Doctoral Dissertation (Shinshu University)

Quantum Computing Model of Neurons and Their Networks

神経細胞とその回路網の量子計算モデル

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Hiroyuki Matsuura

松浦弘幸

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***Abstract*:**

We proposed the positive hypotheses of neural interferences based on physiological knowledge of neurons, ephapse and those quantum effects as engineering models. We thought the neural interferences of axons and synaptic interactions as ephapse are propagated by polaritons, which were a kind of quasi particles. The polaritons were essentially massive vector photon with spin 1. The polaritons, whose particles were relativistic, were strictly governed by Proca equation or quaternary Schrödinger equation. The polaritons were connecting between two ionic currents on phospholipid membrane of neuron. That membrane on their axons was propagating their excitations and action potentials by using polaritons. The Na^+ *currents, into insides of membranes of axons, cause the* K^+ *current's flow to outside of axons, and a series of those processes can generate the quantized polarization waves (polaritons). Various interferences as ephapse, synaptic and the other interactions were intermediated by polaritons. The polaritons were able to go through myelin sheaths by quantum effect. The one polariton makes possible to carry amount of information,* $9.38X10^{12}$ *bits/polariton, at 300 Kelvin. And we recognized to be required at least* $0.693k_BT$ *joules of energy to convey one bit of information. We thought that those quantum interferences were utilizing commonly to adjust our neural and brain's functions.*

It is known, as neural networks are fundamentals of brain's constructions. So, we proposed path integral method in order to calculate quantum probability amplitude for various networks, i.e. Amida lottery, electrical circulations and classical neural systems. Our starting points of new basic theory and calculation methods for quantum bifurcation, quantum circuits, and neural computer are based on path integrals of quantum theory. The problems of classical bifurcation were easily led to Schrödinger equation by considering Nelson's stochastic quantization method. Japanese Amida lottery was a kind of classical bifurcation models because of no interference between each path of lottery. And so we showed how to quantize electric circuits, Amida lottery and complex neural network by applying the method of path integrals. The bifurcation points of Amida lottery corresponded to diffraction point of polariton in quantum theory. Then we constructed the method of quantization of basic circuits as AND, OR and NOT. Moreover, we assumed that we could regard classical switches as scattering potentials (switch's operators). Those were quantization concepts, and those quantized circuits with switch operators corresponded to q-AND, q-NOT, and q-OR circuits. The Proca equation of polariton, which was relativistic field equation, approached to the quaternary Schrödinger equation when the motion of polariton was much slower than light velocity. The kernel K(b,a) for an integral, which was propagator and an

expression of the time development system, was related to an eigenfunction of Schrödinger equation. We found that the neuro-synaptic junctions were regarded as a kind of switch's potential, whose concepts led to quantization of neural networks by using path integrals.

By expanding method of the classical networks to quantum systems with wave equation and path integrals of polaritons, we could obtain both some tools and descriptions for quantum calculations of arbitrary neural circuits. The most important difference between the common (classical) neural network and quantum one are in with or without of an existence of interferences. The quantum system had essentially many interference's relationships in its system, and so its probability was related to the probability amplitude, wave functions and propagators, which were commonly complex functions. On the other hand, the classical probability never contained any interferences since it had in the real number field. And concretely we showed how those quantum methods, whose system contained much interference, were applied to the Bayes' theory, entropy of information theory, and the two-step neural network as a kind of multi channels. Moreover we succeed to obtain approximately output's solution of the quantum network, with mixing many quantum states, and we expressed the feature of network by means of perturbation and path integral description. And we found that our quantum neural network and polariton's model were connected with the common quantum information theory, classical neural system, and we showed quantum network was including some aspects of soft science. Concretely we showed that our methods were closely related to various areas as applications of fuzzy controls, classical neural systems, the classical Information theory and so on.

Keywords: polariton, quasi particles, polarization vectors, sodium ionic currents, potassium ionic currents, wave function, axon, neural network, quantum interferences, ephapse, dielectric materials, Proca equation, quantum circuits, quantum lottery, path integral, Hamiltonian, quaternary Schrödinger equation, superposition, neural network, wave function, propagator, Bayes' theory, entropy, quantum neuron, Schrödinger equation.

1. Artificial Neurons and Phenomena of Ephapse

Many excellent experiments for neuro-function and neural conduction have been performed by usage of micro-needles for neurons, and we have understood notable phenomena of neuro-physiological functions and their structures. Among all, one of the most famous researches is performed by Hodgkin & Huxley, who proposed physiological models based on physical cable theory, ionic currents (Na^+, K^+), local currents and conductions of action potentials [1]. Their model can be able to explain many phenomena of neuro-electrical physiology. In pathological area, Arvanitaki discovered the phenomena of ephapse, which means an existence of interference between many neural axons. When he stimulated one neuron and made action potentials (impulses) arise on the stimulated neuron, that impulses affected on another axon despite of having of no direct connections between two axons. So his discovery and experiments are thought him to make up an artificial synapse and neurons.

However, the ephapse have been believed not to be in the cases of healthy neuro-fibers[11]. It is said that ephapse was found in pathological neural axons, i.e., for examples, neuralgia, causalgia, and what we call, neuron's diseases. So, the axon's or synaptic interferences have been regarded as an evidence of wrong neurons. We have had negative images for the ephapse, whose sign are pathological neuron or symptom of demyelination[26].

We, however, would like to propose a positive hypothesis for ephapse or interferences of neurons in this paper. So I put on those following presuppositions.

Our healthy brain or normal neurons actively utilize electromagnetic interactions, (for examples, leakage current, polarization of membrane, noise current, ephapse, and so on), so as to adjust neuron's functions between each neuron, and so as to accomplish an integration of brain's functions. Note that we do not intend to discuss whether our neuron's model is correct or not, from standpoints of biology. We would like to only discuss from to engineering views and functions.

In the other word, we have an interest in following question: if it were the interferences between each neuron and the brain utilized those weak electromagnetic interactions so as to adjust its functions, we would like to show how neuron's images does change and what biophysical principle governs our neural networks. Moreover, we will propose how to get a mathematical expression of our neural networks. We have been researching for suitable descriptions for those weak electromagnetic nano- or meso- phenomena.

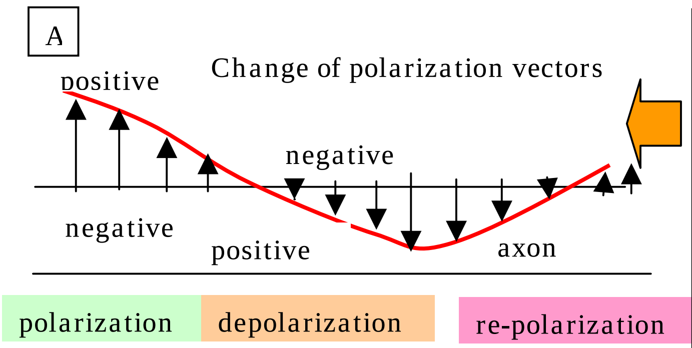
In following section, we intend to mention basic idea and theoretical requests in order to introduce both quantum method and concepts of quasi particles polaritons. Then we show quantum mechanism of neural-conduction based on dielectric of myelin sheath.

We assert in this paper that information for neural interferences as ephapse is propagated by polaritons, which are a kind of quasi particles, i.e., quantized polarization waves. We conclude that polaritons mean massive vector particle with spin 1, and shortly speaking, it is massive photon. Moreover, polaritons are closely related to many ionic currents (Na^+, K^+, Cl^- current) and those channels, when neurons and axons propagate action potentials (impulses). Thus, polaritons run on neural membranes along to axon, and they go easily through myelin sheath by quantum tunnel effects.

We would like to mention the concepts that those quantum interferences are useful to adjust and to harmonize our neural functions and brain's situation [2]-[5]. So, One of our purposes is to study effects of quantum neural-interferences. And our computer (brain), which is constructed by many quantum neurons, and sometimes make mistakes and causes false illusions by quantum effect of polaritons.

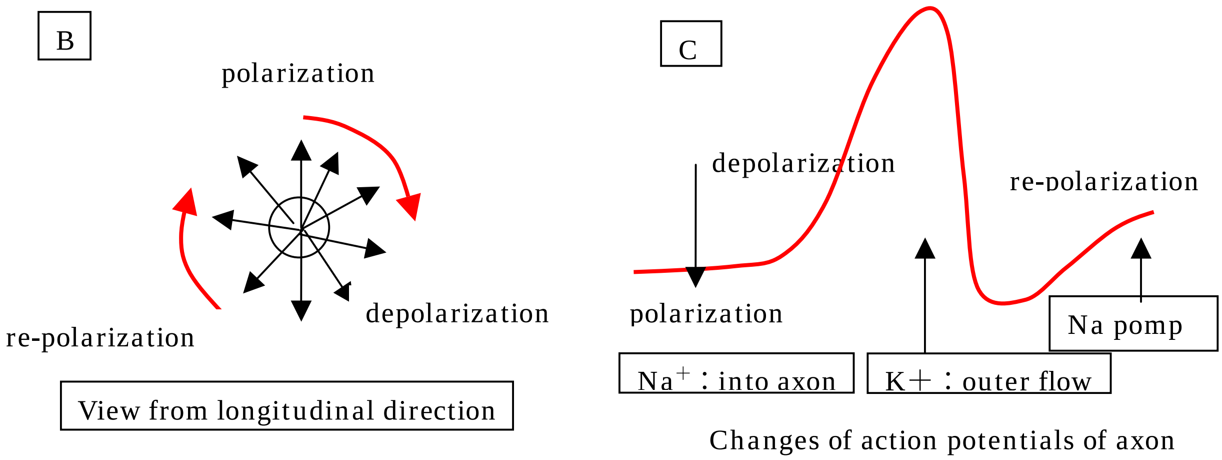
2. Polariton's Model of Neural Conduction

Axons of neurons have a series of polarization's processes: in short, there are the polarization, depolarization and re-polarization by Na^+- and K^+ -currents penetrating axon's membranes.



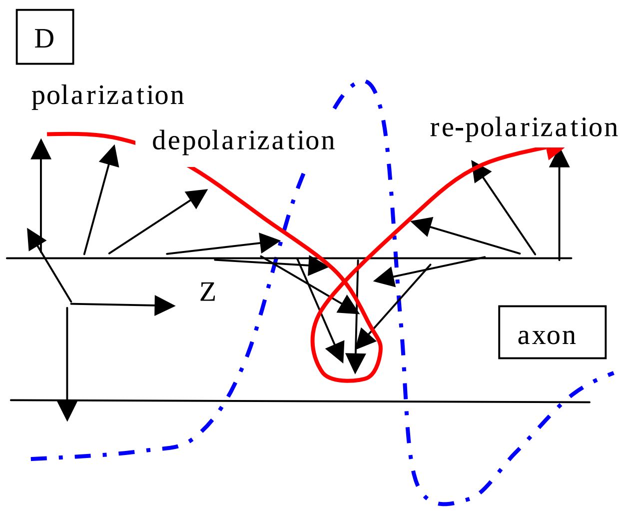
If we observe from this direction of conduction of excitation (action potentials), those processes can be described as rotating vectors. Those phenomena as polarization, depolarization and re-polarization are the quantized rotational vectors, which are quasi particles, what is called, polariton.

If we observe the changes the polarization vectors, we notice approximately to able to describe the changes of action potentials on axons as the rotating polarization vectors (FIGURE 1-A-D). The FIGURE 1-A shows the change of polarization vectors, which mean directions of ionic current and their magnitude. If we observe those polarization vectors, we know it safety to express classically as rotation of those vectors.



(FIGURE 1.-A,B,C,D) Theory of rotating polarization vectors

We regard neural conductions of action potentials (impulse) as propagation of the quantized polarizations vectors, which are correspond to the traveling quasi particles, polariton. Their motions (rotation of vectors and propagating polarization vectors) and the series of processes (polarization -depolarization-repolarization, etc) are caused by mainly ionic currents (Na^+-current, K^+-current, etc.). Those ionic currents are source of polaritons.



The trace of head of polarization vectors are described as spiral curves on the axon. The traveling of the process,polarization-depolarization-repolariztion, makes polarization wave. The quantized polarization waves are quasi particles as called polaritons.

Those currents become sources of polaritons, whose rotating vectors propagate on the neural membrane, and triggers of those two ionic currents arise the polarization waves, and the quantized polarization waves correspond to quasi particles, polaritons.

(A) This figure-A shows the feature of "the changes of magnitude of polarization vectors". According to the conduction of action potentials along to axons, the polarization vectors rapidly change their shapes, directions and magnitude (Fig-A).

(B) The process of conductions of action potentials hypothesizes to shown as rotation of polarization vectors, if we thought the polarization vectors travel along to longitudinal direction of axons (Fig-B).

(C) This picture shows each phase of action potentials, which are mainly generated by those currents, sodium ion's currents, potassium ion's currents and sodium pomp (Fig-C). Those currents are origins of polaritons.

(D) The inverted phase of polarization vectors (depolarization phase, center of Fig,1-D) is pictured, and the polarization vectors are propagating on the membrane of axon.

Those axon's membranes are constructed by phospholipid bilayer, which has characteristics of strong dielectric materials. Those dielectric materials can efficiently conduct the polariztion's waves, or its quantized quasi-particles, polaritons. After all, the quantized and rotating polarization vectors run along to longitudinal direction of axon. The real polaritons are quasi particles covered with a lot of water molecules and ions, which are made by electro-static interactions between bare polariton and waters' molecules (FIGURE 3).

3. Characteristic of Polatriton as Quantum Depolarization Waves

We are able to estimate physical characteristics of quasi polaritons. Considering saltatory conduction of excitations and of action potentials, we can estimate a rage of the existence of polaritons to be almost equals to the width of Ranvier ring, whose length is said to be about 1 μ m (FIGURE 2).

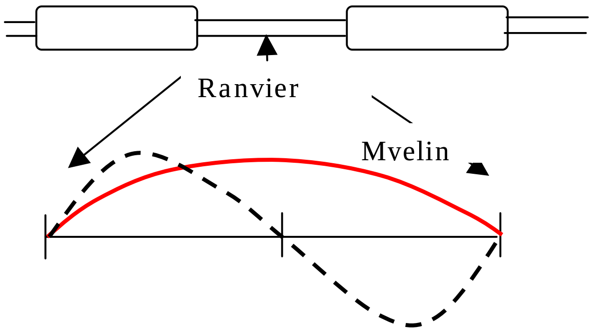


FIGURE 2. Polariton on Ravier Ring

If polaritons exist in the Ranvier Ring, it is reasonable to assume polaritons to be confined between myelin sheaths.
So, we are simply able to apply box type of potential model for the confinement of polaritons. The continuous line is the ground state of polaritons, which are bosons, the dashed line is correspond to the first excited state.

When the wave length of ground state of wave function is considered to be the width of Ranvier ring 1 μ m, the polaritons mass can easily calculate by following relation: the equation says

$$p = \frac{\hbar}{\lambda} = mv, \quad (1)$$

If we adopt the conducting velocity of myelinated axon v = 100m/s, then the wave length of wave function of ground state of polaritons become about equal to the width of Ranvier ring 1 μ m. This calculation for polariton's bare mass results in 6.7×10^{-30}kg. We know, mass of the bare polaritons has at most about ten times as heavy as that of electron mass. And the kinetic energy of a free bare polariton moving along to an axon is estimated as

$$E_K = \frac{1}{2}mv^2 = 2.07 \times 10^{-7} \text{ (eV per a polariton).} \quad (2)$$

That polariton's kinetic energy is so smaller than any specific energies, i.e., thermal energy at 300K = 3.0×10^{-2}eV, ATP hydrolysis = 2.0×10^{-1}eV and etc. (Table 1). And its energy is indicates 10^{-6} times smaller than hydrogen bonds of water molecules, and that energy is ten times larger than kinetic energy of electron moving at 100m/s speeds. Polaritons work as intermediates of electromagnetic interaction by propagation of polarization waves, and so those bare quantized waves (those are quantum particles), which are massive photons, have an average mass 6.7×10^{-30}kg with spin 1.

Those massive photons have serious problems. Generally speaking, biological nanomachines show good efficiency at room temperature, and their input energies almost equal to thermal fluctuation. According to the above common nano-machine's examples, we think that the polariton's kinetic energy should be nearly equal to thermal noise energy. If polaritons are always exposed under water rich circumstance, whose temperature indicates about room temperature T=300K, the energy of thermal noise reaches the value.

$$3/2k_BT = 6.3\times10^{-21}\ \mathrm{J} = 3.9\times10^{-2}\ \mathrm{eV}, \tag{3}$$

Judging from standpoint both Eq.(2) and Eq.(3), we gauss the bare polariton's kinetic energy is almost 10^{-5} times smaller than thermal noise. Those conditions cause serious problems, because of preventing polaritons from normal neural conductions and from traveling action potentials. Thus, the polaritons' kinetic energy is so small that polartions cannot work efficiently under water rich environmental like as human body, since polaritons' motions are interfered with thermal fluctuation and noise. At least, the polaritons, which are against thermal noise, are needed to become 10^5 times heavier than their average mass. Though that mass 6.7×10^{-30}kg is bare polariton's mass, we are able to estimate the quasi polariton mass (dressed mass), which mean the bare polariton to be covered with some ions and water molecules. Thus, the bare polariton is requested to become average 10^5 times heavier than its bare mass (FIGURE 3). Then the bare polariton needs to wear the water molecules, and an average quasi polariton's mass is guessed as

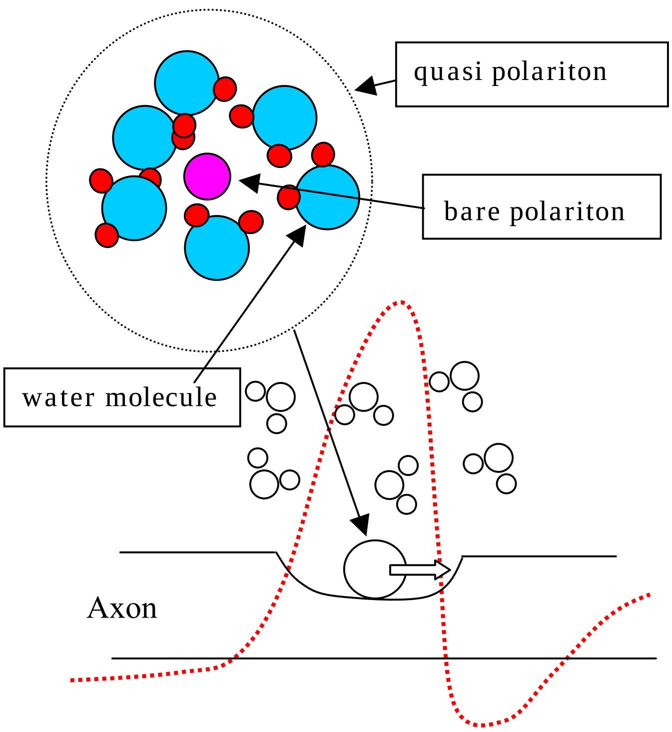
$$m_T \approx \frac{3k_BT}{v^2} = 1.3\times10^{-24}\ \mathrm{(kg)}. \tag{4}$$

And water molecule's mass is 3.1×10^{-26}kg, and the dressed polariton can sufficiently resist the thermal noise under room temperature at 300 Kelvin, if each bare polariton can attract the 41 water molecules at least. Thus, the quasi particle, polariton means

(dressed polariton; quasi polariton) = (bare polariton)+(dressed mass, water molecules).

$$m_T \approx 6.7\times10^{-30}\mathrm{kg} + 1.3\times10^{-24}\ \mathrm{(kg)}. \tag{5}$$

Note that we can detect that dressed polariton's mass, which are covered with many water molecules, however we cannot measure the bare polariton's mass. The polariton, by the quasi particle's mechanism, have energy of polariton as strong as that of thermal noise.



The Fig.3 shows the bare polariton to attract many water molecules by the electrostatic forces. And the bare polariton changes into dressed particle called quasi polariton. It is difficult to measure the mass of the bare polariton.
So, commonly we don't know the bare mass, but we can only detect the quasi(dressed) polariton's mass. The quasi polariton flows along to an axon. And many polaritons are related to various phenomena, i.e., ephapse, neural conduction, tunnel effect and an interference between each neurons.
Thus, polaritons mean quantized polarization waves.

FIGURE 3. Image of quasi particle (Polariton)

According to statistical mechanics, it is said that an order of fluctuation of particles is almost $N^{0.5}$. If we assume the length of human's axon to reach about 1m, and size of water molecule to be 2.0×10^{-10}m (2Å) at its length, about 5.0×10^{9} water molecules exist at the length 1 m per an axon at least. In this case, the particles' average fluctuation is about $N^{0.5}$, i.e., 7.0×10^{4} numbers' water molecules. The fluctuation of 7.0×10^{4} numbers' particles correspond to about 10^{-5}m at length, whose value is lager than the width of Ranvier ring, 1 μm. Since the quasi polariton's size is much smaller than the width of Ranvier ring and both particles' fluctuation and the width of Ranvier ring are less than the value of fluctuation, many of quasi polariotns can occupy their positions on both Ranvier ring. Moreover, that result gives us a suggestion that wave functions of polaritons make an invasion to an interior portion of myelin sheath.
Thus, polariton's momentum fluctuation is given as

(region of Polariton's existence of ground state) < (length of fluctuation of statistics)

$$\Delta p \cong \frac{\hbar}{\Delta x} \rightarrow 1.0^{-29} \text{ (kg m/s).} \tag{6}$$

That mass fluctuation is 1.0^{-31}kg, whose value is hundredth part of bare polariton's mass.

4. Polariton Conveying Information

Generally speaking, the thermal noise is against neural conductions of polariton being a kind of electrical signals. On the other hand, heat generates some sort of undesirable electrical signals. J.B. Johnson, who discovered the electrical fluctuations caused by heat, in terms of a fluctuation voltage produced across a resistor. That fluctuation voltage (noise voltage) is called thermal noise and a hot resistor is a potential source of noise power. In this case, the most noise power N is described as

$$N = k_B TW \tag{7}$$

where k_B is Boltzmann constant, T means temperature of resistor in degree Kelvin, and W is the band width of noise in cycles per second. Obviously the bandwidth W depends only on the properties of our measuring device. Notice that the noise power is given by Eq.(7), where T is temperature of the object. And the thermal noise constitutes a minimum noise, which we should permit, and additional noise sources only make the situation of apparatus and measurement worse. The noise determines the power required to send messages (conduct on axon). And in order to transmit C bits/s, we must have a signal power P related to noise power N by a relation. Referencing Eq.(7), we have

$$C = W \log\left(\frac{1+P}{N}\right) = W \log\left(\frac{1+P}{k_B TW}\right). \tag{8}$$

The P is a given signal power. If the P/k_BTW becomes very small compared with unity, the Eq.(8) gives the following relations: the Eq.(8) becomes

$$C = \frac{1.44P}{k_B T} \tag{9}$$

or

$$P = 0.693 k_B TC\,. \tag{10}$$

The Eq.(10) says that, even when we use a very wide band width, we need at least a power $0.693k_BT$ joule per second to send one bit per second, so that on the average we must use an energy $0.693k_BT$ joule for each bit of information we transmit (C=1). At 300 Kelvin, we obtain the signal power 1.7×10^{-2}eV (per/s)/(bit/s) from Eq.(10). The thermal noise of Eq.(3) is larger than the value of $0.693k_BT$, 1.7×10^{-2}eV (per/s)/(bit/s), and so polariton needs have the same level of energy as or larger than thermal noise in order to convey the neural information according to classical mechanics. However the polariton is a quantum particle and massive photon with spin 1, we should apply quantum effects to the Eq.(7). Herry Nyquist proposed to give an expression for thermal noise applied to all frequencies of light. His expression for thermal noise in a bandwidth W_i was

$$N_i = \frac{\hbar\omega_i W_i}{\exp(\hbar\omega_i / k_B T) - 1}. \tag{11}$$

Quantum effects become important when one polariton energy is comparable to or lager than k_BT. If a polariton energy >> k_BT, then most noise power N_i is given as

$$N_i \approx x(1 + x + x^2 \cdots)\hbar\omega_i W_i = \left[\frac{\hbar\omega_i}{k_B T}\exp\left(-\frac{\hbar\omega_i}{k_B T}\right)\right]k_B T W_i, \qquad x \equiv \exp\left(-\frac{\hbar\omega_i}{k_B T}\right). \tag{12}$$

We take sum for the suffix i and an average of the Eq.(12),

$$\langle N \rangle = \langle E_i W_i \rangle k_B T, \tag{13}$$

Taking the relations Eq.(14), we will obtain the similar expression to classical result

$$\langle E_i \rangle \equiv \left[\frac{\hbar\omega_i}{k_B T}\exp\left(-\frac{\hbar\omega_i}{k_B T}\right)\right], \quad \langle W_i \rangle = const, \tag{14}$$

from the Eq.(13), and if $\langle E_i \rangle = 1$. then Eq.(13) is

$$\langle N \rangle = \langle E_i \rangle k_B T W \Rightarrow \langle N \rangle = k_B T W. \tag{15}$$

Note that the Eq.(15) means approximately a quantum expression of the most noise power which is different from the Eq.(7). And the frequency above, being the exact expression for thermal noise Eq.(11), depart fundamentally from the valid expression at low frequency Eq.(7). It is said that there are the quantum limitations other than the imposed thermal noise as Eq.(11) or Eq.(13). It turns out that ideally $0.693k_BT$ joule per second to send one bit per second is still the limit, and it is impossible to change the above limiting value. The energy per polariton is h ν , and ideally the energy per bit is $0.693k_BT$. (We showed examples of maicro-scopic energy levels of various particles)

TABLE

TABLE.1. Kinetic energy & thermal fluctuation

	Energy (eV)
Polariton's kinetic energy	2.0 X 10-7 eV
Electron's kinetic energy at 100 m/s	3.2 X 10-8 eV
Hydrogen bond	1.0 X 10-1 eV
Thermal energy at 300K	3.0 X 10-2 eV
ATP hydrolysis	2.0 X 10-1 eV

[Nano-machine shows good efficiency at room temperature, and an input energy almost equals to thermal fluctuation.]

Thus, ideally polariton can carry information, and we can know the bits per polariton at 300 Kelvin,

$$\frac{\hbar\omega}{0.693k_BT} = 2.31\times10^{-3}\nu \quad \text{(bits/polariton)}. \tag{16}$$

If we can use frequency of thermal noise, then the polariton carries amount of information, 9.38 X10^{12} bits/polariton, at 300 Kelvin from Eq.(15). And we recognize to be required at least $0.693k_BT$ joules of energy to convey one bit of information.

5. Description of Polariton

Polaritons, having an electromagnetic interaction, should be massive photon with spin1. If the polaritons are traveling along to z-axis, those polaritons having right-handed polarized light are expressed as summation and superposition between state of x-polarized light and that of y-polarized light. This right-handed polarized photon is given as

$$\begin{aligned} \left|\mathbf{E}(z,t)\right\rangle &= E_0\boldsymbol{\varepsilon}_{\mathrm{x}}\exp i(kz-\omega t) + E_0\boldsymbol{\varepsilon}_{\mathrm{y}} i(kz-\omega t+\pi/2) = E_0\boldsymbol{\varepsilon}_{\mathrm{x}}\exp i(kz-\omega t) \\ &+ iE_0\boldsymbol{\varepsilon}_{\mathrm{y}}\exp i(kz-\omega t) = \left|\pi_x\right\rangle \exp i(kz-\omega t) + i\left|\pi_y\right\rangle \exp i(kz-\omega t) \\ &\left|\pi_x\right\rangle = E_0\boldsymbol{\varepsilon}_{\mathrm{x}}, \qquad \left|\pi_y\right\rangle = E_0\boldsymbol{\varepsilon}_{\mathrm{y}} \end{aligned} \tag{17}$$

with the ε_i vectors of polarized light. We attempt to practice normalization right-handed polarized light:

$$\left|\mathbf{E}(z,t)\right\rangle = \frac{1}{\sqrt{2}}(\left|\pi_x\right\rangle + i\left|\pi_y\right\rangle)\exp i(kz-\omega t). \tag{18}$$

We obtain an expression for right-handed polarization state. Using this expression

(18), we practice to differentiate with variable z,

$$\frac{\partial^2 \left| \mathbf{E}(z,t) \right\rangle}{\partial z^2} = -k^2 \left| \mathbf{E}(z,t) \right\rangle, \tag{19}$$

and then we multiply both sides by $-\hbar^2/2m$ and add -V | E(Z,t)>. We notice following relation:

$$E = \hbar\omega = (\hbar k)^2 / 2m + V \tag{20}$$

And we multiple the state vector to both side on Eq.(20). Finally, we obtain Schrödinger equation, which describes motion of three components of polariton, with time dependent factors as shown in Eq.(21).

$$i\hbar \frac{\partial \left| \mathbf{E}(z,t) \right\rangle}{\partial t} = \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \hat{V}(z,t) \right] \left| \mathbf{E}(z,t) \right\rangle. \tag{21}$$

Performing derivations as well as the previous procedure, we obtain the relativistic expression of polariton. We use a relation

$$\hat{E}^2 \left| \mathbf{E}(z,t) \right\rangle = m^2c^4 \left| \mathbf{E}(z,t) \right\rangle + \hat{p}^2 c^2 \left| \mathbf{E}(z,t) \right\rangle + V \left| \mathbf{E}(z,t) \right\rangle \tag{22}$$

$$\because \hat{E} = i\hbar\partial/\partial t, \quad \hat{p} = i\hbar\partial/\partial z,$$

which is named Klein-Gordon equation. And its quantum expression is given as

$$(\hbar\omega)^2 = m^2c^4 + c^2(\hbar k)^2 + V. \tag{23}$$

The Eq.(23) means a relativistic spin 1 (vector) particle moving under potential V. Note that common Klein-Gordon equation has one component, scalar particle, however, the Klein-Gordon equation of Eq.(22) possesses three components vectors. An electromagnetic theory says, in quantum mechanics, that vector potential **A** and scalar potential ϕ is more essential elements than electric field **E** and magnetic field **B.** Thus, according to Maxwell equations, the electromagnetic fields **E** & **B** are related by the vector and scalar potentials **A** & ϕ:

$$\mathbf{B}(x,t) = rot\mathbf{A}(x,t)$$

$$\mathbf{E}(x,t) = -grad\phi - \frac{1}{c}\frac{\partial \mathbf{A}(x,t)}{\partial t} \tag{24}$$

$$A^\mu = (\phi(x,t), \mathbf{A}(x,t))$$

The Eq.(24) teaches that those vectors and scalar potential (**A** & ϕ) obey the Klein-Gordon equation, because **B** & **E** is satisfied with the Klein-Gordon equation as shown in Eq.(22). We introduce strength of an electromagnetic field $F^{\mu\nu}$, whose expression connects quaternary A^μ with both electromagnetic fields **B** & **E.** The $F^{\mu\nu}$ is defined as

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{25}$$

$$\because \mathbf{B}(x,t) = (B^1, B^2, B^3), \quad \mathbf{E}(x,t) = (E^1, E^2, E^3), \quad A^{\mu} = (\phi, \mathbf{A})$$

The polariton of massive photon, quantized particle with spin 1, whose equation of four components is similar to the Klein-Gordon equation of massless photon. The polariton's Lagrangian density is given as

$$\ell = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_{\mu} A^{\mu} - j_{\mu} A^{\mu}, \tag{26}$$

whose expression gives rise to Proca equation (relativistic massive vector's equation), when we apply variational principle for Eq.(26). The Proca equation with an interaction between polariton and current j^{μ}

$$\partial_{\mu} F^{\mu\nu} + m^2 A^{\nu} = j^{\nu} \tag{27}$$

$$\because j^{\nu}(x) = (\rho(\mathbf{x},t), \mathbf{i}(\mathbf{x},t)),$$

is automatically satisfied with Lorentz condition, if that source term $j^{\mu} = 0$ or current conservation law holds correct. (in Eq.(27), we use natural unit system). So under Lorentz condition, the Eq.(27) becomes simply form:

$$(\partial_{\mu} \partial^{\mu} + m^2) A^{\nu} = j^{\nu}. \tag{28}$$

Comparing Eq.(28) with Eq.(22), we notice the corresponding relation between term of V**E**(x,t) and the j^{μ} current. If we consider the current j^{μ} is generated by major two ionic currents, sodium current J_{Na} and potassium current J_K, the total current j^{μ} through axon's membrane becomes as

$$(\partial_{\mu} \partial^{\mu} + m^2) A^{\mu} = j^{\nu}_{Na} + j^{\nu}_{K}. \tag{29}$$

And we notice those currents to be a source of generating many polaritons.

To derive non-relativistic polariton's equation from relativistic equation (29), we need return from the wave function A^{μ} of natural unite to that of MKS unite:

$$A^{\mu}(\mathbf{x},t) = \varphi^{\mu}(\mathbf{x},t) \cdot \exp\left(-\frac{i}{\hbar} mc^2 t \right). \tag{30}$$

Then, we split the time dependent of A^{μ} into two terms, and then the one's term is containing the rest polariton's mass, another is common wave term $\varphi(x,t)$. In the non-relativistic limit, the kinetic energy E_k is so smaller than energy of rest mass that we can reduce it to non relativistic form as

$$E_K = E - mc^2, \quad E' << mc^2 , \tag{31}$$

And non-relativistic kinetic energy E_k means

$$\left| i\frac{\partial \varphi^\mu}{\partial t} \right| \approx E_K \varphi^\mu << mc^2 \varphi^\mu . \tag{32}$$

Hence we have

$$\begin{aligned} &\frac{\partial A^\mu}{\partial t} \approx -i\frac{mc^2}{\hbar}\varphi^\mu \cdot \exp\left(-\frac{i}{\hbar}mc^2 t\right) \\ &\frac{\partial^2 A^\mu}{\partial t^2} \approx \left[-i\frac{2mc^2}{\hbar}\frac{\partial \varphi^\mu}{\partial t} - i\frac{m^2c^4\varphi^\mu}{\hbar^2}\right] \cdot \exp\left(-\frac{i}{\hbar}mc^2 t\right). \end{aligned} \tag{33}$$

Inserting all above approximations into following relativistic relation:

$$p^\mu p_\mu A^\nu + m^2c^2 A^\nu = j^\nu / c , \tag{34}$$

we finally obtain the non-relativistic expression like as Schrödinger equation. That result is non-relativistic polariton's relationship with quaternary components,

$$\begin{aligned} &i\hbar\frac{\partial A^\mu}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]A^\mu \\ &A^\mu = (\phi, \mathbf{A}), \quad j^\nu \hbar^2 / (2mc) \Leftrightarrow \hat{V}A^\nu . \end{aligned} \tag{35}$$

Then A_0 is scalar potential ϕ, and we remove the rest mass term in the non-relativistic limit, the final polariton's equations becomes a set of the quaternary Schrödinger equation:

$$\begin{aligned} &i\hbar\frac{\partial \varphi^0}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\varphi^0 \\ &i\hbar\frac{\partial \varphi^a}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\varphi^a , \\ &\because \varphi^\mu(x) = (\varphi^0(\mathbf{x},t), \varphi^a(\mathbf{x},t)), \quad a = 1,2,3. \end{aligned} \tag{36}$$

Notice that that equation describes a motion of non charged polariton. As a charged polariton is expected to obey to the complex Klein-Gordon equation for electromagnetic interaction. We multiply Eq.(28) by complex conjugate of A_ν, and take the complex conjugate of Eq.(28) and multiple it by A_ν

$$\partial_\mu (A_\nu^* \partial^\mu A_\mu + A_\nu \partial^\mu A^{\nu *}) = j^{\nu *} A_\nu - j_\nu A^{\nu *}. \tag{37}$$

We can define a quaternary current vector J_μ using MKS unit system,

$$J^\mu \equiv \frac{ie\hbar}{2m}(A_\nu^* \partial^\mu A_\mu + A_\nu \partial^\mu A^{\nu *}), \tag{38}$$

and we are able to define the polariron's charge

$$Q \equiv \frac{ie\hbar}{2mc}(A_\nu^* \partial^0 A_\mu + A_\nu \partial^0 A^{\nu *}). \tag{39}$$

Where the Q is time component of A_ν. And the polariton's field A_μ are divided into real part and imaginary part like as Eq.(40)

$$\begin{aligned} A^\mu &= \frac{1}{\sqrt{2}}(A_1^\mu + iA_2^\mu) \\ j^\mu &= j_1^\mu + ij_2^\mu \end{aligned} \tag{40}$$

If the two fields $A_1{}^\mu$ and $A_2{}^\mu$ separately satisfy a Klein-Gordon equation with having the same rest mass m, then the equations can be replaced by one equation for a complex field,

$$\begin{aligned} \left(\partial_\nu \partial^\nu + \frac{m^2c^2}{\hbar^2}\right) A^\mu &= j^\mu \\ \left(\partial_\nu \partial^\nu + \frac{m^2c^2}{\hbar^2}\right) A^{\mu *} &= j^{\mu *} \end{aligned} \tag{41}$$

According to pi-mesons example, should pay attentions for expressions for positive charge's polariton, negative charge's polariton and for neutral particle. And each of equations has following fields:

$$\begin{aligned} A_+^\mu &= A^{\mu *} = \frac{1}{\sqrt{2}}\left(A_1^\mu - iA_2^\mu\right) \\ A_-^\mu &= A^\mu = \frac{1}{\sqrt{2}}\left(A_1^\mu + iA_2^\mu\right) \\ A_0^\mu &= A^\mu = A^{\mu *} \end{aligned} \tag{42}$$

We adopt the same procedure from Eq.(40) to Eq.(42), finally we will reach non-relativistic similar form to Eq.(36). We would like to emphasize that the neutral polariton is characterized by a real eave function, and the charged polaritons have to be represented by complex wave functions.

If the polariton with electric charge q, interacting with both sodium current J_{Na} and potassium current J_K, moves under electromagnetic fields, then a minimal interaction is written as

$$\nabla \to \nabla - \frac{q}{c}\mathbf{G}$$
$$H \to H - qG^0. \tag{43}$$

Then the above relation (43) is inserted into vector's type of Schrödinger equation (35) or (36). Performing after simple calculations, we finally have the complex equation,

$$i\hbar\frac{\partial\varphi^\mu}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right)\varphi^\mu + \frac{q\hbar}{mc}i\left(\mathbf{G}\cdot\nabla + \frac{1}{2}\nabla\cdot\mathbf{G}\right)\varphi^\mu + \left(\frac{q^2}{2mc^2}\mathbf{G}^2 + qG^0\right)\varphi^\mu \tag{44}$$

The complexity of Eq.(44) comes from possession of polariton's electric charge and an electromagnetic interaction, and that equation cannot be reduced to simply form like as Eq.(36) because of containing self-energy of polariton. The neutral polariton can convey only both momentum and energy, and do not carry electromagnetic charge, however, the charged polariton carries its momentum, energy and charged current. We need address as many body problems or quantum field theory since the charged polaritons have many interactions among others.

6. Necessity of Fundamental Equation and Concepts of Quantum Circuits

Hitherto, we have been discussing functions of each neuron, for example, polorization, depolayzation, repolarization and quantization of those processes. At following stage, we would like to refer to some connected neurons systems, what we call, and neural networks.

6.1 Overview of Quantum Theory of Neuron

The models of neurons, their networks and those conducting mechanism are not only important bases of biological brain's functions, but also they have been producing many algorithms and their concepts of soft computing as neuro-fuzzy controls, and as mechanical learning models in many engineering's and information's branches [10]-[13]. However, those models have been based on an independence of each axon of neuron, and so we named those networks as classical ones. We have been hypothesized in classical models that there was not an electromagnetic interference between axons of neurons or synaptic junctions. So, a lot of physiological books say that, each neuron holds independence of each other, and there are not electromagnetic interactions between axons and synapses, because the neurons are governed the law of "all or nothing", and those electromagnetic effects are much small since neurons are covered with lipid nonconductor's membranes, myelin sheath. Action potentials traveling on the axon and the neural processes (polarization, depolarization, repolarization processes), have been believed not to affect on another axon and an ionic current for a long while [1],[10],[29]. They say that each neuron is independent and there is not the interference between each axon of neurons.

According to our hypothesis, however, we have been proposing the other theory and new engineering models accompanied by quantum effect: Each neuron has a lot of interferences caused by polarization of the membrane, leak currents, and ionic currents (Na^+, K^+) and so on. Neurons have many ionic channels, their currents, and polarizations, whose phenomena generate electromagnetic interactions on our brain's surfaces and white matter as we are possible to detect its field by SQUID. Thus, we know that each neuron gives rise to a holistic macro electromagnetic field, and that electromagnetic field governs the function of each neuron [10].

In many previous sections, we referred to another evidence of neural interference, ephapse and the artificial neurons. And we mentioned Prof. Arvanitaki discovered the phenomena of ephapse, which was interference between two neural axons [10]-[13]. When he stimulated one neural axon and generates action potentials, that signal affected on another neuron, despite of defection of direct connections between two neurons. He is said to be the first researcher who made up an artificial neuron. His experiments showed that each neuron had directly neural interferences based on the electromagnetic interactions. Truly, we know that pathological states correspond to neuralgia and causalgia. However we positively assumed that our normal brains always actively utilized those electromagnetic interactions so as to make up our holistic and harmonic neural system. At next steps, we should obtain the basic equations for those electromagnetic interactions of between each of neuron.

We mention those possible forms are the quaternary Schrödinger equations or its relativistic version, what is called, Proca equation. Moreover, an agency for those electromagnetic interactions is polariton, which is a kind of massive photon. The polariton is the quantized polarization wave on dielectric (cell membrane), and it has the spin-value of one (spin 1). From the standpoint of the mesoscopic science, all electromagnetic interactions should be described as elementary processes based on the interactions of massless or massive photons (polariton), because macro electromagnetic phenomena can be reduced to an approximation of quantum electromagnetic dynamics (Q.E.D.). In some previous sections, we referred to the necessity of polariton, and showed the quantization's process for macro electromagnetic phenomena of neurons [10]. The relativistic quantized electromagnetic field of neurons is fundamentally governed by the Proca equation. And we show that the Proca equation can be reduced to the quaternary Schrödinger equation of polariton, since a propagating velocity of the polariton (quantized polarization waves) on neurons was so much slower than that of light in vacuum [10].

6.2 Total Picture of Quantum Network Systems

We attempted to give the descriptions for the polariton's motions on neural axon by using both path integrals and the reduced Proca equation, that was quaternary Schrödinger equation of polariton[10]-[13]. So, we would like to make up the calculating toolbox for polariton's motion, and to show applications for Amida lottery, bifurcations, circuits, scattering problems, and for network systems. In order to describe the polariton's theory (quaternary Schrödinger equation), we think the Feynman's path integral is suitable for the neural conductions and of neuron's interferences. We can automatically introduce quantum effects of polaritons to the network systems, and its expression is much similar to classical mechanical Lagrangian,[4],[38] (Reference to Appendix-1, A1-1, Equivalence to Schrödinger Equation). Moreover, we know that the description of path integral is perfectly equivalent to that of Schrödinger equation. [2]-[4],[6],[19].

At the beginning of section 7, we mention that a bifurcation's problems of decision tree and multi step slit are related to Markov process. So, according to probability's theory, those processes can be expressed as the generalized stochastic equation, i.e., it is Ito equation. Applying Nelson's method, we can reduce that stochastic equation to Schrödinger equation of the wave function ϕ [14], whose process is called the stochastic quantization. On the other hand, Proca equation approximately becomes the quaternary Schrödinger equation of electromagnetic potential (ϕ, $\mathbf{A}$) in the case of the slow polariton's movement [10],[2]-[3]. The polariton, which is massive photon, should obey quaternary Schrödinger equation in non-relativistic area. And the quaternary Schrödinger equations approach to the ordinary Schrödinger equation--- we pay attention to one component's equation of electromagnetic potential--- if a change of the magnetic field is so small (the vector potential $\mathbf{A}$ is constant, or $\delta\mathbf{A}\approx 0$)[10],[4],[2].

Thus, the polariton's motion can approximately expressed by Schrödinger equation of scalar potential ϕ, and that ϕ is related to the bifurcation problems of classical mechanics, information theory and the stochastic equation[6],[4]. After we explained the principle of Feynman path integrals in subsection 7.2, and we calculated an action S for free polariton and for a harmonic oscillator, we apply those path integrals to the descriptions for the Amida lottery and a slit in section 8. They are examples of quantum bifurcation's problems of polariton. In section 9, we discuss some quantum descriptions for simple circuits (for examples, AND-, NOT-, OR-circuit and their complex ones) and switches by using path integral. Then we know the path integral is one of the powerful tools so as to describe the quantum networks and circuits [20]-[23].

The section 10 is mentioned to a perturbation method of Schrödinger equation. Then, we express that our description of neural network based on path integrals automatically lead to perturbation series. Then we mention that the switches of network and circuits are regarded as synaptic junctions or scattering potential of polariton. The section 11 is shared into an explanation of mathematical tools by using path integral's descriptions.

Main theme was to give the ways that we can express the quantum networks containing much interference. Then we described the quantization tools for neural networks, Amida lottery, quantum circuits and many complex diagrams. In our neural networks, the polariton conveyed physical information, and polariton was quantizatied particle of the action potentials (impulses) of neurons [4],[18]. Thus, our description's method and its development mean the quantum theory of network, bifurcation and circuits. For examples, one of great mathematician, R. Penrose said that our brain cell had many micro-turbines, which worked as conductors causing superposition of wave functions. He thought that those wave functions made reduction to only one wave function when we determined something for various problems [24].

We, however, don't intend to discuss whether his theory is true or not, from biological standing points. And we would like to only pick up his concepts that our brain utilizes quantum effect, and that the brain belongs to a kind of quantum circuit. We have been thinking that quantum interferences were playing important roles for our thinking processes.

Therefore, we described the idea of a quantum circuit and new theory for quantum computers of neural computations in following some sections. So we would like to show those quantization-methods of the bifurcation, Amida lottery and decision trees, which contained some fundamental ideas for quantum interferences and the reductions of wave functions.

7.Uncertainty and Superposition

First we would like to discuss a classical bifurcation that contains fundamental problems. The bifurcation is related to both probability and stochastic equations, and its theme leads to Schrödinger equation through Nelson's method, (stochastic quantization) [5],[14].

7.1 Classical bifurcations and Nelson's Stochastic Method

There is much difference between classical bifurcation and quantum bifurcation. The farmer is related to classical probability whose value is always the positive and real number. However, the latter takes complex number, whose function is called probability amplitude.

And the probability amplitude can be connected with a solution of Schrödinger equation. The classical probability cannot automatically expressed interference by superposition principle. However, the probability amplitude essentially contains much interference between each bifurcated branch. And the interference, which arises from superposition principle, plays a lot of important role in our quantum neural theory.

In this section, we would like to show that problems of decision tree can be regarded as a kind of Brownian motion (Markov process), and then we should notice that Brownian motion is governed with Ito equation (general stochastic equation). And according to Nelson's method (stochastic quantization), the Ito equation reaches automatically Schrödinger equation. Thus, the problems of the decision tree can be rewritten into Schrödinger equation of complex function $\chi(X,t)$ by both Fokker-Planck equation and Chapman equation.

At first, we show that small particles (for example electrons or photons) are flowing on the branches of bifurcation-diagram (a kind of decision tree) (Fig.4). We assume that the particles on the diagram diverge for each branch at an equivalent probability, 50%.

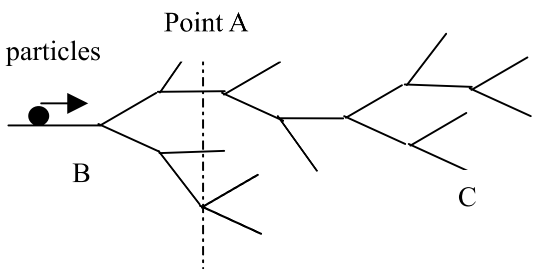


FIGURE 4. Multi-step bifurcation's Problem

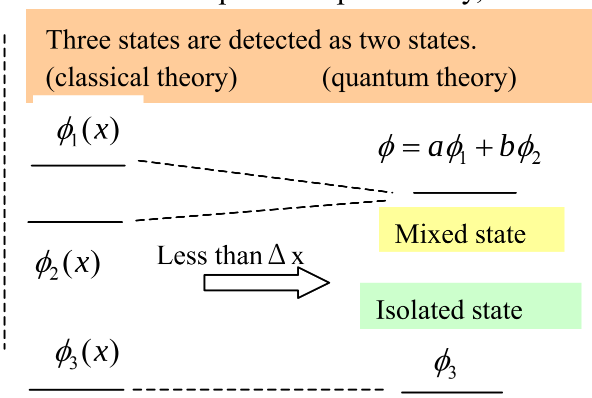


FIGURE 5. Uncertainty and sensitive limitation

When we attempt to deform the branch lines of bifurcation diagram FIGURE 4, then the diagram becomes a following feature: that bifurcation diagram can be represented as the random walk's problem.

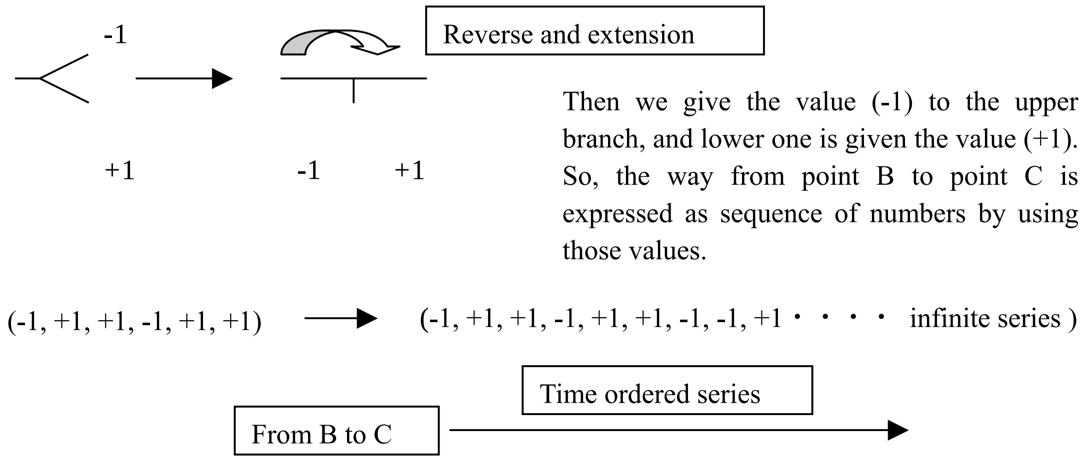


FIGURE 6. Random walk and bifurcation

If we concretely can show the path (from B to C), we obtain a sequence of numbers: the sequence is (-1, +1, +1, -1, +1, +1). If we hypothesize that bifurcations of the diagram make an infinite series, the above finite bifurcation becomes an infinite random walk's problem. So we notice that the infinite sequence is much similar to Markov process or Brownian motion in one dimension. Thus that Brownian motion truly is expressed by stochastic differential equation [4]-[5].

So, We would like to start from a generalized stochastic equation, what is called, Ito equation,

$$dX(t) = b(X(t), t)dt + A(t)dw(t), \tag{45}$$

then the dw(t) has following characteristics of Brownian motion. (deviation A(t): diffusion coefficient, and an average b: drift coefficient).

$$\langle dw \rangle^2 = \langle w(t+\Delta t) - w(t) \rangle^2 = \beta \Delta t \tag{46}$$

$$\langle dw \rangle = \langle w(t+\Delta t) - w(t) \rangle = 0. \tag{47}$$

According to Nelson's stochastic quantization method with stochastic variable X(t), the trace of a particle is divided into two parts. The one is an anterior average derivative, and another is posterior average derivative. Those terms are defined as

$$Df \equiv \lim_{\Delta t \downarrow 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t} \middle| f(s) \text{ is fixed for } s \le t \right\rangle$$ anterior average derivative

(48)

$$D_* f \equiv \lim_{\Delta t \downarrow 0} \left\langle \frac{f(t) - f(t-\Delta t)}{\Delta t} \middle| f(s) \text{ is fixed for } s \ge t \right\rangle .$$ posterior average derivative

(49)

and both average velocities for Brownian motion are calculated as

$$DX(t) = b(X(t), t), \quad D_* X(t) = b_*(X(t), t) . \tag{50}$$

An acceleration a(t) of Brownian particle was defined by Nelson method, and the a(t),

$$a(t) = \frac{1}{2}\left(D_* D + D D_* \right) X(t) , \tag{51}$$

is obtained by performing above derivative for Eq.(50) . We introduce two new variables, u and v: those are

$$v = \frac{1}{2}(b + b_*), \quad u = \frac{1}{2}(b - b_*). \tag{52}$$

Thus, the acceleration a(t) becomes

$$a(t) = -A^2(t) \frac{\beta}{2} \frac{\partial^2 u}{\partial X} + \frac{1}{2} \frac{\partial}{\partial X}\left(v^2 - u^2 \right) + \frac{\partial v}{\partial t} = -\frac{1}{M} \frac{\partial V}{\partial X} . \tag{53}$$

The symbol M means Brownian particle's mass (we think polariton's mass), and the V is potential energy. The Eq.(53) corresponds to Newtonian equation of motion for Brownian particle, and it is said to be mechanical condition. Applying the anterior derivative to Chapman equation, we can define an operator ($A_T f$) of Eq.(55) [14]. The (ρ (X,t_0|Y,t) means probability that the particle which existed in an initial condition (X, t_0) reaches the point Y at time t) [14]. The operator ($A_T f$) is expressed as

$$\left(A_T f \right)(X) = \lim_{\Delta t \downarrow 0} \int dY \cdot f(Y) \frac{\rho(X, t_0 | Y, t + \Delta t) - \rho(X, t_0 | Y, t)}{\Delta t} . \tag{54}$$

Then we can obtain another expression of anterior derivative,

$$\left(A_T f \right)(X) = Df(X) = b(X,t) \frac{\partial f}{\partial X} + A^2 \frac{\beta}{2} \frac{\partial^2 f}{\partial X^2} . \tag{55}$$

We multiple $\rho(X_0, t_0 | X, t)$ to Eq.(55) , and we practice an integration: we finally have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial X}(b\rho) + \frac{\beta}{2} \frac{\partial^2}{\partial X^2}\left(A^2(t) \rho \right). \tag{56}$$

That is Fokker-Planck equation. For the b_*, we have a similar equation:

$$-\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial X}(b_* \rho) + \frac{\beta}{2} \frac{\partial^2}{\partial X^2}\left(A^2(t) \rho \right). \tag{57}$$

We add up both equations, Eq.(56) and Eq.(57), whose equations assimilate with one equation that represents a condition of motion for polariton:

$$\frac{\partial u}{\partial t} = -A^2 \frac{\beta}{2} \frac{\partial^2 v}{\partial X^2} - \frac{\partial}{\partial X}(uv). \tag{58}$$

To unify both condition of mechanics and that of motion, we would like to introduce a complex variable, $\chi(X,t) \equiv u + iv$. We transfer two variables u,v into a single variable χ,

$$\chi \equiv A^2 \beta \frac{\partial}{\partial X} \ln \Psi \tag{59-1}$$

$$\Psi(X,t) \equiv \phi(X,t) \exp\left(-\frac{i}{A^2\beta} \int \eta(t) dt \right). \tag{59-2}$$

and the transcription into single equation is achieved as

$$i \frac{\partial}{\partial t} \phi(X,t) = \left[-\frac{A^2\beta}{2} \frac{\partial^2}{\partial X^2} + \frac{1}{A^2 \beta M} V(X) \right] \phi(X,t). \tag{60}$$

If we take $A^2\beta \to \hbar/M$, we find Eq.(60) to be the common Schrödinger equation. So, the probability density ρ (X,t) is given as

$$\rho(X,t) = \left| \phi(X,t) \right|^2, \tag{61}$$

by a complex probability amplitude ϕ. If we take $t \to it$, Eq.(61) is reduce to Feynman-Kac equation. However, there is difference between Schrödinger equation and Feynman-Kac equation. The Feynman-Kac equation has always real number's solution. On the other hand, the Schrödinger equation almost takes complex number's solution. Thus, the Feynman-Kac equation can describe only classical bifurcation and its probability. However, the Schrödinger equation, whose solution is permitted to have the complex number (probability amplitude), is truly suitable for descriptions of interferences between each quantum state. We should notice that the complex number is an essential characteristic for quantum theory, and that the real number is a character of classical bifurcation problem. And the classical bifurcation's problem is always reduced to Weiner process (Brownian motion) and Markov process. So the classical bifurcation is quantized through the Nelson's method [14]. Thus, the classical stochastic problem can be translated into quantum one by introducing the complex variable and the probability amplitude.

We would like to discuss effects of superposition of the probability amplitude, and we mention those of the sensitive limitation caused by uncertainties. If all paths of FIGURE 5 are governed by uncertainty principle, we find the quantum fluctuations and interferences to exist between each bifurcation's branches. And the fluctuations of position Δx should satisfy the following relation, which is uncertainty:

$$\Delta x \geq \hbar /(\Delta p)\,. \tag{62}$$

So, a path less than the range Δx , is directly governed by effects of quantum mechanics.

We can easily explain the difference between quantum bifurcation and classical one. If particles obey to single-step's bifurcation, a total state vector is written as the superposition and linear combination of all base state vectors. Let's consider two state's model, i.e., those quantum states are ϕ_1 and ϕ_2. If there are those states within uncertainty's range Δx, then a total state ϕ is the summation of the two states:

$$\phi = a\phi_1 + b\phi_2\,. \tag{63}$$

Thus the total probability density of the above state is expressed as

$$|\phi|^2 = |a|^2|\phi_1|^2 + |b|^2|\phi_2|^2 + a^* b\phi^*_1\, \phi_2 + ab^*\phi_1\phi^*_2\,. \tag{64}$$

We notice that the first and second terms of Eq.(64) correspond to classical probability densities. The their and fourth terms, which are expression of quantum effects, mean quantum interference's terms. Uncertainty principle tells us that we cannot detect them as the different two states, if their states do not keep away more than the fluctuations' range Δx from each state (FIGURE 5). As uncertainty of momentum Δp gradually goes to the large value, it is more difficult for us to observe an aspect of bifurcation of particles. So, it will be more clear the difference of both the classical probability and the quantum one.

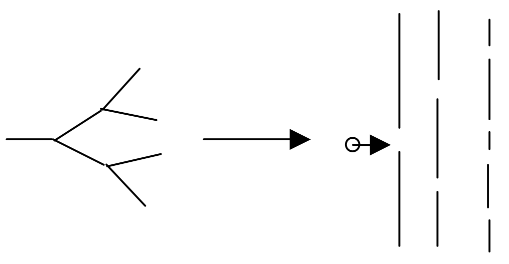


FIGURE 7. Multi-step bifurcations & slits

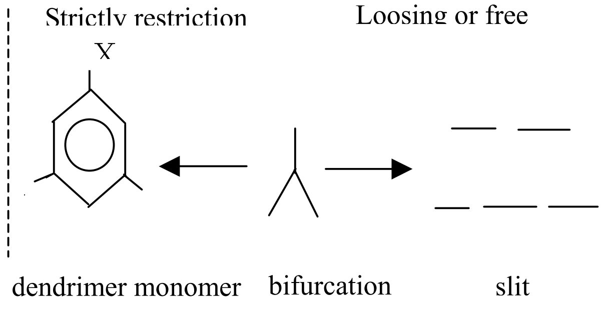


FIGURE 8. Various restricted conditions

(Explanation of FIGURE 7)

To fix particles on the nano-scale conductors (wires), an external force or some potential is impressed on the particles. If it were not for those restricting conditions, many of particles deviate from their paths or conductors, and then they behave as free particles. And we can apply an example of conductors to the model of slits. So we look upon the bifurcation diagrams as multi-step slits when those restricted conditions are going to weaken.

(Explanation of FIGURE 8)
When the range of uncertainty Δ x is nearly equal to sizes of atoms (Å), those processes approach to molecular wires of dendrimer monomers.

7.2 Description of Feynman path integrals

We would like to mention the principle of Feynman path integrals, and intend to apply its method to the motion of free polariton. Subsequently, we describe the scattering problem or the diffraction of the polariton, by its integrals in order to obtain mathematical tools. At the beginning, we consider an action S of particle whose generalized Lagragian has the following form,

$$L = a(t)t^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t). \tag{65}$$

An action S of its motion is given by the time's integral of the Lagrangian between two fixed points, i.e. starting point a and ending point b. We determine the Feynman's kernel K(b,a) that is defined as

$$K(b,a) = \int_a^b \exp\left[\frac{i}{\hbar}S\right]Dx(t), \qquad \because S \equiv \int_a^b L(\dot{x},x,t)\,dt\,, \tag{66}$$

(a: starting point of path, b: ending point of path), (Reference to Appendix-1, A1-2). Here if we attempt to define a quantum action S[x(t)] in an interval [a,b], then the quantum variable x(t) should be divided into two parts. Thus, its variable x(t) is composed of classical path term x_c(t) and quantum fluctuation δ (t), and so we have a relation, $x(t) = x_C(t) + \delta(t)$. And the integral (Dx(t)) should be performed over all paths in the interval [a,b]. Then the action S[x(t)] becomes

$$S[x(t)] = S[x_C(t) + \delta(t)] = \int_{ta}^{tb} dt[a(t)(\dot{x}_C^2 + 2\dot{x}_C\dot{\delta} + \dot{\delta}^2) + \cdots]. \tag{67}$$

If it were not for all δ terms, then Eq.(66) equals to just the classical mechanical action S_c. Notice that S_c contains the only classical variable x_c(t). On the other hand, the quantum action S[x(t)] is composed of two parts. They are the classical action S_c and the second quantum fluctuation's term in Eq.(68),

$$S[x(t)] = S_C[x_C] + \int_{ta}^{tb} dt[a\dot{\delta}^2 + b\delta\dot{\delta} + c\delta^2]. \tag{68}$$

Thus, the kernel K(b,a), which is calculated in [a,b], can be written as

$$K(b,a) = \int_a^b e^{(i/\hbar)Sc[b,a]} \cdot \exp\left[\frac{i}{\hbar}\int_{ta}^{tb}[a\dot{\delta}^2 + b\delta\dot{\delta} + c\delta^2]dt\right]Dx(t). \tag{69}$$

We would like to give an explicit S[x(t)] and kernel of free particle. So the classical action S_c are described as

$$L = m\frac{\dot{x}^2}{2} \quad \text{and} \quad S_C[b,a] = \frac{m}{2}\frac{(x_b - x_a)}{t_b - t_a}. \tag{70}$$

Thus, the kernel of Eq.(70) of the free particle are given as

$$K(b,a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right]. \tag{71}$$

Finally the existence probability of free polaritons at point b, P(b)dx, is becomes

$$P(b)dx = \frac{m}{2\pi\hbar(t_b - t_a)}dx \propto |K(b,a)|^2. \tag{72}$$

Moreover, the wave function of Schrödinger equation Ψ(b) is expressed by the kernel K(b,a), and then we have a simple relation,

$$\psi(b) = \int_{-\infty}^{\infty} K(b,a)\psi(a)\,dx_a, \quad b \equiv (x_b, t_b),\, a \equiv (x_a, t_a). \tag{73}$$

The quantum-polarized waves, which are composed of many photons (there are massive photons), are considered as assembles of harmonic oscillators. The Lagrangian of harmonic oscillator, which means quantum particles of polariton's field, is given as

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2}{2}x^2. \tag{74}$$

Then the kernel is calculated by the same method as the free particle:

$$K = F(T) \cdot \exp\left[\frac{i}{\hbar}S_C\right], \tag{75}$$

$$S_C = \frac{im\omega}{2\hbar\sin\omega T}\left[(x_a^2 + x_b^2)\cos\omega T - 2x_a x_b\right], \tag{76}$$

$$F(T) = \left(\frac{m\omega}{2\pi i\hbar\sin\omega T}\right)^{1/2}. \tag{77}$$

We give some comments on the calculation of path integral. The all paths (branches) of particle is divided into N divisions so as to obtain the kernel of the propagating particle from point a to point b. The kernel means that we find out a particle at an initial point a, and then it goes to the point a to point x_1. Then it goes ahead from x_1 to x_2. Finally the particle from x_{N-1} arrives at an endpoint b. So, the final kernel K(a,b) is given by multi integrals and product of infinitesimal kernels K(i+1,i), i = a, 1,2, ,b.

$$K(b,a) = \int \cdots \int dx_1 \cdots dx_{N-1} K(b, N-1) \cdots K(i+1,i) \cdots K(1,a). \tag{78}$$

When the particles to go ahead from (x_i,t) to (x_{i+1},t+ε) during an infinitesimal time interval ε, an explicit expression of Eq.(78) is

$$K(i+1,i) \equiv \langle i+1 | i \rangle = \exp\left[\frac{i\varepsilon}{\hbar} L(\frac{x_{i+1}-x_i}{\varepsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}) \right], \quad \text{L : Lagrangian.} \tag{79}$$

The second term of kernel K(i+1,i) corresponds to an expression of an inner product using Dirac bra vector < i+1| and ket vector |i >. Moreover, notice that the inner product<B|A>contains a time development operator, U-hat,

$$\langle B | A \rangle \equiv \langle x_B | \hat{U}(t_B, t_A) | x_A \rangle, \quad \hat{U}(t_B, t_A) \equiv \exp\{ -i\hat{H}(t_B - t_A)/\hbar \}, \tag{80}$$
$$B = (x_B, t_B), A = (x_A, t_A)$$

And the above H-hat is Hamiltonian of Schrödinger equation. The motion of particle from point a to point b reduces to the Dirac bra & ket vector description,

$$K(b,a) = \int \cdots \int dx_1 \cdots dx_{N-1} \langle b | N-1 \rangle \cdots \langle i+1 | i \rangle \cdots \langle 1 | a \rangle \equiv \int \cdots \int dx_1 \cdots dx_{N-1} \phi[x(t)] \tag{81}$$
$$\equiv \lim_{\varepsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \prod_{i=1}^{N-1} \langle i+1 | i \rangle$$

The Eq.(81) mentions to take inter product between the (i+1)-th bra and the (i)-th ket vectors and we have got to perform integration over all variables x_i.

8. Description of Quantum Bifurcation and Diffraction

I would like to discuss a relationship between path integral and bifurcation diagram in this section. And I apply the path integral to descriptions of polariton's motion on a slit and on Amida lottery. The path integral is another expression of quantum mechanics, and it is perfectly equivalent of Schrödinger equation. According to path integral, the probability P(a,b) is proportion to the absolute square of kernel K(b,a), i.e. P(b,a)$\propto$|K(b,a)|^2. So, the final amplitude K(b,a) is the sum of contribution of each path ϕ[x(t)],

$$K(b,a) = \sum\nolimits_{over\ all\ path} \phi[x(t)]. \tag{82}$$

The weight of each path is proportional to an exponential of the action S:

$$\phi[x(t)] = const. \times \exp\left(\frac{i}{\hbar} S[x(t)] \right). \tag{83}$$

At first we consider a bifurcation diagram of a single-step slit (FIGURE.5).

$$S[x(t)] = \int L(x, \dot{x}, t)\, dt = \int (T - V)\, dt\,. \tag{84}$$

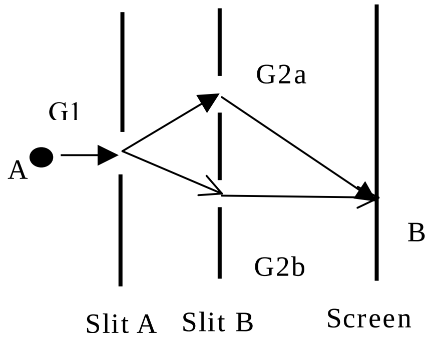


FIGURE 9. Interference of a single-step slit

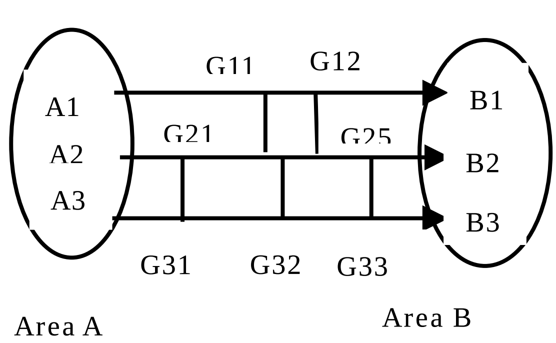


FIGURE 10. Quantum Amida Lottery Circuit

A particle goes through a hole G1 of slit A, and then it experiences the bifurcation by slit B. Finally this particle reaches from the point A to the point B (FIGURE.5). As shown in Eq.(81), the path is written as

$$\phi[x(t)] = \lim_{\varepsilon \to 0} \prod_{i=0}^{N-1} K(i+1, i)\,. \tag{85}$$

I would like to show one example of diffraction in the point x+ α_c at time T. When a free particle goes ahead from the point x to x+ α_c, it is diffracted in the point x+ α_c by a slit. After that diffraction, the particle arrives at a point (x_2,t_2) on the screen. The probability amplitude ϕ [x(t)] with the diffraction becomes

$$\phi[x(t)] = \int_{-b}^{b} d\alpha \langle x_2, t_2 | x_1 + \alpha, T \rangle \langle x_1 + \alpha, T | x_1, t_1 \rangle, \tag{86}$$

at the point (x+ α_c , T). Note that the rage of that integral is limited by an interval [-b,b], which is a size of the hole of slit (not infinite). If we assume a Gaussian slit of the width 2b whose shape is described by exp[- α^2/2b], then we can perform integration of Eq.(86). The result of probability amplitude is given by Eq.(87), since the particle goes through either hole G2-a or G2-b: that result is shown as

$$\phi[x(t)] = \sqrt{\frac{m}{2\pi i\hbar}} \left[T\tau \left(\frac{1}{T} + \frac{1}{\tau} + \frac{i\hbar}{b^2 m} \right) \right]^{-1/2} \times \exp\left[\frac{im}{2\hbar} \left(\frac{x^2}{\tau} + \frac{x_1^2}{T} \right) - \frac{(im/\hbar)^2}{4(im/2\hbar)^2} \cdot W \right] \tag{87}$$

Where W means $\quad W = \dfrac{-x/\tau + x_1/T}{1/\tau + 1/T + (i\hbar / b^2 m)}, \qquad \tau = t_2 - T,\; x = x_2 - x_1\,.$

Finally, the total wave function of FIGURE 9 becomes a summation of both paths, G1 →G2a→B and G1→G2b→B. If there is not an interaction at point G1 on slit A and a particle (polariton) freely goes through the slit G1, the particle obeys complete condition at point G1,

$$\int dx_{G1} \left| G1 \right\rangle \left\langle G1 \right| = 1 . \tag{88}$$

of kernel is calculated by Eq.(88). Thus the total result of amplitude is given as

$$\phi_{all}(x) = \iint dx_{G1} dx_{G2a} \left\langle B \middle| G2a \right\rangle \left\langle G2a \middle| G1 \right\rangle \left\langle G1 \middle| A \right\rangle + \iint dx_{G1} dx_{G2b} \left\langle B \middle| G2b \right\rangle \left\langle G2b \middle| G1 \right\rangle \left\langle G1 \middle| A \right\rangle .$$

Then $\left\langle G_1 \middle| A \right\rangle \equiv \left\langle G_1 \middle| \hat{U}(t_{G1}, t_0) \middle| A_0(t_0) \right\rangle, \qquad \hat{U}(t - t_0) = \exp(-i\hat{H}(t - t_0)/\hbar)$. (89)

Notice that those ket vectors | > in Eq.(89) are not a constant vectors, but they contain the time development factors which are related to Hamiltonian of Schrödinger equation. If a particle has no interaction with all slits, then Eq.(89) simply reduces to free particle's (free polariton) expression from the point A to the B,

$$\phi_{free}(x_{B,A}) = \left\langle B(t) \middle| A(t_0) \right\rangle = K(B, A) = \prod_A^B \left\langle j+1 \middle| j \right\rangle \tag{90-1}$$

, from point A → to point B.

If a single slit is set in the point c and the particle is diffracted at that point c (A< c <B), then a trace of particle has following expression:

$$\phi_{dif}(x_{B,A}) = \int \left\langle B \middle| c \right\rangle \left\langle c \middle| A \right\rangle dx_C = \int K(B,c) K(c,A) dx_C = \int \prod_C^B \left\langle j+1 \middle| j \right\rangle \cdot \prod_A^C \left\langle k+1 \middle| k \right\rangle dx_C$$
$$= \mathrm{A} \rightarrow (\mathrm{c}) \rightarrow \mathrm{B}. \tag{90-2}$$

An Amida lottery is discussed as an example of complex bifurcations and that lottery is a kind of multi-slit(FIGURE 10). So, Japanese Amida lottery is commonly regarded as one of the examples of classical probability problems. To translate the classical lottery into quantum one, we apply the path integral for classical Amida lottery and introduce quantum interferences into classical Amida lottery. So, those processes are a kind of quantization of Amida lottery. As represented in FIGFURE 10, the photon is diffracted at those following points, {G11, G12, G21, G22, G23, G24, G25, G31, G32, G33}. This quantum Amida lottery has a lot of paths so as to go ahead from area A to area B, because of sum for all possible paths.

$$\phi[B1, A] = C_{A11}\varphi[B1, A1] + C_{A21}\varphi[B1, A2] + C_{A31}\varphi[B1, A3] . \tag{91}$$

An each term of right side of Eq.(91) is given by path integrals. The ϕ [A1→B1] is

$$\phi[B1,A1]=\int dx_{G11}dx_{G12}\langle B1|G12\rangle\langle G12|G11\rangle\langle G11|A1\rangle+\int dx_{G24}dx_{G12}dx_{G11} \cdot dx_{G22}dx_{G23}\langle B1|G12\rangle\langle G12|G24\rangle\langle G24|G23\rangle\langle G23|G22\rangle\langle G22|G11\rangle\langle G11|A1\rangle. \quad (92)$$

For ф[B1,A2], we obtain the relation:

$$\begin{aligned}\phi[B1,A2]&=\int dx_{G21}dx_{G22}dx_{23}dx_{24}dx_{12}\langle B1|G12\rangle\langle G12|G24\rangle\langle G24|G23\rangle\\&\cdot\langle G23|G22\rangle\langle G22|G21\rangle\langle G21|A2\rangle+\int dx_{G21}dx_{G22}dx_{G11}dx_{G12}\langle B1|G12\rangle\langle G12|G11\rangle\\&\cdot\langle G11|G22\rangle\langle G22|G21\rangle\langle G21|A2\rangle+\int dx_{G21}dx_{G24}dx_{G23}dx_{G32}dx_{31}dx_{G12}\langle B1|G12\rangle\\&\cdot\langle G12|G24\rangle\langle G24|G23\rangle\langle G23|G32\rangle\langle G32|G31\rangle\langle G31|G21\rangle\langle G21|A2\rangle.\end{aligned} \quad (93)$$

And ф[A3→B1] becomes an expression:

$$\begin{aligned}\phi[B1,A3]&=\int dx_{G12}dx_{G24}dx_{G23}dx_{G32}dx_{G31}\langle B1|G12\rangle\langle G12|G24\rangle\langle G24|G23\rangle\cdot\\&\langle G23|G32\rangle\langle G32|G31\rangle\langle G31|A3\rangle+\int dx_{G12}dx_{G11}dx_{G22}dx_{G21}dx_{G31}\langle B1|G12\rangle\\&\cdot\langle G12|G11\rangle\langle G11|G22\rangle\langle G22|G21\rangle\langle G21|G31\rangle\langle G31|A3\rangle.\end{aligned} \quad (94)$$

We apply the same method to the other paths and full total path, i.e., ф[B2,A] and ф[B3,A]. So, their descriptions are described as

$$\phi[B2,A]=C_{A12}\phi[B2,A1]+C_{A22}\phi[B2,A2]+C_{A32}\phi[B2,A3]. \quad (95)$$

$$\phi[B3,A]=C_{A13}\phi[B3,A1]+C_{A23}\phi[B3,A2]+C_{A33}\phi[B3,A3]. \quad (96)$$

Finally, the total probability amplitude from area A to area B, ф[B,A], is a summation of those paths. Its expression,

$$\phi[B,A]=C_{A1}\phi[B1,A]+C_{A2}\phi[B2,A]+C_{A3}\phi[B3,A], \quad (97)$$

is given by substituting above equations, Eq.(91), Eq.(95) and Eq.(96) into Eq.(97). To observe a part of interferences, we calculate a probability density of ф[A→B1] of Eq.(91).

$$\begin{aligned}\rho[B1,A]&=|\phi[B1,A]|^2=|C_{A11}\phi[B1,A1]|^2+|C_{A21}\phi[B1,A2]|^2+|C_{A31}\phi[B1,A3]|^2\\&+\{C^*_{A11}C_{A21}\phi[B1,A1]^*\phi[B1,A2]+C^*_{A21}C_{A31}\phi[B1,A2]^*\phi[B1,A3]\\&+C^*_{A31}C_{A11}\phi[B1,A3]^*\phi[B1,A1]\}+\{\text{counter terms}\}\end{aligned} \quad (98)$$

Clearly notice that quantum interferences contain those terms {$C^*_{A11}C_{A21}$ ф[B1,A1] * ф[B1,A2] + ・・・}+{counter terms} in Eq.(91) and Eq.(98). In quantum system, we can find also many interferences in following three terms, $|C_{A11}$ ф[A1→B1]$|^2$, $|C_{A21}$ ф[A2→B1]$|^2$, $|C_{A31}$ ф[A3→B1]$|^2$. Because, for example ф[A1→B1], its path is composed of the combination of many small paths, as [A1→G11→G12→B1] and [A1 →G11→G22→G23→G24→G12→B1]. The above those many terms, which vanish in the classical bifurcation problems, represent essential quantum effects and interferences.

Really the classical probability has only one term, $|C_{A21}\,\phi[B1,A2]|^2$, and there is not any interferences of probability (probability amplitude). So normalization condition in that Amida lottery is Eq.(99),

$$\int \phi^*[B,A]\cdot\phi[B,A]dx^1\cdots dx^k = 1\,. \tag{99}$$

And its transitional amplitude from state $\phi[B1,A]$ to state $\phi[B2,A]$ is defined by

$$\langle\phi[B2,A]|\phi[B1,A]\rangle \equiv \int \phi^*[B2,A]\cdot\phi[B1,A]dx^1\cdots dx^k\,. \tag{100}$$

in Eq.(100). After all, that above transitional probability density becomes

$$P([B2,A]\,|[B1,A])dx^1\cdots dx^k = \left|\langle\varphi[B2,A]|\varphi[B1,A]\rangle\right|^2. \tag{101}$$

We can finally obtain the frameworks of quantum bifurcations and interferences by path integral. This section is discussed problems of the diffraction and bifurcations of both the slit and the Amida lottery. We refer to scattering problems of polariton by various potentials in the following section.

9. Switch Operator and Circuit

This section is referred to switch operator, which corresponds to potential (scattering potential) of quantum system. And if we assume switches of circuits and networks as scattering potentials, we can easily express classical circuits (NOT, AND, OR) as quantum ones by path integral.

The particle as photon or polariton goes ahead to point B from point A. And that particle is not diffracted at point c but it is scattered by switch (potential) S at point c. This process is described by the bra and ket expression, and then kernel K(B,A) becomes

$$\phi_C[B,A] = K_C(B,A) = \langle B|\hat{S}_C|A\rangle = \int \langle B|c\rangle S(c)\langle c|A\rangle\, dx_C = \quad \mathrm{A}\rightarrow \mathrm{S} \rightarrow\mathrm{B}. \tag{102}$$

Notice difference between Eq.(102) and Eq.(90-2). The Eq.(102) includes the scattering process by switch potential S at point c, and on the other hand, Eq.(90-2) means the diffraction process at point c. Moreover, Eq.(90-1) simply expresses a free particle having no diffraction process and no scattering potential. So, we show typical three classical circuits which are called as AND-circuit, OR-circuit and NOT-circuit. (FIGURE.11) -(FIGURE.13).

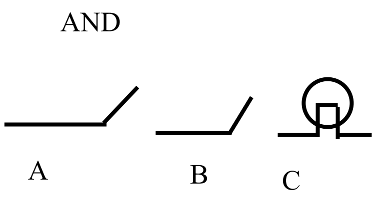


FIGURE 11. AND Circuit

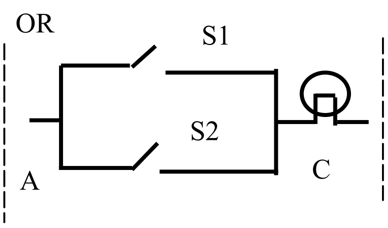


FIGURE 12. OR Circuit

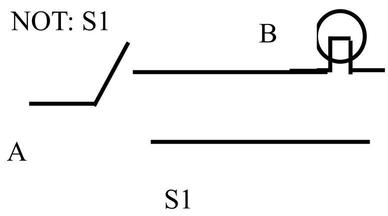


FIGURE 13. NOT Circuit

To obtain quantum description, we apply both rules Eq.(102) and Eq.(90-2) to those circuits. The AND-circuit can change into quantum one, q-AND, whose schema is simply drawn: the particle goes ahead from point A to scattering center S_1, and then it goes to point B. And after scattered by potential S_2, it arrives at final destination, point C.

$$[\, A \rightarrow S_1 \rightarrow B \rightarrow S_2 \rightarrow C \,] .$$

So we can obtain following expressions of quantum circuit FIGURE 11:

$$\phi_{AND}[B,A] = \int K_2(c,b)K_1(b,a)\,dx_B = \int \langle c|\hat{S}_2|b\rangle\langle b|\hat{S}_1|a\rangle\,dx_B . \qquad (103\text{-}1)$$

$$\because \quad K_2(c,b) = \int \langle c|\beta\rangle S_2(\beta)\langle \beta|b\rangle\,dx_\beta = \int K(c,\beta)S_2(\beta)K(\beta,a)\,dx_\beta \text{ ; for } S_2 \qquad (103\text{-}2)$$

$$\because \quad K_1(b,a) = \int \langle b|\alpha\rangle S_1(\alpha)\langle \alpha|a\rangle\,dx_\alpha = \int K(b,\alpha)S_1(\alpha)K(\alpha,a)\,dx_\alpha \text{ ; for } S_1 \qquad (103\text{-}3)$$

Above three equations does not correspond to the expressions of classical AND but they are quantum AND circuit. We would like to label q-AND. The rule of path integral says that an amplitude of different paths works as the additive, and so we can perform superposition of each path (linear combination). So, we apply that rule to classical OR-circuit, which has two parallel switches. So, we can define quantum NOT circuit,

$$\phi_{OR}[C,A] = \langle c|S_1|a\rangle + \langle c|S_2|a\rangle = \int \langle c|\alpha\rangle S_1(\alpha)\langle \alpha|a\rangle dx_\alpha + \int \langle c|\beta\rangle S_2(\beta)\langle \beta|a\rangle dx_\beta . \qquad (104)$$

The OR diagram becomes [A→ S_1 OR S_2) →C]. The NOT circuit is described by a following relation,

$$\phi_{NOT}[B,A] = \langle b|1-\hat{S}_1|a\rangle = \int \langle b|\alpha\rangle\langle\alpha|a\rangle - \langle b|\alpha\rangle S_1(\alpha)\langle\alpha|a\rangle d\alpha\,, \tag{105}$$

whose diagram is [A→ 1-S1 →C]. If three logical gate are combined with each other, for examples NOT, AND & OR circuits (FIG.14) , we can make up complex various quantum circuits and we actually perform to calculate their probability amplitude. Those switch's operators are quite different from common classical switches. Because, the classical switches are always expressed by c-number, but quantum switches take q-number and a potential operator. Those three circuits belong to quantum circuits. An example of combined circuits is showed in diagrams of FIGURE 14.

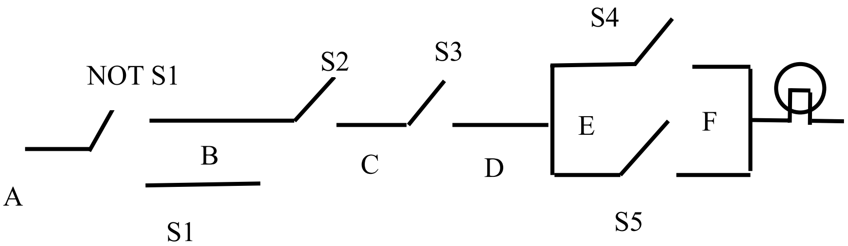


FIGURE 14. Complex circuit (NOT, AND & OR Circuit Φ_{NAO})

Mathematical representation of above figure is given by using multiple integral:

$$\begin{aligned}\phi_{NAO}[F,A] &= \int\cdots\int dx_{\delta} dx_e dx_d dx_{\gamma} dx_C dx_{\beta} dx_b dx_{\alpha}[K(f,\delta)S_4(\delta)K(\delta,e)K(e,d)\\ &\times K(d,\gamma)S_3(\gamma)K(\gamma,c))K(c,\beta)S_2(\beta)K(\beta,b)K(b,\alpha)(1-S_1(\alpha))K(\alpha,a)]\\ &+\int\cdots\int dx_{\varepsilon} dx_e dx_d dx_{\gamma} dx_C dx_{\beta} dx_b dx_{\alpha}[K(f,\varepsilon)S_5(\varepsilon)K(\varepsilon,e)K(e,d)K(d,\gamma)S_3(\gamma)\\ &\times K(\gamma,c))K(c,\beta)S_2(\beta)K(\beta,b)K(b,\alpha)(1-S_1(\alpha))K(\alpha,a)].\end{aligned} \tag{106}$$

Here if propagators (kernels) cause diffractions at points B, C, D, E, then we should perform integration over the slit width. On the other hand, if switch operator S is regarded as a kind of scattering potential, then the range of integral becomes over an infinite range. According to quantum mechanics, physical amount should be described by function of differential operator and time as Hamiltonian: switch operator should be described as

$$\hat{S}_j \equiv \hat{S}_j(\hat{x},\hat{p},t) = S_j(x,-i\hbar\nabla,t)\,. \tag{107}$$

We, as you know, can freely make up an arbitrary circuit by combining those three gates., i.e. those elements are q-AND, q-NOT, and q-OR.

We would like to generalize those quantum gates to m number switch's functions $F_j(S_1,S_2 \cdot\cdot S_N)$, j = 1 to m, whose variables are composed of N number's switch operators. Where F_j means the arbitrary operator's function of N number's switch. Notice that each switch S is an operator and so each F_j is composed of various switch operators. The F_j means operator's function. Thus, we can obtain a generalized description of q-AND switches in this case:

$$\phi_{AND}[C,A] = \int dx_{b1}\cdots dx_{bm}\langle c|b_m\rangle\langle b_m|\hat{F}_m|b_m\rangle\cdots\langle b_1|\hat{F}_1(\hat{S}_1\cdots\hat{S}_N)|a\rangle. \tag{108}$$

We can regard that a circuit has N number's scattering potentials when there are N number switches in its circuit. The rule of switch operator's function $F_j(S)$ is easily generalized as

$$\hat{F}_j \to \int d\chi\cdot|\chi\rangle F_j(S_1(\chi),\cdots)\langle\chi| = \int d\chi\cdot K(\ ,\chi)F_j(S_1(\chi),\cdots)K(\chi,\ ). \tag{109}$$

When N number's switches are connected in parallel, we have a generalized q-OR

$$\begin{aligned}\phi_{OR}[B,A] &= \sum_j^m\langle b|\hat{F}_j(\hat{S}_1,\cdots\hat{S}_N)|a\rangle = \sum_j^m\int d\chi\langle b|\chi\rangle F_j(S_1(\chi),\cdots S_N(\chi))\langle\chi|a\rangle \\ &= \sum_j^m\int d\chi K(b,\chi)F_j(S_1(\chi),\cdots S_N(\chi))K(\chi,a).\end{aligned} \tag{110}$$

Moreover, we given an expression of a multiple q-NOT,

$$\phi_{AND}[C,A] = \int dx_{b1}\cdots dx_{bm}\langle c|b_m\rangle\langle b_m|\hat{F}_m(\hat{\bar{S}}_1\cdots\hat{\bar{S}}_N)|b_m\rangle\cdots\langle b_1|\hat{F}_1(\hat{\bar{S}}_1\cdots\hat{\bar{S}}_N)|a\rangle.$$

$$\because\quad \hat{\bar{S}}_j = 1-\hat{S}_j \qquad j=1,\cdots m \tag{111}$$

Thus the logical switch can be represented by using kernels K(B,A), and so we need perform an integration at each switch points (scattering potential S). And those procedure and consideration naturally lead us to similarity of perturbation methods.

10. Similarity of Perturbation Method and Scattering Form of Switch Potentials

Exactly speaking the massive photon (polariton) is governed by Proca equation. We can reduce Proca equation to quaternary Schrödinger equation [36]-[37]. We can apply quaternary Schrödinger equation to many biological problems since the motion of polariton on neurons is much slower than the velocity of light. The quaternary Schrödinger equation have been described as

$$i\hbar\frac{\partial\phi^0}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+\hat{V}\right]\phi^0$$
$$i\hbar\frac{\partial\phi^a}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+\hat{V}\right]\phi^a \tag{112}$$
$$\because\ \phi^\mu(x)=(\phi^0(\mathbf{x},t),\phi^a(\mathbf{x},t)),\quad a=1,2,3.$$

$$A^\mu(\mathbf{x},t)=\phi^\mu(\mathbf{x},t)\cdot\exp\left(-\frac{i}{\hbar}mc^2t\right). \tag{113}$$

by using quaternary vector potential, A^μ or ϕ^μ. So, the quaternary potential A^μ represents an total electromagnetic field of polariton (massive photon). On the other hand, the ϕ^μ means kinetic parts of the total field A^μ, and the exponential function of Eq.(113) contains longitudinal element of polariton because of having mass term. So the ϕ^0 is scalar potential, and each ϕ^a (a = 1,2,3) is called vector potential of polariton. The rest mass limits the range of an existence of polariton. Moreover, we can reduce the quaternary Schrödinger equation to one component (scalar potential ϕ^0)of Schrödinger equation [33],[36]. If a change of vector potential **A** is so slow or so small, the following derivative of vector potential **A** is nearly equal to zero.

$$\mathbf{B}(x,t)=rot\mathbf{A}(x,t)$$
$$\mathbf{E}(x,t)=-grad\mathbf{A}^0-\frac{1}{c}\frac{\partial\mathbf{A}(x,t)}{\partial t}, \tag{114}$$

From Eq.(112)-(114), the kinetic part of polariton obeys Schrödinger equation of ϕ^0. Then the residual terms become only an electric field as shown in Eq.(115), and quaternary Schrödinger equation has only one component ϕ^0 of polariton's vector potential.

$$\mathbf{E}(x,t)\approx -grad\phi^0. \tag{115}$$

Considering from Eq.(108)-(111), we regard various switch operator's function as a kind of potential. So we add up those switch operators to the potential term of Hamiltonian, and finally we have the following form,

$$i\hbar\frac{\partial\phi^0}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+\hat{V}(\alpha,\beta\cdots;t)+\hat{F}(\hat{S}_1,\cdots,\hat{S}_N)\right]\phi^0. \tag{116}$$

Applying the ordinary perturbation method to Eq.(116), the lowest perturbation's expression with potential term V is given. Comparing the results of Eq.(102)-(105) with perturbation method of quantum mechanics, we can find easily that those expressions of Eq.(102)-(105) are much similar to the first order and the second order term of perturbation method. Thus, the second lowest amplitude of perturbation is described as kernel's expression. As the q-AND circuit has two switch potential terms S1(α) and S2(β), the expression of perturbation is given in Eq.(117) and FIGURE 15.

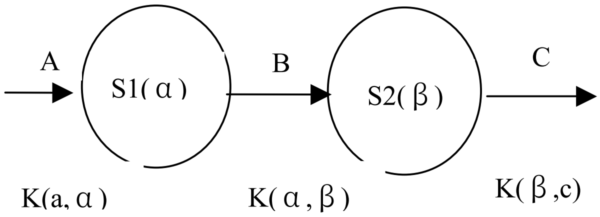


FIGURE 15. Perturbation for Second order Expansion and q-AND circuit

$$\begin{aligned}\phi_{2nd}[C,B,A] &= \left(-\frac{i}{\hbar}\right)^2 \int dx_\beta dx_\alpha \langle c|\beta\rangle S_2(\beta)\langle\beta|\alpha\rangle S_1(\alpha)\langle\alpha|a\rangle \\ &= \left(-\frac{i}{\hbar}\right)^2 \iint dx_\beta dx_\alpha K(c,\beta)S_2(\beta)K(\beta,\alpha)S_1(\alpha)K(\alpha,a).\end{aligned} \qquad (117)$$

We take same procedure for q-OR and q-NOT circuits in order to make up perturbation method of propagation for a particle, polariton. According to the diagrams (FIGURE 16 &17), the q-OR circuit corresponds to the first ordered perturbation of two potentials connected in parallel. The Eq.(102) is similar to the first ordered process of perturbation. We know that the scattering process at point C is given as

$$\begin{aligned}\varphi_C[B,A] &= K_C(B.A) = \int K(b,c)S(c)K(c,a)dx_C = \int \langle b|c\rangle S(c)\langle c|a\rangle dx_C. \\ &= A \rightarrow S_C \rightarrow B\end{aligned} \qquad (118)$$

Applying Eq.(118) for both circuits, q-OR and q-NOT, we can easily address the first order expressions of perturbation.

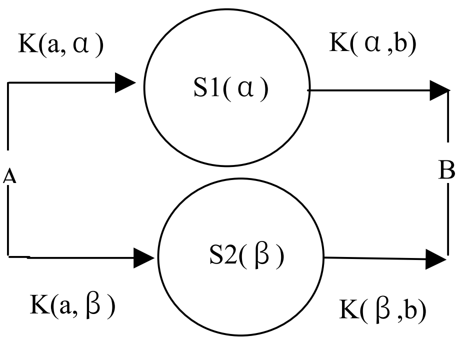


FIGURE 16. Perturbation of q-OR circuit

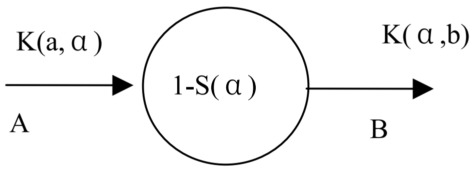


FIGURE 17. Perturbation q-NOT

The FIGURE.16 shows perturbation for first order of two parallel potentials, and we should notice that the point A or B is not diffraction's center but ports of wave function or an appearance of propagator. The first order's perturbation of q-OR becomes

$$\begin{aligned}\phi_{1st}[B,A] &= \left(-\frac{i}{\hbar}\right)\left[\langle c|S_1(\alpha)|a\rangle + \langle c|S_2(\beta)|a\rangle\right] \\ &= \left(-\frac{i}{\hbar}\right)\int dx_\alpha K(c,\alpha)S_1(\alpha)K(\alpha,a) + \left(-\frac{i}{\hbar}\right)\int dx_\beta K(c,\beta)S_2(\beta)K(\beta,a).\end{aligned} \tag{119}$$

And we can know the first ordered amplitude of the switch operator q-NOT,

$$\phi_{1st}[B,A] = \left(-\frac{i}{\hbar}\right)\langle c|(1-S(\alpha))|a\rangle = \left(-\frac{i}{\hbar}\right)\int dx_\alpha K(c,\alpha)(1-S(\alpha))K(\alpha,a)\,. \tag{120}$$

The q-NOT circuit contains only one scattering center, which is a potential (1-S). So, the q-NOT has the first order perturbation as well as the q-OR circuit. According to perturbation method, we find that the q-AND is the second ordered switch system and that both q-NOT and q-OR mean the first ordered switch system. Iterating those procedures, we can easily obtain the higher ordered perturbation expansions. That perturbation series is given as

$$\begin{aligned}K_T[B,A] &= K(B,A) + \left(\frac{-i}{\hbar}\right)\int d\alpha\cdot K(B,\alpha)S_1(\alpha)K(\alpha,A) + \left(\frac{-i}{\hbar}\right)^2\int d\alpha\, d\beta\cdot K(B,\beta) \\ &\times S_2(\beta)K(\beta,\alpha)S_1(\alpha)K(\alpha,A) + \cdots\left(\frac{-i}{\hbar}\right)^m\int\cdots\int d\alpha\, d\beta\cdots K(B,L)S_L(L)\cdots K(\alpha,a)\cdot\end{aligned} \tag{121}$$

Thus, the complex form of kernel, which is propagator or Green's function K_T[B,A], expresses the higher multiple interactions or multi-scattering processes. We notice that the perturbation of K_T[B,A] becomes an infinite series of set of [K(y+1 ,y)S(y)K(y, y-1)]. We would like to apply those rules to constructing a neural network system. The synapses of FIGURE 18 are looked upon as switch's operators or scattering potentials. So, we can rewrite FIGURE 15 as shown in FIGURE.18.

The FIGURE.18 shows the similarity of the three models, and we can describe the propagation of polariton (quantized polarization wave) from one neuron to another neuron through synaptic junction (synapse). If those above neuron-synapse model does not have any diffractions of polaritons at any points and synaptic junctions are expressed as some potentials, the neuron-synapse model enables us to calculate each propagator and total kernel K_T[D,A], (FUGURE.18). That total propagator of polariton is directly given by following expressions: Here is a kernel of FIGURE.18.

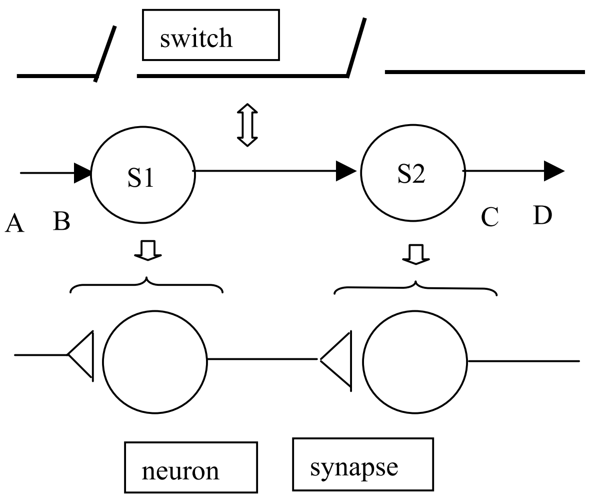


FIGURE 18. Similarity of models

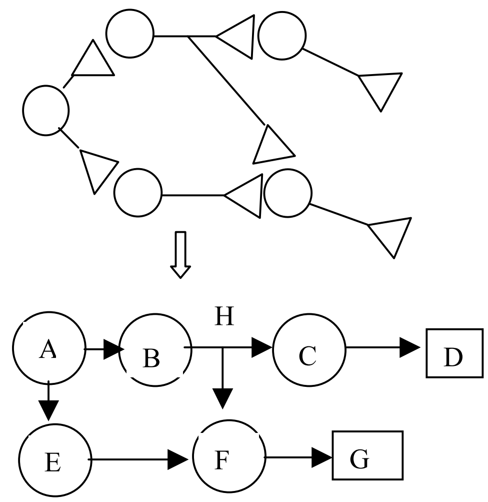


FIGURE 19. Quantum neural network

The expression of FIGURE 18 is given as

$$K_T[D,A] = \int d\eta\, d\xi\, dB\, dC\; K(D,.C)K(C,\xi)S_2(\xi)K(\xi,\eta)S_1(\eta)K(\eta,B)K(B.A)\,. \tag{122}$$

$$K[b,a] = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right] = \langle b | a \rangle\,. \tag{123}$$

Here, K[b,a] means that a free particle goes to point b from point a. The structure of both switch operators S1 and S2 is expressed at functions of each coordinate point (x,y). If we do not have any diffraction's points in both intervals [A,η] and [ξ,D] and the particles are perfectly propagating freely, then both integral dB and dC become equals to 1. Thus, we can remove the integrals of dB and dC from Eq.(122). If a neural network is composed of some neurons as shown in FIGURE.19, then the probability amplitude can be calculated by above calculation procedure. For example, probability amplitude of neuron D is given as

$$\phi_A[D] = \int dCdHdBdA \cdot K(D,C)S_C(C)K(C,H)K(H,B)S_B(B)K(B.A)S_A(A)f(A) \tag{124}$$

The function f(A) of Eq.(124) means an arbitrary wave function. And a free particle has the diffraction at point H and are scattered both points B and C (FIGURE.19). For neuron G, we obtain the probability amplitude (propagator):

$$\phi_A[G] = \int dFdHdBdA \cdot K(G,F)S_F(F)K(F,H)K(H,B)S_B(B)K(B.A)S_A(A)f(A) \qquad (125)$$
$$+ \int dFdEdA \cdot K(G,F)S_F(F)K(F,E)S_E(E)K(E,A)S_A(A)f(A).$$

So, those two wave function, ϕ_A[D] and ϕ_A[G] show that an initial wave function f(A) will arrive at two endpoints D and G, after f(A) was divided into two waves at point A. The f(A) is scattered at many points, A,B,F,E and diffracted at points H,E, by some potentials.

Japanese Amida lottery, which is bifurcation's problem, has many diffraction points as multi-slit. However, Amida lottery does not have any switch's potentials S. On the other hand, quantum circuits and neural networks include both switch's potentials and diffraction's points in their systems.

11. Rules of Calculation for Some Paths

We would like to construct mathematical tools for quantized circuits, neural network and Amida lottery so as to translate classical pictures into quantum ones. Notice that the kernel K[b,a] and inner-product <B|A> are not ordinary wave functions but they describe the time development of propagation satisfying Schrödinger equation. They truly express the propagating motion of a particle from point (A, t_A) to point (B,t_B). Thus, an expression of path integral corresponds to dynamics of particle as well as Newtonian second law of motion. So we would like to summarize important descriptions of the particle's propagation (motion of polariton) in order to calculate probability amplitude for any circuits . If path integral is applied to classical neural networks, then their networks are directly quantized and come to contain a various quantum effects in their systems, i.e. for examples tunnel effects, fluctuations and interferences.

1. free propagation of particle: point A $\rightarrow$ point B.

$$K[B,A] = \langle B|A\rangle = K(b,a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right]. \qquad (126)$$

2. dividing into two parts: two paths are A $\rightarrow$ B and B$\rightarrow$ C. particle is free propagation. B: relay point or diffraction point.

$$K_B[C,A] = \langle C | A \rangle_B = \int dB \cdot K(C,B)K(B,A) = \int dB \langle C | B \rangle \langle B | A \rangle . \quad (127)$$

3. diffraction at point B, slit width δ : A → (B) → C.

$$K[C,A] = \langle C | A \rangle = \int dB \cdot K(C,B)K(B,A) = \int_0^{\delta} dB \langle C | B \rangle \langle B | A \rangle . \quad (128)$$

4. various switch's potentials, for example, synaptic junction and scattering potentials for particles, electromagnetic potentials: A→ B → C.

$$K_B[C,A] = \langle C | A \rangle_B = \langle C | \hat{S}_B | A \rangle = \int dBK(C,B)S(B)K(B,A) = \int dB \cdot \langle C | B \rangle S(B) \langle B | A \rangle .$$

(129)

5. general switch's potentials : A → f:B → C, f:B= f (S1(B), S2(B), ・ ・, Sn(B)).

$$K_B[C,A] = \langle C | A \rangle_B = \langle C | \hat{F}_B | A \rangle = \int dB \cdot K(C,B)F(S_1(B), S_2(B) \cdots S_n(B))K(B,A) \quad (130)$$

$$= \int dB \cdot \langle C | B \rangle F(S_1(B), S_2(B) \cdots S_n(B) \langle B | A \rangle .$$

.

6. abbreviation for line and interaction points: A →B→ C →D→E, then free particle at both points B and D.

$$K_C[E,A] = \langle E | A \rangle_C = \langle E | \hat{S}_C | A \rangle = \int dBdCdD \cdot K(E,D)K(D,C)S(C))K(C,B)K(B,A)$$

$$= \int dC \cdot K(E,C)S(C)K(C,A) \quad (131)$$

7. abbreviation for line, interaction and diffraction: A →(B)→ C → D→E, B: slit or diffraction points. C is scattering point. Notice that we cannot abbreviate dB integral.

$$K_C[E,A] = \langle E | A \rangle_C = \langle E | \hat{S}_C | A \rangle = \int dBdCdD \cdot K(E,D)K(D,C)S(C))K(C,B)K(B,A)$$

$$= \int dBdC \cdot K(E,C)S(C)K(C,B)K(B,A) \quad (132)$$

8. propagation and time-development : initial wave function φ(A)→final state B, $\phi_{A.}$[B].

$$\phi_A[B] = \int dA \cdot K(B,A)\, \phi(A) . \quad (133\text{-}1)$$

$$K[B,A]=\langle B|A\rangle \equiv \langle B;t_B|A;t_A\rangle = \langle B|\hat{U}(t_B,t_A)|A\rangle$$
$$\hat{U}(t_B,t_A)=\exp\left(\frac{-i}{\hbar}\hat{H}(t_B-t_A)\right) \qquad (133\text{-}2)$$

9. relationship between an eigenfunction of Schrödinger equation and its propagator.

$$K[B,A]\equiv \sum_K \psi^{\mu *}(A)\psi^{\mu}(B)\exp[-i\omega_K^{\mu}(t_B-t_A)] \qquad (134)$$

We would like to show an above relationship between the kernel K[B,A] and eigenfunction of Schrödinger equation. The wave function of Schrödinger equation, whose solution is A^{μ} or ϕ^{μ}(or static approximation of polariton ϕ^{0}), can be related to the kernel K[B,A]. The general solution of time dependent quaternary Schrödinger equation is represented as

$$\phi^{\mu}[x]=\sum_J C_J^{\mu}\psi_J^{\mu}(x)\exp(-i\omega_J^{\mu}t). \qquad (135)$$

And the quaternary wave function $\Psi_J{}^{\mu}$, which is an eigenfunction of stationary state, satisfies Eq.(112) or Eq.(116). The $\Psi_J{}^{\mu}$ obeys the quaternary Schrödinger equation: that is

$$\left[-\frac{\hbar^2}{2m}\nabla^2+\hat{V}(\alpha,\beta\cdots;t)+\hat{F}(\hat{S}_1,\cdots,\hat{S}_N)\right]\psi_J^{\mu}(x)=E_J^{\mu}\psi_J^{\mu}(x) \qquad (136)$$
$$\because \quad E^{\mu}=\hbar\omega_J^{\mu}.$$

The wave function at point (A, t_A) is written as

$$\phi^{\mu}[A;t_A]=\sum_J C_J^{\mu}\psi_J^{\mu}(A)\exp(-i\omega_J^{\mu}t_A)\equiv \sum_J a_J^{\mu}\psi_J^{\mu}(A) \qquad (137)$$
$$\therefore C_J^{\mu}=a_J^{\mu}\exp\left(i\omega_J^{\mu}t_A\right).$$

On the other hand, we have a similar expression at point (B,t_B),

$$\phi^{\mu}[B;t_B]=\sum_K C_K^{\mu}\psi_K^{\mu}(B)\exp(-i\omega_K^{\mu}t_B)\equiv \sum_K a_K^{\mu}\psi_K^{\mu}(B)\exp(-i\omega_K^{\mu}t_B+i\omega_K^{\mu}t_A). \qquad (138)$$

Where we substituted Eq.(137) into $C_K{}^{\mu}$ of Eq.(138). The Eq.(137) gives us coefficient $a_K{}^{\mu}$:

$$a_K^{\mu}=\int \psi_K^{\varpi *}(A)\phi^{\mu}[A;t_A]dA. \qquad (139)$$

Substituting Eq.(139) into (137) and comparing that result with Eq.(133), we can obtain an expression of kernel K[B,A].

$$\phi^{\mu}[B;t_B] = \sum_K \{\psi_K^{\mu *}(A)\, \psi_K^{\mu}(B) \exp[-i\omega_K^{\mu}(t_B - t_A)]\} \phi^{\mu}[A;t_A] \tag{140}$$

$$\therefore K[B,A] \equiv \sum_K \psi^{\mu *}(A) \psi^{\mu}(B) \exp[-i\omega_K^{\mu}(t_B - t_A)] \tag{141}$$

The Eq.(141) is shown to be equal to Eq.(136).

12. Polariton's Equation and Rules of Quantum Neural Conduction

In previous section, we made up useful tools for quantum calculation of various networks. We mentioned, heretofore, three quantum expressions, which were both quaternary Schrödinger equation (Proca equation) and Feynman's path integral method.

12.1 Quaternary Schrödinger Equation and Proca Equation

We showed the equation of polaritons on neural axons, and the polarities are exactly governed by Proca equation Eq.(142), which was relativistic one.

$$(\partial_{\mu}\partial^{\mu} + m^2) A^{\mu} = J^{\mu} \tag{142}$$

$$J^{\nu}(x) \equiv (\rho(\mathbf{x},t), i(\mathbf{x},t)) \approx j_{Na}^{\nu} + j_K^{\nu},$$

The symbol m is polariton's mass, and the J^{ν} means the quaternary vector currents. According to classical neural theory like as Hodgkin & Huxley model, the polariton means a quantized polarization wave, which is an impulse from neurons and an action potential. So, the total current j^{μ} is generated by major two ionic currents(sources), which correspond to the sodium current J_{Na} and to the potassium current J_K through neural axon. To derive non-relativistic polariton's equation from relativistic equation (142), we need return from the wave function A^{μ} of natural unite to that of MKS unite:

$$A^{\mu}(\mathbf{x},t) = \varphi^{\mu}(\mathbf{x},t) \cdot \exp\left(-\frac{i}{\hbar} mc^2 t\right) \tag{143}$$

Then, we split the time dependent of A^{μ} into two terms, then the one containing the rest polariton's mass, m. In the non-relativistic limit, the kinetic energy E_k is so small that we can define it as

$$E_K = E - mc^2, \quad E' << mc^2 \tag{144}$$

non-relativistic kinetic energy E_k means

$$\left| i\frac{\partial \varphi^{\mu}}{\partial t} \right| \approx E_K \varphi^{\mu} << mc^2 \varphi^{\mu} \tag{145-1}$$

$$\frac{\partial A^{\mu}}{\partial t} \approx -i\frac{mc^2}{\hbar}\varphi^{\mu}\cdot \exp\left(-\frac{i}{\hbar}mc^2 t\right) \tag{145-2}$$

$$\frac{\partial^2 A^{\mu}}{\partial t^2} \approx \left[-i\frac{2mc^2}{\hbar}\frac{\partial \varphi^{\mu}}{\partial t} - i\frac{m^2c^4\varphi^{\mu}}{\hbar^2}\right]\cdot \exp\left(-\frac{i}{\hbar}mc^2 t\right). \tag{145-3}$$

Inserting this result into following relativistic relation:

$$p^{\mu} p_{\mu} A^{\nu} + m^2 c^2 A^{\nu} = j^{\nu}/c \,., \tag{146}$$

We finally obtain the 4-conponent non-relativistic expressions like as Schrödinger equation. The result is non-relativistic polariton's relationship,

$$\begin{gathered} i\hbar\frac{\partial A^{\mu}}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]A^{\mu} \\ A^{\mu} = (\phi, \mathbf{A}), \quad j^{\nu}\hbar^2/(2mc) \Leftrightarrow \hat{V}A^{\nu}. \end{gathered} \tag{147}$$

We reach the final polariton's equation with 4-conponents. The motion of polaritons is described by above 4-conponents' equations: they are scalar potential $A_0 = \phi$ and vector potential is **A**. If the quaternary vector potential of electromagnetic field of polaritons are having **A** = constant or **A** changing much slowly (i.e., stationary magnetic field), then the Eq.(147) becomes common Schrödinger equation for polariton with only having the scalar potential ϕ,

$$\begin{gathered} i\hbar\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\phi(\mathbf{x},t) = \hat{H}\phi(\mathbf{x},t) \\ \because \mathbf{B}(\mathbf{x},t) = rot\, A(\mathbf{x},t) \approx 0, \quad \mathbf{E}(x,t) = -grad\ \phi(x,t). \end{gathered} \tag{148}$$

To simply our problem we discuss the near static magnetic field being accompanied with scalar potential case, whose quaternary solution nearly equals to $A^{\mu} = (\phi$, Constant **A**).

12.2 Diagrams Expression

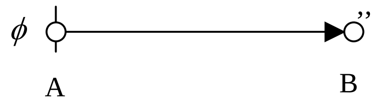
We would like to propose how to describe diagrams of relationships between path integral and networks. Those rules have following expressions.

(1) The solution ϕ of Eq.(148) are written down by using kernel K(B,A) of ϕ for free

propagation of polariton ($\phi(\mathbf{A})$ is an initial condition) : A^{ν} : quaternary vector potential.

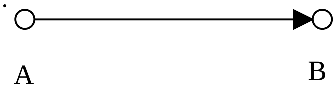
$$i\hbar\frac{\partial\phi}{\partial t}=\hat{H}\phi \quad\Rightarrow\quad \phi(\mathbf{x},t)=\int K(B,A)\phi(A)dA \quad\Rightarrow\quad A^{\mu}=(\phi(\mathbf{x},t),\text{constant Ac}) \qquad (149)$$

$$\because B\equiv(\mathbf{x},t),\quad A\equiv(\mathbf{x}_0,t_0),$$



The K(B,A) of free polariton is represented as

$$K(B,A)=\left[\frac{2\pi i\hbar(t-t_0)}{m}\right]^{-1/2}\exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right]={}_t\langle B|A\rangle_0 \,. \qquad (150)$$

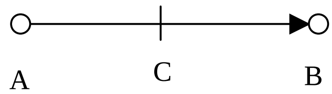


And the position B becomes

$$x(t)=x_0+\frac{x-x_0}{t-t_0}(t-t_0) \qquad (151)$$

(2) If the kernel $K_C(B,A)$ is divided into two parts by a relay's point C, then its kernel,

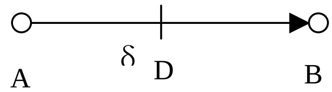
$$K_C(B,A)\equiv{}_t\langle B|_C A\rangle_0=\int K(B,C)K(C,A)dC \qquad (152)$$



is given by Feynman path integral.

If the polariton is diffracted by potentials at point D, then we have a similar relation with using slit width δ :

$$K_C(B,A)\equiv{}_t\langle B|_C A\rangle_0=\int_0^{\delta} K(B,D)K(D,A)dD\,. \qquad (153)$$



The kernel K(B,A) should be governed with Schrödinger equation:

$$i\hbar\frac{\partial K(B,A)}{\partial t}=\hat{H}K(B,A) \quad . \qquad (154)$$

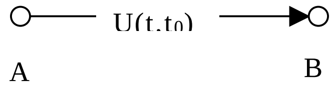
(3)When a state vector $|\phi(t)>$ is projected into x-axis of Cartesian coordinate, the wave function $\phi(\mathbf{x},t)$ has an expression,

$$\phi(\mathbf{x},t)\equiv\langle x|\phi(t)\rangle,\quad \because|\phi(t)\rangle=U(t,t_0)|\phi(t_0)\rangle \,. \qquad (155)$$

(4) When we substitute Eq.(155) into Eq.(149), an explicit description of unitary operator $U(t,t_0)$ obeys the same Schrödinger equation. The unitary operator,

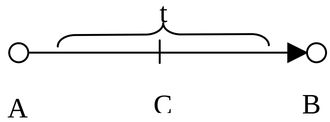
$$U(t,t_0)=\exp\left(-i\hat{H}(t-t_0)/\hbar\right), \qquad (156)$$

is finally applied for the kernel K(B,A), so the time-development's form of kernel becomes

$$K(B,A) = {}_t\langle B|A\rangle_0 = \langle B|\hat{U}(t,t_0)|A\rangle .$$ 

(157)

(5)The special case of kernel,

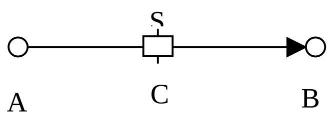
$$K_C(B,A) = {}_t\langle B|{}_C A\rangle_t = \langle B|\hat{U}(t,t_C)\hat{U}(t_C,t)|A\rangle .$$ 

(158)

equals to this delta function at fixed time t, and we have

$$\int K(B,C)K(C,A)dC = \int {}_t\langle B|C\rangle_{tc}\langle C|A\rangle_t dC = \langle B|A\rangle_t = \delta(\mathbf{x}-\mathbf{x}_0)$$
$$\because \int dX\, |X\rangle\langle X| = 1. \qquad (159)$$

(6) If the free polariton is scattered by general potentials V as being observed in atomic structures or by switch function S of electronic circuit at point C, we have a similar scattering representation to the diffraction's Eq.(153) by using Eq.(157):

$$K_C(B,A) \equiv {}_t\langle B|\hat{S}_C|A\rangle_0 = \int K(B,C)S(C)K(C,A)dC$$
$$= \int \langle B|\hat{U}(t,t_C)|C\rangle S(C)\langle C|\hat{U}(t_C,t_0)|A\rangle dC. \qquad (160)$$


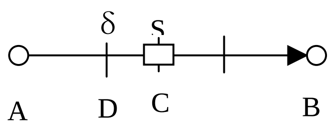
(7) When the scalar potential of polariton is governed by φ of that quaternary Schrödinger equation-(148), then a time-development state |φ(t)> of the formal expression for Eq.(149) is

$$|\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi(0)\rangle . \qquad (161\text{-}1)$$

And completeness of the eigen-state vector $|\Psi_i(t)\rangle$, which is applied for Eq.(159), leads us to the kernel expression of proper wave function $\Psi_i(x,t)$.

$$K(B,A) = {}_t\langle B|A\rangle_0 = \sum_j \psi_j(\mathbf{x})\psi_j^*(\mathbf{x}_0)\exp\left(-iE_j(t-t_0)\right) \qquad (161\text{-}2)$$
$$\because |\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi(0)\rangle, \quad \sum_j |\psi_j(\mathbf{x})\rangle\langle\psi_j^*(\mathbf{x}_0)| = 1, \quad \langle x|\phi(t)\rangle = \phi(x,t).$$

(8) Both Rules of the diffraction at point D and the potential scattering at point C are described by the form of path integral, and then we have the kernel $K_{DC}(B,A)$:

$$K_{DC}(B,A) = {}_t\langle B|\hat{S}_C|{}_D A\rangle_0 = \int dDdCdEK(B,E)K(E,C)S(C)K(C,D)K(D,A) \qquad (162\text{-}1)$$
$$= \int_0^{\delta}\int dCK(B,C)S(C)K(C,A).$$


If we use those kernels descriptions, we can transform many classical neural networks into quantum neural ones.   For example, we would like to obtain a quantum expression of the network by applying above relations for following classical neural network,

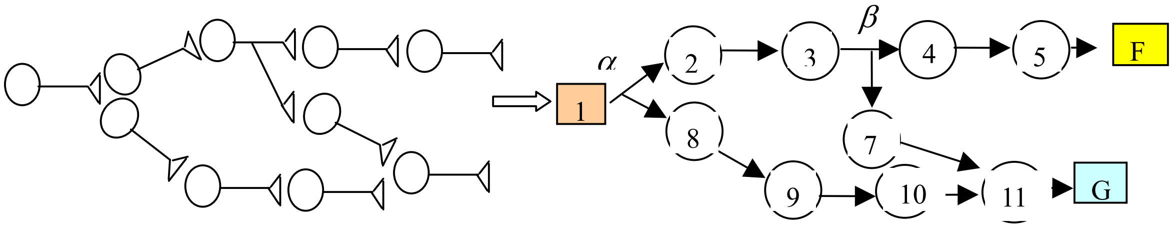
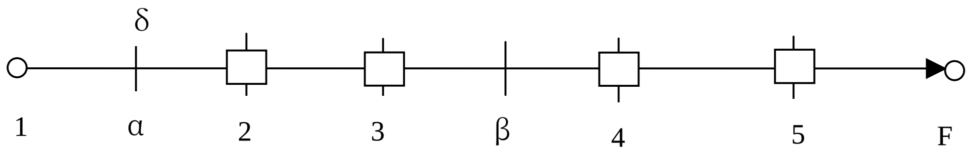


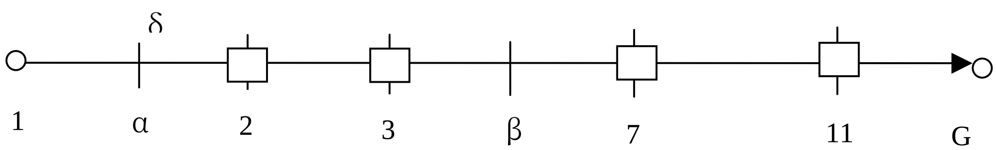
FIGURE 20. Quantum calculation of neural network

We represent diagrams of three paths of FIGURE 20, which are constructed by above pictures (path 1, path 2 and path 3).   Those are following diagrams.

(Diagram of Path 1)



(Diagram of Path 2)



(Diagram of Path 3)

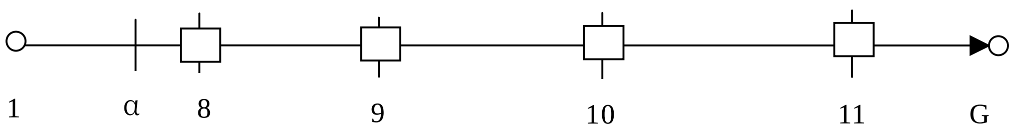


FIGURE 21. Diagrams of three paths without current source

According to those diagrams, we can easily obtain expressions of kernel of path integral.

When an action potential, which is quantized polarization vector (polariton in our models), conducts from neuron-1 to neuron-5 (point F) or to neuron-11 (point G), we are able to calculate the state of wave function at the point F or the point G. In the other word, an initial wave function $\Psi(1)$ propagates from the point-1 to the point F or point G, and our methods enable to know the final wave function $\Psi(F)$ or $\Psi(G)$. The $\Psi(F)$ is given as

$$\psi(F) = \int K(F,1)\psi(1)dx_1, \qquad K(F,1) \equiv K(F,x_1) \tag{162-2}$$

from using Eq.(149). And if we can write down the expression of the kernel K(F,1), the final result of wave function at the point F:

$$K(F,1) = \int dx_5 \cdots dx_1 d\beta d\alpha K(F,5)S(5)K(5,4)S(4)K(4,\beta)K(\beta,3)S(3)K(3,2)S(2)K(2,\alpha)K(\alpha,1).$$

We apply the same method for the point G, and the wave function $\Psi(G)$ at point G becomes the sum of two different paths, which are both 1⇒2⇒3⇒7⇒11⇒G & 1⇒8⇒9⇒10⇒11⇒G. The one path is shown as

$$K_A(G,1) = \int dx_{11} \cdot\cdot dx_1 d\beta d\alpha K(G,11)S(11)K(11,7)S(7)K(7,\beta)K(\beta,3)S(3)K(3,2)S(2)K(2,\alpha)K(\alpha,1)$$

and another is

$$K_B(G,1) = \int dx_{11} \cdot\cdot dx_1 d\alpha K(G,11)S(11)K(11,10)S(10)K(10,9)S(9)K(9,8)S(8)K(8,\alpha)K(\alpha,1).$$

So, notice that the final wave function $\Psi(G)$ is given as the sum of two paths,

$$K(G,1) = K_A(G,1) + K_B(G,1) \qquad \therefore \psi(G,1) = \int K(G,1)\psi(1). \tag{163}$$

Then diagram of FIGURE 19 is pictured as shown in FIGURE 22. In this case, ϕ is a sources of current or generator of wave function.

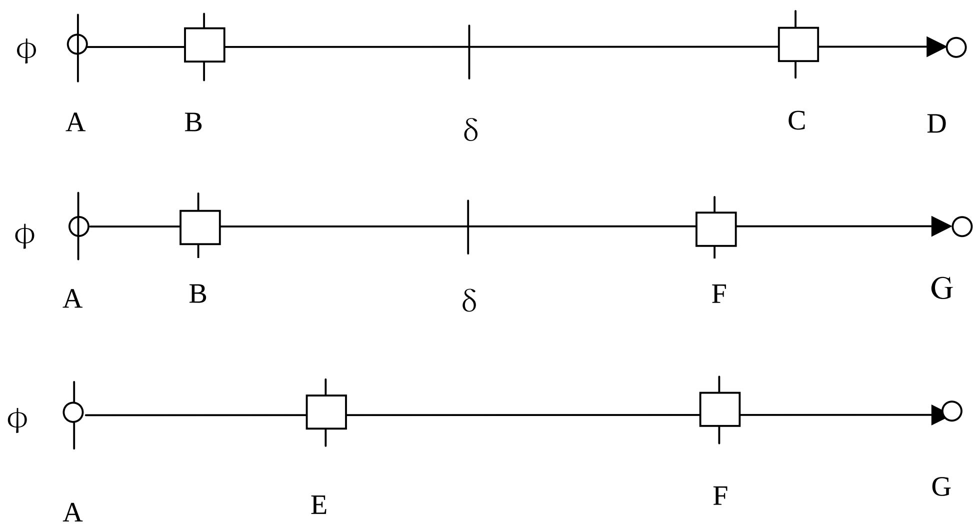


FIGURE 22. Diagram of paths with current source

Thus, we can rewrite various classical neural networks into the quantum ones by using above formulas, and those expressions are not the static expressions of quantum state but they are dynamic descriptions of the propagations and the time developments of systems, which correspond to polariton conductions and their motions.

12.3 Relationship between Theory of Quantum Information and Polariton

The following many sections are shared an explanation and practical calculations of some statistical theory, Bayes' theory and entropy by using quantum mechanics. By applying those previous mathematical tools to statistical problems, we could transform various neural networks and Beyes' form into quantum styles [23],[27]-[28].

To show the differences between classical information theory and quantum information theory, we attempt concretely to calculate the classical and quantum Bayes' theory, entropy, and outputs of neural networks. And we would like to express those cases by applying polariton's theory and tools developed in previous sections. The Bayes' theory is applied for many network theory and control systems. So, many excellent reports and books are published in the region of Information science [2],[11]. As you know, Bayes ' sstatistics, which is often used in an inferential of causality, is said to be subjective probability when the Bayes' method is compared with normal probability theory[26]-[30]. The classical mechanics has essentially an apparent pathway between causes and effects, and it is deterministic method. However, the causality of quantum mechanics is essentially probabilistic phenomena since its time development of state is governed by the complex probability amplitude of Schrödinger equation and Proca equation[5],[13]. We have already shown, our polariton's neural theory can be described by massive relativistic equation - Proca equation, or its reduction style of non-relativistic quaternary Schrödinger equations. We know that quantum theory has an interference of phenomena, mixing principle of each pure state, superposition and tunnel effects. Common Bayes' theory, we call it classical theory, is not considered interferences of phenomena between each event. In the other word, all events are independent of each other (no superposition). We think it interesting to research how the quantum interference affects on the classical Bayes' theory, the entropy and information. So one of the purpose of following some sections are that we show a concrete expression of quantum Bayes' form, instead of classical Bayes' theory, by using a basic set of orthogonal state vectors for simple model. And we clearly describe the differences between classical Bayes' theory and quantum form. In the secondary stage, we compared their entropies of both systems. In the two-step's neural networks of multiple channels, we could approximately obtain a solution by means of perturbation method and path integrals.

Finally, we would like to point out similarities of formal descriptions between soft scientific theories and quantum control systems. The first example referred to similarity of between neural network control and quantum neuro system, and the second is refer to similarities of between fuzzy probability and quantum expectation values.

13 Bayes' Theory and Its Quantum Expression by State Vectors

We would like to mention both the famous classical Bayes' theory and our style of quantum Bayes' form.

13.1 Classical Form and Quantum Form

When we know a final result for an event B, the Bayes' probability is defined as the ratio that an event A_k (where k = 1 to N) arises. Then we have the common formula of Bayes':

$$P_{Cl}(A_K|B) = \frac{P(B|A_K)\cdot P(A_K)}{P(B)}, \qquad \because P(B) = \sum_K^n P(B|A_K)\cdot P(A_K). \tag{164}$$

We are able to regard $P(A_K)$ as a probability of occurrence of event A, and $P(B|A_K)$ means to be a correspondence probability when initial probability is $P(A_K)$. The probability $P(B|A_K)$ represents a condition that an event A_K is propagated to the state B, when the event A_K took place at an occurrence probability $P(A_K)$. So, the symbol $P(B|A_K)$ is regarded as a kind of classical propagator of probability $P(A_K)$, or transitional probability. We are commonly regarding Eq.(164) as the theory of classical Bayes' theory. And we attempted to expand the propagator's concepts from the classical standpoint into the quantum mechanical one. To expand from the above classical Bayes' theory to the quantum versions, we need a rule that the classical Bayes' theory should be reproduced by an expectation value of quantum operator's equations if their expectation value are calculated. The expectation value of quantum Maxwell equations (quantum electrodynamics) has to obey to the rule of the classical Maxwell equations. Thus, $P(A_K)$ and $P(B|A_K)$ should be regarded as operators of quantum expression, and those eigen functions of both operators should be regarded as complex probability amplitudes. Performing to re-interpret classical relation into quantum one, we would like to show one of the simplest cases of quantum expressions.

Notice that the simplest quantum form is given as following forms:

$$\langle A_K | \hat{P}(A|B) | B \rangle \equiv \frac{\langle B | \hat{P}(B|A) \cdot \hat{P}(A) | A_K \rangle}{\sum_j^n \langle B | \hat{P}(B|A) \cdot \hat{P}(A) | A_j \rangle}. \tag{165}$$

The quantum form is similar to classical Bays' theory; however, all probabilities' relations are not c-numbers but q-numbers of operators in quantum Bayes'.
One of initial state vectors is $|A_K>$, and the final state vector is represented as $|B>$. The Eq.(165) should be more simplified by a relationship between the initial vectors and the final vector (FIGURE 23). We know, the FIGURE 23 mentions that quantum neural network FIGURE 23-A is similar to natural neural one, FIGURE 23-B. And some quantum neural networks are composed of many axons and many synapses, which cause the quantum interferences. In order to calculate the Eq.(165), we would like to introduce some rules that define eigen state vectors having the completeness and orthonormality.

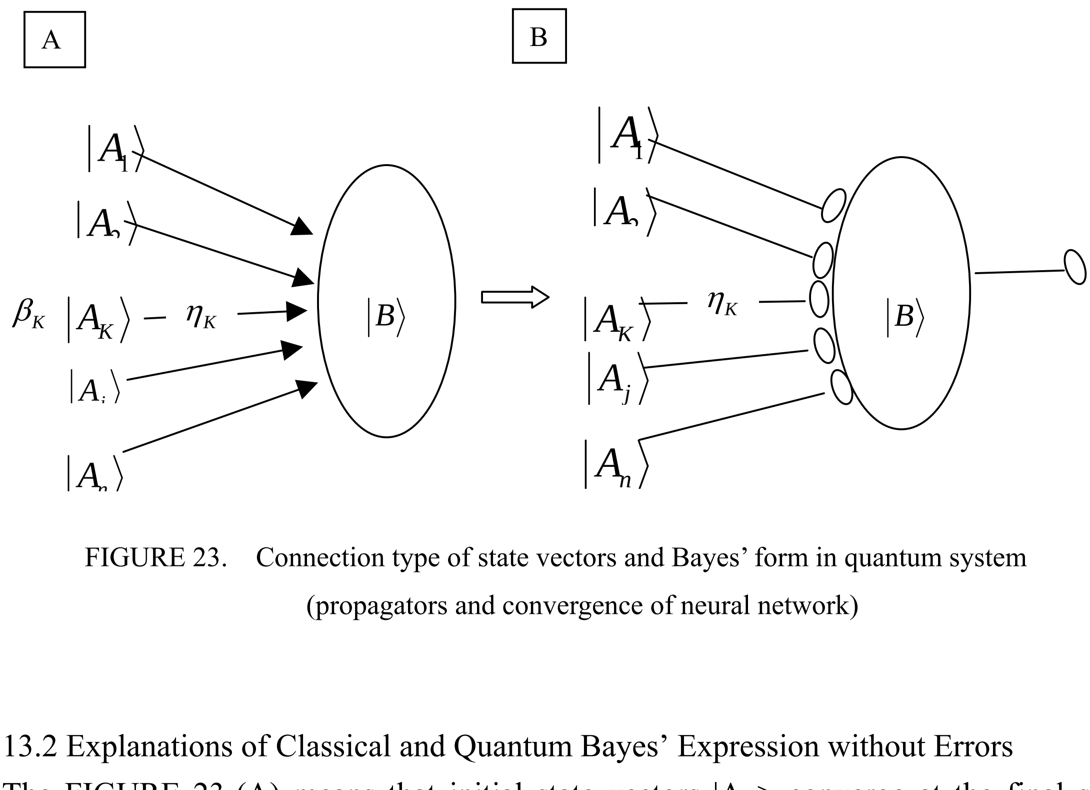


FIGURE 23. Connection type of state vectors and Bayes' form in quantum system (propagators and convergence of neural network)

13.2 Explanations of Classical and Quantum Bayes' Expression without Errors

The FIGURE 23-(A) means that initial state vectors $|A_K>$ converge at the final state vector $|B>$, and the each characters β_K is probability amplitude of occurrence of the corresponding initial state vectors $|A_K>$. The η -hat, which is described by P(B|A)-hat, is a propagating operator meaning a transitional state from A to B. The η -hat determines the propagating conduction's rate of state vectors. That figure 23-(B) shows a connectional type of many neurons, which is ordinary called as "convergence style of connection". Those two figure are so similar to each other that classical Bayes' form can almost translate into the convergence type of connection of quantum neural networks. For reducing Eq.(165) into simpler expressions, we introduce the following relation being used in ordinary quantum mechanics: we have completeness for bra & ket vectors,

$$\sum_j^n \left| A_j \right\rangle \left\langle A_j \right| = 1 . \tag{166}$$

Utilizing Eq.(166) and substituting it into Eq.(165), we are able to rewrite the numerator of Eq.(165), and we obtain

$$\left\langle B \middle| \hat{P}(B \mid A) \cdot \hat{P}(A) \middle| A_K \right\rangle = \sum_j^n \left\langle B \middle| \hat{P}(B \mid A) \middle| A_j \right\rangle \left\langle A_j \middle| \hat{P}(A) \middle| A_K \right\rangle . \tag{167}$$

We should note that the second term <A_j|P(A)-hat|A_K> of the r.h.s. Eq.(167) is the occurrence amplitude of event A_K at the state vector |A_K>. (^, -hat: operator). The state P(A)-hat|A_K> transit to any states |A_j> by the potential operator P(A_K)-hat, and finally the total occurrence amplitude becomes <A_j|P(A)-hat|A_K>. The first term <B|P(B|A)-hat|A_j> of Eq.(167) corresponds to the transitional and propagator's amplitude. For simplifying those expressions, we set the some rules. The initial N-numbers' vectors make a complete and orthogonal set {|A_K>, k = 1,N}.

$$\left\langle A_j \middle| A_K \right\rangle = \delta_{jK} . \tag{168}$$

So an arbitrary vectors |M> can be expanded by those initial vectors.

$$\left| M \right\rangle = \sum_j^n a_j \left| A_j \right\rangle \tag{169}$$

The initial state vectors |A_j> are in some pure states at start point t = 0, and then we assume that those vectors are satisfied with eigen equations.

$$\hat{P}(A) \left| A_j \right\rangle = \beta_j \left| A_j \right\rangle$$
$$\because \hat{P}(A) \equiv \hat{P}(x, -i\hbar\, \partial / \partial x). \tag{170}$$

Even if signals or information are propagating their communication channels and those processes are free from mistakes, we cannot escape an attenuation, exhaustion, dissipation at various junctions (neuro-synaptic junction, joining, etc.). So, mixture of various state occurs at final states |B>. Then the propagating states are expressed as

$$\hat{P}(B|A) \left| A_j \right\rangle \equiv \hat{\eta}(A) \left| A_j \right\rangle = \eta_j \left| A_j \right\rangle . \tag{171}$$

Both Eq.(170) and Eq.(171) tell us that two operators, those P(A)-hat and η (A)-hat are commutative each other. So we know

$$[\hat{P}(A),\hat{\eta}(A)] \equiv \hat{P}\hat{\eta} - \hat{\eta}\hat{P} = 0 . \tag{172}$$

And we notice that final state vector |B> is not in pure state, but in a mixed state, and final state is given by superposition of initial pure states |A>. We give an expansion of the final state by using superposition of the initial state, |A>. The mixed state |B>,

$$\left|B\right\rangle = \sum_j^n C_j \left|A_j\right\rangle . \tag{173}$$

becomes summing up possible state vectors $|A_K>$ by using Eq.(169).
Applying the Eq.(170)-(171) to Eq.(167), the numerator of Eq.(165) can be expressed by a simple relation,

$$\left\langle B\middle|\hat{P}(B\mid A)\hat{P}(A)\middle|A_K\right\rangle = \sum_j^n \left\langle B\middle|\hat{P}(B\mid A)\middle|A_j\right\rangle\left\langle A_j\middle|\hat{P}(A)\middle|A_K\right\rangle = C_K^*\beta_K\eta_K , \tag{174}$$

and the same procedure are practiced to the denominator of Eq.(165), whose result is

$$\sum_K^n \left\langle B\middle|\hat{P}(B\mid A)\hat{P}(A)\middle|A_K\right\rangle = \sum_K^n \left\langle B\middle|\hat{P}(B\mid A)\middle|A_K\right\rangle\beta_K = \sum_j^n C_j^*\beta_j\eta_j . \tag{175}$$

Finally, we obtain quantum Bayes' from: it is not probability but probability amplitude.

$$\left\langle A_K\middle|\hat{P}(A\mid B)\mid B\right\rangle = \frac{\left\langle B\middle|\hat{P}(B\mid A)\cdot\hat{P}(A)\mid A_K\right\rangle}{\sum_j^n\left\langle B\middle|\hat{P}(B\mid A)\cdot\hat{P}(A)\mid A_j\right\rangle} = \frac{C_K^*\beta_K\eta_K}{\sum_j^n C_j^*\beta_j\eta_j} . \tag{176}$$

The above result, Eq.(176) is almost similar to the classical calculation-Eq.(164), however, Eq.(176) has complex coefficient C_K whose complex number causes an essential difference between the classical Bayes' theorem and the quantum Bayes' one of Eq.(176). So, the classical Bayes' probability has real numbers, on the other hand quantum Bayes' form becomes complex numbers. Thus, the quantum Bayes' form applied for polaritons on neurons has a lot of interferences among polaritons and neural networks. The Eq.(176) shows polaritons to possess the phase and complex numbers, which mean to arise quantum effect interferences and probability amplitude. However, the classical Bayes' form has real numbers, which directly mean the probability or probability density, and the real values can not cause the interferences between each neuron. We would like to develop the calculation of Eq.(176) by using wave function. The state vector $|A_K>$ obeys Eq.(149),

$$i\hbar\frac{\partial}{\partial t}\left|A_K\right\rangle = \hat{H}\left|A_K\right\rangle . \tag{177}$$

Projecting to coordinate system, we have Schrödinger equation for the wave function,

$$i\hbar\frac{\partial}{\partial t}\langle x_K | A_K\rangle = \hat{H}\langle x_K | A_K\rangle \Rightarrow \phi_K(x_K, t_K) \equiv \langle x_K | A_K\rangle . \tag{178}$$

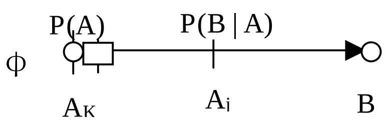
We will reach the simple form of the numerator of Eq.(165) by using Eq.(171), Eq.(173) and Eq.(178):

$$\begin{aligned}
&\langle B|\hat{P}(B\,|\,A)\cdot\hat{P}(A)|A_K\rangle = \int\langle B|\hat{P}(B\,|\,A)\hat{P}(A)|x_j\rangle\langle x_j|A_K\rangle dx_j \\
&= \int\langle B|\hat{P}(B\,|\,A)|x\rangle\langle x|\hat{P}(A)|x_j\rangle\phi_K(x_j)dx_j dx = \int\langle B|\hat{P}(B\,|\,A)|x_j\rangle\beta_j\phi_K(x_j)dx_j \\
&= \int\sum_q^n C_q^*\langle A_q|x_j\rangle\eta_j\delta(x_q - x_j)\beta_j\phi_K(x_j)dx_j dx_q = \int C_j^* p_j\eta_j\phi_j^*(x_j)\phi_K(x_j)dx_j \\
&\therefore \langle B|\hat{P}(B\,|\,A)\cdot\hat{P}(A)|A_K\rangle = C_j^*\beta_j\eta_j, \qquad (\because \int\phi_j^*(x_j)\phi_K(x_j)dx = \delta_{jK})
\end{aligned} \tag{179}$$

We can show that the result of state vectors calculation, Eq.(176), perfectly agrees to Eq.(179) of representation of the wave function. The state vector $|A_K>$ obeying under Schrödinger equation can be looked upon as an explicit wave function $\phi(x_K,t_K)$.

We would like to give the propagators and wave function at point B by using an initial wave function at point A_K. The function at A_K means the initial polariton $\phi_K(x_K,t_K) = \phi(A_K)$, whose wave function is produced by the generating operator P(A)-hat of probability amplitude. And polariton's function reaches the scattering center at point A_j, and it is scattered here, and then the scattered polaritons travels on axon until a state |B>. After all the final state of wave function $\phi(x_B, t_B) = \phi(B)$ can be described as

$$\begin{aligned}
&\phi(x_B,t_B) = \phi(B) = \iint K(B,A_j)P(B|A_j)K(A_j|A_K)P(A_K)\phi_K(A_K)dx_K dx_j \\
&= \iint\langle B|A_j\rangle P(B|A_j)\langle A_j|A_K\rangle P(A_K)\phi_K(A_K)dx_K dx_j = \int C_K^*\eta_K\beta_K\phi_K(A_K)dx_K \\
&\because \eta_j = P(B|A_j) = \langle A_j|\hat{P}(B|A)|A_j\rangle, \quad \beta_k = \langle A_K|\hat{P}(A)|A_K\rangle
\end{aligned} \tag{180}$$



by using Eq.(149), Eq.(161) and Eq.(173).

We can estimate an appearance of the wave propagation of $\phi_K(A_K)$ at final state |B> based on Eq.(180). The $\phi_K(A_K)$ changes that initial phase by affection of C_K containing the mixing state |B>. The normalized inner product of state vectors |B> is calculated

$$\langle B|B\rangle = \sum_j^n |C_j|^2 = 1 . \tag{181}$$

Then we have the normalized vector |B> to be expressed by superposition of many pure state vectors. That description is

$$\left|B\right\rangle = \sum_j^n \frac{1}{\sqrt{n}} \exp\left(i\theta_j\right)\left|A_j\right\rangle = \sum_j^n C_j\left|A_j\right\rangle. \tag{182}$$

Operating bra vector <x| from the l.h.s., so as to obtain a coordinate expression, the practical expressions are given as

$$\left\langle x\right|B\left.\right\rangle = \sum_j^n \frac{1}{\sqrt{n}}\left\langle x\right|A_j\left.\right\rangle \cdot \exp\left(i\theta_j\right), \tag{183}$$

$$\left\langle x\right|A_j\left.\right\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{i(p_j x - E_j t_j)}{\hbar}\right). \tag{184}$$

Note that the Eq.(184) clearly obeys to Schrödinger equation(150) as shown in Eq.(178). After all, quantum Bayes' from which is probability amplitude, is not probability, we have ϕ

$$\phi(A_K|B) = \left\langle A_K\right|\hat{P}(A|B)\left|B\right\rangle = \frac{\beta_K\eta_K \cdot \exp\left(i\theta_K\right)}{\sum_j^n \beta_K\eta_K \cdot \exp\left(i\theta_j\right)}. \tag{185}$$

Thus, probability becomes

$$P_q(A_K \mid B) = \left|\phi(A_K|B)\right| = \frac{|\beta_K|\cdot|\eta_K|}{\sqrt{\sum_j^n|\beta_j|^2|\eta_j|^2 + 2\,\mathrm{Re}(Z)}} \approx \frac{|\beta_K|\cdot|\eta_K|}{\sum_j^n|\beta_j|\cdot|\eta_j|\cdot(1+N)(1+M)}$$

$$\because \mathrm{Re}(Z) = \sum_{j>K}^n \mathrm{Re}\left[\beta_j^*\beta_K\eta_j^*\eta_K \cdot \exp i(\theta_j - \theta_K)\right], \quad N \equiv -\sum_{j\neq K}^n|\beta_j\eta_j|\cdot|\beta_K\eta_K|\Big/\big(\sum_K^n|\beta_K\eta_K|\big)^2,$$

$$M \equiv \frac{\mathrm{Re}(Z)}{\sum_j^n|\beta_j|^2|\eta_j|^2}. \tag{186}$$

Those results are rewritten by using

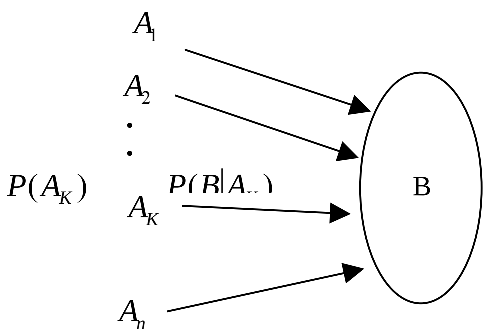
$$|\beta_K| \to P(A_K), \quad |\eta_K| \to P(A_K \mid B). \tag{187}$$

Because, both β and η mean eigen values of operators and so they are a kind of probability amplitudes. And the quantum Bayes' probability,

$$P_Q(A_K \mid B) = \frac{P(B \mid A_K)\cdot P(A_K)}{\{\sum_j^n P(B \mid A_j)\cdot P(A_j)\}\cdot(1+N+M+MN)} \tag{188}$$

is written as similar to classical Bayes' expression except the term of Re(Z), N and M. So the additional terms N and M are effects of the quantum interferences. All of pure states are mixing each other, and the new mixed state |B> is generated at the above junction of FIGURE 23. If we apply Eq.(188) for quantum neural networks, the mixed state |B> represents the states of an information around neuro-synaptic junctions, or each of axon's interferences (or cables of artificial neurons, ephapse).

We would like to obtain the entropy of both occurrence probabilities of the classical case and quantum one, since entropy is one of the most important elements of information theory. Both of the classical occurrence probability and the propagating probability have their values of real numbers $P(A_K)$ and of non-negative ones $P(B|A_K)$. On the other hand, quantum case does not mean direct probability, but the quantum form corresponds to eigen values of operator $P(A_K)$-hat and $P(B|A_K)$-hat, and their counter probability amplitudes, β_K and η_K. The FIGURE 24 shows concepts of the occurrence probabilities, quantum occurrence operators, and aspects of the propagation of the probabilities, and its quantum version of network's path (they are really communication paths or axons).



When the occurrence probability is $P(A_K)$ at point A_K, and the information propagates from A_k to B, the entropy is defined as

$$H(B|A_K) = -P(A_K)P(B|A_K)\log_2 P(A_K)P(B|A_K)$$

.

FIGURE 24. Classical propagation

Thus, total entropy from all of A to B is given as

$$H(B \mid A) = -\sum_k^n \alpha_k \lambda_k \log_2(\alpha_k \lambda_k) \qquad (189)$$
$$\because \alpha_k \equiv P(A_k), \quad \lambda_k \equiv P(B|A_k).$$

If we pay only attention to occurrence probability, its entropy is calculated by the result:

$$H(A) = -\sum_k^n \alpha_k \log_2 \alpha_k \qquad (190)$$

Notice the α_k to be the real and positive number.

However, the amplitude of entropy of quantum system $\sigma_q(B|A_K)$, which is not always real number, is defined as

$$\begin{aligned}\sigma_q(B \mid A_K) &\equiv \left\langle B \middle| \hat{P}(B \mid A)\hat{P}(A) \cdot \log(\hat{P}(B \mid A)\hat{P}(A)) \middle| A_K \right\rangle \\ &= -\sum_j^n \left\langle B \middle| \hat{P}(B \mid A)\hat{P}(B \mid A) \middle| A_j \right\rangle \cdot \left\langle A_j \middle| \log\{\hat{P}(B \mid A)\hat{P}(A)\} \middle| A_K \right\rangle \\ &= -C_K^* \beta_K \eta_K \cdot \log_2(\beta_K \eta_K),\end{aligned} \qquad (191)$$

by using eigen-state vectors, Eq.(170)-(171), and by taking expectation values of operators. Moreover, the amplitude of entropy for A_K is $\sigma_q(A_K)$, which is related to the state A_K :

$$\sigma_q(A_K) \equiv -\langle A_j | \hat{P}(A) \log_2\{\hat{P}(A)\} | A_K \rangle \delta_{jK} = -\beta_K \log_2 \beta_K = -|\beta_K| e^{i\gamma_K} (\log_2 |\beta_K| + \log_2 e^{i\gamma_K})$$
$$\approx -|\beta_K| e^{i\gamma_K} (\log_2 |\beta_K| + 1.443 i\gamma_K) \qquad \because -\pi \le \mathrm{Im}(\beta_K) \le \pi, \quad \arg(\beta_K) = \gamma_K \tag{192}$$

by using eigen-state vectors, Eq.(170)-(171), and its result is the operation of expectation values. Then the entropy of the occurrence of quantum system, $H_q(A)$ for A_K, is written as

$$H_q(A_K) \equiv -\left[\sigma_q^*(A_K)\sigma_q(A_K)\right]^{1/2} = -|\beta_K| \cdot \left\{(\log_2|\beta_K|)^2 + (\gamma_K/\ln 2)^2\right\}^{1/2}$$
$$\approx -|\beta_K| \log_2 |\beta_K| \cdot (1 + J_K) = -(1 + J_K) \cdot P(A_K) \log_2 P(A_K) \qquad \because J_K \equiv \frac{1.041\gamma_K^2}{(\log_2|\beta_K|)^2}. \tag{193}$$

We sum up the entropy of each pure state to obtain the total entropy of probabilities:

$$H_{Tq}(A_K) = -\sum_K^n |\beta_K| \cdot \left\{(\log_2|\beta_K|)^2 + (\gamma_K/\ln 2)^2\right\}^{1/2} \approx -\sum_K^n (1 + J_K) \cdot P(A_K) \log_2 P(A_K) \tag{194}$$

Comparing the above result of Eq.(194) with the classical result of Eq.(190), we immediately find the J_K-term to be added to $P(A_K)\log_2 P(A_K)$, whose additional term is directly generated by a phase of the wave function of the A_k, and the phase affects on the occurrence probability, and it gives rise to an interference, reflexive interaction and transitional action.

However, if both operators, $P(A_K)$-hat and $P(B|A_K)$-hat, are Hermitian and the counter states belong to their pure states, then their eigen values become real numbers since the their phase γ_k reduce to zero. Thus, the quantum result of Eq.(194) perfectly coincides with the classical case of Eq.(190). Using those relations for Eq.(191), the total entropy amplitude of the final state |B> is given as $\sigma_{Tq}(B|A_K)$, which is not pure state but it is clearly the mixed state superposed by many pure states. the $\sigma_{Tq}(B|A_K)$ is

$$\sigma_{Tq}(B \mid A) = -\sum_K^n \sigma_q(B \mid A_K) = -\sum_K^n C_K^* \beta_K \eta_K \cdot \log_2(\beta_K \eta_K). \tag{195}$$

And according to the method of Eq.(187)-(193), the total entropy is calculated by those equations, we obtain the final result of the quantum expression corresponding to classical relation (183):

$$
\begin{aligned}
H_{Tq}(B\mid A) &= -\left[\sum_K^n \sigma_{Tq}^*(B\mid A_K)\sigma_{Tq}(B\mid A_K)\right]^{1/2} \\
&= -\left[\sum_K^n \left|C_K\beta_K\eta_K\right|^2 \{(\log_2\left|\beta_K\eta_K\right|)^2 + (\tau_K\log_2 e)^2\} + 2\sum_{J>K}^n \mathrm{Re}(W_{J,K})\right]^{1/2} \\
&\approx -\sum_K^n \left|C_K\right|\cdot\left|\beta_K\eta_K\right|\cdot\log_2\left|\beta_K\eta_K\right|(1+D)(1+B+E) \\
&= -\sum_J^n P(B\mid A_J)P(A_J)\cdot\left(\log_2 P(B\mid A_J)P(A_J)\right)\cdot\left|C_J\right|(1+D)(1+B+E)\cdot
\end{aligned}
\tag{196}
$$

$$
\because \beta_K = \left|\beta_K\right|e^{i\gamma_K}, \quad \eta_K = \left|\eta_K\right|e^{i\delta_K}, \quad \tau_K \equiv (\gamma_K + \delta_K),
$$

$$
W_{J,K} = \sum_{J\neq K}^n C_K^* C_J \beta_K \beta_J^* \eta_K \eta_J^* \cdot \log_2(\beta_K\eta_K)\cdot\log_2(\beta_J^*\eta_J^*).
$$

$$
D = -\frac{\sum_{J\neq K}^n \left|C_J C_K \beta_J \beta_K \eta_J \eta_K\right| \log_2\left|\beta_K\eta_K\right|\cdot\log_2\left|\beta_J\eta_J\right|}{\left(\sum_J^n \left|C_J\beta_J\eta_J\right|\cdot\log_2\left|\beta_J\eta_J\right|\right)^2}
$$

$$
B = \frac{\left(\sum_K^n \tau_K \log_2 e\right)^2}{\sum_K^n \left|C_K^*\beta_K\eta_K\right|^2\cdot\left(\log_2\left|\beta_K\eta_K\right|\right)^2}, \quad E = \frac{\sum_{J>K}^n \mathrm{Re}(W_{J,K})}{\sum_K^n \left|C_K^*\beta_K\eta_K\right|^2\left(\log_2\left|\beta_K\eta_K\right|\right)^2}.
$$

Notice that entropy of quantum system has a lot of complex additional terms whose effects arise from much interference and a mixture of the pure states. Comparing Eq.(196) with classical entropy Eq.(190), we find the same expression of term, $P(B|A)P(A)\log_2 P(B|A)P(A)$ which means classical effect, and the other residual terms are corresponding to much interference of quantum system. Considering of both results Eq.(193) and Eq.(196), we can conclude that generally the entropy of quantum system is greater than that of classical system since the other residual terms are additional and non-negative. Thus, the quantum interference between pure states makes out an increase of entropy larger than the case of the classical system.

13.3 Multi Classical and Quantum Channels with Errors

We are discussing quantum channel without noise and its Bayes' form, and here-from we would like to study the channels with multi-dimensional channels with errors in this subsection. Now we have two channels, whose one is classical case and another means quantum system as shown in FIGURE 24-1 and 4-2. According to explanation of previous section, the $P(A_s)$ and the $P(B_j|A_s)$ correspond to the occurrence of probability of an event A_s and the propagating probability from the event A_s to the final result B_j. Thus, we know the classical channels of Bayes' form:

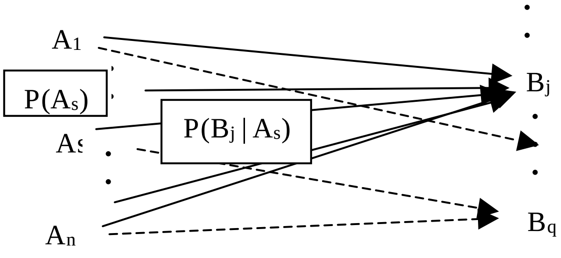


FIGURE 24-1. Classical multi channels

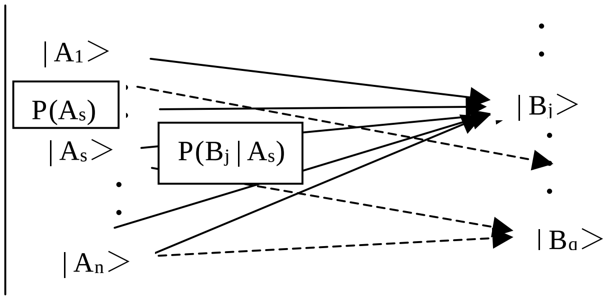


FIGURE 24-2. Quantum multi channels

$$P_{Cl}(A_K|B_j) = \frac{P(B_j|A_K)\cdot P(A_K)}{P(B)}, \qquad \because P(B_j) = \sum\nolimits_K^n P(B_j|A_K)\cdot P(A_K). \tag{197}$$

That representation is a Bayes' probability of multi channels as same as Eq.(164). On the other hand, quantum case is acquired by practicing to change those probabilities into the corresponding quantum operators, $P(A_s)$-hat and $P(B_j|A_s)$-hat. On the other hand, the classical event A_s is translated into a state vector $|A_s\rangle$. The simplest multi quantum channels are given as following forms of FIGURE 24-2:

$$\left\langle A_K\right|\hat{P}(A\,|\,B)\,|\,B_j\rangle \equiv \frac{\left\langle B_j\right|\hat{P}(B\,|\,A)\cdot\hat{P}(A)\,|\,A_K\rangle}{\sum\nolimits_j^n\left\langle B_j\right|\hat{P}(B\,|\,A)\cdot\hat{P}(A)\,|\,A_j\rangle}. \tag{198}$$

We would like to introduce both classical and quantum expressions of error's propagating probability, $1\text{-}P(B_j|A_s)$ and $1\text{-}P(B_j|A_s)$-hat, into our Eq.(197) or Eq.(198). Thus, we define the similar rules to simplify quantum calculations and observations as previous subsection.

1. Base set: the state vectors $|A_s\rangle$, (s = 1to n) make a complete set, and they are in pure state. States vectors $|B_j\rangle$, (s = 1to q) are in not pure states but they belong to the mixed states of all pure $|A_s\rangle$.

2. Orthonormality of base set: the pure state vectors hold on orthonormality.

$$\left\langle A_j(t_j)\middle| A_K(t_K)\right\rangle = \delta_{jK}\delta(t_j - t_s)\,. \tag{199}$$

3.An eigen function and eigen state, and propagating operators. The probability of occurrence of state A_s becomes as

$$\hat{P}(A)\left| A_j\right\rangle = \beta_j\left| A_j\right\rangle. \tag{200}$$

4. Propagating operators with errors and correct propagation in quantum channels: If the correct probability P(B_j|A_s)-hat is in state |A_s>, then the error probability's operator is expressed as 1-P(B_j|A_s)-hat. We have the p-numbers correct channels, and so the rests (n-p) numbers are in wrong. Then the correct and wrong propagating case are

$$\text{Correct case: } \hat{P}(B|A)\big|A_j\big\rangle \equiv \hat{\eta}(A)\big|A_j\big\rangle = \xi_j \eta_j \big|A_j\big\rangle, \quad 1 \le j \le p, \tag{201-1}$$

$$\text{Wrong case: } \{1-\hat{P}(B|A)\}\big|A_j\big\rangle \equiv \hat{\eta}^{w}(A)\big|A_j\big\rangle = \xi_j (1-\eta_j)\big|A_j\big\rangle. \tag{201-2}$$

The propagating operator η-hat commonly conveys probability amplitude of a correct information and ξ means a conduction's rate of propagating processes, however sometimes we fails to transmit the correct information from |A_s> to |B_j>. We assume that the p numbers channels are in correct states and the other (n-p) numbers channels propagate the signals to be wrong. Our propagating operator of neuron's model is to have four effects, which mainly contain neural conductions, ephapse among axons, thermal noise, and interferences nearby synaptic junction. And errors are induced by various interference and noise. The correct propagating operators η (A)-hat is composed of those factors:

η (A) = (neural conduction)+(ephapse)+ (noise & attenuation)+(synaptic interferences).

5. The each final state |B_j>, (j = 1 ~ q) is written down as summing up pure initial states. Thus, the |B_j>, (j = 1 ~ q), is mixed and superposed by a lot of pure states |A_s>. So, final mixed states enable to be expanded by n-numbers bases of orthonormal pure states.

So we have some final states written down as

$$\text{A final state of B: } \big|B_j\big\rangle = \sum_s^q C_s^j \big|A_s\big\rangle. \quad 1 \le j \le q \tag{202}$$

As we assume that the p channels are in correct and the others (n-p) are in wrong conditions, the numerator of Eq.(198) becomes by applying Eq.(199)-(201),

$$\big\langle B_j \big|\hat{P}(B \mid A)\cdot \hat{P}(A)\big|A_s\big\rangle = \big\langle B_j \big|\hat{P}(B \mid A)\cdot \beta_s \big|A_s\big\rangle = \beta_s \big\langle B_j \big|\hat{P}(B \mid A)\big|A_s\big\rangle = C_s^{*j}\beta_s \xi_s \eta_s. \tag{203}$$

The other the denominator's Eq.(179) is given by same way as Eq.(180), except an existence of both channels being correct and wrong. We can decide the expression of denominator,

$$\sum_s^n \langle B^j | \hat{P}(B|A)\hat{P}(A) | A_s \rangle = \sum_s^n \langle B | \hat{P}(B|A) | A_s \rangle \beta_s = \sum_{s=p+1}^n C_s^{*j} \beta_s \xi_s (1-\eta_s)$$

$$+\sum_{s=1}^p C_s^{*j} \beta_s \xi_s \eta_s = \sum_{s=1}^p C_s^{*j} \beta_s \xi_s \eta_s + \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} (1-\eta_{p+s}). \qquad (204)$$

and final quantum Bayes ’ form for state $|B_j>$ becomes

$$\langle A_s | \hat{P}(A|B) | B_j \rangle \equiv \frac{C_s^{*j} \beta_s \xi_s \eta_s}{\sum_{s=1}^p C_s^{*j} \beta_s \xi_s \eta_s + \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} (1-\eta_{p+s})}. \qquad (205)$$

The denominator of Eq.(205) is similar to that of Eq.(176) except to the second error’s term. So, the first term represents the correct propagating amplitude, the second term is the case of the wrong (an error) propagation or communication. According to previous subsection 13-2, we easily notice that the result has complex interferences between correct channels (i.e. axons of neurons) and wrong ones, because of taking absolute value of Eq.(205). They are two types of interferences: one type belongs to each of correct channel, and another is in wrong channels. Moreover, we find that a new interference Z_q, by using Eq.(186), appears in the probability of quantum Bayes' P_Q as shown in Eq.(205):

$$P_Q(A_s|B_j) = \frac{\left| C_{s=1}^{*j} \beta_s \xi_s \eta_s \right|}{\left(\left| \sum_{s=1}^p C_{s=1}^{*j} \beta_s \xi_s \eta_s \right|^2 + \left| \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} (1-\eta_{p+s}) \right|^2 + Z_q \right)^{1/2}}$$

$$Z_q \equiv \left(\sum_{s=1}^p C_{s=1}^{*j} \beta_s \xi_s \eta_s \right) \cdot \left(\sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} (1-\eta_{p+s}) \right) \qquad (206)$$

That Z_q says an existence of interferences in between correct channels and wrong channels. Next we calculate both an amplitude of entropy for all paths $\sigma_{tA}(B_j|A)$, from A_s (s = 1,n) to B_j, and finally we obtain the total amplitude of entropy for the mixed state for all B_j, (j = 1,q). That is described by the symbol $\sigma(B|A)$. We know the result $\sigma(B_j|A_s)$ by Eq.(176):

$$\sigma_{tA}(B_j|A) \equiv \sum_s^n \sigma(B_j|A_s) = \langle B_j | \hat{P}(B|A)\hat{P}(A) \cdot \log(\hat{P}(B|A)\hat{P}(A)) | A_s \rangle$$

$$= -\sum_{s=1}^p C_s^{*j} \beta_s \xi_s \eta_s \cdot \log_2(\beta_s \eta_s) - \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} \overline{\eta}_{p+s} \cdot \log_2(\beta_{p+s} \overline{\eta}_{p+s}), \qquad (207)$$

$$\because \overline{\eta}_{p+q} \equiv 1-\eta_{p+s},$$

And then $\sigma(B|A)$ is expressed as

$$\sigma(B|A) \equiv \sum_j^q \sigma_{tA}(B_j|A) = -\sum_j^q \left(\sum_{s=1}^p C_s^{*j} \beta_s \xi_s \eta_s \cdot \log_2(\beta_s \eta_s) \right)$$

$$-\sum_j^q \left(\sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} \overline{\eta}_{p+s} \cdot \log_2(\beta_{p+s} \overline{\eta}_{p+s}) \right) \qquad (208)$$

(ξ means a conduction’s rate of propagating processes). The total entropy H(B|A) from state A to state B is calculated as

$$H(B|A) \equiv -\left[\sigma^*(B|A)\cdot\sigma(B|A)\right]^{1/2}$$
$$= -\left|\sum_j^q \left(\sum_{s=1}^p C_s^{*j}\beta_s\xi_s\eta_s \cdot \log_2(\beta_s\eta_s)\right) + \sum_j^q \left(\sum_{s=1}^{n-p} C_{p+s}^{*j}\beta_{p+s}\xi_{p+s}\overline{\eta}_{p+s} \cdot \log_2(\beta_{p+s}\overline{\eta}_{p+s})\right)\right|. \tag{209}$$

From Eq.(209), we find not only interferences of correct channels and that of wrong ones, but also a lot of interferences between correct and wrong channels, which is truly quantum effects without being in classical systems. In following section, we would like to discuss an approximate solution's method, being generally called perturbation theory.

14. Perturbation Method for Multi Quantum Channels

In previous sections, I proposed various concepts, i.e. quantum bifurcations, quantum Amida lots, and quantum, quantum circuits, quasi-particle-polaritons, and quantum neural conductions (hypothesis of polaritons)[1]-[5],[9],[17]. Though we adopt mathematical expressions for model of polaritons, we would like to emphasize that quantum interferences play an important roles in our information system in order to adjust and to maintain homeostatic states of neural networks and brains [6]-[8]. Some important examples of polaritons are neural conductions and the coupling relationship between ionic currents (Na+, K+, Cl^-) and quasi particles (polaritons). Polaritons connect between many ionic currents of neural activities by many quantum interferences. For examples, they are polarization waves, the carried charges, their momentum and energies. And we showed polaritons, being massive photons, are governed by Proca equation, whose form is reduced into quaternary Schrödinger equation [17]. As far as magnetic field changes so slowly, its vector potential **A** takes nearly equal to constant value. Then the polariotns can be regarded as their motions (neural conduction: maximum velocity 100m/s) being much low velocity, comparing with light velocity. Under those conditions, we simply have only to consider motions of the scalar component of the polaritons. Finally, instead of both Proca equation and the quaternary Schrödinger equations, we have only to consider the common scalar Schrödinger equation, which has the only scalar potential ϕ of polaritons as Eq.(147). The time development of a final state $|B_j>$, with the scalar potential ϕ and constant vector potential **A,** can be described as

$$i\hbar\frac{\partial\left|B_j(t)\right\rangle}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\left|B_j(t)\right\rangle = \hat{H}\left|B_j(t)\right\rangle. \tag{210}$$

We know various patterns of connections of neural networks, i.e. convergence as shown in FIGURE 25, divergence, recurrent and so on. So we discuss the time development of the final state $|B_j>$ by applying perturbation method for Eq.(210).

14.1 Perturbation of Time Dependent for Final Sate B_j

We would like to show an approximate method of FIGURE 25, and discuss physical descriptions from the middle layer to the output's ones. The FIGURE 25 contains a convergence's type of neural network when we pay attention to one neuron $|B_j>$. And the neurons of the first layers are connected with those of the second layers. Each neuron of first layer's, $|A_s>$, (s = 1,m), has N-number's pure states, and each neuron of second layer's, $|B_j>$, (j = 1,q) is in a mixed state. Polaritons are approximately governed by Schrödinger equation Eq.(210), and the final mixed state $|B_j>$, with an initial sate at time t_1, have the following expression

$$\left\langle x \middle| B_j(t) \right\rangle = \exp\left(-i \frac{\hat{H}t}{\hbar} \right) \left\langle x \middle| B_j(t_1) \right\rangle \equiv \left\langle x \middle| \hat{U}(t) \middle| B_j(t_1) \right\rangle . \tag{211}$$

Generally we hypothesize that its non-perturbation part can be exactly solved, when we use a complete base set of N-number's pure state vectors. So we have following relationship,

$$i\hbar \frac{\partial}{\partial t} \left| A_s(t) \right\rangle = \hat{H}_0 \left| A_s(t) \right\rangle, \quad s = 1,2 \cdots N \tag{212}$$

and $\left| A_s(t) \right\rangle = \left| A_s \right\rangle \left| t \right\rangle$.

As a base set of pure state vectors is assumed to have the solutions of the non perterbation, then each state vector $|A_s(t)>$ is regarded as an exact solution of Eq.(212). Each pure state's expression is written down as

$$\left| A_s(t) \right\rangle = \exp\left(\frac{-i\varepsilon_s t}{\hbar} \right) \left| A_s \right\rangle = \left(\frac{-i\hat{H}_0 t}{\hbar} \right) \left| A_s \right\rangle . \quad (t = 1, m) \quad ; |As> : \text{spatial component} \tag{213}$$

The Eq.(212) directly gives us one of final state vectors $|B_j>$ belonging to the second layer. The general solution of Eq.(199) is given by expanding and superposing of Eq.(213), we have

$$\left| B_j(t) \right\rangle = \sum_s^m C_s^j(t) \left| A_s(t) \right\rangle = \sum_s^m C_s^j(t) \exp\left(\frac{-i\varepsilon_s t}{\hbar} \right) \left| A_s \right\rangle . \tag{214}$$

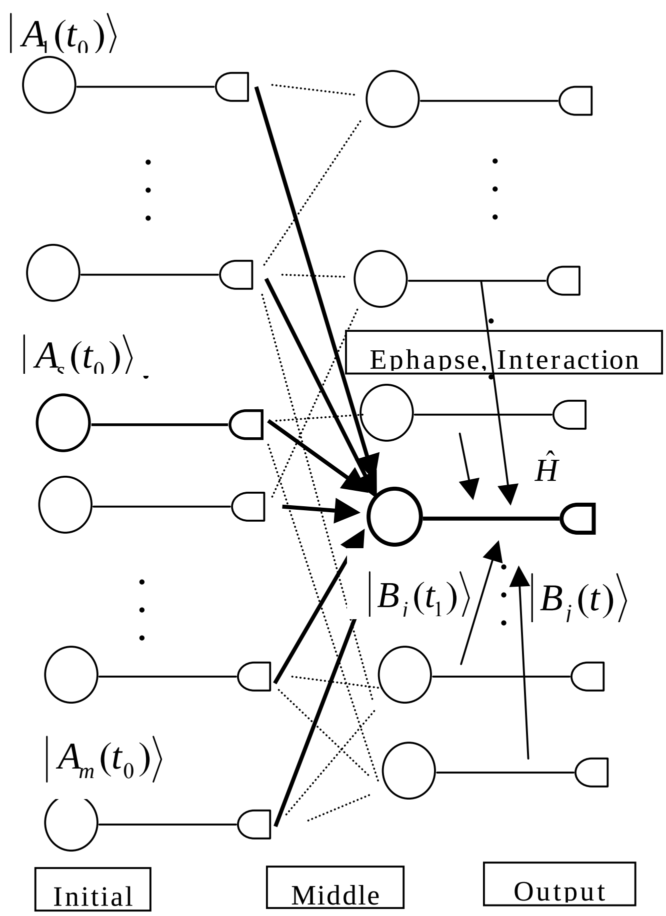


FIGURE 25. Quantum neural network and its Interactions

We consider all combinations between first layers' state vectors and second layers' ones as you know in branch of information theory. One different point is to quantum interference with each vector. And note that you find out network system to contain both types, i.e., convergence and divergence. It is convergence type if you pay attention to only second layer's state vectors, $|B_j(t)\rangle$, i.e,. connection of bold real lines. If you regard those state vectors as classical neuron array, you find a familiar neural network system. The motion of polaritons are described by Hamiltonian, which are separated into two parts. One is the part which we can solve exactly, and another part contains a complex interactions, and we cannot be easily solved. If we take perturbation method, Hamiltonian separated into both parts: we can get an exact solution of term H_0, and another is difficult to solve the Schrödinger equation for part H_I.

Then making an orthonormalization for basic vectors done, we have

$$\langle A_j | A_K \rangle = \delta_{jK} . \tag{215}$$

The full Hamiltonian H-hat is divided into two parts, and the one is a non-perturbation term H_0, another has a perturbation effect H_I:

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_I \tag{216}$$

Substituting Eq.(214) into (210) and, we take an inner product for them by $\langle A_p|$, (p = 1,m): each coefficient is given as

$$C_p^j(t) = \frac{1}{i\hbar}\lambda \sum_s^m C_s^j(t) \langle A_p | \hat{H}_I(t) | A_s \rangle \cdot \exp\left[-\frac{it}{\hbar}(\varepsilon_s - \varepsilon_p) \right]. \tag{217}$$

Hitherto we do not introduce any approximating methods, and now we are going to practice to perturbation series. That expansion of coefficients $C^j_p(t)$ is represents as

$$C_p^j(t) = C_p^{j(0)}(t) + \lambda^{(1)} C_p^{j(1)}(t) + \lambda^{(2)} C_p^{j(2)}(t) + \cdots + \cdots \lambda^{(w)} C_p^{j(w)}(t) \cdots \tag{218}$$

We take the ordered (m+1)-th term into consideration, and gather the first ordered terms:

$$\begin{aligned} C_p^{j(1)}(t) &= \frac{1}{i\hbar}\int_0^t \sum_s^m \alpha_s^{j(0)} \langle A_P | \hat{H}_I(t) | A_s \rangle \exp\frac{-it_1}{\hbar}(\varepsilon_s - \varepsilon_p)\, dt_1 \\ C_p^{j(0)}(t) &= C_p^{j(0)}(0) = \alpha_s^{j(0)} = const. \end{aligned} \tag{219}$$

So, the second ordered perturbation term is described as

$$\begin{aligned} C_p^{j(2)}(t) &= \left(\frac{1}{i\hbar}\right)^2 \sum_W^m \int_0^t \{\int_0^{t2} \sum_s^m \alpha_s^{j(0)} \langle A_P | \hat{H}_I(t) | A_s \rangle \exp\frac{-it_1}{\hbar}(\varepsilon_s - \varepsilon_p)\, dt_1 \} \langle A_p | \hat{H}_I(t_2) | A_W \rangle \\ &\times \exp\frac{-it_1}{\hbar}(\varepsilon_W - \varepsilon_p)\, dt_2. \end{aligned} \tag{220}$$

Those results are substituted into Eq.(214), and finally we reach the approximate solution of polariton's with perturbation expansion for state |Bj>:

$$\begin{aligned} |B_j(t)\rangle &= \sum_s^m C_s^j(t) \exp\left(\frac{-i\varepsilon_s t}{\hbar}\right) |A_s\rangle \approx \sum_p^m \Big(\Big[\alpha_p^{j(0)} + \Big\{ \frac{1}{i\hbar}\int_0^t \sum_s^m \alpha_s^{j(0)} \langle A_p | \hat{H}_I(t_1) | A_s \rangle \\ &\times \exp\frac{-it_1(\varepsilon_s - \varepsilon_p)}{\hbar} dt_1 \Big\} + \Big\{ \left(\frac{1}{i\hbar}\right)^2 \sum_W^m \int_0^t \Big(\int_0^{t2} \sum_s^m \alpha_s^{j(0)} \langle A_W | \hat{H}_I(t_1) | A_s \rangle \\ &\times \exp\frac{-it_1(\varepsilon_s - \varepsilon_p)}{\hbar} dt_1 \Big) \times \langle A_p | \hat{H}_I(t_2) | A_W \rangle \exp\frac{-it_1(\varepsilon_s - \varepsilon_p)}{\hbar} dt_2 \Big\} \Big] |A_p\rangle \exp\frac{-i\varepsilon_p t}{\hbar} \Big). \end{aligned} \tag{221}$$

The r.h.s of Eq.(221) means the zero-th ordered, the first ordered and second ordered perturbation term for an exact Hamiltonian H-hat. If we pay attention to the processes of neural conduction with having an interacting Hamiltonian H_I-hat (i.e. the processes of polarization, depolarization, and Na pomp, phenomena of the ephapse, etc.), and if the H_0 -hat means free polariton's motion, we can obtain an approximating solutions based on perturbation method Eq.(221). As the final state $|B_j>$ consists of many pure states $|A_s>$, the rate $b^j_s(t)$ of probability amplitude for $|A_s>$ is defined as

$$b_s^j(t) = \langle A_s | B_j \rangle = b_s^j \exp\left(-i\frac{\varepsilon_s t}{\hbar}\right), \quad \because \langle A_s | A_K \rangle = \delta_{sK}. \quad \text{from Eq.(124).} \tag{222-1}$$

So, total amplitude of rate R^j becomes summing up all numbers' of possible states,

$$R^j = \sum_s^m b_s^j(t) = \sum_s^m \langle A_s | B_j \rangle = \sum_s^m b_s^j \exp\left(-i\frac{\varepsilon_s t}{\hbar}\right). \tag{222-2}$$

If the probability amplitude of an occurrence for an event B^j is in proportion to the above total amplitude R^j, whose probability is expressed by $|R^j|$. Then the variation of energies caused by perturbation H_I-hat, from state A_s to B^j, is calculated by the relation, which is called as transition amplitude between those states:

$$\delta E_{js}^{B} = \left\langle B_j(t_1) \middle| \hat{H}_I \middle| A_s(t_0) \right\rangle = \sum\nolimits_s^m b_s^j \left\langle B_j \middle| \hat{H}_I \middle| A_s \right\rangle \exp\left(\frac{-i(\varepsilon_s - \varepsilon_p)}{\hbar} \right). \tag{223}$$

Then it is important to notice, the above expression is much similar to the second term of Eq.(185) if we regard the interaction operator H_I-hat as the propagating operator $P(B^j|A_s)$-hat. So that reason enable to translate the relationship of Eq.(223) into an energy propagating expression of the probability amplitude when we translate the $P(B^j|A_s)$-hat into the H_I-hat. When an initial state $|A_s>$, which is in pure state A_s, we assume that the amplitude of the occurrence obeys to Eq.(199). So, we would like to discuss the propagation and relations between an initial phase, $|A_s>$ and a middle phase as shown in FIGURE 25, in next subsection.

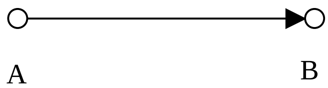
14.2 Propagation from initial phase to middle phase

Practically to calculate the propagation amplitude from state A_s to B_j, we should adopt path integral formula and the propagation of plane wave $\phi_s(x)$. That eigen function is $\phi_s(x)$:

$$\phi_s(x) = \frac{1}{\sqrt{L}} \exp(ik_s x). \quad \text{free plane wave for state vector, } |A_s> \tag{224}$$

Then we can have a result of kernel for free propagation with applying Eq.(234) to Eq.(161), (162) and Eq.(149). The Kernel $K(B_j|A_s)$, $B_j(x_1, t_1)$ and $A_s(x_0,t_0)$, is given as

$$K(B_j|A_s) = \frac{1}{L} \sum\nolimits_{ks} \exp\{ik_s(x_1 - x_0) - i\varepsilon_s(t_1 - t_0)/\hbar\}. \tag{225}$$



$$\varepsilon_s = \frac{\hbar^2 k_s^2}{2m}. \quad \text{the symbol m means quasi polariton's mass (dressed polariton).} \tag{226}$$

Changing sum of k_s into an integral, the final result,

$$K(B_j, A_s) = \left[\frac{m}{2\pi i\hbar(t_1 - t_0)} \right]^{1/2} \exp\left[\frac{im(x_1 - x_0)^2}{2\hbar(t_1 - t_0)} \right] = {}_{t1}\left\langle B_j \middle| A_s \right\rangle_0 . \tag{227}$$

are corresponding to the Eq.(150). And after $(t_1\text{-}t_0)$, the wave function, with taking an initial condition of source for $|A_s(x_0,t_0)> = \Psi_s(x_0,t_0)$, is described as

$$\psi(B_j)=\int K(B_j,A_s)\psi(A_s)\,dA_s=\int\left[\frac{m}{2\pi i\hbar(t_1-t_0)}\right]^{1/2}\exp\left[\frac{im(x_1-x_0)^2}{2\hbar(t_1-t_0)}\right]\psi(A_s)\,dA_s$$
$$=\int\left[\frac{m}{2\pi i\hbar(t_1-t_0)}\right]^{1/2}\exp\left[\frac{im(x_1-x_0)^2}{2\hbar(t_1-t_0)}\right]\cdot\left(\frac{1}{\sqrt{L}}\exp\left(ik_s x_0\right)\right)dx_0$$
$$=\frac{1}{\sqrt{L}}\exp\left(\frac{-i\hbar k_s^2(t_1-t_0)}{2m}\right)\cdot\exp\left(ik_s x_1\right). \tag{228}$$

.

That result says the propagation to change the phase of wave function when the plane wave arrives at the point B_j, though its momentum is conserved at same wave number k_s. After all, the time development $|B_j\rangle$, by using path integral, is written at arbitrary time by unitary operator U-hat of Eq.(211):

$$\left\langle x\middle|B_j(t)\right\rangle=\exp\left(-i\frac{\hat{H}t}{\hbar}\right)\left\langle x_1\middle|B_j(t_1)\right\rangle\equiv\frac{1}{\sqrt{L}}\exp\left(-i\frac{\hat{H}t}{\hbar}\right)\cdot\exp\left(\frac{-i\hbar k_s^2(t_1-t_0)}{2m}+ik_s x_1\right),$$
$$\because\ \psi(B_j)=\left\langle x\middle|B_j(t)\right\rangle,\quad t\equiv t-t_1. \tag{229}$$

Above expression of $\langle x|B_j(t)\rangle$ is truly the result of an output phase of FIGURE 25. So, summing up all initial conditions, we obtain a total probability amplitude of transition states from $|A_s\rangle$, (s = 1,m) to $|B_j\rangle$. We would like to introduce the correct conducting operator η_a-hat, (a = 1,p), and wrong conducting case $(1-\eta_a)$-hat, (a = p+1,m), and ξ-hat is a conduction rate under propagating processes. And all of them are quantum operators:

$$\Psi(B_j)\equiv\frac{1}{\sqrt{L}}\exp\left(-i\frac{\hat{H}(t-t_1)}{\hbar}\right)\cdot\left\{\sum_{a=1}^{p}\hat{\eta}_a\hat{\xi}_a\exp\left(\frac{-i\hbar k_a^2(t_1-t_0)}{2m}+ik_a x_1\right)\right.$$
$$\left.+\sum_{a=1}^{m-p}(1-\hat{\eta}_a)\hat{\xi}_a\left(\frac{-i\hbar k_a^2(t_1-t_0)}{2m}+ik_a x_1\right)\right\} \tag{230}$$

Those operators are almost q-numbers in some cases, but sometimes we notice that they are c-numbers, or control functions (potentials). One of the simple examples of ξ-hat equals to 1 (no consumption on channels), and the η-hat corresponds to Gauss function (Gauss slits) or step function. In Gauss function, the Gauss slit is given by the following form,

$$\hat{\eta}_a=C\exp(-\alpha x^2)\,. \tag{231}$$

The Gauss slit is divided into many cases by both conditions of C and the η_a. Thus, we would like to show all possible cases:

(1) If $0 < C < 1$, then we have p number's correct channels and the (m-p) number's wrong ones.

(2) All channels are correct, if the $\eta_a = 1$.

(3)All channels are wrong, if $\eta_a = 0$.

(4) If C is over 1, whose η_a are composed a lot of four parts, i.e. $\eta_a = 0$, $0 < \eta_a < 1$, $\eta_a = 1$, and $\eta_a > 1$, and so we should discuss to divide into four parts.

(a) if the part of $\eta_a = 0$, then $1 - \eta_a$ equals to 1, so then all channels are wrong.

(b) if the η_a is in the range, $0 < \eta_a < 1$, then Eq.(89) has two channels: The one channel corresponds the correct part η_a and another is the wrong one, $1 - \eta_a$.

(c) if the part of $\eta_a = 1$, then the wrong channel $1 - \eta_a = 0$, then all channels are correct.

(d) if the part of $\eta_a > 1$, then the part of $\eta_a > 1$ is able to amplify an input wave function during passing the channel of the slit. On the other hand, the wrong channel , $(1 - \eta_a)$, has negative value, and we notice that a sign of the wave function is inversed from positive into negative. We intend to regard the reversing parts as inhibitory potential or inhibitory neurons

Thus, the above phenomena show that we are able to control the communication channels by means of making gate, slit and some functions.

In next section, the similarity of between the quantum neural network and classical one are pointed out, and finally we refer to the method of quantum expectation value, whose operation is likely to that of fuzzy probability.

15. Application of Quantum Neuron

We would like to show two examples of simple application of the quantum neural systems. One is an example of quantum neural network, which looks like classical neural network's model, another is probability of fuzzy set theory called the fuzzy probability[34]-[35].

15.1 Quantum Neural Network

The classical neural networks are described as famous following relations: if inputs signal X_j, $(j = 1, N)$, weighted by W_{KJ}, are added to the K-th neuron, then the changes of activity of membrane potential U_k are commonly expressed as

$$U_K = \sum_j^N W_{Kj} X_j - h_j \,. \tag{232}$$

A classical output Y_K is determined by propagator function f(・) and the potential U_K. Thus the Y_K becomes output of the classical networks:

$$Y_K = f(U_K) = \frac{1}{1+\exp(-aU_K)} \,. \tag{233}$$

On the other hand, if we pay attention to a quantum neural network, its networks can be written by the same manner to classical network, and then the state $|A_{BK}(t)\rangle$ is

$$\left|A_K^B(t)\right\rangle = \sum_j^N C_{Kj}(t)\left|A_j(t)\right\rangle = \sum_j^N \{C_{Kj}(t)\exp\left(\frac{-i\varepsilon_j t}{\hbar}\right)\left|A_j\right\rangle - h_j\left|A_j(0)\right\rangle\}, \qquad \because \left|A_j(0)\right\rangle = const. \tag{234}$$

The weight W_{KJ} and signal X_j correspond to the weight C_{Kj} (coefficient) of superposition of the quantum state vector $|A_j(t)\rangle$, and the final state $|A_K{}^B(t)\rangle$ is regarded as the classical potential term U_K. The classical output Y_K is determined by propagator function f(・) and potential U_K. By the same reason, the quantum outputs are given by the following relation

$$\begin{aligned}\Phi_K &= \left\langle x\middle|A_K^B(t)\right\rangle = \sum_j^N C_{Kj}(t)\left\langle x\middle|A_j(t)\right\rangle - h_j C_{j0} = \sum_j^N C_{Kj}(t)\exp\left(\frac{-i\varepsilon_j t}{\hbar}\right)\left\langle x\middle|A_j\right\rangle - h_j C_{j0}, \\ &= \sum_j^N \{C_{Kj}(t)\exp\left(\frac{-i(\mathbf{px}+\varepsilon_j t)}{\hbar}\right) - h_j C_{j0}, \qquad \because C_{j0} = \left\langle x\middle|A_j(0)\right\rangle.\end{aligned} \tag{235}$$

in the projection of the coordinate space. So, we easily find, the classical output Y_k can be replaced by the quantum expression Ψ_K. Thus, we have an equation of

$$\Psi_K = f(\Phi_K) = \frac{1}{1+\exp(-a\Phi_K)} \,. \tag{236}$$

Two expressions of output functions are much similar to each other, however, the quantum outputs truly contain various quantum effects which are essentially difference from the classical networks, because the quantum output function Ψ_K allows complex number's functions, and it does not means the probability but corresponds to the probability amplitude. The other hand, the parameters of classical networks Y_k, U_K and X_j, are quite real numbers since they dose not have interferences among others..

15.2 Fuzzy Probability and Quantum Neuron

We would like to refer to an example of a fuzzy probability by taking up a dice. The A is defined as the set of numbers of the dice

Set X: $X = \{1, 2, 3, 4, 5, 6\}$. (237)

We consider a fuzzy event as an elements of set A taking nearly equal to the value 6, which means the fuzzy probability $P_E(\approx 6)$. To calculate the fuzzy probability $P(\approx 6)$, it is necessary to introduce a membership function of the set A. For example, each element of the membership function is given as A(X), (X = 1,6),

$$A(1) = 0,\ A(2) = 0.1,\ A(3) = 0.3,\ A(4) = 0.6,\ A(5) = 0.9,\ A(6) = 1. \tag{238}$$

Then we can calculate the fuzzy probability by using probability P(X), since we are having the membership function. Thus, the fuzzy probability $P_E(\approx 6)$ is obtained by procedure,

$$P_E(\approx 6) = A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6). \tag{239}$$

We assume that the dice has an equivalent probability for each value:

$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$. So we have final result $P_E(\approx 6) = 0.483$. According to common probability method, the probability, we can obtain the value 5 or 6 of the dice, has the same expression,

$$A(1) = 0,\ A(2) = 0,\ A(3) = 0,\ A(4) = 0,\ A(5) = 1,\ A(6) = 1. \tag{240}$$

Thus, we have

$$\begin{aligned} P_E(5 \vee 6) &= A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6) \\ &= 1 \times 1/6 + 1 \times 1/6 = 1/3. \end{aligned} \tag{241}$$

Hitherto based on the above discussion, both probabilities, $P_E(X_J)$ can be written down by using the probability density $P_\rho(X)$ and membership function $F_J(X)$ for $X=X_J$,

$$P_E(\approx X_J) = \int_{all\ X} P_\rho(X) F_J(X) dX\ . \tag{242}$$

In order to expand Eq.(241) by regarding sub index J, we consider a set of membership function F_J, and that of probability density P_ρ. We make the inner products of the elements:

$$\begin{aligned} &F = \{F_1(X), F_2(X) \cdots F_M(X)\}.,\quad P_\rho = \{P_{\rho 1}(X), P_{\rho 2}(X) \cdots P_{\rho M}(X)\} \xrightarrow[product]{} \\ &I = \left\{ \int_{all\ X} P_{\rho 1}(X) F_1(X) dX,\ \int_{all\ X} P_{\rho 2}(X) F_2(X) dX, \cdots \int_{all\ X} P_{\rho M}(X) F_M(X) dX \right\}. \end{aligned} \tag{243}$$

So we have an expression of fuzzy probability of two variables, when we regard the indexes of $P_\rho(\cdot)$ and $F_J(\cdot)$ as function of variable X, y:

$$P_E(\approx y) = \int P_\rho(X, y) F(X, y)\, dX\ . \tag{244}$$

That is the fuzzy probability when it takes the value to be about y. Thus, we find that those equations from Eq.(242) to Eq.(244) show the fuzzy probability, and we notice that those description of the expectation value have mathematically some similarities between fuzzy system and quantum one. According to the quantum mechanics, its probability density $P_E(\approx X)$ is defined as $|\Psi|^2$, it is possible to translate the fuzzy probability into quantum language. Then an expectation is be, according to quantum mechanics,

$$\left\langle F_J(\approx X_J)\right\rangle = \int P_\rho(X)F_J(X)\,dX = \int \Psi^*(X)F_J(X)\Psi(X)\,dX\ . \tag{245}$$

Notice that the fuzzy probability Eq.(242), by the membership function, has similarity to the expectation value of quantum mechanics. Thus, we can estimate the various physical quantities and the controls of quantum neural networks, since the fuzzy probability is a kind of quantum probability. The fuzzy probability $P_E(\approx X)$ can directly be translated into the expectation value of membership function $\left\langle F_J(X)\right\rangle$. And we find the Fuzzy membership function $F_J(X)$ to correspond to a physical observable, which can be translated into the operator of physical quantity $F_J(X)$-hat. If the polariton, conducting on axon, has an eigen value E_J and eigen function Ψ_J belonging to Schrödinger equation-(149), then the quantum mechanical expectation of the membership function (strictly speaking, that is a membership operator) is given by

$$\left\langle \hat{F}_J(\approx X_J,P)\right\rangle = \int \psi_J^*(X)\hat{F}_J(X,P)\psi_J(X)dX, \qquad \because\ \hat{F}_J(X,P)\equiv F_J(X,-i\hbar\nabla)\ . \tag{246}$$

After all, those equations, from Eq.(242) to Eq.(246), show the similarity of the fuzzy probability and the quantum description of the expectation process (Reference to Appendix-2. A2-1 & A2-2, we mention relationship between Choquet Integral and Quantum mechanical expectation).

16. Summary and Conclusion

We would like to show summary, conclusion, new model of quantum neuron and its network, and quantum probability.

16.1 Ionic Current and Role of Polariton

We proposed a hypothesis of polariton for quantum neural conduction's theory on artificial axons[36]. The polariton, which means a quasi particle, is considered to be real object, which carries momentum, energy, impulse, charge current, and those various quantum interferences. The model of polariton is described as the quantized polarization wave, being generated by an action potential of neural membrane on axons and ionic currents. The polariton flew from neural body to synapse along to the axon. The phenomenon is commonly known to be the neural conduction based on classical physiology. However, we think classical process, (polarization- depolarization -repolarization), is can be quantized and described as the rotation of the quantized polarization vector. The classical conduction is translated as propagation of rotational quantized vector, whose phenomenon is equivalent to the propagation of polariton. We think, the quantized polarization wave gives rise to phenomena of the neuro-interferences, (for examples, ephapse, causalgia, neuralgia), various neural activities. The propagation of the quantized vector is described as conduction of polariton. The polariton is an essential carrier of neural information, conduction and interference of each neuron. The polariton is a kind of quasi particle. The polariton has various physical quantities: for examples, mass about 1.3×10^{-24}kg, spin 1, massive photon, positive, neutral and negative charge, and so on.

Polariton is a kind of the agent of information. If we can use frequency of thermal noise, then the polariton carries amount of information, 9.38×10^{12} bits/polariton, at 300 Kelvin . And we recognize to be required at least $0.693k_BT$ joules of energy to convey one bit of information.

To resist the thermal fluctuation and noise, each bare polariton need attract about 41 water molecules, and that phenomenon is known as hydration. Commonly we are only able to measure and to observe the physical characteristics of the hydrated polariton, which means quasi polariton. We think, that quasi mechanism is an important idea that, it is said nano machine to attain an excellent efficiency by using same magnitude of energy as the thermal noise at room temperature. When the polariton is in the ground state, whose state means the wavelength of polariton lies in almost 1 μm, and its range of existence is between 0.6 μm and 10 μm. The polariton satisfies the quaternary Schrödinger equation and complex Klein-Gordon equation. Strictly speaking, the polaritons motion is given in Proca field, with massive vector photon. Both inflow and outflow, which are both sodium ionic current and potassium ionic current through neural membrane, cause the neural conduction along to axon. And the arised polarization wave, which travels along to axon, conveys action potential as an excitation's impulse. Polariton is the quantized polarization wave. Generally speaking, an inflow of sodium ionic current causes an outflow of potassium ionic current from soma. Then the polariton electrically connects both ionic currents, and those currents are sources of polariton. Polariton is a real particle like as an electron, anion and cation, and it is a dressed and medium particle being caused by rotation of polarization's phase. The rapid communication of information lies in a quantum tunnel effect, and polariton gives rise to the tunnel current in myelin sheath. Both sodium and potassium ionic current are truly sources of the polariton's generation, and those currents make the many polaritons arise on the dielectric phospholipid membrane of neuron. Those polaritons act on Ranvier ring, and they affect on neighbor neurons, whose phenomena are defined as quantum neural interference. This phenomena is called ephapse, which is a physiological action based on quantum interference caused by many polaritons. And we think, they regularly work as a physiological functional adjustor, and that ephapse contributes to maintenance of homeostasis of neural networks and brain.

Macroscopic phenomena show us that each neuron receives an influence of the fluctuant electromagnetic field as shown in magneto-encephalogram. For examples, each neuron is subject to electromagnetic phenomena like as an induced electromotive force, leak current and so on. Those holistic electromagnetic effects of brain give rise to the many polaritons at the microscopic level. We believe that those effects modify various activities of neural networks like as cooperation, disaffection, divergence, and convergence. Dr. Shams reported in 2000, the sound induced by flash, which was a kind of illusions.

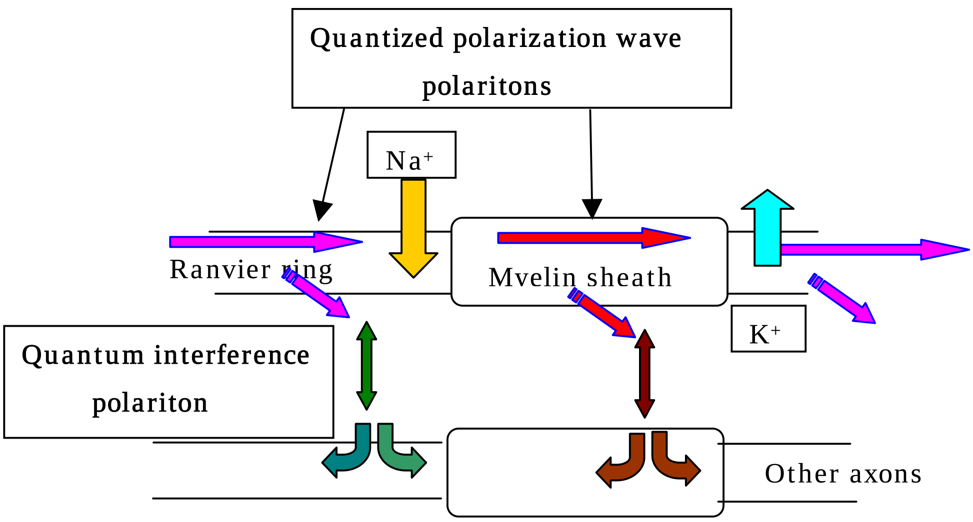


FIGURE 26. Na^{+},K^{+}, ionic currents and roles of polaritons

When the one short pulse of light was illuminated to our eyes and our ears were simultaneously twice stimulated by short duration-sounds for a short intervals, then our brains felt that the electric lamp twice put on a light. Though a visual area is away from an auditory area and both areas have anatomically independent routes of neural conductions, the stimulations of visual area affected on auditory area. We think, those phenomena to be examples of the illusion and of macroscopic neural interference. In the other word, polaritons of visual area affected on the neurons of auditory area and polaritons caused both those mistakes and illusions of two neural areas.

16.2 Quantization of Circuits and Expressions

We showed new basic theory of calculation methods for quantum bifurcation, quantum circuits, and neural computation by using path integrals of quantum theory[32],[37].

At the beginning, we showed that a decision tree can be regarded as a kind of Brownian motion (Markov process), and then the motion was governed with Ito equation (general stochastic equation). And according to Nelson's method (stochastic quantization), the Ito equation finally reached Schrödinger equation. Thus, we knew that problems of classical bifurcation were easily led to Schrödinger equation by considering Nelson's stochastic quantization method. The second example was Japanese Amida lottery, which was a kind of classical bifurcation models because of no interference between each path of lottery. However, we introduced a lot of diffraction points to Amida lottery, and we showed the calculating method of quantum amplitude by path integrals. That path integral was a quantization method of Amida lottery, which contained a lot of diffraction points. If we regarded classical bifurcation points as diffraction points and we summed up the probability amplitudes of all possible paths, we could translate the classical bifurcations into quantum interferences and diffraction's problems of networks.

We discussed the method of quantization of basic circuits as AND, OR and NOT. Those classical circuits did not have any quantum effects. For examples there were not quantum effects as the superposition and probability interference in those classical circuits. In order to perform quantization of those circuits, we adopted the path integral to above three basic circuits. We mentioned that we could regard classical switches as scattering potentials (switch's operators). So, that was quantization concepts, and those quantized circuits with switch operators corresponded to q-AND, q-NOT, and q-OR circuits. Moreover, we succeeded to show the calculation's methods of complex quantum circuits and neural networks by path integrals. The switch of each circuit was looked upon as switch's potential of Hamiltonian. Thus, Hamiltonian operator H could be described as

H = (kinetic energy T) + (potential energy V) + (Switch's potential $F(S_1,S_2, \quad S_N)$).

The Hamiltonian was connected to quaternary Schrödinger equation since the wave function was related to the motion of polariton as massive photon. Exactly speaking, the motion of polariton should be prescribed by Proca equation of relativistic kinematics. However, the Proca equation approached to the quaternary Schrödinger equation when the motion of polariton was much slower than light velocity.

The kernel K(b,a), which was propagator and an expression of the time development of system, was related to an eigenfunction of Schrödinger equation. And we found that the q-OR was similar to the first ordered perturbation of two potential scattering problems. The q-AND was shown to have similarity to the second ordered perturbation of single particle. It is important to notice that the wave function $\phi(x,t)$ was an expression of a situation of wave in the point x at time t, and its expression was static. The kernel K(B,A), however, truly represented the motion of the particle from point (A,t_A) to point (B,t_B), and so its expression was dynamical. Finally, we found that the neuro-synaptic junctions were regarded as a kind of switch's potential, whose concepts led to quantization of neural networks by using path integrals.

We think that quantum interference plays an essential role among many neural networks in our brain. The normal neuron actively utilizes various interferences so as to adjust each neural function through leak polatirons from neural axons and synaptic junctions,

16.3 Concrete Expression of Information Theory

We, at first, showed the expressions of motion of polaritons based on Proca equation. And we can reduce Proca equation into quaternary Schrödinger equation. We can have the only scalar potential ϕ by ignoring vector potential **A** of magnetic fields, if polariton's mass is so large and their motions on axons are so slow. The interferences among many neurons can be expressed by description of path integrals instead of wave equation, and the method of path integral is closely related to Feynman kernel, whose expressions represent an appearance of motion and propagation of polaritons. We attempted to compare classical Bayes' theorem with quantum Bayes' form[31]. The quantum Bayes' expression is given as q-number's operator, though counter observable and eigen values are real numbers. On the other hand, the classical Bayes' form is described by c-number and the observable is also real numbers. An essential difference between the quantum expression and the classical one in both Bayes' forms is whether the occurrence probability (amplitude) and the propagating probability (propagator) take complex numbers or not. So, the classic system is related to real number, which always directly means the probability. On the other hand, the quantum closely is connected to complex number. For example, its wave function describes the probability amplitude, which always takes complex number, and so the probability is given as absolute values of the probability amplitude.

Thus, quantum Bayes' form contains much interference between each quantum states vectors. However, there are not interferences between each of the event based on the classical Bayes' theory. And, we showed that result of the quantum Bayes' form is equivalent to the classical Bayes' theorem if it were not for the interferences between each quantum state vector, which means that pure states are changed into many mixed states by interactions, interferences and potential scatterings. We calculated values of entropy by both types, which were classical system and quantum's one. The quantum entropy, which compared with classical expression, had some interference terms. We can conclude that generally the entropy of quantum system is greater than that of classical system since the other residual terms are additional and non-negative. Thus, the quantum interference between pure states makes out an increase of entropy larger than the case of the classical system.

Those interferences combines many states so as to make up new mixed states as well as quantum Bayes' theory. And we applied both of Schrödinger equation and path integral so as to calculate an output power of each neuron for hierarchic neural networks. We showed those networks contained much interference, and we succeeded to obtain approximately solutions of an output expression from each neuron, by perturbation method and path integral method. We obtain the possibility of two types of neurons by tuning the width of Gauss slit on multi channel quantum networks. So we find that the some type of neuron works as amplifier, and another type is regarded as the inhibitive. Moreover, we discuss the similarity of both quantum network and classical one. The essential differences between both systems are whether there are operations as the superposition and the interferences or not. The quantum states are requested the superposition of polariton's wave function (propagators) and we can estimate an overlapping coefficient. On the other hand, a total output power of classical neurons is determined by summing up each of an input signal and a weighted factor, however the interferences of wave function never exist in classical system. The quantum neurons can control their networks as the classical networks do. The only difference of both networks is whether there are interferences between each path or not. There is another similarity between in the fuzzy probability (Choquet Integral) and the expectation values of quantum theory. Then, we know, the fuzzy probability is described by inner products and summations between ordinary probabilities and corresponding values of their membership function. On the other hand, an expectation of quantum theory was commonly calculated by using wave functions and some potential. We showed that the quantum expectations have the same descriptions with the fuzzy probability, if the membership function is regarded as the corresponding potentials of the wave function. The differences between those descriptions were pointed that the fuzzy probability should be the real number's probability (probability density), on the contrary the quantum description was given by the wave function (probability amplitude), which ordinarily took the complex number. And the common probability density is governed by Fokker-Planck equation with the scalar density function ϕ. However, the wave function of the polariton should be essentially described by the quaternary Schrödinger equation $A^{\mu}(\phi,\mathbf{A})$, except the slow change of magnetic field (i.e., vector potential **A** is almost constant). The polariton obeys ordinary Schrödinger equation of the one component ϕ, which is scalar potential of electric field. We think that both our quantum neural network and polariton's model contain a common quantum information theory, its computation method, and classical neural system. And our quantum descriptions are related to various areas, for examples, applications for fuzzy controls, classical neural systems, the classical Information theory and so on.

17.4 Further Development

We would like to refer to development of polariton's neurons and network. We obtained the quaternary Schrödinger equation for polariton's characteristics by reduction of Proca equation. The Eq.(27) and Eq.(34) contain both currents J^{μ}, which are Na^{+} currents and K^{+} currents. So, we are going to solve those coupled equations when total J^{μ} current is given. We think that it is important to analyze conditions of axon's membranes when both currents generate the polaritons, because of polariton's source.

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[Ref.End]

Appendix-1

Equivalence to Schrödinger Equation

A1-1 From Path Integral to Schrödinger Equation

According to relations from Eq.(133-1) and Eq.(133-2), a wave function $\phi_A[B]$ is able to be expressed by both kernel K(B,A) and an initial $\phi(A)$. We write down those two relations. The rule No. 8 refers to between the propagation and its time-development. The Eq.(133-1) means that an initial wave function $\phi(A)$ goes to a final state B, $\phi_{A.}[B]$.

$$\phi_A[B] = \int dA \cdot K(B,A)\,\phi(A)\,. \tag{133-1}$$

$$\begin{aligned} K[B,A] &= \langle B | A \rangle \equiv \langle B; t_B | A; t_A \rangle = \langle B | \hat{U}(t_B, t_A) | A \rangle \\ \hat{U}(t_B, t_A) &= \exp\left(\frac{-i}{\hbar} \hat{H}(t_B - t_A) \right). \end{aligned} \tag{133-2}$$

We would like to consider its time development of minute time-span, and so, we take A = (x+ η ,t) and B = (x,t+ ε). So, we have

$$\phi(x, t+\varepsilon) = \int K(x, t+\varepsilon; x+\eta, t)\,\phi(x+\eta, t)\,d(x+\eta)\cdot . \tag{A1.1}$$

Using the following both Eq.(79) and Eq.(84-2),

$$K(i+1, i) \equiv \langle i+1 | i \rangle = \exp\left[\frac{i\varepsilon}{\hbar} L\left(\frac{x_{i+1}-x_i}{\varepsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right) \right], \quad \text{L : Lagrangian.} \tag{79}$$

$$S[x(t)] = \int L(x, \dot{x}, t)\,dt \equiv \int (T - V)\,dt = \int \left(\frac{m}{2}\dot{x}^2 - V\right). \tag{84-2}$$

(T: kinetic energy, V: potential energy, S: action, and L: Lagrangian, and Ref. Appendix-1, A1-2). We show an introduction to Eq.(79), Eq.(84), Eq.(64) and Eq.(65)).From those three equations, we can easily obtain the Feynman kernel's representation, K(x, t+ ε ; x+ x+ η ,t) for minute time-span ε :

$$\begin{aligned} K(x, t+\varepsilon; x+\eta, t) &= \frac{1}{A} \exp\left[\frac{i\varepsilon}{\hbar} L\left(\frac{2x+\eta_i}{2}, \frac{-\eta}{\varepsilon}, \frac{2t+\varepsilon}{2}\right) \right] \\ &= \frac{1}{A} \exp\left[\frac{i\varepsilon}{\hbar} \left\{ \frac{m\eta^2}{2\varepsilon^2} - V(x+\eta/2) \right\} \right] \cong \frac{1}{A} \left[1 - \frac{i\varepsilon}{\hbar} V\left(x + \frac{\eta}{2}\right) \right] \exp\left(\frac{i}{\hbar} \frac{m\eta^2}{2\varepsilon} \right). \end{aligned} \tag{A1.2}$$

(A: normalization constant). Expanding ϕ with ε , η and substituting Eq.(A1.2) into Eq.(A1-1), the Eq.(A1-1) gives an expression,

$$\phi(x,t)+\varepsilon\frac{\partial\phi}{\partial t}\cong\int d\eta\cdot\frac{\exp\left(\frac{i}{\hbar}\frac{m\eta^2}{2\varepsilon}\right)}{A}\left[1-\frac{i\varepsilon}{\hbar}V(x)\right]\cdot\left[\phi(x,t)+\eta\frac{\partial\phi}{\partial x}+\frac{\eta^2}{2}\frac{\partial^2\phi}{\partial x^2}\right]\cdot. \quad \text{(A1.3)}$$

Comparing r.h.s with l.h.s of Eq.(A1.3), we know the ϕ have to satisfy the following relation,

$$\phi(x,t)=\int d\eta\frac{\phi(x,t)}{A}\exp\left(\frac{i}{\hbar}\frac{m\eta^2}{2\varepsilon}\right). \quad \text{(A1.4)}$$

We can determine the normalization constant by Gaussian integral, and then is given as

$$A=\sqrt{\frac{2i\pi\hbar\varepsilon}{m}}. \quad \text{(A1.5)}$$

Moreover, the Eq.(A1.3) teach us the following relation:

$$\varepsilon\frac{\partial\phi}{\partial t}=-\frac{i\varepsilon}{\hbar}V(x)+\left[\int_{-\infty}^{\infty}d\eta\exp\left(\frac{i}{\hbar}\frac{m\eta^2}{2\varepsilon}\right)\frac{1}{A}\eta^2\right]\cdot\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}+\text{(integral , odd function of } \eta\text{)} \quad \text{(A1.6)}$$

Performing Gaussian integral, we can finally obtain the final form to be famous for Schrödinger equation, since an integral of odd function of η goes to zero. Through the above procedure, we recognize that Feynman path integral is equivalent to Schrödinger equation. Thus, a path integral expression of polariton on network system means to be equal to an expression of wave function, and path integral of Feynman automatically contains a lot of quantum effects.

Secondly, we would like to mention relationship between classical mechanics and quantum theory. First of all, its theory is said to be Ehrenfest's theorem by which we are able to describe the relation between quantum mechanical expectation and Newtonian mechanics. The theorem says the quantum mechanical expectation for momentum and position obeys the Newtonian mechanics [38]. Using wave function ϕ, we know that those two equations for momentum's and position's expectation, <p> and <x>, which are

$$\frac{d}{dt}\langle x\rangle\equiv\frac{d}{dt}\langle\phi|x|\phi\rangle=\frac{1}{m}\int\phi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\phi\,dx=\frac{1}{m}\langle p\rangle, \quad \text{(A1.7-1)}$$

$$\frac{d}{dt}\langle p\rangle=\int\phi\left(-\frac{\partial V(x)}{\partial x}\right)\phi^*dx=\left\langle-\frac{\partial V(x)}{\partial x}\right\rangle=\langle F\rangle. \quad \text{(A1.7-2)}$$

(V(x) = potential so, the < F > corresponds to force term.). We have already known the Feynman path integral to be equivalent to Schrödinger equation. Moreover, its path integral has to satisfy the Ehrenfest's theorem.

Secondly, the those equations, from Eq.(64) to Eq.(68), teach us relationship between path integral and the action S. Adopting both the Eq.(67) and the uncertainty principle Eq.(6), we are able to appreciate the magnitude or the order of quantum fluctuation Eq.(68). From Eq.(67), the action S[x(t)] is

$$S[x(t)] = S_C[x_C] + \int_{ta}^{tb} dt[a(t)\dot{\delta}^2 + b(t)\delta\dot{\delta} + c(t)\delta^2]. \tag{67}$$

The first term of the r.h.s means classical action S_C and the second term corresponds the action of quantum fluctuation. The Eq.(6) tells us the uncertainty principle,

$$\Delta p = m\dot{\delta} \cong \frac{\hbar}{\delta} \to 1.0^{-29} \text{ (kg m/s)}. \tag{A1.8}$$

And so Eq.(A1.8) is applyed to the second term of Eq.(67), then we have

$$\begin{aligned}\int_{ta}^{tb} dt[a(t)\dot{\delta}^2 + b(t)\delta\dot{\delta} + c(t)\dot{\delta}^2] &= \int_{ta}^{tb} dt[a(t)\left(\frac{\hbar}{m\delta}\right)^2 + b(t)\delta\left(\frac{\hbar}{m\delta}\right) + c(t)\delta^2] \\ &\geq \int_{ta}^{tb} dt \cdot \sqrt[3]{a(t)b(t)c(t)}\,\frac{\hbar}{m} \approx order\left[\int_{ta}^{tb} dt \cdot a(t)\frac{\hbar}{m}\right] \equiv order\left[S_q\right]\end{aligned} \tag{A1.9}$$

Since the a(t) corresponds to a coefficient of kinetic energetic term, so the order of function exp(・) of Eq.(68) can estimated at about mass's order of particle. The result is

$$\begin{aligned}order\left\{\exp\left[\frac{i}{\hbar}\int_{ta}^{tb}[a\dot{\delta}^2 + b\delta\dot{\delta} + c\delta^2]dt\right]\right\} &\approx order\left[\exp\left\{\frac{i}{m}\int a(t)dt\right\}\right] \\ \approx order\left[\exp\left(i\int_{ta}^{tb} dt\right)\right] &= order\left[\exp i(t_b - t_a)\right].\end{aligned} \tag{A1.10}$$

Thus, the term of $\exp[i(t_b\text{-}t_a)]$ shows so much high frequency, since the above macroscopic time interval $(t_b\text{-}t_a)$ is much larger than quantum mechanical time interval (transitional time or tunnel effect time, et al. Their time interval is said less than 10^{-9}s)). Thus, an integral of the much high oscillational term is nearly equal to zero, and thus the Eq.(68) goes to

$$\begin{aligned}K(b,a) &\approx \int_a^b e^{(i/\hbar)Sc[b,a]} Dx(t) \cdot order\left[\exp\{i(t_b - t_a)\}\right] \to \int_a^b e^{(i/\hbar)Sc[b,a]} Dx(t) \\ &\to \int_a^b e^{(i/\hbar)Sc[b,a]} dx(t).\end{aligned} \tag{A1.11}$$

Finally, notice that path integral becomes to common line integral on the classical trajectory of particle since $\exp\{i(t_b\text{-}t_a)\} \to 0$. And then the classical action $S_C[b,a]$ has much important role, and the classical trajectory of particle has generally more weight than its quantum fluctuation.

The classical action is defined as the time integral for Lagrangian:

$$S_C[b,a] \equiv \int_a^b L(\dot{x},x,t)\,dt\,, \tag{A1.12-1}$$

So, we practice variation for Eq.(A1.12), and then we reach an equation of Eular-Lagrange of Eq.(A1.12):

$$\delta S_C[b,a] \equiv \int_a^b \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right) dt = \int_a^b \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x dt = 0\,. \tag{A1.12-2}$$

That equation corresponds to Newtonian equation of motion of analytical mechanics. Moreover, Lagrangian decides Hamiltonian according to following procedure.

$$L(x,\dot{x},t) = T - V \equiv p\dot{x} - H(p,x)\,. \tag{A1.13}$$

The first quantization means to introduce an operator into classical Hamiltonian,

$$H(p,x) \to H\left(-i\hbar \frac{d}{dx}, x \right). \tag{A1.14}$$

And then we can obtain well-known Schrödinger equation applying wave function. If we would like to adopt the action S (or classical Lagrangian) for its system's description, it is possible to choose path integral form, whose variables are c-number (cf. an operator is described by q-number). Polariton, photon and the other bosons are governed by common algebra; however, electron, neutron and proton (fermions) need to be expressed by Grassmann algebra.

Finally we can assert the sentence: "We can automatically introduce quantum effect of polaritons to the network systems, and its expression is much similar to classical mechanical Lagrangian."

A1-2 Way to Feynman Kernel

We would like simply to show you the way to Feynman kernel from the unitary operator of Eq.(133-1) and Eq.(133-2). The original path integral's idea, whose staring points are based on classical Lagrangian, has been begun by Paul, Dirac [39],[40]. And R. Feynman has developed the Dirac's method, and he wrapped up modern path integral formalism [4],[38],[41]. The purpose of an appendix A1-2 is to show the general way to Eq.(65) from Eq.(64) and . From Eq.(133-2), we know the unitary operator.

$$K[B,A] = \langle B | A \rangle \equiv \langle B; t_B | A; t_A \rangle = \langle B | \hat{U}(t_B,t_A) | A \rangle$$

$$\hat{U}(t_B,t_A) = \exp\left(\frac{-i}{\hbar} \hat{H}(t_B - t_A) \right). \tag{133-2}$$

When we divide the macro time interval (t_B-t_A) into N numbers' minute spans ε = (t_B-t_A)/N, then we have

$$\hat{U}(t_B,t_A)=\exp\left(\frac{-i}{\hbar}\hat{H}(t_B-t_A)\right)=\left(e^{-i(\hat{T}+\hat{V})\varepsilon/\hbar}\right)^N \cong \left(e^{-i\hat{T}\varepsilon/\hbar}e^{-i\hat{V}\varepsilon/\hbar}\right)^N . \tag{A1.15}$$

Substituting Eq.(A1-15) into Eq.(A1.15) is substituted into the kernel of Eq.(133-2), we find the propagator

$$K(B,A)\cong \lim_{N\to\infty}\left\langle x_B,t_B\left|\left(e^{-i\hat{T}\varepsilon/\hbar}e^{-i\hat{V}\varepsilon/\hbar}\right)^N\right|x_A,t_A\right\rangle . \tag{A1.16}$$

Adopting completeness of position's bra and ket vector, the above kernel is expressed as

$$K(B,A)\cong \lim_{N\to\infty}\int dx_1 dx_2\cdots dx_{N-1}\prod_{j=0}^{N-1}\left\langle x_{j+1}\left|e^{-i\hat{T}\varepsilon/\hbar}e^{-i\hat{V}\varepsilon/\hbar}\right|x_j\right\rangle , \tag{A1.17}$$

As well as position's vector, we use the same completeness for momentum vector, we have

$$\begin{aligned}K(B,A)&\cong \lim_{N\to\infty}\int dx_1\cdots dx_{N-1}\int dp_0\cdots dp_{N-1}\prod_{j=0}^{N-1}\left\langle x_{j+1}\left|\exp\{-i\varepsilon p_j^2/2m\hbar\}\right|p_j\right\rangle \\ &\quad\times\left\langle p_j\left|\exp\{-i\,\varepsilon V_j/\hbar/2m\hbar\}\right|x_j\right\rangle \qquad\qquad (A1.18)\\ &=\lim_{N\to\infty}\int dx_1\cdots dx_{N-1}\int dp_0\cdots dp_{N-1}\prod_{j=0}^{N-1}\exp\left[-(i/\hbar)\varepsilon\left\{\left(p_j^2/2m\right)+V(x)\right\}\right]\left\langle x_{j+1}\mid p_j\right\rangle\left\langle p_j\mid x_j\right\rangle \\ &=\lim_{N\to\infty}\left(\frac{1}{2\pi\hbar}\right)\int dx_1\cdots dx_{N-1}\int dp_0\cdots dp_{N-1}\cdot\exp\left[(i/\hbar)\varepsilon\sum_{j=0}^{N-1}\left[p_j\left(x_{j+1}-x_j\right)/\varepsilon-\left(p_j^2/2m\right)-V(x_j)\right]\right]\end{aligned}$$

And when N→∞, the function exp[・] of Eq.(A1.18) goes to

$$\varepsilon\sum_{j=0}^{N-1}\left[p_j\frac{(x_{j+1}-x_j)}{\varepsilon}-\frac{p_j^2}{2m}-V(x_j)\right]\cong\int_{tA}^{tb}dt\cdot\left[p\dot{x}-H\right]\equiv S(x_B,t_B;x_A,t_A) \quad . \tag{A1.19}$$

$$\because L(x,\dot{x},t)=\frac{m}{2}\dot{x}^2-V(x)=p\dot{x}-H(p,x)$$

The Eq.(A1.19) clearly mentions the path integral to be closely related to Lagrangian and Hamiltonian of the classical mechanics. Performing Gaussian integral for momentum p_j of Eq.(A1.19), finally the kernel (propagator) of Eq.(A1.18) is given as

$$K(B,A)=\lim_{N\to\infty}\left(\frac{m}{2i\pi\hbar\varepsilon}\right)^{N/2}\int dx_1\cdots dx_{N-1}\exp\left[\frac{i\varepsilon}{\hbar}\sum_{j=0}^{N-1}\left\{\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2-V(x_j)\right\}\right] . \tag{A1.20}$$

Then the above function exp(・) is reduced to Lagrangian form:

$$\sum_{j=0}^{N-1}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2-V(x_j)\right]\cong\int_{tA}^{tB}\left[\frac{m}{2}\dot{x}(t)^2-V(x(t))\right]dt=\int_{tA}^{tB}Ldt=S . \tag{A1.21}$$

After all, we obtain the path integral form,

$$
\begin{aligned}
K(B,A) &= \int Dx(t)\exp\left[\frac{i}{\hbar}\int_{tA}^{tB} L(x,\dot{x},t)dt\right] = \int Dx(t)\exp\left[\frac{i}{\hbar}S\right] \\
&= \int Dx(t)Dp(t)\exp\left[\frac{i}{\hbar}\int_{tA}^{tB} p\dot{x} - H(p,x)dt\right]
\end{aligned}
\tag{A1.22}
$$

Thus, we recognize that this result, Eq.(A1.22), perfectly coincides with Eq.(65), and the classical Lagrangian form leads us to Feynman path integral.

Appendix-2

Choquet Integral and Quantum Mechanical Expectation

A2-1 Fizzy probability and Choquet Integral

We showed that the fuzzy probability $P_E(\approx 6)$ was obtained by Eq.(239),

$$P_E(\approx 6) = A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6)\,. \tag{239}$$

If we consider an independent variable X to have continuity, then the Eq.(239) is described by an integral form,

$$P_E(\approx 6) = \sum_X P(X)\cdot A(X) \cong \int P(X)\cdot A(X)\,dX\,. \tag{A2.1}$$

as we wrote in Eq.(242). The Choquet Integral of this case, the value A(X) means to be Fuzzy measure and the probability P(X) corresponds to its counter grade. So we are able to have an expression for the Choquet Integral (FIGURE A-1), [42],[43].

$$\begin{aligned} P_{Chouquet}(\approx 6) \equiv (C)\int_X f\,d\mu = (P(1)-P(0))\cdot \sum_{X=1}^{X=6} A(X) + (P(2)-P(1))\cdot \\ \sum_{X=2}^{X=6} A(X) + (P(3)-P(2))\cdot \sum_{X=3}^{X=6} A(X) + (P(4)-P(3))\cdot \sum_{X=4}^{X=6} A(X) \\ +(P(5)-P(4))\cdot \sum_{X=5}^{X=6} A(X) + (P(6)-P(5))\cdot \sum_{X=6}^{X=6} A(X) \end{aligned} \tag{A2.2}$$

And we define as P(0) = 0.

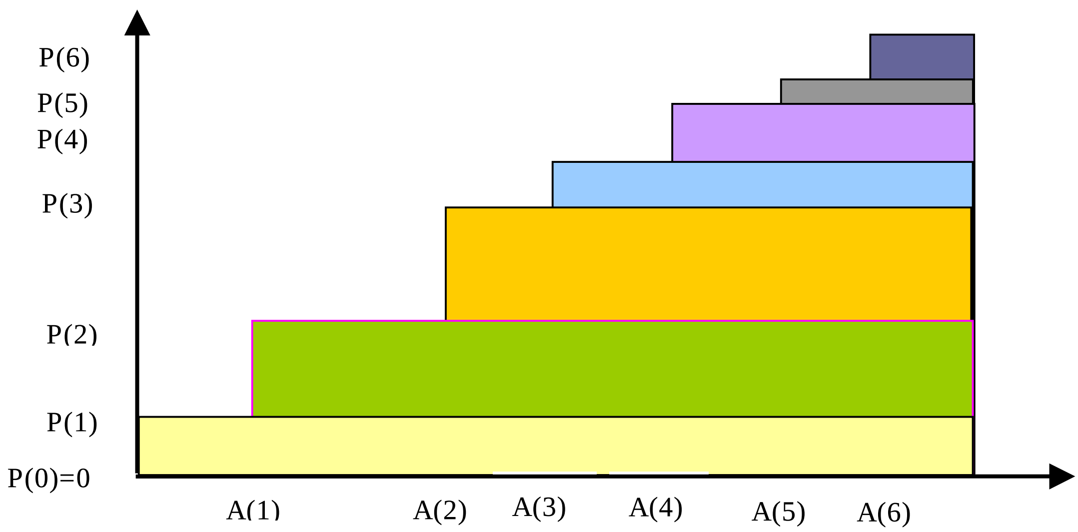

FIGURE A-1. Choquet Integral of Fuzzy Set Theory

Simplifying the Eq.(A2.2), we immediately notice that the fuzzy probability $P_E(\approx 6)$ of Eq.(239) or Eq.(A2.1) is equal to the results of Eq.(A2.2), which is the definition of Choquet Integral in real number's area.

A2-2 Difference between Fuzzy Integral and Quantum Integral

Calculating Fuzzy probability (Choquet Integral or Sugano Integral), all functions, A(X), P(X) and its variable X are always real numbers. And we never encounter to complex numbers under its calculation process. However, the quantum mechanical expectation is essentially different from those Fuzzy integrals except the similarity of formal style (reference to Eq.(242) and Eq.(245)). The wave function (probability amplitude) Ψ of Eq.(245) generally means complex function. However, its expectation and variable X have to take real values, because the expectation should be observable and X is coordinate of our space. In Eq.(245), we assume that the Ψ takes a plane wave exp(-ikX), and we adopt its complex conjugate wave function Ψ^*= exp{i(k+⊿k)X} with slight difference of momentum. And if $F_j(X)$ (i.e. A(X)) is momentum operator, then the quantum mechanical expectation becomes

$$\begin{aligned}\langle F_J(\approx X_J)\rangle &= \int P_\rho(X)F_J(X)\,dX = \int \Psi^*(X)F_J(X)\Psi(X)\,dX \\ &= \int e^{ikX}\left(-i\hbar\frac{d}{dX}\right)e^{-i(k+\Delta k)X}dX = -\int_{-\infty}^{\infty} e^{-i\Delta k X}\cdot\hbar(k+\Delta k)dX = -2\pi\delta(\Delta k)\cdot\hbar(k+\Delta k)\end{aligned} \qquad \text{(A2.3)}$$

So, where $\delta(\Delta k)$ means Dirac delta function. The Δk is nearly to zero, and then $\delta(\Delta k)$ becomes a very sharp function, and we perform an integral for Eq.(A2.3) at near to zero. We have the result:

$$\int\langle F_J(\approx X_J)\rangle d(\Delta k) = -\int_{-\varepsilon}^{\varepsilon} 2\pi\delta(\Delta k)\cdot\hbar(k+\Delta k)\cdot d(\Delta k) = -2\pi\hbar k\,. \qquad \text{(A2.4)}$$

It is much important to notice that the result of calculation is not infinite, but it becomes a finite value. In the case of Choquet Integral, we can adopt Ψ= cos(kX), and Ψ^* is cos{(k+Δk)X}, and moreover, $F_j(X)$ means momentum operator. And we obtain the calculating result of Eq.(242):

$$\begin{aligned}P_E(\approx X_J) &= \int_{all\ X} P_\rho(X)F_J(X)dX = \int_{-\infty}^{\infty}\cos(kX)\cdot\left(-i\hbar\frac{d}{dX}\right)\cos(k+\Delta k)XdX \\ &= i\hbar\int_{-\infty}^{\infty}\cos(kX)\cdot(k+\Delta k)\sin\{(k+\Delta k)X\}dX = 0.\end{aligned} \qquad \text{(A2.5)}$$

The result of Eq.(A.4) takes always zero value because of orthogonality of trigonometric function. If the above $F_j(X)$ takes real number A, its result becomes divergence and infinite,

$$P_E(\approx X_J) = \int_{all\ X} P_\rho(X)F_J(X)dX = \int_{-\infty}^{\infty} \cos(kX)\cdot(A)\cos\{(k+\Delta k)X\}dX = \infty\,. \qquad \text{(A2.6)}$$

if the Δ k is much near to zero. And if the Δ k is not equal to zero, we always obtain zero momentum, and those results are not significant. Thus, if we adopt probability amplitude Ψ which is complex number, we should naturally be led to quantum mechanical expectation so as to prevent from giving a nonsensical result, instead of Choquet Integral or Fuzzy probability.

[END ALL]