

Exact Heisenberg operator solutions for multi-particle quantum mechanics

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Abstract

Exact Heisenberg operator solutions for independent ‘sinusoidal coordinates’ as many as the degree of freedom are derived for typical exactly solvable multi-particle quantum mechanical systems, the Calogero systems based on any root system. These Heisenberg operator solutions also present the explicit forms of the *annihilation-creation operators* for various quanta in the interacting multi-particle systems. At the same time they can be interpreted as *multi-variable generalisation* of the *three term recursion relations* for multi-variable orthogonal polynomials constituting the eigenfunctions.

1 Introduction

Modern quantum physics is virtually unthinkable without annihilation-creation operators, which are defined as the positive/negative energy parts of the *Heisenberg field operator solutions* of a *free* field theory, an infinite collection of independent harmonic oscillators. These annihilation-creation operators map an eigenstate of a free Hamiltonian into another, but not connecting those of a full theory. Our knowledge of the Heisenberg operator solutions of a full interacting theory, on the other hand, is quite limited in spite of the central role played by the Heisenberg operator solutions in field theory in general. In the so-called exactly

solvable quantum field theories, factorised S -matrices and some of the correlation (Green's) functions are the highest achieved points up to now.

Following our embryonic work [1] on the construction of exact Heisenberg operator solutions for various degree one quantum mechanics, we present in this paper a modest first step in the quest of deriving exact Heisenberg operator solutions for a family of interacting multi-particle dynamics. These are the Calogero systems [2], which are integrable multi-particle dynamics based on root systems. For the theories based on the classical root systems, the A , B , C and D series, the number of particles can be as large as wanted, but not infinite as in field theories. A complete set of exact Heisenberg solutions for 'sinusoidal coordinates' [1, 3], as many as the degree of freedom, is derived elementarily in terms of the universal Lax pair [4, 5, 6], which is a well-established solution mechanism for classical and quantum Calogero-Sutherland-Moser systems [2, 7, 8] based on any root system [9]. Explicit forms of various annihilation-creation operators are obtained as the positive/negative energy parts of the Heisenberg operator solutions. They map an eigenvector of the full Hamiltonian into another. These sinusoidal coordinates and the corresponding annihilation-creation operators provide multi-variable generalisation of the three term recursion relations [1] of orthogonal polynomials constituting the eigenfunctions of the Calogero Hamiltonian.

This paper is organised as follows. In section two, the rudimentary facts and notation of Calogero systems based on any root system are recapitulated together with the universal Lax pair matrices. In section three the complete set of exact Heisenberg operator solutions is derived quite elementarily based on generating functions constructed from the universal Lax matrices. Remarks on the special features of the D -type theories are given at the end of the section. The final section is for a summary and comments for further research. The Appendix gives the list of the preferred sets of weight vectors for explicit representations of Lax matrices based on the exceptional and the non-crystallographic root systems.

2 Calogero Systems

In this paper we will derive *exact Heisenberg operator solutions for the Calogero systems*. They are one-dimensional multi-particle dynamics with inverse (distance)² potential inside a harmonic confining potential. They have a remarkable property that they are exactly solvable at the classical [10] and quantum [2, 6] levels. The exact quantum solvability has been shown in the *Schrödinger picture*, as the entire energy spectrum is known and the corresponding

eigenfunctions can be constructed explicitly by a finite number of algebraic processes (the lower triangularity of the Hamiltonian in certain basis) [6]. The exact quantum solvability in the *Heisenberg picture* of the Calogero systems will be demonstrated in the next section by constructing the explicit Heisenberg operator solutions for the independent ‘sinusoidal coordinates’ as many as the degree of freedom. Let us briefly recapitulate the essence of the quantum Calogero systems [2, 6] together with appropriate notation necessary in this paper.

2.1 Quantum Hamiltonian

A Calogero system is a Hamiltonian dynamics associated with a root system [4, 9] Δ of rank r , which is a set of vectors in \mathbb{R}^r with its standard inner product. Its dynamical variables are the coordinates $\{q_j\}$ and their canonically conjugate momenta $\{p_j\}$, satisfying the canonical commutation relations¹:

$$[q_j, p_k] = i\delta_{jk}, \quad [q_j, q_k] = [p_j, p_k] = 0, \quad j, k = 1, \dots, r. \quad (2.1)$$

These will be denoted by vectors in \mathbb{R}^r ,

$$q \stackrel{\text{def}}{=} (q_1, \dots, q_r), \quad p \stackrel{\text{def}}{=} (p_1, \dots, p_r), \quad p \cdot q \stackrel{\text{def}}{=} \sum_{j=1}^r p_j q_j, \quad p^2 \stackrel{\text{def}}{=} \sum_{j=1}^r p_j^2, \quad q^2 \stackrel{\text{def}}{=} \sum_{j=1}^r q_j^2. \quad (2.2)$$

The momentum operator p_j acts as a differential operator

$$p_j = -i \frac{\partial}{\partial q_j}, \quad j = 1, \dots, r.$$

The ‘factorised’ Hamiltonian is

$$\mathcal{H}(p, q) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{j=1}^r \left(p_j - i \frac{\partial W}{\partial q_j} \right) \left(p_j + i \frac{\partial W}{\partial q_j} \right), \quad (2.3)$$

$$= \frac{1}{2} p^2 + \frac{\omega^2}{2} q^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g_{|\rho|} (g_{|\rho|} - 1) |\rho|^2}{(\rho \cdot q)^2} - E_0, \quad (2.4)$$

$$W(q) \stackrel{\text{def}}{=} -\frac{\omega}{2} q^2 + \sum_{\rho \in \Delta_+} g_{|\rho|} \log |\rho \cdot q|, \quad g_{|\rho|} > 0, \quad \omega > 0. \quad (2.5)$$

The summation is over the set of positive roots Δ_+ , with $\Delta = \Delta_+ \cup (-\Delta_+)$. The real *positive* coupling constants $g_{|\rho|}$ are defined on orbits of the corresponding Coxeter group, *i.e.*

¹ For the *A*-type theory, it is customary to consider A_{r-1} and to embed all the roots in \mathbb{R}^r . This is accompanied by the introduction of one more degree of freedom, q_r and p_r . The genuine A_{r-1} theory corresponds to the relative coordinates and their momenta, and the extra degree of freedom is the center of mass coordinate and its momentum.

they are identical for roots of the same length. The Hamiltonian is invariant under the finite reflection group (Coxeter group, or Weyl group) generated by the set of roots Δ :

$$\mathcal{H}(s_\alpha(p), s_\alpha(q)) = \mathcal{H}(p, q), \quad \forall \alpha \in \Delta, \quad (2.6)$$

with the reflection s_α defined by

$$s_\alpha(x) \stackrel{\text{def}}{=} x - (\alpha^\vee \cdot x)\alpha, \quad \forall x \in \mathbb{R}^r, \quad \alpha^\vee \stackrel{\text{def}}{=} 2\alpha/|\alpha|^2. \quad (2.7)$$

Obviously the Heisenberg equations of motion for the coordinates $\{q_j\}$ are trivial

$$i[\mathcal{H}, q_j] = \frac{d}{dt}q_j = p_j, \quad j = 1, \dots, r, \quad (2.8)$$

whereas those for the canonical momenta $\{p_j\}$ have the same form as the Newton equations:

$$\frac{d^2}{dt^2}q_j = \frac{d}{dt}p_j = i[\mathcal{H}, p_j] = -\omega^2 q_j + \sum_{\rho \in \Delta_+} \frac{g_{|\rho|}(g_{|\rho|} - 1)|\rho|^2 \rho_j}{(\rho \cdot q)^3}, \quad j = 1, \dots, r. \quad (2.9)$$

The hard repulsive potential $\sim 1/(\rho \cdot q)^2$ near the reflection hyperplane $H_\rho \stackrel{\text{def}}{=} \{q \in \mathbb{R}^r \mid \rho \cdot q = 0\}$ is insurmountable at the quantum level as well as the classical. Thus the motion is always confined within one Weyl chamber. This feature allows us to constrain the configuration space to the principal Weyl chamber (Π : set of simple roots)

$$PW \stackrel{\text{def}}{=} \{q \in \mathbb{R}^r \mid \alpha \cdot q > 0, \quad \alpha \in \Pi\}, \quad (2.10)$$

without loss of generality.

The positive semi-definite form of the factorised Hamiltonian (2.3) simply allows the determination of the ground state wavefunction:

$$\mathcal{H}\phi_0(q) = 0, \quad \phi_0(q) \stackrel{\text{def}}{=} e^{W(q)} = \prod_{\rho \in \Delta_+} |\rho \cdot q|^{g_{|\rho|}} \cdot e^{-\frac{\omega}{2}q^2}, \quad (2.11)$$

which is real and obviously square integrable

$$\int_{PW} \phi_0^2(q) d^r q < \infty. \quad (2.12)$$

The constant part E_0 of the Hamiltonian (2.3) is usually called the *ground state energy*

$$E_0 \stackrel{\text{def}}{=} \omega \left(\frac{r}{2} + \sum_{\rho \in \Delta_+} g_{|\rho|} \right). \quad (2.13)$$

The excited energy spectrum is *integer spaced* and is *independent of the coupling constants* $\{g_{|\rho|}\}$ [6]:

$$\mathcal{H}\phi_{\mathbf{n}}(q) = \mathcal{E}_{\mathbf{n}}\phi_{\mathbf{n}}(q), \quad \mathbf{n} \stackrel{\text{def}}{=} (n_1, \dots, n_r), \quad n_j \in \mathbb{N} \stackrel{\text{def}}{=} \mathbb{Z}_{\geq 0}, \quad (2.14)$$

$$\mathcal{E}_{\mathbf{n}} \stackrel{\text{def}}{=} \omega N_{\mathbf{n}}, \quad N_{\mathbf{n}} \stackrel{\text{def}}{=} \sum_{j=1}^r n_j f_j. \quad (2.15)$$

Here $\{f_j\}$ are the integers related to the *exponents* $\{e_j\}$ of Δ :

$$f_j \stackrel{\text{def}}{=} 1 + e_j. \quad (2.16)$$

They indicate the degrees where independent Coxeter invariant polynomials exist. The set of integers

$$F_{\Delta} \stackrel{\text{def}}{=} \{f_1, f_2, \dots, f_r\} \quad (2.17)$$

is shown in Table I for each root system Δ .

Δ	F_{Δ}	Δ	F_{Δ}
A_{r-1}	$\{2, 3, \dots, r, 1\}$	E_8	$\{2, 8, 12, 14, 18, 20, 24, 30\}$
B_r	$\{2, 4, 6, \dots, 2r\}$	F_4	$\{2, 6, 8, 12\}$
C_r	$\{2, 4, 6, \dots, 2r\}$	G_2	$\{2, 6\}$
D_r	$\{2, 4, \dots, 2r-2, r\}$	$I_2(m)$	$\{2, m\}$
E_6	$\{2, 5, 6, 8, 9, 12\}$	H_3	$\{2, 6, 10\}$
E_7	$\{2, 6, 8, 10, 12, 14, 18\}$	H_4	$\{2, 12, 20, 30\}$

Table I: The set of integers $F_{\Delta} = \{f_1, f_2, \dots, f_r\}$ for which independent Coxeter invariant polynomials exist.²

Excited states eigenfunctions have the following general structure:

$$\phi_{\mathbf{n}}(q) = \phi_0(q)P_{\mathbf{n}}(q), \quad (2.18)$$

in which $P_{\mathbf{n}}(q)$ is a Coxeter invariant polynomial in $\{q_j\}$ of degree $N_{\mathbf{n}}$.

² For A_{r-1} root system, $f_r = 1$ corresponds to the degree of freedom for the center of mass coordinate. The B_r and C_r Calogero systems are equivalent.

2.2 Quantum Lax Pair

The derivation of the Heisenberg operator solutions depends heavily on the universal Lax pair which applies to any root system. The universal Lax pair operators [5, 6] are

$$L(p, q) \stackrel{\text{def}}{=} p \cdot \hat{H} + X(q), \quad X(q) \stackrel{\text{def}}{=} i \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{(\rho \cdot \hat{H})}{\rho \cdot q} \hat{s}_\rho, \quad (2.19)$$

$$M(q) \stackrel{\text{def}}{=} -\frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{|\rho|^2}{(\rho \cdot q)^2} \hat{s}_\rho + \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{|\rho|^2}{(\rho \cdot q)^2} \times I, \quad (2.20)$$

in which I is the identity operator and $\{\hat{s}_\alpha : \alpha \in \Delta\}$ are the reflection operators of the root system. They act on a set of \mathbb{R}^r vectors $\mathcal{R} \stackrel{\text{def}}{=} \{\mu^{(k)} \in \mathbb{R}^r \mid k = 1, \dots, d\}$, permuting them under the action of the reflection group. The vectors in \mathcal{R} form a basis for the representation space \mathbf{V} of dimension d . The operator M satisfies the relation [5, 6]

$$\sum_{\mu \in \mathcal{R}} M_{\mu\nu} = \sum_{\nu \in \mathcal{R}} M_{\mu\nu} = 0, \quad (2.21)$$

which is essential for deriving quantum conserved quantities and annihilation-creation operators. The matrix elements of the operators $\{\hat{s}_\alpha : \alpha \in \Delta\}$ and $\{\hat{H}_j : j = 1, \dots, r\}$ are defined as follows:

$$(\hat{s}_\rho)_{\mu\nu} \stackrel{\text{def}}{=} \delta_{\mu, s_\rho(\nu)} = \delta_{\nu, s_\rho(\mu)}, \quad (\hat{H}_j)_{\mu\nu} \stackrel{\text{def}}{=} \mu_j \delta_{\mu\nu}, \quad \rho \in \Delta, \quad \mu, \nu \in \mathcal{R}. \quad (2.22)$$

The Lax equation

$$i[\mathcal{H}, L] = \frac{d}{dt}L = [L, M] \quad (2.23)$$

is equivalent to the Heisenberg equation of motion for $\{q_j\}$ and $\{p_j\}$ for the Hamiltonian (2.4) without the harmonic confining potential, *i.e.*, $\omega = 0$. It should be emphasised that the l.h.s. of (2.23) is a quantum commutator, whereas the r.h.s. is a matrix commutator as well as quantum. The full Heisenberg equations of motion with the harmonic confining potential read

$$i[\mathcal{H}, L^\pm] = \frac{d}{dt}L^\pm = [L^\pm, M] \pm i\omega L^\pm, \quad (2.24)$$

in which M is the same as before (2.20), and L^\pm and Q are defined by

$$L^\pm \stackrel{\text{def}}{=} L \pm i\omega Q, \quad Q \stackrel{\text{def}}{=} q \cdot \hat{H}, \quad (L^+)^\dagger = L^-, \quad (2.25)$$

with L, \hat{H} as earlier (2.19), (2.22). One direct consequence of the Lax pair equation (2.24) is the existence of a wide variety of *quantum conserved quantities*:

$$\frac{d}{dt} \text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k}) = 0, \quad \epsilon_j \in \{+, -\}, \quad \forall k \in 2\mathbb{N}, \quad \sum_{j=1}^k \epsilon_j = 0, \quad (2.26)$$

in which $\text{Ts}(A)$ denotes the *total sum* of a matrix A [5, 6] with suffices $\mu, \nu \in \mathcal{R}$:

$$\text{Ts}(A) \stackrel{\text{def}}{=} \sum_{\mu, \nu \in \mathcal{R}} A_{\mu\nu}. \quad (2.27)$$

This is a simple outcome of the property (2.21) of the matrix M [5, 6, 11, 12]. The quantum conserved quantities are the simplest example of more general results:

$$\begin{aligned} i[\mathcal{H}, \text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k})] &= \frac{d}{dt} \text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k}), \quad \forall k \in \mathbb{N}, \\ &= i\omega m \text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k}), \quad \sum_{j=1}^k \epsilon_j = m. \end{aligned} \quad (2.28)$$

In other words, $\text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k})$ shifts the eigenvalue of \mathcal{H} by m unit (or, ωm) when it acts on any eigenstate of \mathcal{H} . Namely such operators are all candidates of *annihilation-creation operators* for \mathcal{H} :

$$e^{i\mathcal{H}t} \text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k}) e^{-i\mathcal{H}t} = \text{Ts}(L^{\epsilon_1} L^{\epsilon_2} \cdots L^{\epsilon_k}) e^{i\omega m t}, \quad \sum_{j=1}^k \epsilon_j = m. \quad (2.29)$$

Note that the order of L^+ and L^- is immaterial.

The results and statements in this subsection are *universal*, *i.e.* they are valid for any root system Δ and for any choice of the Lax pair matrices, *i.e.* the choice of \mathcal{R} . For deriving the explicit forms of the Heisenberg operator solutions, however, it is convenient to choose as \mathcal{R} the set of the least dimensions for each Δ . For the classical root systems, A, B and D , they are the set of vector weights³:

$$A_{r-1} \quad : \quad \mathcal{R} \stackrel{\text{def}}{=} \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r\}, \quad d \stackrel{\text{def}}{=} \#\mathcal{R} = r, \quad (2.30)$$

$$B_r, D_r \quad : \quad \mathcal{R} \stackrel{\text{def}}{=} \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \dots, \pm\mathbf{e}_r\}, \quad d \stackrel{\text{def}}{=} \#\mathcal{R} = 2r, \quad (2.31)$$

in which $\{\mathbf{e}_j\}$ are the orthonormal basis of \mathbb{R}^r , $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$. We list in the Appendix the preferred choice of \mathcal{R} for the exceptional and the non-crystallographic root systems.

³ To be more precise, for B_r they should be called the set of short roots.

3 Exact Heisenberg Solutions

Now let us proceed to derive the explicit Heisenberg operator solutions for various ‘sinusoidal coordinates’ [1, 3] as many as the degree of freedom. For simplicity of presentation, let us fix the angular frequency of the harmonic confining potential as unity, $\omega = 1$ hereafter. We have already known one exact Heisenberg operator solution. In [1] it was shown that the quadratic invariant

$$\eta^{(1)} \propto q^2 = \sum_{j=1}^r q_j^2 \quad (3.1)$$

is the simplest sinusoidal coordinate:

$$[\mathcal{H}, [\mathcal{H}, \eta^{(1)}]] = 4(\eta^{(1)} - c_1 \mathcal{H} - c_2), \quad c_1, c_2 : \text{const}, \quad (3.2)$$

where c_1 is given by $\eta^{(1)} = c_1 q^2$ and $c_2 = c_1 E_0$. As shown in [1] the exact Heisenberg solution

$$e^{i\mathcal{H}t} \eta^{(1)} e^{-i\mathcal{H}t} \quad (3.3)$$

is easily evaluated. It should be noted that any root system has a degree 2 Coxeter invariant $2 \in F_\Delta$ (Table I), which is proportional to $\eta^{(1)}$. In fact, $\eta^{(1)}$ is a sinusoidal coordinate [1] for a Hamiltonian \mathcal{H}_{ge} more general than the Calogero system; a general homogeneous potential of degree -2 within a confining harmonic potential [13]:

$$\mathcal{H}_{\text{ge}} = \frac{1}{2} \sum_{j=1}^r (p_j^2 + q_j^2) + V(q), \quad \sum_{j=1}^r q_j \frac{\partial}{\partial q_j} V(q) = -2V(q), \quad (3.4)$$

which satisfies $[\mathcal{H}_{\text{ge}}, [\mathcal{H}_{\text{ge}}, q^2]] = 4(q^2 - \mathcal{H}_{\text{ge}})$.

For the Calogero system based on any root system Δ , we will derive the explicit forms of the Heisenberg operator solutions:

$$e^{i\mathcal{H}t} \eta^{(j)} e^{-i\mathcal{H}t}, \quad j = 1, \dots, r, \quad (3.5)$$

for a complete set of sinusoidal coordinates defined by

$$\{\eta^{(1)}, \eta^{(2)}, \dots, \eta^{(r)}\}, \quad \eta^{(j)} \stackrel{\text{def}}{=} \text{Ts}(Q^{f_j}) = \text{Tr}(Q^{f_j}), \quad f_j \in F_\Delta. \quad (3.6)$$

In (3.6), the matrix Q (2.25) is diagonal therefore its total sum (Ts) is the same as the trace (Tr). Let us note that any Coxeter invariant polynomial in $\{q_j\}$ can be expressed as

a polynomial in $\{\eta^{(1)}, \eta^{(2)}, \dots, \eta^{(r)}\}$. It is easy to verify $\eta^{(1)} \propto q^2$ for an root system. The higher sinusoidal coordinates for the classical root systems are:

$$A_{r-1} : \eta^{(k)} \propto \sum_{j=1}^r q_j^{k+1}, \quad k = 1, \dots, r-1; \quad \eta^{(r)} \propto \sum_{j=1}^r q_j, \quad (3.7)$$

$$B_r, D_r : \eta^{(k)} \propto \sum_{j=1}^r q_j^{2k}, \quad k = 1, \dots, r, \quad (3.8)$$

except for $\eta^{(r)}$ for D_r which reads

$$\eta^{(r)} \propto \prod_{j=1}^r q_j. \quad (3.9)$$

See the remark at the end of the section. As shown in Table I, all the integers $\{f_j\}$ are *even* for B_r and D_r , except for $f_r = r$ for *odd* r of D_r . This is also related to the fact that

$$\text{Tr}(Q^{2l+1}) = \sum_{j=1}^r q_j^{2l+1} + \sum_{j=1}^r (-q_j)^{2l+1} = 0,$$

for the Lax operators based on the vector weights (2.31).

The sinusoidal coordinates take various different forms for the exceptional and non-crystallographic root systems. Note that the overall normalisation of $\{\eta^{(k)}\}$ is immaterial.

The derivation of exact Heisenberg operator solutions is quite elementary. Let us introduce a *generating function* of the total sum of the homogeneous polynomials in L^+ and L^- :

$$G^{(j)}(s) \stackrel{\text{def}}{=} \text{Ts}((L^+ + sL^-)^{f_j}), \quad s \in \mathbb{C}, \quad f_j \in F_\Delta, \quad (3.10)$$

which is a polynomial in s of degree f_j

$$G^{(j)}(s) = \sum_{l=0}^{f_j} b_{f_j; f_j-2l} s^l, \quad b_{f_j; l}^\dagger = b_{f_j; -l}. \quad (3.11)$$

The coefficient $b_{f_j; f_j-2l}$ is the total sum of a *completely symmetric product* consisting of $f_j - l$ times L^+ and l times L^- and it can be explicitly evaluated. As shown in (2.29), we obtain

$$e^{i\mathcal{H}t} b_{f_j; f_j-2l} e^{-i\mathcal{H}t} = b_{f_j; f_j-2l} e^{i(f_j-2l)t}. \quad (3.12)$$

Namely $b_{f_j; f_j-2l}$ is either an *annihilation operator* ($l > f_j/2$) being the *positive energy part* or a *creation operator* ($l < f_j/2$) being the *negative energy part* or a conserved quantity ($l = f_j/2$), the constant part. The conserved quantities are in general not in involution

$$[b_{f_j; 0}, b_{f_k; 0}] \neq 0, \quad j \neq k. \quad (3.13)$$

On the other hand, for $s = -1$ we obtain

$$L^+ - L^- = 2iQ$$

and

$$\begin{aligned} G^{(j)}(-1) &= (2i)^{f_j} \text{Ts}(Q^{f_j}) = (2i)^{f_j} \eta^{(j)} = \sum_{l=0}^{f_j} (-1)^l b_{f_j; f_j-2l}, \\ \eta^{(j)} &= (2i)^{-f_j} \sum_{l=0}^{f_j} (-1)^l b_{f_j; f_j-2l}. \end{aligned} \quad (3.14)$$

Thus we arrive at the main result of the paper; *the complete set of exact Heisenberg operator solutions* for the ‘sinusoidal coordinates’ $\{\eta^{(j)}\}$:

$$\eta^{(j)}(t) \stackrel{\text{def}}{=} e^{i\mathcal{H}t} \eta^{(j)} e^{-i\mathcal{H}t} = (2i)^{-f_j} \sum_{l=0}^{f_j} (-1)^l b_{f_j; f_j-2l} e^{i(f_j-2l)t}, \quad j = 1, 2, \dots, r. \quad (3.15)$$

This clearly shows that $\eta^{(j)}(t)$ is a superposition of various sinusoidal motions. The trivial fact that these ‘sinusoidal coordinates’ commute among themselves

$$\eta^{(j)} \eta^{(k)} = \eta^{(k)} \eta^{(j)}, \quad j \neq k = 1, \dots, r \quad (3.16)$$

is translated into the commutation relations among the annihilation-creation operators

$$\sum_{\substack{l, m \\ l+m: \text{fixed}}} [b_{f_j; f_j-2l}, b_{f_k; f_k-2m}] = 0. \quad (3.17)$$

Among the annihilation-creation operators belonging to $\eta^{(j)}$, the two extreme ones corresponding to $l = 0$ and $l = f_j$ have a special meaning. They consist of L^+ (L^-) only

$$b_{f_j; f_j} = \text{Ts}((L^+)^{f_j}), \quad b_{f_j; -f_j} = \text{Ts}((L^-)^{f_j}), \quad b_{f_j; f_j}^\dagger = b_{f_j; -f_j} \quad (3.18)$$

and commute among themselves

$$[b_{f_j; f_j}, b_{f_k; f_k}] = 0, \quad [b_{f_j; -f_j}, b_{f_k; -f_k}] = 0, \quad j, k = 1, \dots, r, \quad (3.19)$$

as is clear from (3.17). These special annihilation-creation operators have been known for some time [6, 9, 12, 14].

Since the l -th term in (3.15) is annihilated by $d/dt - i(f_j - 2l)$, we obtain a linear differential equation with constant coefficients satisfied by $\eta^{(j)}(t)$:

$$\prod_{\substack{l=-f_j \\ \text{step } 2}}^{f_j} \left(\frac{d}{dt} - il \right) \cdot \eta^{(j)}(t) = 0. \quad (3.20)$$

Equivalently this can be rewritten as

$$\prod_{\substack{l=-f_j \\ \text{step } 2}}^{f_j} (\text{ad}(\mathcal{H}) - l) \cdot \eta^{(j)} = 0. \quad (3.21)$$

Here $\text{ad}(\mathcal{H})$ denotes a commutator $\text{ad}(\mathcal{H})X \stackrel{\text{def}}{=} [\mathcal{H}, X]$ for any operator X . For even f_j case, the factor $d/dt - i(f_j - 2l)$ for $l = f_j/2$ could be omitted as it annihilates the conserved quantity. Then we obtain a differential equation of order f_j

$$\prod_{\substack{l=-f_j, \neq 0 \\ \text{step } 2}}^{f_j} \left(\frac{d}{dt} - il \right) \cdot (\eta^{(j)}(t) - 2^{-f_j} b_{f_j;0}) = 0, \quad (3.22)$$

$$\prod_{\substack{l=-f_j, \neq 0 \\ \text{step } 2}}^{f_j} (\text{ad}(\mathcal{H}) - l) \cdot (\eta^{(j)} - 2^{-f_j} b_{f_j;0}) = 0, \quad (3.23)$$

instead of $f_j + 1$ for odd f_j case. In [1], for a wide class of solvable quantum systems with one degree of freedom, we discussed the ‘closure relation’

$$[\mathcal{H}, [\mathcal{H}, \eta]] = \eta R_0(\mathcal{H}) + [\mathcal{H}, \eta] R_1(\mathcal{H}) + R_{-1}(\mathcal{H}) \quad (3.24)$$

with $R_i(\mathcal{H})^\dagger = R_i(\mathcal{H})$. By introducing $\alpha_\pm(\mathcal{H})$ as $R_1(\mathcal{H}) = \alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H})$ and $R_0(\mathcal{H}) = -\alpha_+(\mathcal{H})\alpha_-(\mathcal{H})$, this closure relation is rewritten as

$$(\text{ad}(\mathcal{H}) + \alpha_+(\mathcal{H}))(\text{ad}(\mathcal{H}) + \alpha_-(\mathcal{H}))(\eta + R_0(\mathcal{H})^{-1} R_{-1}(\mathcal{H})) = 0. \quad (3.25)$$

Eqs. (3.21) and (3.23) are multi-particle generalisation of this relation.

When a sinusoidal coordinate $\eta^{(j)}$ is multiplied to an eigenvector $\phi_{\mathbf{n}}$ of \mathcal{H} ,

$$(2i)^{f_j} \eta^{(j)} \phi_{\mathbf{n}} = \sum_{l=0}^{f_j} (-1)^l b_{f_j; f_j-2l} \phi_{\mathbf{n}}, \quad (3.26)$$

the l -th term belongs to the eigenspace of \mathcal{H} with the eigenvalue $\mathcal{E}_{\mathbf{n}} + f_j - 2l$. Thus these $f_j + 1$ terms are all orthogonal to each other. This is a *multi-variable generalisation* of the

three-term recursion relation of the orthogonal polynomials of one variable [1]. Through a similarity transformation in terms of the ground state wavefunction $\phi_0(q)$ (2.11), let us define

$$\tilde{L} \stackrel{\text{def}}{=} \phi_0^{-1} \circ L \circ \phi_0, \quad \tilde{L}^\pm \stackrel{\text{def}}{=} \phi_0^{-1} \circ \tilde{L}^\pm \circ \phi_0 = \tilde{L} \pm iQ, \quad (3.27)$$

$$\tilde{b}_{f_j;f_j-2l} \stackrel{\text{def}}{=} \phi_0^{-1} \circ b_{f_j;f_j-2l} \circ \phi_0, \quad j = 1, \dots, r, \quad l = 0, 1, \dots, f_j. \quad (3.28)$$

Needless to say the identity

$$\phi_0^{-1} \circ \text{Ts}(A^n) \circ \phi_0 = \text{Ts}((\phi_0^{-1} \circ A \circ \phi_0)^n)$$

holds for any matrix A consisting of operators. Then one can present the corresponding results for the multi-variable orthogonal polynomials $\{P_{\mathbf{n}}(q)\}$ (2.18) constituting the eigenvectors:

$$(2i)^{f_j} \eta^{(j)} P_{\mathbf{n}}(q) = \sum_{l=0}^{f_j} (-1)^l \tilde{b}_{f_j;f_j-2l} P_{\mathbf{n}}(q). \quad (3.29)$$

The operators \tilde{L} , \tilde{L}^\pm and $\tilde{b}_{f_j;f_j-2l}$ are closely related to the Dunkl operators [6, 15].

The annihilation-creation operators provide an algebraic solution method of the Calogero systems. The entire Hilbert space is generated by the multiple application of creation operators on the ground state wavefunction $\phi_0(q) = e^{W(q)}$:

$$\prod_{j=1}^r \prod_{\substack{0 < l \leq f_j \\ l \equiv f_j \pmod{2}}} b_{f_j;l}^{n_{(j,l)}} \cdot \phi_0, \quad \forall n_{(j,l)} \in \mathbb{N}, \quad (3.30)$$

whereas ϕ_0 is destroyed by all the annihilation operators

$$b_{f_j;l} \phi_0 = 0, \quad -f_j \leq l < 0, \quad l \equiv f_j \pmod{2}. \quad (3.31)$$

Obviously the above states (3.30) are over-complete and the orthogonality of various eigenvectors belonging to the same degenerate eigenspace is not guaranteed. For example, a complete basis of the Hilbert space is given by using $b_{f_j;f_j}$ only [6]

$$\prod_{j=1}^r b_{f_j;f_j}^{n_j} \cdot \phi_0, \quad \forall n_j \in \mathbb{N}. \quad (3.32)$$

Remark on D -type Theory As noticed above, the D -type theory requires separate treatment due to the special $f_r = r$. We need Lax pairs based on the *spinor* and *anti-spinor* weights:

$$\mathcal{R} \stackrel{\text{def}}{=} \left\{ \pm \frac{1}{2} \mathbf{e}_1 \pm \frac{1}{2} \mathbf{e}_2 \pm \cdots \pm \frac{1}{2} \mathbf{e}_r \right\}, \quad d \stackrel{\text{def}}{=} \#\mathcal{R} = 2^{r-1}. \quad (3.33)$$

Here the the number of $-$ signs is *even* (*odd*) for the spinor (anti-spinor) weights. They both form 2^{r-1} dimensional representations of the Lie-algebra D_r , and these representations are called *minimal*. The minimal weights for the A , D and E root systems have played an important role in constructing simple Lax pair representations [4].

For D_{odd} we simply use the Lax matrix $L_{(\text{sp})}^{\pm}$ ($L_{(\text{as})}^{\pm}$) based on either the spinor (sp) or the anti-spinor (as) weights and proceed in the same way as above:

$$\begin{aligned} G_{(\text{sp})}^{(r)}(s) &\stackrel{\text{def}}{=} \text{Ts}((L_{(\text{sp})}^+ + s L_{(\text{sp})}^-)^r), \quad s \in \mathbb{C}, \quad f_r = r, \\ &= \sum_{l=0}^r b_{(\text{sp})r;r-2l} s^l. \end{aligned} \quad (3.34)$$

Then we obtain the expansion of $\eta^{(r)}$ in terms of the annihilation-creation operators and the corresponding exact Heisenberg operator solution

$$G_{(\text{sp})}^{(r)}(-1) = (2i)^r \text{Ts}(Q_{(\text{sp})}^r) = (2i)^r \eta^{(r)} = \sum_{l=0}^r (-1)^l b_{(\text{sp})r;r-2l}, \quad (3.35)$$

$$\eta^{(r)} = (2i)^{-r} \sum_{l=0}^r (-1)^l b_{(\text{sp})r;r-2l} \propto q_1 q_2 \cdots q_r, \quad (3.36)$$

$$\eta^{(r)}(t) \stackrel{\text{def}}{=} e^{i\mathcal{H}t} \eta^{(r)} e^{-i\mathcal{H}t} = (2i)^{-r} \sum_{l=0}^r (-1)^l b_{(\text{sp})r;r-2l} e^{i(r-2l)t}. \quad (3.37)$$

For D_{even} , the situation is slightly more complicated, partly because of the existence of another sinusoidal coordinate $\eta^{(r/2)}$, which is a Coxeter invariant polynomial of $\{q_j\}$ of degree r , too. Here we prepare the Lax matrices based on both the spinor $L_{(\text{sp})}^{\pm}$ and the anti-spinor weights $L_{(\text{as})}^{\pm}$, since

$$\eta^{(r)} \propto \text{Ts}(Q_{(\text{sp})}^r) - \text{Ts}(Q_{(\text{as})}^r) \propto q_1 q_2 \cdots q_r. \quad (3.38)$$

It is quite elementary to verify (3.36) and (3.38). Let us introduce two ‘generating functions’

$$G_{(\text{sp})}^{(r)}(s) \stackrel{\text{def}}{=} \text{Ts}((L_{(\text{sp})}^+ + s L_{(\text{sp})}^-)^r) = \sum_{l=0}^r b_{(\text{sp})r;r-2l} s^l, \quad (3.39)$$

$$G_{(\text{as})}^{(r)}(s) \stackrel{\text{def}}{=} \text{Ts}((L_{(\text{as})}^+ + s L_{(\text{as})}^-)^r) = \sum_{l=0}^r b_{(\text{as})r;r-2l} s^l. \quad (3.40)$$

Next we introduce the sinusoidal coordinate $\eta^{(r)}$ as their difference at $s = -1$,

$$\begin{aligned}\eta^{(r)} &\stackrel{\text{def}}{=} (2i)^{-r} (G_{(\text{sp})}^{(r)}(-1) - G_{(\text{as})}^{(r)}(-1)) = \text{Ts}(Q_{(\text{sp})}^r) - \text{Ts}(Q_{(\text{as})}^r) \\ &= (2i)^{-r} \sum_{l=0}^r (-1)^l (b_{(\text{sp}) r;r-2l} - b_{(\text{as}) r;r-2l}).\end{aligned}\tag{3.41}$$

We obtain the corresponding exact Heisenberg operator solution

$$\eta^{(r)}(t) \stackrel{\text{def}}{=} e^{i\mathcal{H}t} \eta^{(r)} e^{-i\mathcal{H}t} = (2i)^{-r} \sum_{l=0}^r (-1)^l (b_{(\text{sp}) r;r-2l} - b_{(\text{as}) r;r-2l}) e^{i(r-2l)t}.\tag{3.42}$$

This completes the derivation of the exact Heisenberg operator solutions for the D -type Calogero systems.

4 Summary and Comments

A complete set of exact Heisenberg operator solutions, as many as the degree of freedom, is constructed for the Calogero systems based on any root system, including the exceptional and non-crystallographic ones. Based on the complete set, one can write down the Heisenberg operator solution $e^{i\mathcal{H}t} A e^{-i\mathcal{H}t}$ for any operator A expressible as a polynomial in the sinusoidal coordinates $\{\eta^{(j)}\}$. This is the first demonstration of the *exact solvability of multi-particle quantum mechanics* in the *Heisenberg picture*. At the same time, these Heisenberg operator solutions provide the explicit forms of various *annihilation-creation operators*, as the *positive* and *negative energy parts*. Their commutation relations are, in general, quite involved. As in the simplest case of degree one quantum mechanics [1], these sinusoidal coordinates and their expansion into the annihilation-creation operators provide the explicit forms of the *multi-variable generalisation of the three term recursion relations* for the orthogonal polynomials constituting the multi-variable eigenfunctions. The derivation of the Heisenberg operator solutions is a simple consequence of the *universal* Lax pair, which manifests the quantum integrability of Calogero systems based on *any root system*.

Let us conclude this paper with a few comments on possible future directions of the present research. For better understanding of multi-particle quantum mechanics in general, it is desirable to enlarge the list of exact Heisenberg operator solutions. The obvious candidates are: the Sutherland systems [7] having trigonometric potentials, various super-symmetric generalisations of the Calogero-Sutherland systems [5, 11], the Ruijsenaars-Schneider-van-

Diejen systems [16] which are ‘discrete’ counterparts of the Calogero-Sutherland systems, and the (affine) Toda molecules.

The newly found annihilation-creation operators suggest an interesting possibility of introducing *multi-particle coherent states* as common eigenstates of certain annihilation operators. It is a good challenge to construct explicit examples of such multi-particle coherent states having mathematically elegant structure and/or practical use.

A completely integrable system, including the Calogero-Sutherland-Moser systems, has the so-called *hierarchy structure*. It is characterised by the existence of mutually involutive conserved quantities

$$\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_r, \quad [\mathcal{H}_j, \mathcal{H}_k] = 0, \quad j, k = 1, \dots, r, \quad (4.1)$$

which could be adopted as independent Hamiltonians generating different but compatible time-flows; t_1, t_2, \dots, t_r , as many as the degree of freedom. It is a good challenge to construct common Heisenberg operator solutions to all the flows of the hierarchy

$$e^{i \sum_{j=1}^r \mathcal{H}_j t_j} \tilde{\eta}^{(k)} e^{-i \sum_{j=1}^r \mathcal{H}_j t_j}, \quad k = 1, \dots, r. \quad (4.2)$$

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Appendix: The Preferred Choice of \mathcal{R}

Here we list, for the exceptional and the non-crystallographic root systems, the set \mathcal{R} to be used for the explicit evaluation of the Heisenberg operator solutions. They are of the lowest dimensionality.

1. E_6 : The weights of **27** (or $\overline{27}$) dimensional representation of the Lie algebra. They are minimal representations.
2. E_7 : The weights of **56** dimensional representation of the Lie algebra. This is a minimal representation.
3. E_8 : The set consisting of all 240 roots.

4. F_4 : Either of the set consisting of all 24 long roots or 24 short roots.
5. G_2 : Either of the set consisting of all 6 long roots or 6 short roots.
6. $I_2(m)$: The set consisting of the vertices R_m of the regular m -gon

$$R_m = \{v_j = (\cos(2k\pi/m + t_0), \sin(2k\pi/m + t_0)) \in \mathbb{R}^2 \mid k = 1, \dots, m\}, \quad (\text{A.1})$$

in which $t_0 = 0$ ($\pi/2m$) for m even (odd).

7. H_3 : The set consisting of all 30 roots.
8. H_4 : The set consisting of all 120 roots.

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