Yukawa Institute Kyoto

DPSU-07-4
YITP-07-64
arXiv:0710.2209[hep-th]
October 2007

q-oscillator from the q-Hermite Polynomial

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Abstract

By factorization of the Hamiltonian describing the quantum mechanics of the continuous q-Hermite polynomial, the creation and annihilation operators of the q-oscillator are obtained. They satisfy a q-oscillator algebra as a consequence of the shape-invariance of the Hamiltonian. A second set of q-oscillator is derived from the exact Heisenberg operator solution. Now the q-oscillator stands on the equal footing to the ordinary harmonic oscillator.

PACS : 03.65.-w, 03.65.Ca, 03.65.Fd, 02.30.Ik, 02.30.Gp, 02.20.Uw

1 Introduction

In this Letter, the explicit forms of the generators of a q-oscillator algebra are derived from the quantum mechanical Hamiltonian [1, 2] of the q-Hermite polynomial [3], the q-analogue of the Hermite polynomial constituting the eigenfunctions of the harmonic oscillator. This is in sharp contrast to the common approach to q-oscillators [4], which assumes certain forms of the algebras without any dynamical/analytical contents behind them. On the other hand, the ordinary harmonic oscillator algebra generated by the annihilation/creation operators has rich analytical structure of differential operators related with the classical analysis of the Hermite polynomial together with the coherent and squeezed states, etc.

Since the annihilation/creation operators of the harmonic oscillator and their algebra are the cornerstone of modern quantum physics, their good deformation is bound to play an important role, as evidenced by the representation theory of the quantum groups in terms of the q-oscillators. Thus our new results are expected to enrich the subject by stimulating the interplay between (quantum) algebra and analysis through new coherent/squeezed states etc, which would find applications in quantum optics and quantum information theory. Here we discuss only Rogers' q-Hermite polynomial [3], or the so-called continuous q-Hermite polynomial [5, 6] for the parameter range $0 < q < 1$. Like the Hermite polynomial, the q-Hermite polynomial has no parameter other than q.

This Letter is organized as follows. The factorized Hamiltonian for the q-Hermite polynomial is presented and the q-oscillator commutation relation is shown to be a simple consequence of their structure. After brief exploration of the eigenfunctions, the exact Heisenberg operator solution [7] is presented. A second set of q-oscillator algebra is derived from the explicit forms of the annihilation/creation operators which are the positive/negative energy parts of the exact Heisenberg operator solution. These q-oscillators reduce to the ordinary harmonic oscillator in the $q \to 1$ limit. Relationship to various forms of q-oscillator algebras is explained. The Letter concludes with some historical comments and a summary.

2 Hamiltonian for the q-Hermite polynomial

The Hamiltonian of the 'discrete' quantum mechanics for one degree of freedom has the general structure [2, 1]

$$\mathcal{H} \stackrel{\text{def}}{=} \sqrt{V(x)}\, e^{\gamma p} \sqrt{V(x)^*} + \sqrt{V(x)^*}\, e^{-\gamma p} \sqrt{V(x)} - V(x) - V(x)^* \tag{1}$$

$$= \sqrt{V(x)}\, q^{D} \sqrt{V(x)^*} + \sqrt{V(x)^*}\, q^{-D} \sqrt{V(x)} - V(x) - V(x)^*, \tag{2}$$

in which $x \in \mathbb{R}$ is the coordinate and $p = -i\partial_x$ is the conjugate momentum. The constant γ in the present case is $\gamma \stackrel{\text{def}}{=} \log q$, $0 < q < 1$ and the potential function for the dynamics of the q-Hermite polynomial is given by

$$V(x) \stackrel{\text{def}}{=} \frac{1}{(1-z^2)(1-qz^2)}, \quad z \stackrel{\text{def}}{=} e^{ix}, \tag{3}$$

with $D = p = -i\partial_x = z\frac{d}{dz}$. It is a special case of the Askey-Wilson polynomial [1, 5]. The Hamiltonian is factorized as

$$\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}, \tag{4}$$

$$\mathcal{A}^\dagger \stackrel{\text{def}}{=} -i\big(\sqrt{V(x)}\, q^{D/2} - \sqrt{V(x)^*}\, q^{-D/2}\big), \tag{5}$$

$$\mathcal{A} \stackrel{\text{def}}{=} i\big(q^{D/2}\sqrt{V(x)^*} - q^{-D/2}\sqrt{V(x)}\,\big). \tag{6}$$

With the explicit form of the potential function V, (3), it is straightforward to derive the q-oscillator commutation relation

$$\mathcal{A}\mathcal{A}^\dagger - q^{-1}\mathcal{A}^\dagger\mathcal{A} = q^{-1} - 1. \tag{7}$$

Sometimes it is written as $[\mathcal{A}, \mathcal{A}^\dagger]_{q^{-1}} = q^{-1} - 1$ with the standard notation $[A, B]_c \stackrel{\text{def}}{=} AB - cBA$. We also have

$$[\mathcal{H}, \mathcal{A}]_q = (q-1)\mathcal{A}, \quad [\mathcal{H}, \mathcal{A}^\dagger]_{q^{-1}} = (q^{-1} - 1)\mathcal{A}^\dagger. \tag{8}$$

The q-oscillator commutation relation (7) is also a consequence of the *shape invariance* without shifting parameter [8] among the general Askey-Wilson potentials [1, 5]. One could also say that the commutation relation of the harmonic oscillator $aa^\dagger - a^\dagger a = 1$ is a manifestation of the shape-invariance.

The groundstate wavefunction ϕ_0 is annihilated by the operator $\mathcal{A}$:

$$\mathcal{A}\phi_0 = 0 \implies \phi_0(x) \stackrel{\text{def}}{=} \sqrt{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}, \tag{9}$$

in which the standard notation of q-Pochhammer symbol $(a\,; q)_n$ is used:

$$(a\,; q)_n \stackrel{\text{def}}{=} \prod_{k=1}^{n} (1 - aq^{k-1}) = (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{10}$$

including the limiting case $n \to \infty$. With this choice of the groundstate wavefunction, we can show that the Hamiltonian (1) is hermitian with respect to the inner product $(f, g) = \int_0^\pi f(x)^* g(x) dx$ in the Hilbert space $L^2[0, \pi]$ [9]. By using the factorization (4) and the q-oscillator relation (7), it is straightforward to demonstrate that $(\mathcal{A}^\dagger)^n \phi_0$ is an eigenstate of the Hamiltonian with the geometric sequence spectrum:

$$\mathcal{H}(\mathcal{A}^\dagger)^n \phi_0 = \mathcal{E}_n (\mathcal{A}^\dagger)^n \phi_0, \quad \mathcal{E}_n \stackrel{\text{def}}{=} q^{-n} - 1. \tag{11}$$

3 The q-Hermite polynomial

The analytical approach to the Schrödinger equation

$$\mathcal{H}\phi_n = \mathcal{E}_n\phi_n, \tag{12}$$

which is a *difference* equation instead of a second order differential equation, goes as follows. By similarity transformation in terms of the groundstate wavefunction ϕ_0, one introduces

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(x)(q^D - 1) + V(x)^*(q^{-D} - 1), \tag{13}$$

which acts on the polynomial part of the eigenfunction $P_n(\eta(x))$:

$$\phi_n(x) = \phi_0(x)P_n(\eta(x)). \tag{14}$$

It is elementary to show

$$\widetilde{\mathcal{H}}\,(z + 1/z)^n = (q^{-n} - 1)(z + 1/z)^n + \text{lower order terms in } z + 1/z, \tag{15}$$

since the residues at $z = \pm 1$, $z = \pm q^{\pm 1/2}$, and $z = \pm q^{\mp 1/2}$ all vanish. Thus one can find the eigenpolynomial in $\eta(x) = \cos x = (z + 1/z)/2$, which is called the continuous q-Hermite polynomial introduced by Rogers [3, 5]

$$\widetilde{\mathcal{H}}\,H_n(\cos x|q) = \mathcal{E}_n H_n(\cos x|q), \tag{16}$$

$$H_n(\cos x|q) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \frac{(q\,;q)_n}{(q\,;q)_k(q\,;q)_{n-k}}\, e^{i(n-2k)x},$$

$$H_0 = 1, \quad H_1(\cos x|q) = 2\cos x. \tag{17}$$

It has a definite parity. Reflecting the orthogonality of the eigenfunctions of the Hamiltonian $\mathcal{H}$, $(\phi_n, \phi_m) \propto \delta_{nm}$, it is orthogonal with respect to the weight function $\phi_0(x)^2$:

$$\int_0^\pi \phi_0(x)^2 H_n(\cos x|q) H_m(\cos x|q) dx = \delta_{n\,m} \frac{2\pi}{(q^{n+1}; q)_\infty}, \tag{18}$$

satisfying the three term recurrence relation

$$2\eta H_n(\eta|q) = H_{n+1}(\eta|q) + (1 - q^n)H_{n-1}(\eta|q). \tag{19}$$

The action of the creation $\mathcal{A}^\dagger$ and annihilation $\mathcal{A}$ operators on the polynomial $H_n(\cos x|q)$ is

$$\widetilde{\mathcal{A}}^\dagger \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{A}^\dagger \circ \phi_0, \quad \widetilde{\mathcal{A}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{A} \circ \phi_0, \tag{20}$$

$$\widetilde{\mathcal{A}}^\dagger = q^{-\frac{1}{2}} \frac{-1}{z - z^{-1}} \big(z^{-2} q^{D/2} - z^2 q^{-D/2}\big), \tag{21}$$

$$\widetilde{\mathcal{A}} = \frac{-1}{z - z^{-1}} \big(q^{D/2} - q^{-D/2}\big), \tag{22}$$

$$\widetilde{\mathcal{A}}^\dagger H_n(\cos x|q) = q^{-(n+1)/2} H_{n+1}(\cos x|q), \tag{23}$$

$$(\widetilde{\mathcal{A}}^\dagger)^n 1 = q^{-n(n+1)/4} H_n(\cos x|q), \tag{24}$$

$$\widetilde{\mathcal{A}} H_n(\cos x|q) = (q^{-n/2} - q^{n/2}) H_{n-1}(\cos x|q). \tag{25}$$

The similarity transformed $\widetilde{\mathcal{A}}$ (22) is proportional to the divided difference operator.

4 Heisenberg operator solution

The harmonic oscillator is a typical example for which the Heisenberg operator solution is known and the annihilation/creation operators can also be extracted as the positive/negative frequency parts of the Heisenberg operator solution. The situation is parallel but slightly different for the q-oscillator. The exact Heisenberg operator solution is derived and its positive/negative frequency parts give another set of annihilation/creation operators $a^{(\pm)}$ which are closely related to $\mathcal{A}$ and $\mathcal{A}^\dagger$. (For the general theory of exact Heisenberg operator solutions, see [7, 10] for systems of single degree of freedom and [11] for a class of multi-particle dynamics.)

We start from the *closure relation*

$$[\mathcal{H}, [\mathcal{H}, \cos x]\,] = \cos x\, R_0(\mathcal{H}) + [\mathcal{H}, \cos x]\, R_1(\mathcal{H}), \tag{26}$$

$$R_0(\mathcal{H}) \stackrel{\text{def}}{=} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (\mathcal{H} + 1)^2, \tag{27}$$

$$R_1(\mathcal{H}) \stackrel{\text{def}}{=} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (\mathcal{H} + 1), \tag{28}$$

which can be readily verified. This relation enables us to express any multiple commutator

$$[\mathcal{H}, [\mathcal{H}, \cdots, [\mathcal{H}, \cos x] \cdots]]$$

as a linear combination of the operators $\cos x$ and $[\mathcal{H}, \cos x]$ with coefficients depending on the Hamiltonian $\mathcal{H}$ only. Thus we arrive at the exact Heisenberg operator solution for the *sinusoidal coordinate* $\eta(x) \stackrel{\text{def}}{=} \cos x$ [7]:

$$e^{it\mathcal{H}} \cos x \, e^{-it\mathcal{H}} = \cos x \, \frac{q \, e^{i\alpha_+(\mathcal{H})t} + e^{i\alpha_-(\mathcal{H})t}}{1+q} + [\mathcal{H}, \cos x] \, \frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{(q^{-1}-q)(\mathcal{H}+1)}, \tag{29}$$

$$\alpha_\pm(\mathcal{H}) = (q^{\mp 1} - 1)(\mathcal{H}+1). \tag{30}$$

This simply means that the coordinate $\cos x$ undergoes sinusoidal motions with frequencies $\alpha_\pm(\mathcal{H})$.

While factorization of Hamiltonian is known to provide the annihilation/creation operators only for the harmonic oscillator and the q-oscillator, the authentic definition of the annihilation/creation operators is through the positive/negative frequency parts of the Heisenberg operator solution [7] ($\eta = \cos x$):

$$e^{it\mathcal{H}} \cos x \, e^{-it\mathcal{H}} = a^{(+)} \, e^{i\alpha_+(\mathcal{H})t} + a^{(-)} \, e^{i\alpha_-(\mathcal{H})t}, \tag{31}$$

$$a^{(\pm)} = \frac{\pm 1}{q^{-1}-q}\Big([\mathcal{H}, \eta]_{q^{\pm 1}} + (1-q^{\pm 1})\eta\Big)(\mathcal{H}+1)^{-1},$$

$$a^{(-)\dagger} = a^{(+)}. \tag{32}$$

Their action on the full eigenfunction is ($\phi_n(x) \stackrel{\text{def}}{=} \phi_0(x) H_n(\cos x|q)$):

$$a^{(-)}\phi_n = \tfrac{1}{2}(1-q^n)\phi_{n-1}, \quad a^{(+)}\phi_n = \tfrac{1}{2}\phi_{n+1}, \tag{33}$$

to be compared with

$$\mathcal{A}\phi_n = q^{-\frac{n}{2}}(1-q^n)\phi_{n-1}, \quad \mathcal{A}^\dagger\phi_n = q^{-\frac{n+1}{2}}\phi_{n+1}. \tag{34}$$

From these and (18), it is easy to check the hermiticity

$$(\phi_{n-1}, a^{(-)}\phi_n) = (a^{(+)}\phi_{n-1}, \phi_n), \tag{35}$$

$$(\phi_{n-1}, \mathcal{A}\phi_n) = (\mathcal{A}^\dagger\phi_{n-1}, \phi_n). \tag{36}$$

They satisfy commutation relations

$$[a^{(-)}, a^{(+)}] = \tfrac{1}{4}(1-q)(\mathcal{H}+1)^{-1}, \tag{37}$$

$$[\mathcal{H}, a^{(\pm)}] = (q^{\mp 1} - 1)a^{(\pm)}(\mathcal{H}+1). \tag{38}$$

By removing the Hamiltonian from the r.h.s. they can be cast into another q-oscillator form

$$a^{(-)}a^{(+)} - q a^{(+)}a^{(-)} = \tfrac{1}{4}(1-q), \tag{39}$$

$$\mathcal{H}a^{(\pm)} - q^{\mp 1}a^{(\pm)}\mathcal{H} = (q^{\mp 1} - 1)a^{(\pm)}. \tag{40}$$

It should be noted that the q-oscillator relations (39)-(40) also hold for the continuous big q-Hermite polynomial [5, 7]. We will report on this topic elsewhere.

The two types of creation-annihilation operators are closely related with each other [7]

$$a^{(+)} = \mathcal{A}^\dagger X, \quad a^{(-)} = X^\dagger \mathcal{A}, \tag{41}$$

with

$$X = -\frac{i}{2}q\big(z\sqrt{V(x)}\,q^{D/2} - z^{-1}\sqrt{V(x)^*}\,q^{-D/2}\big)(\mathcal{H}+1)^{-1}, \tag{42}$$

$$X^\dagger = \frac{i}{2}q(\mathcal{H}+1)^{-1}\big(q^{D/2}z^{-1}\sqrt{V(x)^*} - q^{-D/2}z\sqrt{V(x)}\,\big), \tag{43}$$

and the operators X and $X^\dagger$ map the eigenfunction ϕ_n to itself:

$$X\phi_n = \frac{1}{2}q^{(n+1)/2}\phi_n, \quad X^\dagger\phi_n = \frac{1}{2}q^{(n+1)/2}\phi_n. \tag{44}$$

The structure of these operators is better understood by the similarity transformation in terms of the groundstate wavefunction ϕ_0

$$\widetilde{X} \stackrel{\text{def}}{=} \phi_0^{-1}\circ X\circ\phi_0, \quad \widetilde{X}^\dagger \stackrel{\text{def}}{=} \phi_0^{-1}\circ X^\dagger\circ\phi_0. \tag{45}$$

In fact, their actions on polynomials $\{H_n(\cos x|q)\}$ are essentially identical:

$$\widetilde{X} = \frac{1}{2}q^{\frac{1}{2}}\frac{-1}{z-z^{-1}}(z^{-1}q^{D/2} - zq^{-D/2})(\widetilde{\mathcal{H}}+1)^{-1}, \tag{46}$$

$$\widetilde{X}^\dagger = \frac{1}{2}q^{\frac{1}{2}}(\widetilde{\mathcal{H}}+1)^{-1}\frac{-1}{z-z^{-1}}(z^{-1}q^{D/2} - zq^{-D/2}). \tag{47}$$

The main part of $\widetilde{X}$ and $\widetilde{X}^\dagger$, defined by

$$\mathcal{D}^q = \frac{-1}{z-z^{-1}}(z^{-1}q^{D/2} - z\,q^{-D/2}), \tag{48}$$

was also introduced by Atakishiyev-Klimyk [12] eq(9). It satisfies the relation

$$\mathcal{D}^q H_n(\cos x|q) = q^{-n/2}H_n(\cos x|q), \tag{49}$$

and it factorizes $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}+1$:

$$(\mathcal{D}^q - 1)(\mathcal{D}^q + 1) = \widetilde{\mathcal{H}}, \quad (\mathcal{D}^q)^2 = \widetilde{\mathcal{H}}+1. \tag{50}$$

The coherent state of the harmonic oscillator is defined as the eigenvector of the annihilation operator; $a\psi = \alpha\psi$, which is the generating function of the Hermite polynomials.

We encounter a parallel situation here. The eigenvector of the operator $a^{(-)}$, $a^{(-)}\psi(x;\alpha) = \alpha\psi(x;\alpha)$, is given by

$$\psi(x\,;\alpha) = \phi_0(x)\sum_{n=0}^{\infty}\frac{(2\alpha)^n}{(q\,;q)_n}H_n(\cos x|q) \tag{51}$$

$$= \phi_0(x)\frac{1}{(2\alpha\,e^{ix};q)_\infty(2\alpha\,e^{-ix};q)_\infty}. \tag{52}$$

The second factor is the generating function of the q-Hermite polynomials [5, 6]. The coherent state defined by the other annihilation operator $\mathcal{A}$, $\mathcal{A}\psi'(x;\alpha) = \alpha\psi'(x;\alpha)$, has a similar structure:

$$\psi'(x\,;\alpha) = \phi_0(x)\sum_{n=0}^{\infty}\frac{\alpha^n q^{\frac{1}{4}n(n+1)}}{(q\,;q)_n}H_n(\cos x|q). \tag{53}$$

5 Limit to the ordinary harmonic oscillator

The q-oscillators reduce to the ordinary harmonic oscillator in the $q \to 1$ limit. To show this, let us introduce two parameters (L and c) and a new coordinate x':

$$x = \frac{\pi}{2} - \frac{\pi}{L}x' \ \Big(\Rightarrow\ -\frac{L}{2} < x' < \frac{L}{2}\Big), \quad q = e^{-\frac{2\pi}{cL}}. \tag{54}$$

The momentum operator conjugate to x' is $p' = -i\frac{d}{dx'} = -\frac{\pi}{L}p$. Then the desired limit is obtained by setting $L = \pi c$ and taking $c \to \infty$ limit:

$$c^2\mathcal{H} \to x'^2 + p'^2 - 1, \quad c^2\mathcal{E}_n \to 2n, \tag{55}$$

$$\left.\begin{matrix} c\mathcal{A}^\dagger \\ c\mathcal{A} \end{matrix}\right\} \to x' \mp ip', \quad ca^{(\pm)} \to \frac{1}{2}(x' \mp ip'), \quad \left.\begin{matrix} X^\dagger \\ X \end{matrix}\right\} \to \frac{1}{2}, \tag{56}$$

$$c\cos x \to x' \ \ (-\infty < x' < \infty), \quad c^4R_0(\mathcal{H}) \to 4, \quad c^2R_1(\mathcal{H}) \to 0, \tag{57}$$

$$\frac{(q\,;q)_\infty\phi_0(x)^2}{2\sqrt{\pi}\,c} \to e^{-x'^2}, \tag{58}$$

$$c^nH_n(\cos x|q) = c^nH_n\big(\sin\frac{x'}{c}\ \big|\ e^{-\frac{2}{c^2}}\big) \to H_n(x'). \tag{59}$$

Here we have used the Jacobi's triple product identity [6] and its modular transformation property (the S-transformation) for deriving (58), and the three term recurrence relations for (59).

6 Other forms of q-oscillators

Here we will discuss the relationship between our intrinsic q-oscillator algebra (7)-(8) and those introduced purely algebraically for quantum group representations around 1989-90 [4].

First let us introduce the number operator $\mathcal{N}$ through the energy spectrum formula (11),

$$(\mathcal{H}+1)^{\mp 1} = q^{\pm\mathcal{N}}, \quad \mathcal{N}\phi_n = n\phi_n, \quad n \in \mathbb{Z}_+, \tag{60}$$

which counts the level from the groundstate. Several different forms of q-oscillator algebras are introduced, among which we list two typical ones:

$$bb^\dagger - q^{-1}b^\dagger b = q^{\mathcal{N}}, \tag{61}$$

$$bb^\dagger - q\, b^\dagger b = q^{-\mathcal{N}}. \tag{62}$$

If we define b and $b^\dagger$ by

$$b = \frac{\mathcal{A}\, q^{\mathcal{N}/4}}{(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{\frac{1}{2}}}, \quad b^\dagger = \frac{q^{\mathcal{N}/4}\mathcal{A}^\dagger}{(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{\frac{1}{2}}}, \tag{63}$$

it is straightforward to verify

$$bb^\dagger - q^{-\frac{1}{2}}b^\dagger b = q^{\mathcal{N}/2}, \tag{64}$$

which becomes (61) by identification $q \to q^2$. Likewise, the q-oscillator algebra of $a^{(\pm)}$ (39) is related to (62) by similar transformations.

7 Comments and summary

Some historical comments are in order. There were attempts to relate q-oscillator algebras to the difference equation of the q-Hermite polynomial. None of them is based on a Hamiltonian, thus hermiticity is not manifest and the logic for factorization is unclear. Here we list a few such attempts. Atakishiyev and Suslov in 1990 [13] wrote down an algebra

$$bb^+ - q^{-1}b^+ b = 1, \quad H = b^+ b, \tag{65}$$

which is related to our q-oscillator algebra (7) by a similarity transformation

$$\sqrt{q^{-1}-1}\begin{pmatrix} b \\ b^+ \end{pmatrix} = \frac{1}{\sqrt{\sin x}} \circ \begin{pmatrix} \mathcal{A} \\ \mathcal{A}^\dagger \end{pmatrix} \circ \sqrt{\sin x}. \tag{66}$$

Floreanini, LeTourneux and Vinet presented in 1994 [14] a q-oscillator algebra ($\widetilde{\mathcal{N}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{N} \circ \phi_0$)

$$A_- A_+ - q^{-1}A_+ A_- = 1, \quad KA_\pm = q^{\mp\frac{1}{2}}A_\pm K, \quad K = q^{-\widetilde{\mathcal{N}}/2}, \tag{67}$$

which is in our notation

$$A_+ = -\widetilde{\mathcal{A}}^\dagger, \quad A_- = \frac{-1}{q^{-1}-1}\,\widetilde{\mathcal{A}}, \quad K = \mathcal{D}^q. \tag{68}$$

In 2003 Borzov and Damaskinsky [15] wrote down

$$a_q^- a_q^+ - q\, a_q^+ a_q^- = 1, \tag{69}$$

starting from the three term recurrence relation of the q-Hermite polynomial and defining the annihilation/creation operators in their own way.

In summary: we have derived two q-oscillator algebras (7) and (39) from the Hamiltonian of the q-Hermite polynomial (4)–(6) [1, 7, 10], which is a special case of the Askey-Wilson polynomial [5, 6]. The generators are genuine annihilation/creation operators and the hermiticity is manifest.

Acknowledgments

This work is supported in part by Grants-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, No.18340061 and No.19540179.

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