^{\Delta^{op}}$.

Theorem 4.1.10 ([DP95]). *There exists a cofibrantly generated model structure on $\text{Set}^{\Delta^{op}}$ which*
$$I' = \{\partial\Delta[n] \longrightarrow \Delta[n] \mid 0 \le n \le 2\}$$
is a set of generating cofibrations,
$$J' = \{\Lambda_j^n \longrightarrow \Delta[n], \Lambda_k^3 \longrightarrow \partial\Delta[3] \mid 0 < n \le 2, 0 \le j \le n, 0 \le k \le 3\}$$
is a set of generating trivial cofibrations, and the class of weak equivalences is the class of weak 1-equivalences. We denote the category of simplicial sets equipped with the above model structure by $\text{Set}_1^{\Delta^{op}}$.

Another way to define the 1-type model structure on Cat is to use the transfer principle in Theorem 3.4.7. We can transfer the above 1-type model structure on $\text{Set}^{\Delta^{op}}$ to Cat using the pair of adjoint functors $c : \text{Set}^{\Delta^{op}} \Longleftrightarrow \text{Cat} : N$.

Theorem 4.1.11. *The category Cat admits a model structure which I is a set of generating cofibrations, J is a set of generating trivial cofibrations, and the class of weak equivalences is the class of weak 1-equivalences, where I and J are given in Notation 1.1.12.*

Proof. We verify that the conditions in Theorem 3.4.7 are satisfied for
$$c : \text{Set}_1^{\Delta^{op}} \Longleftrightarrow \text{Cat} : N.$$

The domain of any functor in $c(I')$ or $c(J')$ is a small category consists of finite objects and finite morphisms, thus $c(I')$ and $c(J')$ permit small object argument. On the other hand, every functor in $c(J')$ is a trivial cofibration in Cat_1. Since trivial cofibrations in a model category are closed under retracts, pushouts and sequential colimits, a relative $c(J')$-cell is a weak 1-equivalence. Hence, N sends relative $c(J')$-cells to weak 1-equivalences. We can obtain a cofibrantly generated model structure on Cat with the set of generating cofibrations $c(I')$ and the set of generating trivial cofibrations $c(J')$ by Theorem 3.4.7. Finally, we have $c(I')$-inj$=$ I-inj and $c(J')$-inj $= J$-inj, therefore I is a set of generating cofibrations and J is a set of generating trivial cofibrations of the model category. $\qquad\square$

Corollary 4.1.12. *The model structure given in Theorem 4.1.11 coincides with the 1-type model structure on Cat given in Definition 4.1.1.*

Proof. The class I generates cofibrations in Cat_{JT} and Cat_1. The two model structures have the same classes of weak equivalences and cofibrations, thus these model structures are equal. $\qquad\square$

4.2 Fibrations and coverings

In this section, we characterize fibrations in $\mathbf{Cat_1}$. By observing the set of generating trivial cofibrations in $\mathbf{Cat_1}$, it turns out that fibrations in $\mathbf{Cat_1}$ coincide with categories fibered and cofibered in groupoids. Also we can see a covering in $\mathbf{Cat}$ as a special case of fibrations.

Definition 4.2.1. A functor $F : C \longrightarrow D$ is called a category fibered in groupoids if the following two conditions are satisfied:

1. For every object x in C and every morphism $f : y \longrightarrow F(x)$ in D, there exists a morphism $g : x' \longrightarrow x$ in C such that $F(g) = f$.

2. For every morphism $f : x' \longrightarrow x''$ in C and every object x in C, the map

$$\alpha : C(x, x') \longrightarrow C(x, x'') \times_{D(F(x), F(x''))} D(F(x), F(x'))$$

given by $g \mapsto (f \circ g, F(g))$ is bijective.

By reversing the above morphisms, we can define categories cofibered in groupoids.

Proposition 4.2.2. *A functor $F : C \longrightarrow D$ is a category fibered and cofibered in groupoids if and only if it has the right lifting property with respect to J.*

Proof. The first condition of categories fibered and cofibered in groupoids corresponds to the lifting property for J_1. The map α is surjective if and only if F has the right lifting property with respect to J_2. Finally, the map α is injective if and only if F has the right lifting property with respect to J_3. □

Theorem 4.2.3. *The 1-type model category $\mathbf{Cat_1}$ consists of the following structure. If $f : X \longrightarrow Y$ is a functor, then*

1. *f is a weak equivalence if and only if it is a weak 1-equivalence,*

2. *f is a cofibration if and only if it is injective on the set of objects,*

3. *f is a fibration if and only if it is a category fibered and cofibered in groupoids.*

Proof. By Corollary 4.1.12 and Proposition 4.2.2. □

Theorem 4.2.4 ([Lur09]). *A functor p is a fibration in $\mathbf{Cat_1}$ if and only if Np is a Kan fibration.*

Corollary 4.2.5. *A category G is a groupoid if and only if $N(G)$ is a Kan complex.*

Proof. By Proposition 4.1.7, groupoids coincide with fibrant objects in $\mathbf{Cat_1}$, and Kan complexes coincide with fibrant objects in $\mathbf{Set}_K^{\Delta^{op}}$. Theorem 4.2.4 implies that G is a groupoid if and only if $N(G)$ is a Kan complex since N preserves terminal object. □

Lemma 4.2.6. *If $p : E \longrightarrow B$ is a covering, then p is a fibration in $\mathbf{Cat_1}$.*

Proof. By the definition of coverings, p has the right lifting property with respect to J_1. Suppose that we have the following commutative diagram

$$
\begin{array}{ccc}
I_2 & \xrightarrow{f} & E \\
{\scriptstyle j_2}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\
[2] & \xrightarrow{g} & B.
\end{array}
$$

We obtain a morphism $\alpha : f(0) \longrightarrow f(2)$ over $g(0 \longrightarrow 1) : g(0) \longrightarrow g(1)$ by the right lifting property of p. We have $p\,(f(2 \longrightarrow 1) \circ \alpha) = g(0 \longrightarrow 2)$ and $f(2 \longrightarrow 1) \circ \alpha = f(0 \longrightarrow 2)$ by the unique lifting property, then p has the right lifting property with respect to J_2. The unique lifting property also implies that p has the right lifting property with respect to J_3, similarly. $\qquad\square$

Definition 4.2.7. Let $f : X \longrightarrow Y$ be a functor, then the category of fiber $f^{-1}(y)$ over $y \in Y_0$ is defined as the subcategory of X define by $f^{-1}(y)_0 = f^{-1}(y)$ and $f^{-1}(y)(a,b) = p^{-1}(1_y)$.

Proposition 4.2.8. *A functor $p : E \longrightarrow B$ is a covering if and only if p is a fibration in Cat_1 and the category of fiber $p^{-1}(b)$ is discrete for any $b \in B_0$.*

Proof. Let $p : E \longrightarrow B$ be a covering, then p is a fibration by Lemma 4.2.6, and every fiber has the only identity morphisms by the unique lifting property. Conversely, let p be a fibration with discrete fibers. Since p is a fibration, p has the right lifting property with respect to J_1. We will show that uniqueness of the lifting. For the following commutative diagram

$$
\begin{array}{ccc}
* & \xrightarrow{e} & E \\
\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\
[1] & \xrightarrow{f} & B
\end{array}
$$

we assume that $g, h : [1] \longrightarrow E$ satisfy $p \circ g = p \circ h = f$ and $g(0) = h(0) = e(*)$. The right lifting property of p with respect to J_2 implies that there exists $w : g(1) \longrightarrow h(1)$ such that $w \circ g = h$ and $p \circ w = 1_{f(1)}$. It follows that w is a morphism in $p^{-1}(f(1))$. However, $p^{-1}(f(1))$ has only identity morphisms, thus $w = 1$. Therefore, $g = h$. $\qquad\square$

4.3 A factorization of morphisms in the 1-type model category

The 1-type model category Cat_1 is equipped with a pair of factorizations of morphisms satisfying the set of axioms of model structure given by the small object argument [Hir03], [Hov99]. It is obtained by taking many pushouts and sequential colimits. This section gives another factorization of Cat_1 which induces the groupoidification in Definition 2.1.1 and universal covers in Definition 2.2.7.

Definition 4.3.1. Let $f : X \longrightarrow Y$ be a functor. A category E_f is defined by

$$(E_f)_0 = \{(x, y_*) \in X_0 \times \mathrm{Mor}(\pi(Y))_0 \mid f(x) = y_0\}$$

and

$$E_f((x, y_*), (x', y'_*)) = \{(g_*, g) \in \pi(X)(x, x') \times Y(y_n, y'_m) \mid y'_* \circ f(g_*) = g \circ y_* \in \pi(Y)\}$$

where y_n and y'_m are the last objects of y_*, y'_*, respectively. When $X = *$, the category E_f is precisely $\widehat{Y}$ in Definition 2.2.7. Define a functor $j : X \longrightarrow E_f$ by $j(x) = (x, 1_{f(x)})$ and $p : E_f \longrightarrow Y$ by $p(x, y_*) = y_n$, then we have $f = p \circ j$.

Proposition 4.3.2. *The functor* $p : E_f \longrightarrow Y$ *is a fibration in* Cat_1.

Proof. Suppose that we have the following commutative diagram

$$
\begin{array}{ccc}
* & \xrightarrow{\ \alpha\ } & E_f \\
{\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle p} \\
[1] & \xrightarrow[\ \beta\ ]{} & Y.
\end{array}
$$

Let $\alpha(*) = (x, y_*)$, then $\beta(0 \longrightarrow 1) : \beta(0) = y_n \longrightarrow \beta(1)$. Define a functor $\gamma : [1] \longrightarrow E_f$ by $\gamma(0) = \alpha(*) = (x, y_*)$, $\gamma(1) = (x, \beta(0 \longrightarrow 1) \circ y_*)$ and $\gamma(0 \longrightarrow 1) = (1, \beta(0 \longrightarrow 1))$. It makes the above diagram commutative, thus p has the right lifting property with respect to J_1.

Next, suppose that we have the following commutative diagram

$$
\begin{array}{ccc}
I_2 & \xrightarrow{\ \alpha\ } & E_f \\
{\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle p} \\
[2] & \xrightarrow[\ \beta\ ]{} & Y.
\end{array}
$$

The image of α describes the diagram

$$(x', y'_*) \xrightarrow{\ (g_*, g)\ } (x, y_*) \xleftarrow{\ (h_*, h)\ } (x'', y''_*)$$

in E_f. Since $\beta(1 \longrightarrow 2)$ is a morphism from the last object of y'_* to y''_*, the morphism

$$(h_* \circ g_*^{-1}, \beta(1 \longrightarrow 2)) : (x', y'_*) \longrightarrow (x'', y''_*)$$

gives a functor $[2] \longrightarrow E_f$ making the above diagram commutative. Thus, p has the right lifting property with respect to J_2.

Finally, suppose that we have the following commutative diagram

$$
\begin{array}{ccc}
CS^1 & \xrightarrow{\ \alpha\ } & E_f \\
{\scriptstyle j_3}\downarrow & & \downarrow{\scriptstyle p} \\
[2] & \xrightarrow[\ \beta\ ]{} & Y.
\end{array}
$$

The image of α describes the diagram

$$(x, y_*) \xrightarrow{(g_*, g)} (x', y'_*) \overset{(h_*, h)}{\underset{(h'_*, h')}{\rightrightarrows}} (x'', y''_*)$$

in E_f. Since $h_* \circ g_* = h'_* \circ g_*$ in $\pi(X)$,

$$h_* = h_* \circ g_* \circ g_*^{-1} = h'_* \circ g_* \circ g_*^{-1} = h'_*$$

in $\pi(X)$. Moreover, β implies that $h = h'$, then $(h_*, h) = (h'_*, h')$ and it gives a functor $[2] \longrightarrow E_f$ making the above diagram commutative, thus p has the right lifting property with respect to J_3. $\qquad\square$

Proposition 4.3.3. *The functor* $j : X \longrightarrow E_f$ *is a trivial cofibration in* Cat_1.

Proof. It is obvious that j is a cofibration, thus it suffices to prove that j is a weak 1-equivalence. A functor $r : \pi E_f \longrightarrow \pi X$ is defined by $r(x, y_*) = x$ and $r(g_*, g) = g_*$. This is an inverse functor of $\pi j : \pi X \longrightarrow \pi E_f$, therefore j is a weak 1-equivalence by Lemma 4.1.5. $\qquad\square$

Corollary 4.3.4. *The factorization given in Definition 4.3.1 satisfies the factorization axiom of model structure in Definition 3.1.1.*

Corollary 4.3.5. *Let C be a small category and let $q : C \longrightarrow *$ be the unique morphism to the terminal object $*$ in Cat. Then q is factored as $C \xrightarrow{j} E_q \xrightarrow{p} *$. The trivial cofibration $j : C \longrightarrow E_q$ is the canonical inclusion $C \longrightarrow \pi(C)$ associated to the groupoidification in Definition 2.1.1.*

Corollary 4.3.6. *Let $(C, *)$ be a pointed connected category and let $k : * \longrightarrow C$ be the embedding functor to the base point. Then k is factored as $* \xrightarrow{j} E_k \xrightarrow{p} C$. The fibration $p : E_k \longrightarrow C$ is the universal cover $\widehat{C} \longrightarrow C$ in Definition 2.2.7.*

4.4 Relations between the 1-type model category and other model categories

The model category Cat_1 is related to other model categories by the following pairs of adjoint functors

$$\mathrm{Grd} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathrm{Cat} \underset{c}{\overset{N}{\rightleftarrows}} \mathrm{Set}^{\Delta^{op}} \underset{S_*}{\overset{|-|}{\rightleftarrows}} \mathrm{Top}.$$

Proposition 4.4.1. *The pair of adjoint functors*

$$\pi : \mathrm{Cat}_1 \Longleftrightarrow \mathrm{Grd}_A : \iota$$

is a Quillen equivalence.

Proof. Since π preserves weak equivalences and cofibrations, Proposition 3.1.7 implies that (π, ι) is a Quillen adjunction. Furthermore, (π, ι) is a Quillen equivalence since the canonical inclusion $C \longrightarrow \pi(C)$ is a weak 1-equivalence for any small category C by Proposition 4.1.4. $\qquad\square$

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Proposition 4.4.2. *The pair of adjoint functors*

$$c : \mathrm{Set}_K^{\Delta^{op}} \Longleftrightarrow \mathrm{Cat}_1 : N$$

is a Quillen adjunction.

Proof. The functor N preserves fibrations by Proposition 4.2.4. For a cofibration $i : A \longrightarrow B$ in $\mathrm{Set}_K^{\Delta^{op}}$, the map on the set of vertexes $i_0 : A_0 \longrightarrow B_0$ is injective. The map on the set of objects of $ci : cA \longrightarrow cB$ is $i_0 : A_0 \longrightarrow B_0$, thus ci is a cofibration. It follows that c preserves cofibrations, hence (c, N) is a Quillen adjunction. $\qquad\square$

We will prove that Cat_1 is Quillen equivalent to $\mathrm{Set}_1^{\Delta^{op}}$. Recall the definition of fundamental groups of simplicial sets (Kan complexes) in Definition 1.3.19. Let $(K, *)$ be a pointed Kan complex. Two 1-simplices $x, y \in K_1$ satisfying $d_i(x) = d_i(y) = *$ for $i = 0, 1$ are homotopic, denoted by $x \simeq y$, if there exists a 2-simplex $z \in K_2$ such that $d_0 z = *$, $d_1 z = x$ and $d_2 z = y$. The fundamental group $\pi_1(K, *)$ is given by

$$\{x \in K_1 \mid d_i(x) = *, i = 0, 1\}/ \simeq .$$

Note that $\pi_1(K, *)$ is equal to the fundamental group of cK.

Lemma 4.4.3. *If K is a Kan complex, then cK is a groupoid.*

Proof. A morphism of cK from a to b is a class of sequence $e_1 e_2 \cdots e_n$ of 1-simplices of K. There exists $e \in K_1$ satisfying $e = e_1 e_2 \cdots e_n$ in cK since K is a Kan complex. Furthermore, there exists $d \in K_1$ such that $de = a$ and $ed = b$, thus all morphisms of cK are invertible. $\qquad\square$

Proposition 4.4.4. *The counit map $\eta : K \longrightarrow NcK$ is a weak 1-equivalence in $\mathrm{Set}^{\Delta^{op}}$ if K is a Kan complex.*

Proof. It is obvious that $\eta_* : \pi_0(X) \longrightarrow \pi_0(NcX)$ is an isomorphism since $X_0 \cong (NcX)_0$. The category cK is a groupoid by Lemma 4.4.3, hence the map η_* on fundamental groups is the following isomorphism

$$\pi_1(K, *) = cK(*, *) \cong cNcK(*, *) \cong \pi_1(NcK, *).$$

$\qquad\square$

Corollary 4.4.5. *The counit map $\eta : K \longrightarrow NcK$ is a weak 1-equivalence in $\mathrm{Set}^{\Delta^{op}}$ for any K.*

Proof. By the factorization in $\mathrm{Set}_K^{\Delta^{op}}$, there exists a Kan complex RK with a trivial cofibration $i : K \longrightarrow RK$ in $\mathrm{Set}_K^{\Delta^{op}}$ for any simplicial set K. The following diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ \eta\ } & NcK \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle Nci} \\
RK & \xrightarrow[\ \eta\ ]{} & NcRK
\end{array}
$$

is commutative, and $\eta : RK \longrightarrow NcRK$ is a weak 1-equivalence by Proposition 4.4.4 and also i is a weak 1-equivalence. By Proposition 4.4.2,

$$c : \mathrm{Set}_K^{\Delta^{\mathrm{op}}} \Longleftrightarrow \mathrm{Cat}_1 : N$$

is a Quillen adjunction, therefore c preserves trivial cofibrations by Proposition 3.1.7. Therefore, ci is a weak 1-equivalence in $\mathbf{Cat}$ and Nci is a weak 1-equivalence in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$. The diagram follows that $\eta : X \longrightarrow NcX$ is a weak 1-equivalence. $\quad\square$

Theorem 4.4.6. *The pair of adjoint functors*

$$c : \mathrm{Set}_1^{\Delta^{\mathrm{op}}} \Longleftrightarrow \mathrm{Cat}_1 : N$$

is a Quillen equivalence.

Proof. By Theorem 4.2.3, (c, N) is a Quillen adjunction. Suppose X is a cofibrant object in $\mathrm{Set}_1^{\Delta^{\mathrm{op}}}$ and G is a fibrant object in Cat_1 and $f : X \longrightarrow NG$ is a weak equivalence in $\mathrm{Set}_1^{\Delta^{\mathrm{op}}}$. We show that the map $f^\sharp : cX \longrightarrow G$ given by $cX \xrightarrow{cf} cNG \cong G$ is a weak equivalence in Cat_1. In the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & NG \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \cong} \\
NcX & \xrightarrow[Ncf]{} & NcNG,
\end{array}
$$

the map η is a weak 1-equivalence from Corollary 4.4.5, so is Ncf. Thus, cf and $f^\sharp$ are weak equivalences in Cat_1. Conversely, it is obvious that f is a weak equivalence in $\mathrm{Set}_1^{\Delta^{\mathrm{op}}}$ if $f^\sharp$ is a weak equivalence in Cat_1. $\quad\square$

4.5 The 1-type model category and stacks

Categories (co)fibered in groupoids appearing in the definition of fibrations in Cat_1 are closely related to stacks. A stack is a generalized notion of sheaf for a category. Although there are several definitions of stacks, Hollander gave one of them in terms of homotopy limits [Hol07].

Definition 4.5.1 (Stack). Let (C, J) be a site (see [MM92]). A functor (or presheaf) $F : C^{\mathrm{op}} \longrightarrow \mathbf{Grd}$ is a stack if

$$F(X) \longrightarrow \mathrm{holim} F(U_*)$$

(see in Definition 3.3.1) is an equivalence of categories for any cover $\{U_i \longrightarrow X\}$.

In order to deal with stacks using homotopy theory, Hollander consider a model structure on the category $\mathbf{Grd}^{C^{\mathrm{op}}}$ of functors valued in groupoids and the category $\mathcal{F}(C)$ of categories fibered in groupoids on a small category C. These categories are related to each other by the Grothendieck construction and the section functor.

The Grothendieck construction appearing in Definition 2.2.6 can be defined for a functor $F : C^{\mathrm{op}} \longrightarrow \mathbf{Cat}$. It gives a functor from the category $\mathbf{Cat}^{C^{\mathrm{op}}}$ of functors on C^{op} valued in small categories to the overcategory $\mathbf{Cat} \downarrow C$.

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Definition 4.5.2. For a functor $F : C^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, the Grothendieck construction $\mathrm{Gr}F$ is a category consisting of

$$\cdot (\mathrm{Gr}F)_0 = \{(x, a) \mid x \in C_0, a \in (Fx)_0\}$$

and

$$\mathrm{Gr}F((x, a), (y, b)) = \{(f, u) \mid f \in C(x, y), u \in Fx(a, Ffb)\}.$$

It is equipped with the canonical projection $p_F : \mathrm{Gr}F \longrightarrow C$. This correspondence gives a functor $\mathrm{Gr} : \mathrm{Cat}^{C^{\mathrm{op}}} \longrightarrow \mathrm{Cat} \downarrow C$. There exists a right adjoint functor of Gr. The section functor

$$\Gamma : \mathrm{Cat} \downarrow C \longrightarrow \mathrm{Cat}^{C^{\mathrm{op}}}$$

is defined by

$$\Gamma_p(x) = \{s : C \downarrow x \longrightarrow E \mid p \circ s = q\}$$

for any functor $p : E \longrightarrow C$, where $q : C \downarrow x \longrightarrow C$ is the canonical projection.

Proposition 4.5.3. $\mathrm{Gr} : \mathrm{Cat}^{C^{\mathrm{op}}} \Longleftrightarrow \mathrm{Cat} \downarrow C : \Gamma$ *is a pair of adjoint functors.*

Proof. For a functor $F : C^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ and a functor $p : E \longrightarrow C$, a natural isomorphism

$$\alpha : \mathrm{Cat}^{C^{\mathrm{op}}}(F, \Gamma_p) \longrightarrow \mathrm{Cat} \downarrow C(\mathrm{Gr}F, E)$$

is defined by $\alpha(t)(x, a) = t_x(a)(1_x)$ for $(x, a) \in (\mathrm{Gr}F)_0$. $\square$

Remark 4.5.4. The above pair of adjoint functors induces $\mathrm{Gr} : \mathrm{Grd}^{C^{\mathrm{op}}} \Longleftrightarrow \mathcal{F}(C) : \Gamma$.

Let us consider the following model categories. At this time, we do not need a site, but only a small category.

Definition 4.5.5. Let C be a small category, then we consider the following model categories.

- $\mathrm{Cat}_1^{C^{\mathrm{op}}}$ is the category of functors on C^{op} valued in small categories with the projective model structure which has objectwise weak equivalences (resp. fibrations) in Cat_1 as weak equivalences (resp. fibrations).

- $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ is the category of functors on C^{op} valued in groupoids with the projective model structure which has objectwise weak equivalences (resp. fibrations) in Grd_A as weak equivalences (resp. fibrations).

- $\mathcal{F}(C)_H$ is the category of categories fibered in groupoids with the induced model structure from $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ through the pair of adjoint functors (Gr, Γ), namely, f is a weak equivalence (resp. fibration) if Γf is a weak equivalence (resp. fibration) in $\mathrm{Grd}_A^{C^{\mathrm{op}}}$.

Theorem 4.5.6. *The above three model categories* $\mathrm{Cat}_1^{C^{\mathrm{op}}}$, $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ *and* $\mathcal{F}(C)_H$ *are Quillen equivalent to each other.*

Proof. Hollander shows that $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ and $\mathcal{F}(C)_H$ are Quillen equivalent by the pair of adjoint functors (Gr, Γ) in [Hol07]. Moreover, $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ and $\mathrm{Cat}_1^{C^{\mathrm{op}}}$ are Quillen equivalent since Grd_A and Cat_1 are Quillen equivalent by Proposition 4.4.1. $\square$

Let us consider a small category C with a Grothendieck topology J to discuss stacks. Hollander uses the left Bousfield localization of $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ or $\mathcal{F}(C)_H$ with respect to a class of morphisms induced by homotopy colimits (see in Definition 3.3.1) of covers in J.

Definition 4.5.7. Let (C, J) be a site. For an object $X \in C_0$, sometimes we identify it with the functor $C(-, X) : C^{\mathrm{op}} \longrightarrow \mathrm{Set} \subset \mathrm{Cat}$ by Yoneda lemma. For a cover $\{U_i \longrightarrow X\}$ on $X \in C_0$, the nerve U_* is the cosimplicial object in $\mathrm{Grd}^{C^{\mathrm{op}}}$

$$U_n = \coprod U_{i_0} \times_X U_{i_1} \times_X \cdots \times_X U_{i_n}.$$

Let S_J be the set of morphisms

$$S_J = \{\mathrm{hocolim} U_* \longrightarrow X \mid \{U_i \longrightarrow X\} \colon \mathrm{cover}\},$$

the left Bousfield localization of $\mathrm{Grd}_A^{C^{\mathrm{op}}}$ with respect to S_J is denoted by $\mathrm{Grd}_L^{C^{\mathrm{op}}}$. Similarly, the left Bousfield localization of $\mathrm{Cat}_1^{C^{\mathrm{op}}}$ with respect to S_J is denoted by $\mathrm{Cat}_L^{C^{\mathrm{op}}}$.

Also we can define a set S'_J of morphisms in $\mathcal{F}(C)$ by

$$S'_J = \{\mathrm{hocolim}\,(C \downarrow U_*) \longrightarrow C \downarrow X \mid \{U_i \longrightarrow X\} \colon \mathrm{cover}\}.$$

The left Bousfield localization of $\mathcal{F}(C)$ with respect to S'_J is denoted by $\mathcal{F}(C)_L$.

The following theorem is induced by Theorem 4.5.6

Theorem 4.5.8. *The above three model categories $\mathrm{Grd}_L^{C^{\mathrm{op}}}$, $\mathrm{Cat}_L^{C^{\mathrm{op}}}$ and $\mathcal{F}(C)_L$ are Quillen equivalent to each other.*

Theorem 4.5.9 ([Hol07]). *A functor on C^{op} valued in groupoids is a stack if and only if it is fibrant in $\mathrm{Grd}_L^{C^{\mathrm{op}}}$.*

Proof. A fibrant object in $\mathrm{Grd}_L^{C^{\mathrm{op}}}$ coincides with an S_J-local object. Hence F is fibrant if and only if

$$\mathrm{Grd}^{C^{\mathrm{op}}}(X, F) \longrightarrow \mathrm{Grd}^{C^{\mathrm{op}}}(\mathrm{hocolim} U_*, F) \cong \mathrm{holim}\,\mathrm{Grd}^{C^{\mathrm{op}}}(U_*, F)$$

is an equivalence of categories. The Yoneda lemma for 2-categories implies that $F(X) \longrightarrow \mathrm{holim} F(U_*)$ is an equivalence of categories. $\qquad\square$

We introduce one more model structure on the overcategory $\mathrm{Cat} \downarrow C$. induced from Cat_1.

Definition 4.5.10. For a small category C, $\mathrm{Cat}_1 \downarrow C$ is the overcategory $\mathrm{Cat} \downarrow C$ with a model structure, in which a map is weak equivalence, fibration or cofibration if it is one of Cat_1.

Theorem 4.5.11. *The pair of adjoint functors*

$$\mathrm{Gr} : \mathrm{Cat}_1^{C^{\mathrm{op}}} \Longleftrightarrow \mathrm{Cat}_1 \downarrow C : \Gamma$$

is a Quillen adjunction.

Proof. The Grothendieck construction sends the set of generating cofibrations of $\mathrm{Cat}_1^{C^{op}}$ to

$$\{1 \times i : C \downarrow x \times X \longrightarrow C \downarrow x \times Y \mid i \in I\}$$

in $\mathrm{Cat} \downarrow C$. These morphisms are cofibrations of Cat_1, hence Gr preserves cofibrations. Similarly, Gr sends the set of generating trivial cofibrations of $\mathrm{Cat}_1^{C^{op}}$ to trivial cofibrations of $\mathrm{Cat}_1 \downarrow C$. Hence, Gr preserves trivial cofibrations, and (Gr, Γ) is a Quillen adjunction. $\square$

Bibliography

[And78]   D. W. Anderson. Fibrations and geometric realizations. Bull. Amer. Math. Soc. 84 (1978), no. 5, 765–788.

[CGT04]   C. Casacuberta, M. Golasinski and A. Tonks. Homotopy localization of groupoids. Forum Math. 18 (2006), no. 6, 967–982.

[CRS12]   Claude Cibils, Maria Julia Redondo and Andrea Solotar. The intrinsic fundamental group of a linear category. Algebr. Represent. Theory 15 (2012), no. 4, 735–753.

[DP95]   Carmen Elvira-Donazar and Luis-Javier Hernandez-Paricio. Closed model categories for the n-type of spaces and simplicial sets. Math. Proc. Cambridge Philos. Soc. 118 (1995), no. 1, 93–103.

[GZ67]   P. Gabriel and M. Zisman. Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35 Springer-Verlag New York, Inc., New York 1967 x+168 pp.

[Hig05]   P. J. Higgins. Categories and groupoids. Repr. Theory Appl. Categ. No. 7 (2005), 1–178.

[Hir03]   Philip S. Hirschhorn. Model categories and their localizations. Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, 2003. xvi+457 pp.

[Hol07]   Sharon Hollander. A homotopy theory for stacks. arXiv:math/0110247v2.

[Hov99]   Mark Hovey. Model categories. Mathematical Surveys and Monographs, 63. American Mathematical Society, Providence, RI, 1999. xii+209 pp.

[JT91]   André Joyal and Myles Tierney. Strong stacks and classifying spaces. Category theory (Como, 1990), 213–236, Lecture Notes in Math., 1488, Springer, Berlin, 1991.

[KL01]   G. M. Kelly and Stephen Lack. V-Cat is locally presentable or locally bounded if V is so. Theory Appl. Categ. 8 (2001), 555–575.

[Lur09]   Jacob Lurie. Higher topos theory. Annals of Mathematics Studies, 170. Princeton University Press, Princeton, NJ, 2009. xviii+925 pp.

[Mac98]   S. Mac Lane. Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

[May92]   J. P. May. Simplicial objects in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. viii+161 pp.

[May99]   J. P. May. A concise course in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. x+243 pp.

[Min02]   Elias Gabriel Minian. Cat as a Λ-cofibration category. J. Pure Appl. Algebra 167 (2002), no. 2-3, 301–314.

[MM92]   S. MacLane and I. Moerdijk. Sheaves in geometry and logic. A first introduction to topos theory. Corrected reprint of the 1992 edition. Universitext. Springer-Verlag, New York, 1994. xii+629 pp.

[Qui67]   Daniel G. Quillen. Homotopical algebra. Lecture Notes in Mathematics, No. 43 Springer-Verlag, Berlin-New York 1967 iv+156 pp.

[Qui68]   Daniel G. Quillen. The geometric realization of a Kan fibration is a Serre fibration. Proc. Amer. Math. Soc. 19 1968 1499–1500.

[Qui73]   Daniel G. Quillen. Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[Rap10]   George Raptis. Homotopy theory of posets. Homology, Homotopy Appl. 12 (2010), no. 2, 211–230.

[Rez00]   Charles Rezk. A model category for categories. preprint. 2000.

[Str72]   Arne Strøm. The homotopy category is a homotopy category. Arch. Math. (Basel) 23 (1972), 435–441.

[Tho79]   R. W. Thomason. Homotopy colimits in the category of small categories. Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 91–109.

[Tho80]   R. W. Thomason. Cat as a closed model category. Cahiers Topologie Géom. Différentielle 21 (1980), no. 3, 305–324.