

Exact solution in the Heisenberg picture and annihilation-creation operators

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Abstract

The annihilation-creation operators of the harmonic oscillator, the basic and most important tools in quantum physics, are generalised to most solvable quantum mechanical systems of single degree of freedom including the so-called ‘discrete’ quantum mechanics. They admit exact Heisenberg operator solution. We present unified definition of the annihilation-creation operators ($a^{(\pm)}$) as the positive/negative frequency parts of the exact Heisenberg operator solution.

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Introduction

The annihilation-creation operators are the simplest solution method for quantum mechanical systems. In this Letter we provide unified definition of annihilation-creation operators for most solvable quantum mechanical systems of one degree of freedom. A quantum mechanical system is called *solved* or *solvable* if the entire energy spectrum $\{\mathcal{E}_n\}$ and the corresponding eigenvectors $\{\phi_n\}$, $\mathcal{H}\phi_n = \mathcal{E}_n\phi_n$ are known [1]. This is the solution in the Schrödinger picture. We will show that they also possess exact Heisenberg operator solutions. The annihilation-creation operators are defined as the positive/negative frequency parts of the Heisenberg operator solution and they are hermitian conjugate to each other. This method also applies

to the ‘discrete’ quantum mechanical systems, which are deformations of quantum mechanics obeying certain difference equations; see full paper [2]. Our results will be translated to those of the corresponding orthogonal polynomials. In particular, the exact Heisenberg operator solution of the ‘sinusoidal coordinate’ corresponds to the so-called *structure relation* [3] for the orthogonal polynomials, including the Askey-Wilson and Meixner-Pollaczek polynomials [4].

Heisenberg Operator Solution

We will focus on the discrete energy levels, finite or infinite in number, which are non-degenerate in one dimension. For the majority of the solvable quantum systems [1], the n -th eigenfunction has the following general structure $\phi_n(x) = \phi_0(x)P_n(\eta(x))$, in which $\phi_0(x)$ is the *ground state* wavefunction and $P_n(\eta(x))$ is an orthogonal polynomial of degree n in a real variable η .

Our main claim is that this $\eta(x)$ undergoes a ‘*sinusoidal motion*’ under the given Hamiltonian \mathcal{H} , at the classical as well as quantum level. The latter is simply the exact Heisenberg operator solution. To be more specific, at the classical level we have

$$\{\mathcal{H}, \{\mathcal{H}, \eta\}\} = -\eta R_0(\mathcal{H}) - R_{-1}(\mathcal{H}). \quad (1)$$

The two coefficients R_0 and R_{-1} are, in general, polynomials in the Hamiltonian \mathcal{H} . This leads to a simple sinusoidal time-evolution:

$$\begin{aligned} \eta(x; t) &= \sum_{n=0}^{\infty} ((-t)^n/n!) (\text{ad } \mathcal{H})^n \eta \\ &= -\{\mathcal{H}, \eta\}_0 \frac{\sin[t\sqrt{R_0(\mathcal{H}_0)}]}{\sqrt{R_0(\mathcal{H}_0)}} - R_{-1}(\mathcal{H}_0)/R_0(\mathcal{H}_0) \\ &\quad + (\eta(x)_0 + R_{-1}(\mathcal{H}_0)/R_0(\mathcal{H}_0)) \cos[t\sqrt{R_0(\mathcal{H}_0)}], \end{aligned} \quad (2)$$

in which $\text{ad } \mathcal{H} X = \{\mathcal{H}, X\}$ and the subscript 0 means the initial value (at $t = 0$). The corresponding quantum expression is

$$[\mathcal{H}, [\mathcal{H}, \eta]] = \eta R_0(\mathcal{H}) + [\mathcal{H}, \eta] R_1(\mathcal{H}) + R_{-1}(\mathcal{H}), \quad (3)$$

in which $R_1(\mathcal{H})$ is the quantum effect. The exact Heisenberg operator solution reads

$$\begin{aligned}
e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} &= \sum_{n=0}^{\infty} ((it)^n/n!) (\text{ad } \mathcal{H})^n \eta \\
&= [\mathcal{H}, \eta(x)] \frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})} - R_{-1}(\mathcal{H})/R_0(\mathcal{H}) \\
&\quad + (\eta(x) + R_{-1}(\mathcal{H})/R_0(\mathcal{H})) \times \frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})}, \tag{4}
\end{aligned}$$

in which ad is now a commutator $\text{ad } \mathcal{H} X = [\mathcal{H}, X]$, instead of the Poisson bracket. The two ‘‘frequencies’’ are

$$\begin{aligned}
\alpha_{\pm}(\mathcal{H}) &= (R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2 + 4R_0(\mathcal{H})})/2, \\
\alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H}) &= R_1(\mathcal{H}), \\
\alpha_+(\mathcal{H})\alpha_-(\mathcal{H}) &= -R_0(\mathcal{H}). \tag{5}
\end{aligned}$$

If the quantum effects are neglected, *i.e.* $R_1 \equiv 0$ and $\mathcal{H} \rightarrow \mathcal{H}_0$, we have $\alpha_+ = -\alpha_- = \sqrt{R_0(\mathcal{H}_0)}$, and the above Heisenberg operator solution reduces to the classical one (2). When the exact operator solution (4) is applied to ϕ_n , the r.h.s. has only three time-dependence, $e^{i\alpha_{\pm}(\mathcal{E}_n)t}$ and a constant. Thus the l.h.s. can only have two non-vanishing matrix elements when sandwiched by ϕ_m , except for the obvious ϕ_n corresponding to the constant term. In accordance with the above general structure of the eigenfunctions, they are $\phi_{n\pm 1}$; that is $\langle \phi_m | \eta(x) | \phi_n \rangle = 0$, for $m \neq n \pm 1, n$. This imposes the following conditions on the energy eigenvalues

$$\mathcal{E}_{n+1} - \mathcal{E}_n = \alpha_+(\mathcal{E}_n), \quad \mathcal{E}_{n-1} - \mathcal{E}_n = \alpha_-(\mathcal{E}_n).$$

These conditions, together with their ‘hermitian conjugate’ ones, combined with the ground state energy $\mathcal{E}_0 = 0$ determine the entire discrete energy spectrum as shown by Heisenberg and Pauli. In this letter we adopt the factorised Hamiltonian:

$$\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}/2, \quad \mathcal{A}\phi_0 = 0 \Rightarrow \mathcal{H}\phi_0 = 0, \quad \mathcal{E}_0 = 0.$$

The consistency of the procedure requires that the coefficient of $e^{i\alpha_-(\mathcal{H})t}$ on the r.h.s. should vanish when applied to the ground state ϕ_0 :

$$-[\mathcal{H}, \eta(x)]\phi_0 + (\eta(x)\alpha_+(0) - R_{-1}(0)/\alpha_-(0))\phi_0 = 0,$$

which is the equation determining the ground state eigenvector ϕ_0 in the Heisenberg picture.

Thus we arrive at a *dynamical and unified definition* of the **annihilation-creation operators**:

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} = a^{(+)}(\mathcal{H}, \eta)e^{i\alpha_+(\mathcal{H})t} - R_{-1}(\mathcal{H})/R_0(\mathcal{H}) + a^{(-)}(\mathcal{H}, \eta)e^{i\alpha_-(\mathcal{H})t}, \quad (6)$$

$$\begin{aligned} a^{(\pm)} &= a^{(\pm)}(\mathcal{H}, \eta) \\ &\stackrel{\text{def}}{=} \left(\pm[\mathcal{H}, \eta(x)] \mp (\eta(x) + R_{-1}(\mathcal{H})/R_0(\mathcal{H}))\alpha_{\mp}(\mathcal{H}) \right) / (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})). \end{aligned} \quad (7)$$

When acting on the eigenvector ϕ_n , they read

$$a^{(\pm)}\phi_n(x) = \frac{\pm 1}{\mathcal{E}_{n+1} - \mathcal{E}_{n-1}} \left([\mathcal{H}, \eta(x)] + (\mathcal{E}_n - \mathcal{E}_{n\mp 1})\eta(x) + \frac{R_{-1}(\mathcal{E}_n)}{\mathcal{E}_{n\pm 1} - \mathcal{E}_n} \right) \phi_n(x). \quad (8)$$

By using the three-term recursion relation of the orthogonal polynomial P_n

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta)$$

on the l.h.s. of (6), we arrive at

$$a^{(+)}\phi_n = A_n\phi_{n+1}, \quad a^{(-)}\phi_n = C_n\phi_{n-1}. \quad (9)$$

Based on these relations, it is easy to show that $a^{(\pm)}$ are *hermitian conjugate* to each other. Sometimes it is convenient to introduce $a'^{(\pm)}$ with a different normalisation

$$\begin{aligned} a'^{(\pm)} &\stackrel{\text{def}}{=} a^{(\pm)}(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})) \\ &= \pm[\mathcal{H}, \eta(x)] \mp (\eta(x) + R_{-1}(\mathcal{H})/R_0(\mathcal{H}))\alpha_{\mp}(\mathcal{H}), \end{aligned} \quad (10)$$

which are no longer hermitian conjugate to each other.

In the literature there is a quite wide variety of proposed annihilation and creation operators [5]. Historically most of these operators are connected to the so-called *algebraic theory of coherent states*, which are defined as the eigenvectors of the annihilation operator (AOCS, Annihilation Operator Coherent State). Therefore, for a given potential or a quantum Hamiltonian, there could be as many coherent states as the definitions of the annihilation operators. In our theory, on the contrary, the annihilation-creation operators are uniquely determined except for the overall normalisation. In terms of the simple parametrisation $\psi = \sum_{n=0}^{\infty} c_n \phi_n(x)$ the equation $a^{(-)}\psi = \lambda\psi$, $\lambda \in \mathbb{C}$ can be solved with the help of the formula (9)

$$\psi = \psi(\lambda, x) = \phi_0(x) \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{k=1}^n C_k} \cdot P_n(\eta(x)). \quad (11)$$

To the best of our knowledge, the ‘sinusoidal coordinate’ was first introduced in a rather broad sense for general (not necessarily solvable) potentials as a useful means for coherent state research by Nieto and Simmons [6]. In [2] the necessary and sufficient condition for the existence of the ‘sinusoidal coordinate’ (3) is analysed within the context of ordinary quantum mechanics $\mathcal{H} = p^2/2 + V(x)$. It turns out that the potential can be expressed in terms of the $\eta(x)$ and $d\eta/dx$

$$V(x) = \left(r_0^{(0)} \eta^2/2 + r_{-1}^{(0)} \eta + c \right) / (d\eta/dx)^2 - r_1/8. \quad (12)$$

The parameters appear in $R_0(\mathcal{H}) = r_0^{(1)}\mathcal{H} + r_0^{(0)}$, $R_1(\mathcal{H}) = r_1$ and $R_{-1}(\mathcal{H}) = r_{-1}^{(1)}\mathcal{H} + r_{-1}^{(0)}$ and c is the constant of integration. These potentials are all *shape invariant* [7]. The Kepler problems in various coordinates and the Rosen-Morse potential are not contained in (12), though they are shape invariant and solvable.

We give three typical examples. See [2] for more.

Pöschl-Teller potential

The Hamiltonian of the Pöschl-Teller potential, the eigenvalues and the eigenfunctions are ($0 < x < \pi/2$):

$$\begin{aligned} \mathcal{H} &\stackrel{\text{def}}{=} (p - ig \cot x + ih \tan x)(p + ig \cot x - ih \tan x)/2, \\ \mathcal{E}_n &= 2n(n + g + h), \quad g, h > 0, \quad \eta(x) = \cos 2x, \\ \phi_n(x) &= (\sin x)^g (\cos x)^h P_n^{(\alpha, \beta)}(\cos 2x), \end{aligned} \quad (13)$$

in which $P_n^{(\alpha, \beta)}(\eta)$ is the Jacobi polynomial and $\alpha = g - 1/2$, $\beta = h - 1/2$. The classical solution of the initial value problem is ($\mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} + (g + h)^2/2$):

$$\begin{aligned} \cos 2x(t) &= \left(\cos 2x(0) + \frac{g^2 - h^2}{2\mathcal{H}'_0} \right) \cos[2t\sqrt{2\mathcal{H}'_0}] \\ &\quad - p(0) \sin 2x(0) \frac{\sin[2t\sqrt{2\mathcal{H}'_0}]}{\sqrt{2\mathcal{H}'_0}} - \frac{g^2 - h^2}{2\mathcal{H}'_0}. \end{aligned} \quad (14)$$

The corresponding quantum expressions are

$$[\mathcal{H}, [\mathcal{H}, \cos 2x]] = \cos 2x (8\mathcal{H}' - 4) + 4[\mathcal{H}, \cos 2x] + 4(\alpha^2 - \beta^2), \quad (15)$$

with $\alpha_{\pm}(\mathcal{H}) = 2 \pm 2\sqrt{2\mathcal{H}'}$. The annihilation and creation operators are

$$a^{(\pm)}/2 = a^{(\pm)} 2\sqrt{2\mathcal{H}'} = \pm \sin 2x \frac{d}{dx} + \cos 2x \sqrt{2\mathcal{H}'} + \frac{\alpha^2 - \beta^2}{\sqrt{2\mathcal{H}' \pm 1}}. \quad (16)$$

When applied to the eigenvector ϕ_n as $2\mathcal{E}_n + (g + h)^2 = (2n + g + h)^2$, we obtain:

$$a'^{(-)}/2 \phi_n = \frac{4(n + \alpha)(n + \beta)}{2n + \alpha + \beta} \phi_{n-1}, \quad (17)$$

$$a'^{(+)} / 2 \phi_n = \frac{4(n + 1)(n + \alpha + \beta + 1)}{2n + \alpha + \beta + 2} \phi_{n+1}. \quad (18)$$

Deformed harmonic oscillator

The deformed harmonic oscillator is a simplest example of shape invariant ‘discrete’ quantum mechanics. The Hamiltonian of ‘discrete’ quantum mechanics studied in this letter has the following form [8] (with some modification for the Askey-Wilson case):

$$\mathcal{H} \stackrel{\text{def}}{=} \left(\sqrt{V(x)} e^p \sqrt{V(x)^*} + \sqrt{V(x)^*} e^{-p} \sqrt{V(x)} - V(x) - V(x)^* \right) / 2. \quad (19)$$

The eigenvalue problem for \mathcal{H} , $\mathcal{H}\phi = \mathcal{E}\phi$ is a difference equation, instead of a second order differential equation. Let us define S_{\pm} , T_{\pm} and \mathcal{A} by

$$\begin{aligned} S_+ &\stackrel{\text{def}}{=} e^{p/2} \sqrt{V(x)^*}, & S_- &\stackrel{\text{def}}{=} e^{-p/2} \sqrt{V(x)}, \\ T_+ &\stackrel{\text{def}}{=} S_+^\dagger S_+ = \sqrt{V(x)} e^p \sqrt{V(x)^*}, \\ T_- &\stackrel{\text{def}}{=} S_-^\dagger S_- = \sqrt{V(x)^*} e^{-p} \sqrt{V(x)}, \\ \mathcal{A} &\stackrel{\text{def}}{=} i(S_+ - S_-), & \mathcal{A}^\dagger &= -i(S_+^\dagger - S_-^\dagger). \end{aligned} \quad (20)$$

Then the Hamiltonian is factorized

$$\begin{aligned} \mathcal{H} &= (T_+ + T_- - V(x) - V(x)^*)/2 \\ &= (S_+^\dagger - S_-^\dagger)(S_+ - S_-)/2 = \mathcal{A}^\dagger \mathcal{A}/2. \end{aligned} \quad (21)$$

The potential function $V(x)$ of the deformed harmonic oscillator is $V(x) = a + ix$, $-\infty < x < \infty$, $a > 0$. As shown in some detail in our previous paper [8], it has an equi-spaced spectrum ($\mathcal{E}_n = n$, $n = 0, 1, 2, \dots$) and the corresponding eigenfunctions are a special case of the Meixner-Pollaczek polynomial $P_n^{(a)}(x; \frac{\pi}{2})$ [4],

$$\phi_0(x) = \sqrt{\Gamma(a + ix)\Gamma(a - ix)}, \quad \eta(x) = x, \quad (22)$$

$$\phi_n(x) = \phi_0(x) P_n(x), \quad P_n(x) \stackrel{\text{def}}{=} P_n^{(a)}(x; \frac{\pi}{2}), \quad (23)$$

which could be considered as a deformation of the Hermite polynomial.

The Poisson bracket relations are $\{\mathcal{H}, x\} = -\sqrt{a^2 + x^2} \sinh p$, $\{\mathcal{H}, \{\mathcal{H}, x\}\} = -x$, leading to the harmonic oscillation,

$$x(t) = x(0) \cos t + \sqrt{a^2 + x^2(0)} \sinh p(0) \sin t, \quad (24)$$

which endorses the naming of the deformed harmonic oscillator. The corresponding quantum expressions are also simple: $[\mathcal{H}, x] = -i(T_+ - T_-)/2$, $[\mathcal{H}, [\mathcal{H}, x]] = x$,

$$e^{it\mathcal{H}} x e^{-it\mathcal{H}} = x \cos t + i[\mathcal{H}, x] \sin t = x \cos t + (T_+ - T_-)/2 \sin t. \quad (25)$$

The annihilation and creation operators are

$$a'^{(\pm)} = 2a^{(\pm)} = x \pm [\mathcal{H}, x] = x \mp i(T_+ - T_-)/2. \quad (26)$$

These operators were also introduced in [9] by a different reasoning from ours. The action of the annihilation creation operators on the eigenvectors is

$$a'^{(-)}\phi_n = (n + 2a - 1)\phi_{n-1}, \quad a'^{(+)}\phi_n = (n + 1)\phi_{n+1}. \quad (27)$$

From these it is easy to verify the $\mathfrak{su}(1, 1)$ commutation relations including the Hamiltonian \mathcal{H} :

$$[\mathcal{H}, a'^{(\pm)}] = \pm a'^{(\pm)}, \quad [a'^{(-)}, a'^{(+)}] = 2(\mathcal{H} + a). \quad (28)$$

The coherent state (11), is simply obtained from the formula (27) and $a'^{(-)} = 2a^{(-)}$:

$$\psi(x) = \phi_0(x) \sum_{n=0}^{\infty} \frac{(2\lambda)^n}{(2a)_n} P_n^{(a)}\left(x; \frac{\pi}{2}\right), \quad (29)$$

which has a concise expression in terms of the hypergeometric function ${}_1F_1$

$$\psi(x) = \phi_0(x) e^{2i\lambda} {}_1F_1\left(\begin{matrix} a + ix \\ 2a \end{matrix} \middle| -4i\lambda\right). \quad (30)$$

Askey-Wilson polynomial

The Askey-Wilson polynomial belongs to the so-called q -scheme of hypergeometric orthogonal polynomials [4]. It has four parameters a_1, a_2, a_3, a_4 on top of q ($0 < q < 1$), and is considered as a three-parameter deformation of the Jacobi polynomial. As a dynamical system, it could be called a deformed Pöschl-Teller potential. The quantum-classical correspondence has some subtlety because of another ‘classical’ limit $q \rightarrow 1$.

The Hamiltonian of the Askey-Wilson polynomial has a bit different form from (19):

$$\mathcal{H} \stackrel{\text{def}}{=} \left(\sqrt{V(z)} q^D \sqrt{V(z)^*} + \sqrt{V(z)^*} q^{-D} \sqrt{V(z)} - V(z) - V(z)^* \right) / 2, \quad (31)$$

with a potential function $V(z)$, $z = e^{ix}$, $0 < x < \pi$:

$$V(z) = \frac{\prod_{j=1}^4 (1 - a_j z)}{(1 - z^2)(1 - qz^2)}, \quad D \stackrel{\text{def}}{=} z \frac{d}{dz} = -i \frac{d}{dx} = p.$$

We assume $-1 < a_1, a_2, a_3, a_4 < 1$ and $a_1 a_2 a_3 a_4 < q$. The eigenvalues and eigenfunctions are [8]:

$$\begin{aligned} \mathcal{E}_n &= (q^{-n} - 1)(1 - a_1 a_2 a_3 a_4 q^{n-1})/2, \\ \phi_0(x) &= \sqrt{\frac{(z^2; q)_\infty}{\prod_{j=1}^4 (a_j z; q)_\infty} \frac{(z^{-2}; q)_\infty}{\prod_{j=1}^4 (a_j z^{-1}; q)_\infty}}, \\ \eta(x) &= (z + z^{-1})/2 = \cos x, \quad \phi_n(x) = \phi_0(x) P_n(\cos x), \\ P_n(\eta) &\stackrel{\text{def}}{=} p_n(\eta; a_1, a_2, a_3, a_4 | q), \end{aligned} \quad (32)$$

in which $p_n(\eta; a_1, a_2, a_3, a_4 | q)$ is the Askey-Wilson polynomial [4].

The classical sinusoidal motion (2) holds with the Hamiltonian ($\gamma = \log q$):

$$\begin{aligned} \mathcal{H}_c &= \sqrt{V_c(z) V_c(z)^*} \cosh \gamma p - (V_c(z) + V_c(z)^*)/2, \\ V_c(z) &= \prod_{j=1}^4 (1 - a_j z) / (1 - z^2)^2. \end{aligned} \quad (33)$$

The coefficients in the classical expression are

$$\begin{aligned} R_0(\mathcal{H}_c) &= \gamma^2 (\mathcal{H}_c^2 + c_1 \mathcal{H}_c + c_2), \\ R_{-1}(\mathcal{H}_c) &= -\gamma^2 (c_3 \mathcal{H}_c + c_4), \end{aligned}$$

with $c_1 = 1 + b_4$, $c_2 = (1 - b_4)^2/4$, $c_3 = (b_1 + b_3)/4$, $c_4 = (1 - b_4)(b_1 - b_3)/8$. Here we use the abbreviation

$$b_1 \stackrel{\text{def}}{=} \sum_{1 \leq j \leq 4} a_j, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_j a_k a_l, \quad b_4 \stackrel{\text{def}}{=} \prod_{j=1}^4 a_j.$$

The corresponding quantum expressions are

$$\begin{aligned} R_0(\mathcal{H}) &= q(q^{-1} - 1)^2 \left((\mathcal{H}')^2 - (1 + q^{-1})^2 b_4/4 \right), \\ R_1(\mathcal{H}) &= q(q^{-1} - 1)^2 \mathcal{H}', \quad \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} + (1 + q^{-1} b_4)/2, \\ R_{-1}(\mathcal{H}) &= -q(q^{-1} - 1)^2 \left((b_1 + q^{-1} b_3) \mathcal{H}/4 + (1 - q^{-2} b_4)(b_1 - b_3)/8 \right). \end{aligned} \quad (34)$$

The two frequencies are:

$$\alpha_{\pm}(\mathcal{H}) = (q^{-1} - 1) \left((1 - q)\mathcal{H}' \pm (1 + q)\sqrt{(\mathcal{H}')^2 - q^{-1}b_4} \right) / 2. \quad (35)$$

The annihilation-creation operators are:

$$a^{(\pm)} = \left(\pm(q^{-1} - 1)(z^{-1}(1 - qz^2)T_+ + z(1 - qz^{-2})T_-) / 4 \right. \\ \left. \mp \cos x \alpha_{\mp}(\mathcal{H}) \pm R_{-1}(\mathcal{H})\alpha_{\pm}(\mathcal{H})^{-1} \right) / (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})). \quad (36)$$

Their effects on the eigenvectors are:

$$a^{(-)}\phi_n = \frac{(1 - q^n) \prod_{1 \leq j < k \leq 4} (1 - a_j a_k q^{n-1})}{2(1 - b_4 q^{2n-2})(1 - b_4 q^{2n-1})} \phi_{n-1}, \quad (37)$$

$$a^{(+)}\phi_n = \frac{1 - b_4 q^{n-1}}{2(1 - b_4 q^{2n-1})(1 - b_4 q^{2n})} \phi_{n+1}. \quad (38)$$

The coherent state is

$$\psi(x) = \phi_0(x) \sum_{n=0}^{\infty} \frac{(2\lambda)^n}{(q; q)_n} \frac{(a_1 a_2 a_3 a_4; q)_{2n}}{\prod_{1 \leq j < k \leq 4} (a_j a_k; q)_n} P_n(\cos x). \quad (39)$$

Conclusions We have shown that most solvable quantum mechanics of one degree of freedom have exact Heisenberg operator solution. The annihilation-creation operators ($a^{(\pm)}$) are defined as the positive/negative frequency parts of the exact Heisenberg operator solution. These ($a^{(\pm)}$) are hermitian conjugate to each other. This method also applies to the so-called ‘discrete’ quantum mechanics whose eigenfunctions are deformations of the classical orthogonal polynomials known as the Askey-scheme of hypergeometric orthogonal polynomials.

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