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Dynamical Theory of Generalized Matrices

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We propose a generalization of spin algebra using multi-index objects and a dynamical system analogous to matrix theory. The system has a solution described by generalized spin representation matrices and possesses a symmetry similar to the volume preserving diffeomorphism in the p-brane action.

§1. Introduction

Group theoretical analysis has been applied successfully to a wide range of physical systems, because they are often invariant under certain transformations, and such symmetry transformations, in many cases, form a group. Matrices can represent the action of such group elements. Among physical quantities, the spin variables [on which representation matrices of $su(2)$ operate] have played important roles. Relativistic particles are classified with respect to two kinds of spin variables, because the Lorentz algebra is essentially specified by $su(2)\times su(2)$.[1] The spin variables and their extensions appear in the non-commutative geometry, which is considered to represent a possible description of space-time at a fundamental level.[2] For example, the fuzzy 2-sphere constitutes a non-commutative space whose coordinates are inherently representation matrices of the spin algebra.[3] This space is used in the matrix description of a spherical membrane.[4],[5] It also appears as a solution of matrix theory and the matrix model with a Chern-Simons-like term.[6],[7] Models related to higher-dimensional fuzzy spheres have been examined in various contexts.[8]–[12] Hence, it is a challenge to explore the generalization of spin algebra and representation matrices in order to unveil yet unknown systems.

Recently, a generalization of spin algebra based on three-index objects has been proposed, and the connection between triple commutation relations and uncertainty relations has been investigated.[13] This algebra can be generalized using an n-fold product as the multiplication operation and an n-fold commutator among n-index objects, as discussed below. Such n-index objects are called 'n-th power matrices', which are interpreted as generalizations of ordinary matrices, and a new type of mechanics has been proposed based on them.[14],[15] This type of mechanics can be regarded as a generalization of Heisenberg's matrix mechanics. It has interesting properties, but it is not yet clear whether it is applicable to real physical systems nor what physical meaning many-index objects possess. In an attempt to realize a breakthrough with regard to the physical application of generalized matrices, we shift our focus to other systems. With the expectation that studying the analogous

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systems represented by matrix theory and the matrix model will provide some information, it is interesting to explore symmetry properties in dynamical systems of generalized matrices, while keeping their classical counterparts in mind. One possible classical analog is the system constituted by p-brane.[17)]

In this paper, we propose a generalization of spin algebra using n-th power matrices, and a dynamical system analogous to matrix theory. This system has a solution described by generalized spin representation matrices and possesses a symmetry similar to the volume-preserving diffeomorphism in the p-brane action.

This paper is organized as follows. In the next section, we give a definition of generalized spin algebras, generalized spin representation matrices and a variant of a fuzzy sphere. We study a generalization of matrix theory based on generalized matrices in §3. Section 4 is devoted to conclusions and discussion. In Appendix A, we define n-th power matrices, an n-fold product, an n-fold commutator and two kinds of trace operations. As we see from the definition of the n-fold product, we do not use the Einstein summation rule that repeated indices are summed, to avoid confusion. In Appendix B, we study transformation properties of hermitian n-th power matrices. We explain the classical analog of generalized spin algebra in Appendix C, and the framework of classical p-branes in Appendix D.

§2. Generalized spin algebra

First, we review the spin algebra $su(2)$. This algebra is defined by

$$[J^a, J^b]_{mn} = i\hbar \sum_c \varepsilon^{abc} (J^c)_{mn}, \tag{2·1}$$

where J^a $(a = 1, 2, 3)$ are spin representation matrices, $\hbar$ is the reduced Planck constant, and ε^{abc} is the Levi-Civita symbol. Matrices in the adjoint representation are the 3×3 matrices given by

$$(J^a)_{mn} = -i\hbar\varepsilon^{amn}, \tag{2·2}$$

where each of the indices m and n runs from 1 to 3.

Let us generalize the spin algebra defined by (2·1) using hermitian n-th power matrices. (See Appendix A for the definition of hermitian n-th power matrices.) In analogy to (2·2), we define the $(n+1) \times (n+1) \times \cdots \times (n+1)$ matrices that we consider as follows:

$$(J^a)_{l_1 l_2 \cdots l_n} = -i\hbar_{(n)} \varepsilon^{a l_1 l_2 \cdots l_n}, \quad (K^a)_{l_1 l_2 \cdots l_n} = \hbar_{(n)} |\varepsilon^{a l_1 l_2 \cdots l_n}|, \tag{2·3}$$

where $\varepsilon^{a l_1 l_2 \cdots l_n}$ is the $(n+1)$-dimensional Levi-Civita symbol, each of the indices a and l_i $(i = 1, 2, \cdots, n)$ runs from 1 to $n+1$, and $\hbar_{(n)}$ is a new physical constant. Hereafter, $\hbar_{(n)}$ is set to 1 for simplicity. We find that the generalized matrices J^a and K^a form the algebra

$$[J^{a_1}, \cdots, J^{a_{n-2j}}, K^{a_{n-2j+1}}, \cdots, K^{a_n}] = (-1)^j i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_{n+1}} J^{a_{n+1}}, \tag{2·4}$$

$$[J^{a_1}, \cdots, J^{a_{n-2j-1}}, K^{a_{n-2j}}, \cdots, K^{a_n}] = (-1)^{j+1} i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_{n+1}} K^{a_{n+1}}, \tag{2·5}$$

for an even integer n and $j = 0, 1, \cdots, n/2$, and

$$[J^{a_1}, \cdots, J^{a_{n-2j}}, K^{a_{n-2j+1}}, \cdots, K^{a_n}] = (-1)^{j+1} i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_{n+1}} K^{a_{n+1}}, \quad (2\cdot6)$$

$$[J^{a_1}, \cdots, J^{a_{n-2j-1}}, K^{a_{n-2j}}, \cdots, K^{a_n}] = (-1)^{j+1} i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_{n+1}} J^{a_{n+1}}, \quad (2\cdot7)$$

for an odd integer n and $j = 0, 1, \cdots, (n-1)/2$. Here, the indices l_i are omitted, and the n-fold commutator is defined by (A·4).

There exists the following subalgebra of the algebra defined by (2·4) and (2·5) whose elements are $G^a = (J^1, \cdots, J^{n+1})$ for an even integer n:

$$[G^{a_1}, G^{a_2}, \cdots, G^{a_n}]_{l_1 l_2 \cdots l_n} = i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_n a_{n+1}} (G^{a_{n+1}})_{l_1 l_2 \cdots l_n}. \quad (2\cdot8)$$

Similarly, there exists a subalgebra of the algebra defined by (2·6) and (2·7) whose elements consist of a suitable set of J^{a_p} and K^{a_q}. For example, the elements $G^a = (J^1, \cdots, J^n, K^{n+1})$ for an odd integer n form the algebra given by

$$[G^{a_1}, G^{a_2}, \cdots, G^{a_n}]_{l_1 l_2 \cdots l_n} = -i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_n a_{n+1}} (G^{a_{n+1}})_{l_1 l_2 \cdots l_n}. \quad (2\cdot9)$$

We refer to the algebra defined by (2·8) and (2·9) as a 'generalized spin algebra' and collectively write

$$[G^{a_1}, G^{a_2}, \cdots, G^{a_n}]_{l_1 l_2 \cdots l_n} = (-1)^n i \sum_{a_{n+1}} \varepsilon^{a_1 a_2 \cdots a_n a_{n+1}} (G^{a_{n+1}})_{l_1 l_2 \cdots l_n}. \quad (2\cdot10)$$

We refer to the elements of the generalized spin algebra as 'generalized spin representation matrices'. We explain the classical analog of generalized spin algebra using a generalization of Hamiltonian dynamics in Appendix C. Filippov also proposed a generalization of Lie algebra using vectors in the n-dimensional Euclidean space as elements and the vector product as the multiplication operation.[18),*)] In that realization, the basis vectors form an analog of the generalized spin algebra (2·10). Xiong obtained an algebra that is essentially equivalent to the algebra (2·8), using the n-th power matrices $(T_a)_{i_1 i_2 \cdots i_{2m}} \equiv \varepsilon_{a i_1 i_2 \cdots i_{2m}}$, $(a = 1, \cdots, N = 2m+1)$.[20)] We have generalized the construction to the case with an arbitrary integer N.

The elements G^a satisfy the so-called 'fundamental indentity',

$$[[G^{a_1}, \cdots, G^{a_n}], G^{a_{n+1}}, \cdots, G^{a_{2n-1}}]_{l_1 l_2 \cdots l_n}$$
$$= \sum_{i=1}^{n} [G^{a_1}, \cdots, [G^{a_i}, G^{a_{n+1}}, \cdots, G^{a_{2n-1}}], \cdots, G^{a_n}]_{l_1 l_2 \cdots l_n}. \quad (2\cdot11)$$

This identity is regarded as an extension of the Jacobi identity.

*) See also Ref. 19) for a generalization of Lie algebra.

For later convenience, here we present several formulae for the generalized spin representation matrices G^a. By using (2·10) and the relation $\sum_{a_1,\cdots,a_n} \varepsilon^{aa_1\cdots a_n}\varepsilon^{ba_1\cdots a_n} = n!\delta_{ab}$, we obtain the formula

$$(G^a)_{l_1 l_2\cdots l_n} = \frac{-i}{n!}\sum_{a_1,a_2,\cdots,a_n} \varepsilon^{aa_1a_2\cdots a_n}[G^{a_1}, G^{a_2}, \cdots, G^{a_n}]_{l_1l_2\cdots l_n}$$
$$= -i\sum_{a_1,a_2,\cdots,a_n} \varepsilon^{aa_1a_2\cdots a_n}(G^{a_1}G^{a_2}\cdots G^{a_n})_{l_1l_2\cdots l_n}. \quad (2·12)$$

From (2·12), we derive the formulae

$$\sum_a \mathrm{Tr}_{(2)}(G^a)^2 \equiv \sum_a \sum_{l_1,\cdots,l_{n-1},l_n} (G^a)_{l_1\cdots l_{n-1}l_n}(G^a)_{l_1\cdots l_n l_{n-1}}$$
$$= -\frac{1}{n!}\sum_{a_1,\cdots,a_n}\sum_{l_1,\cdots,l_{n-1},l_n} [G^{a_1},\cdots,G^{a_n}]_{l_1\cdots l_{n-1}l_n}[G^{a_1},\cdots,G^{a_n}]_{l_1\cdots l_n l_{n-1}}$$
$$= -\frac{i}{n!}\sum_{a,a_1,\cdots,a_n}\sum_{l_1,\cdots,l_{n-1},l_n} \varepsilon^{aa_1\cdots a_n}(G^a)_{l_1\cdots l_{n-1}l_n}[G^{a_1},\cdots,G^{a_n}]_{l_1\cdots l_n l_{n-1}}$$
$$= -i\sum_{a,a_1,\cdots,a_n}\sum_{l_1,\cdots,l_{n-1},l_n} \varepsilon^{aa_1\cdots a_n}(G^a)_{l_1\cdots l_{n-1}l_n}(G^{a_1}\cdots G^{a_n})_{l_1\cdots l_n l_{n-1}}, \quad (2·13)$$

where $\mathrm{Tr}_{(2)}$ is the second kind of trace operator defined by (A·17).

The coordinates X^i of a fuzzy 2-sphere are defined by the matrices J^i $(i = 1, 2, 3)$ in the spin j representation as[3]

$$(X^i)_{mn} = \frac{R}{\sqrt{j(j+1)}}(J^i)_{mn}, \quad (2·14)$$

where R is regarded as the radius of the fuzzy 2-sphere. The coordinates X^i satisfy the relations

$$[X^i, X^j]_{mn} = i\frac{R}{\sqrt{j(j+1)}}\sum_k \varepsilon^{ijk}(X^k)_{mn}, \qquad \sum_i (X^i)^2_{mn} = R^2\delta_{mn}, \quad (2·15)$$

where each of the indices m and n runs from 1 to $2j+1$. Similarly, the coordinates X^i $(i = 1, 2, \cdots, 2k+1)$ of a fuzzy $2k$-sphere are the (tensor products of) matrices which satisfy the relations[9]

$$[X^{i_1}, X^{i_2}, \cdots, X^{i_{2k}}]_{mn} = i\zeta \sum_{i_{2k+1}} \varepsilon^{i_1i_2\cdots i_{2k}i_{2k+1}}(X^{i_{2k+1}})_{mn}, \quad (2·16)$$

$$\sum_i (X^i)^2_{mn} = R^2\delta_{mn}, \quad (2·17)$$

where ζ is a constant parameter. These fuzzy spheres are typical examples of a non-commutative space.

Now we propose a variant of a fuzzy sphere based on hermitian n-th power matrices X^i $(i = 1, 2 \cdots, n+1)$. The variables X^i are interpreted as the coordinates that satisfy the relations

$$[X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} = i\eta \sum_{i_{n+1}} \varepsilon^{i_1 i_2 \cdots i_n i_{n+1}} (X^{i_{n+1}})_{l_1 l_2 \cdots l_n}, \tag{2·18}$$

$$\sum_{i} \sum_{l_1, \cdots, l_{n-2}, k} (X^i)_{l_1 \cdots l_{n-2} l_{n-1} k} (X^i)_{l_1 \cdots l_{n-2} k l_n} = R^2 \delta_{l_{n-1} l_n}, \tag{2·19}$$

where η is a constant parameter. The variables describing this new kind of n-dimensional space are, in general, non-commutative and non-associative for the n-fold product (A·2). The above relations (2·18) and (2·19) are invariant under the rotation

$$(X^i)_{l_1 l_2 \cdots l_n} \to (X^i)'_{l_1 l_2 \cdots l_n} = \sum_j O^i_j (X^j)_{l_1 l_2 \cdots l_n}, \tag{2·20}$$

where O^i_j are elements of $(n+1)$-dimensional orthogonal group. We assume that an infinitesimal rotation is generated by the transformation

$$\delta (X^i)_{l_1 l_2 \cdots l_n} = \sum_j \theta^{ij} (X^j)_{l_1 l_2 \cdots l_n} = i[\Theta_1, \cdots, \Theta_{n-1}, X^i]_{l_1 l_2 \cdots l_n}, \tag{2·21}$$

where $\theta^{ij} (= -\theta^{ji})$ are infinitesimal parameters and Θ_k $(k = 1, \cdots, n-1)$ are the "generators" of rotations. If the generators Θ_k are given by $\Theta_k = \sum_i \theta^{(k)}_i X^i$, then θ^{ij} is written

$$\theta^{ij} = -\eta \sum_{i_1, \cdots, i_{n-1}} \varepsilon^{i_1 \cdots i_{n-1} ij} \theta^{(1)}_{i_1} \cdots \theta^{(n-1)}_{i_{n-1}}, \tag{2·22}$$

where $\theta^{(k)}_i$ are infinitesimal parameters.

§3. Dynamical system of generalized matrices

In this section, we study a generalization of matrix theory using hermitian n-th power matrices. We write down the Lagrangian, the Hamiltonian, the equation of motion, and a solution in terms of generalized spin representation matrices, and study their symmetry properties.

Let us study the system described by the following Lagrangian:

$$\begin{aligned} L = &\frac{1}{2} \sum_i \sum_{l_1, l_2, \cdots, l_n} (D_0 X^i)_{l_1 l_2 \cdots l_n} (D_0 X^i)_{l_2 l_1 \cdots l_n} \\ &+ \frac{\alpha}{n \cdot n!} \sum_{i_1, i_2, \cdots, i_n} \sum_{l_1, l_2, \cdots, l_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_2 l_1 \cdots l_n} \\ &- \beta \sum_i \sum_{l_1, l_2, \cdots, l_n} (X^i)_{l_1 l_2 \cdots l_n} (X^i)_{l_2 l_1 \cdots l_n} \\ &- \gamma \frac{2i}{n+1} \sum_{i, i_1, i_2, \cdots, i_n} \sum_{l_1, l_2, \cdots, l_n} f^{i i_1 i_2 \cdots i_n} (X^i)_{l_1 l_2 \cdots l_n} (X^{i_1} X^{i_2} \cdots X^{i_n})_{l_2 l_1 \cdots l_n}, \end{aligned} \tag{3·1}$$

where $X^i = X^i(t)$ $(i = 1, 2, \cdots, N)$ are time-dependent hermitian n-th power matrices, α, β and γ are real parameters, and $f^{ii_1i_2\cdots i_n}$ are real antisymmetric parameters. The covariant time derivative D_0 is defined by

$$\begin{aligned}(D_0X^i)_{l_1l_2\cdots l_n} &\equiv \frac{d}{dt}(X^i(t))_{l_1l_2\cdots l_n} + i[A_1, \cdots, A_{n-1}, X^i(t)]_{l_1l_2\cdots l_n} \\ &= \frac{d}{dt}(X^i(t))_{l_1l_2\cdots l_n} + i \sum_{m_1,m_2,\cdots,m_n} \mathcal{A}(t)^{m_1m_2\cdots m_n}_{l_1l_2\cdots l_n}(X^i(t))_{m_1m_2\cdots m_n}, \end{aligned} \tag{3·2}$$

where A_k $(k = 1, \cdots, n-1)$ are hermitian n-th power matrices and $\mathcal{A}(t)$ is the "gauge field" of time. The first and second lines in (3·2) are similar to (D·13) and (D·8), respectively. Let us require that the Leibniz rule with regard to the covariant time derivative hold for the n-fold commutator $[B_1(t), \cdots, B_n(t)]$ as follows:

$$(D_0[B_1(t), \cdots, B_n(t)])_{l_1l_2\cdots l_n} = \sum_{l=1}^{n}[B_1(t), \cdots, D_0B_l(t), \cdots, B_n(t)]_{l_1l_2\cdots l_n}. \tag{3·3}$$

This requirement is satisfied for an arbitrary matrix $(A_1)_{l_1l_2}$ with respect to the usual commutator $[B_1(t), B_2(t)]$, but it is not necessarily satisfied for arbitrary n-th power matrices $(A_k)_{l_1l_2\cdots l_n}$ with respect to the n-fold commutator for $n \geq 3$. We find that the Leibniz rule (3·3) holds for an arbitrary $B_l(t)$ $(l = 1, 2, \cdots, n)$ if the matrices A_k are normal n-th power matrices and the antisymmetric object defined by $A(t)_{l_1l_2\cdots l_n} \equiv (-1)^{n-1}(\widetilde{A_1 \cdots A_{n-1}})_{l_1l_2\cdots l_n}$ satisfies the cocycle condition

$$(\delta A(t))_{m_0m_1\cdots m_n} \equiv \sum_{i=0}^{n}(-1)^i A(t)_{m_0m_1\cdots \hat{m}_i\cdots m_n} = 0, \tag{3·4}$$

where the index $\hat{m}_i$ is omitted. [Also, see (A·6) for the definition of $(\widetilde{A_1 \cdots A_{n-1}})$.] Then, the covariant time derivative (3·2) is written

$$(D_0X^i)_{l_1l_2\cdots l_n} \equiv \frac{d}{dt}(X^i(t))_{l_1l_2\cdots l_n} + iA(t)_{l_1l_2\cdots l_n}(X^i(t))_{l_1l_2\cdots l_n}. \tag{3·5}$$

In the case $n = 2$, $\alpha = 1$ and $\beta = \gamma = 0$, the Lagrangian (3·1) is reduced to the bosonic part of BFSS matrix theory by setting $R = gl_s = 1$.[21)] Here, R is the compactification radius, g is the string coupling constant, and l_s is the string length scale. The term with γ is regarded as a generalization of the Myers term. It is known that the Myers term appears in the case of a background antisymmetric field.[6)] The BFSS matrix theory describes a system of D0-branes, and it has been conjectured that it provides a microscopic description of M-theory in the light-front coordinates.[*)] The Lagrangian of the matrix theory is derived through the dimensional reduction of a (9+1)-dimensional super Yang-Mills Lagrangian to a (0+1)-dimensional Lagrangian. The matrix theory is also interpreted as a regularization of supermembrane theory.[4)]

There are several proposals for a "discretization" or quantization of a p-brane system.[19),23),24)] Our realization employing n-th power matrices is one of these,

*) See Ref. 22) for a comprehensive review of matrix theory.

because the first and second terms in (3·1) can be regarded as counterparts to (D·10). In our system with $n \geq 3$, it is not clear whether there exist such interesting physical implications as in the BFSS matrix theory.

The Hamiltonian is given by

$$\begin{aligned} H = &\frac{1}{2}\sum_i \sum_{l_1,l_2,\cdots,l_n} (\Pi^i)_{l_1 l_2 \cdots l_n} (\Pi^i)_{l_2 l_1 \cdots l_n} \\ &- \frac{\alpha}{n \cdot n!} \sum_{i_1,i_2,\cdots,i_n} \sum_{l_1,l_2,\cdots,l_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_2 l_1 \cdots l_n} \\ &+ \beta \sum_i \sum_{l_1,l_2,\cdots,l_n} (X^i)_{l_1 l_2 \cdots l_n} (X^i)_{l_2 l_1 \cdots l_n} \\ &+ \gamma \frac{2i}{n+1} \sum_{i,i_1,i_2,\cdots,i_n} \sum_{l_1,l_2,\cdots,l_n} f^{i i_1 i_2 \cdots i_n} (X^i)_{l_1 l_2 \cdots l_n} (X^{i_1} X^{i_2} \cdots X^{i_n})_{l_2 l_1 \cdots l_n}, \end{aligned} \tag{3·6}$$

where Π^i is the canonical momentum conjugate to X^i.

The following equation of motion is derived from the Lagrangian (3·1):

$$\begin{aligned} &(D_0^2 X^i)_{l_1 l_2 \cdots l_n} + \frac{2\alpha}{n!} \sum_{i_1,\cdots,i_{n-1}} [X^{i_1}, \cdots, X^{i_{n-1}}, [X^{i_1}, \cdots, X^{i_{n-1}}, X^i]]_{l_1 l_2 \cdots l_n} \\ &+ 2\beta (X^i)_{l_1 l_2 \cdots l_n} + 2i\gamma \sum_{i_1,i_2,\cdots,i_n} f^{i i_1 i_2 \cdots i_n} (X^{i_1} X^{i_2} \cdots X^{i_n})_{l_1 l_2 \cdots l_n} = 0. \end{aligned} \tag{3·7}$$

Now we consider the case that $f^{a_1 a_2 \cdots a_{n+1}} = \varepsilon^{a_1 a_2 \cdots a_{n+1}}$ (where $a_k, k = 1, 2, \cdots, n+1$), and other components of $f^{i i_1 i_2 \cdots i_n}$ vanish. In this case, we find the non-trivial solution

$$(X^a)_{l_1 l_2 \cdots l_n} = \xi (G^a)_{l_1 l_2 \cdots l_n}, \quad (X^q)_{l_1 l_2 \cdots l_n} = 0, \quad \mathcal{A}(t)^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} = 0, \tag{3·8}$$

where G^a ($a = 1, 2, \cdots, n+1$) are generalized spin representation matrices and $q = n+2, \cdots, N$. The parameter ξ depends on α, β and γ as

$$\xi = \left(\frac{\gamma \pm \sqrt{\gamma^2 - 4\alpha\beta}}{2\alpha} \right)^{\frac{1}{n-1}}. \tag{3·9}$$

This solution is interpreted as the counterpart of the n-brane solution in the BFSS matrix theory. For simplicity, we consider the case with $\beta = 0$ and $\alpha = \gamma = 1$. Then we have the solution with $\xi = 1$,

$$\begin{aligned} &(X^a)_{l_1 l_2 \cdots l_n} = (x^a + v^a t)\delta_{l_1 l_2 \cdots l_n} + (G^a)_{l_1 l_2 \cdots l_n}, \\ &(X^q)_{l_1 l_2 \cdots l_n} = (x^q + v^q t)\delta_{l_1 l_2 \cdots l_n}, \quad \mathcal{A}(t)^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} = 0, \end{aligned} \tag{3·10}$$

where x^a, v^a, x^q and v^q are constants and $\delta_{l_1 l_2 \cdots l_n} = \delta_{l_1 l_2} \cdots \delta_{l_{n-1} l_n}$. The Hamiltonian takes a negative value for this solution:

$$H = \frac{1-n}{n(n+1)} \sum_a \sum_{l_1,l_2,\cdots,l_n} (G^a)_{l_1 l_2 \cdots l_n} (G^a)_{l_2 l_1 \cdots l_n}. \tag{3·11}$$

Hence the energy eigenvalue of this vacuum is lower than that of the trivial solution, i.e., $X^i = \mathcal{A}(t) = 0$.

Next, we study the symmetry properties of the above system. (See Appendix B for discussion of the transformation properties of hermitian n-th power matrices.) The system for $n = 2$ is invariant under the time dependent unitary transformation

$$(X^i(t))_{l_1 l_2} \to (X^i(t))'_{l_1 l_2} = \sum_{m_1, m_2} U(t)_{l_1 m_1} (X^i(t))_{m_1 m_2} U(t)^\dagger_{m_2 l_2}, \qquad (3\cdot12)$$

$$\begin{aligned}(A_1(t))_{l_1 l_2} \to (A'_1(t))_{l_1 l_2} &= \sum_{m_1, m_2} U(t)_{l_1 m_1} (A_1(t))_{m_1 m_2} U(t)^\dagger_{m_2 l_2} \\ &\quad + i \sum_m \frac{d}{dt} U(t)_{l_1 m} \cdot U(t)^\dagger_{m l_2},\end{aligned} \qquad (3\cdot13)$$

where $U(t)_{lm}$ is an arbitrary unitary matrix. Infinitesimal transformations are given by

$$\delta(X^i(t))_{l_1 l_2} = i[\Lambda(t), X^i(t)]_{l_1 l_2}, \qquad (3\cdot14)$$

$$\delta(A_1(t))_{l_1 l_2} = -\frac{d}{dt}\Lambda(t)_{l_1 l_2} + i[\Lambda(t), A_1(t)]_{l_1 l_2}, \qquad (3\cdot15)$$

where $\Lambda(t)$ is the hermitian matrix related to $U(t)$ as $U(t) = \exp(i\Lambda(t))$. The transformations (3·12) and (3·14) can be rewritten as

$$(X^i(t))_{l_1 l_2} \to (X^i(t))'_{l_1 l_2} = \sum_{m_1, m_2} R(t)^{m_1 m_2}_{l_1 l_2} (X^i(t))_{m_1 m_2}, \qquad (3\cdot16)$$

$$\delta(X^i(t))_{l_1 l_2} = i \sum_{m_1, m_2} \lambda(t)^{m_1 m_2}_{l_1 l_2} (X^i(t))_{m_1 m_2}, \qquad (3\cdot17)$$

respectively. Here $R(t)$ and $\lambda(t)$ are the "transformation matrices" given by $R(t)^{m_1 m_2}_{l_1 l_2} = U(t)_{l_1 m_1} U(t)^*_{l_2 m_2}$ and $\lambda(t)^{m_1 m_2}_{l_1 l_2} = \Lambda(t)_{l_1 m_1}\delta_{l_2 m_2} - \Lambda(t)_{m_2 l_2}\delta_{l_1 m_1}$. They are related as $R(t) = \exp(i\lambda(t))$. (Note that here we use different notation for the transformation matrix, $(\lambda(t))$, from that in Appendix B, $(r^{(\Lambda)})$.) In terms of $R(t)$ and $\lambda(t)$, the finite and infinitesimal transformations of $\mathcal{A}(t)$ are given by

$$\begin{aligned}\mathcal{A}(t)^{m_1 m_2}_{l_1 l_2} \to \mathcal{A}'(t)^{m_1 m_2}_{l_1 l_2} &= \sum_{n_1, n_2} \sum_{k_1, k_2} R(t)^{n_1 n_2}_{l_1 l_2} \mathcal{A}(t)^{k_1 k_2}_{n_1 n_2} R(t)^{-1}{}^{m_1 m_2}_{k_1 k_2} \\ &\quad + i \sum_{n_1, n_2} \frac{d}{dt} R(t)^{n_1 n_2}_{l_1 l_2} \cdot R(t)^{-1}{}^{m_1 m_2}_{n_1 n_2},\end{aligned} \qquad (3\cdot18)$$

$$\begin{aligned}\delta\mathcal{A}(t)^{m_1 m_2}_{l_1 l_2} &= -\frac{d}{dt}\lambda(t)^{m_1 m_2}_{l_1 l_2} - i \sum_{n_1, n_2} \mathcal{A}(t)^{n_1 n_2}_{l_1 l_2} \lambda(t)^{m_1 m_2}_{n_1 n_2} \\ &\quad + i \sum_{n_1, n_2} \lambda(t)^{n_1 n_2}_{l_1 l_2} \mathcal{A}(t)^{m_1 m_2}_{n_1 n_2},\end{aligned} \qquad (3\cdot19)$$

respectively. Here we have $\mathcal{A}(t)^{m_1 m_2}_{l_1 l_2} = A_1(t)_{l_1 m_1}\delta_{l_2 m_2} - A_1(t)_{m_2 l_2}\delta_{l_1 m_1}$, and $R(t)^{-1}$ is the inverse of the transformation matrix $R(t)$. These matrices satisfy the relations

$$\sum_{n_1, n_2} R(t)^{n_1 n_2}_{l_1 l_2} R(t)^{-1}{}^{m_1 m_2}_{n_1 n_2} = \sum_{n_1, n_2} R(t)^{-1}{}^{n_1 n_2}_{l_1 l_2} R(t)^{m_1 m_2}_{n_1 n_2} = \delta_{l_1 m_1}\delta_{l_2 m_2}. \qquad (3\cdot20)$$

Now let us study the extension of the unitary transformation of $X^i(t)$ and $\mathcal{A}(t)$ to the case with $n \geq 3$. First we consider the infinitesimal transformations of $X^i(t)$ generated by the set of "generators" Λ_k $(k = 1, 2, \cdots, n-1)$ through the n-fold commutator defined as follows:

$$\begin{aligned}\delta(X^i(t))_{l_1 l_2 \cdots l_n} &= i[\Lambda_1, \cdots, \Lambda_{n-1}, X^i(t)]_{l_1 l_2 \cdots l_n} \\ &= i \sum_{m_1, m_2, \cdots, m_n} \lambda(t)_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} (X^i(t))_{m_1 m_2 \cdots m_n}. \end{aligned} \tag{3·21}$$

The expression (3·21) is similar to (D·18) and (D·14). Under the transformation (3·21), the covariant time derivative $D_0 X^i$ transforms covariantly,

$$\delta(D_0 X^i)_{l_1 l_2 \cdots l_n} = i \sum_{m_1, m_2, \cdots, m_n} \lambda(t)_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} (D_0 X^i)_{m_1 m_2 \cdots m_n}, \tag{3·22}$$

if $\mathcal{A}(t)$ transforms simultaneously as

$$\begin{aligned}\delta \mathcal{A}(t)_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} &= -\frac{d}{dt} \lambda(t)_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} - i \sum_{k_1, k_2, \cdots, k_n} \mathcal{A}(t)_{l_1 l_2 \cdots l_n}^{k_1 k_2 \cdots k_n} \lambda(t)_{k_1 k_2 \cdots k_n}^{m_1 m_2 \cdots m_n} \\ &\quad + i \sum_{k_1, k_2, \cdots, k_n} \lambda(t)_{l_1 l_2 \cdots l_n}^{k_1 k_2 \cdots k_n} \mathcal{A}(t)_{k_1 k_2 \cdots k_n}^{m_1 m_2 \cdots m_n}. \end{aligned} \tag{3·23}$$

The expression (3·23) is similar to (D·15). We can show that the first and third terms in (3·1) are invariant under the above infinitesimal transformations (3·21) and (3·23). However, the second and fourth terms in (3·1) are not necessarily invariant, because the n-fold commutator $[X^{i_1}, X^{i_2}, \cdots, X^{i_n}]$ transforms as

$$\begin{aligned}\delta[X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} &= \sum_{k=1}^{n} [X^{i_1}, \cdots, \delta(X^{i_k}), \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} \\ &= i \sum_{p} \sum_{(j_1, \cdots, j_n)} \sum_{k} \mathrm{sgn}(P) (X^{j_1})_{l_1 \cdots l_{n-1} k} \\ \cdots \sum_{m_1, m_2, \cdots, m_n} &\lambda(t)_{l_1 \cdots l_{n-p} k l_{n+2-p} \cdots l_n}^{m_1 m_2 \cdots m_n} (X^{j_p})_{m_1 m_2 \cdots m_n} \cdots (X^{j_n})_{k l_2 \cdots l_n} \end{aligned} \tag{3·24}$$

under the transformation (3·21), and the transformation (3·24) is not always covariant form. If $[X^{i_1}, X^{i_2}, \cdots, X^{i_n}]$ transforms covariantly, i.e.,

$$\begin{aligned}&\delta[X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} \\ &\quad = i \sum_{m_1, m_2, \cdots, m_n} \lambda(t)_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{m_1 m_2 \cdots m_n}, \end{aligned} \tag{3·25}$$

our entire system possesses local symmetry.

We now discuss the case in which the covariant time derivative is given by (3·5). In this case, $(D_0 X^i)$ is invariant under the transformation of $X^i(t)$ and $A(t)$ given by

$$\delta(X^i(t))_{l_1 l_2 \cdots l_n} = i[\Lambda_1, \cdots, \Lambda_{n-1}, X^i(t)]_{l_1 l_2 \cdots l_n}$$

$$= i\Lambda(t)_{l_1 l_2 \cdots l_n} (X^i(t))_{l_1 l_2 \cdots l_n}, \tag{3·26}$$

$$\delta A(t)_{l_1 l_2 \cdots l_n} = -\frac{d}{dt}\Lambda(t)_{l_1 l_2 \cdots l_n}, \tag{3·27}$$

where Λ_k $(k = 1, \cdots, n-1)$ are real normal n-th power matrices, and $\Lambda(t)$ is a real antisymmetric object defined by

$$\Lambda(t)_{l_1 l_2 \cdots l_n} \equiv (-1)^{n-1} (\Lambda_1 \widetilde{\cdots \Lambda}_{n-1})_{l_1 l_2 \cdots l_n}. \tag{3·28}$$

Here, we require that the function $\Lambda(t)$ has the property

$$(\delta\Lambda(t))_{m_0 m_1 \cdots m_n} \equiv \sum_{i=0}^{n} (-1)^i \Lambda(t)_{m_0 m_1 \cdots \hat{m}_i \cdots m_n} = 0. \tag{3·29}$$

When the antisymmetric objects $A(t)$ and $\Lambda(t)$ are treated as n-th power matrices, the transformation (3·27) is rewritten

$$\begin{aligned}\delta A(t)_{l_1 l_2 \cdots l_n} = -\frac{d}{dt}\Lambda(t)_{l_1 l_2 \cdots l_n} &- i[A_1, \cdots, A_{n-1}, \Lambda(t)]_{l_1 l_2 \cdots l_n} \\ &+ i[\Lambda_1, \cdots, \Lambda_{n-1}, A(t)]_{l_1 l_2 \cdots l_n}.\end{aligned} \tag{3·30}$$

Note that the last two terms in (3·30) are canceled out. The expression (3·30) is similar to (D·19). The finite versions of the transformations (3·26) and (3·27) are given by

$$(X^i(t))_{l_1 l_2 \cdots l_n} \to (X^i(t))'_{l_1 l_2 \cdots l_n} = e^{i\Lambda(t)_{l_1 l_2 \cdots l_n}} (X^i(t))_{l_1 l_2 \cdots l_n}, \tag{3·31}$$

$$A(t)_{l_1 l_2 \cdots l_n} \to A(t)'_{l_1 l_2 \cdots l_n} = A(t)_{l_1 l_2 \cdots l_n} - \frac{d}{dt}\Lambda(t)_{l_1 l_2 \cdots l_n}, \tag{3·32}$$

respectively. It is easy to see that the Lagrangian (3·1) is invariant under the transformations (3·26) and (3·27) or (3·31) and (3·32). If $A(t)$ is a coboundary, i.e., $A(t) = \delta\Omega(t)$, there is the extra symmetry according to which $A(t)$ is invariant under the transformation

$$\begin{aligned}\Omega(t)_{m_1 m_2 \cdots m_{n-1}} &\to \Omega'(t)_{m_1 m_2 \cdots m_{n-1}} \\ &= \Omega(t)_{m_1 m_2 \cdots m_{n-1}} + (\delta\Theta(t))_{m_1 m_2 \cdots m_{n-1}},\end{aligned} \tag{3·33}$$

where $\Theta(t)$ is an $(n-2)$-th rank antisymmetric object.

Next we discuss a generalization of the unitary transformations (3·16) and (3·18), which are given by

$$\begin{aligned}(X^i(t))_{l_1 l_2 \cdots l_n} &\to (X^i(t))'_{l_1 l_2 \cdots l_n} \\ &= \sum_{m_1, m_2, \cdots, m_n} R(t)^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} (X^i(t))_{m_1 m_2 \cdots m_n},\end{aligned} \tag{3·34}$$

$$\begin{aligned}\mathcal{A}(t)^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} &\to \mathcal{A}'(t)^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} \\ &= \sum_{n_1, n_2, \cdots, n_n} \sum_{k_1, k_2, \cdots, k_n} R(t)^{n_1 n_2 \cdots n_n}_{l_1 l_2 \cdots l_n} \mathcal{A}(t)^{k_1 k_2 \cdots k_n}_{n_1 n_2 \cdots n_n} R(t)^{-1}{}^{m_1 m_2 \cdots m_n}_{k_1 k_2 \cdots k_n} \\ &\quad + i \sum_{n_1, n_2, \cdots, n_n} \frac{d}{dt} R(t)^{n_1 n_2 \cdots n_n}_{l_1 l_2 \cdots l_n} \cdot R(t)^{-1}{}^{m_1 m_2 \cdots m_n}_{n_1 n_2 \cdots n_n},\end{aligned} \tag{3·35}$$

where $R(t)$ is a "transformation matrix" and $R(t)^{-1}$ is its inverse. In the case that $R(t)$ can be factorized into a product of matrices as

$$R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} = V^{m_1}_{l_1} V^{m_2}_{l_2} \cdots V^{m_n}_{l_n}, \tag{3·36}$$

the first and third terms in (3·1) are invariant under the transformation in the case that V^m_l is an orthogonal matrix O^m_l. We find that the Lagrangian (3·1) is invariant under the discrete transformation for which $V^m_l = \delta^{\sigma(m)}_l$. Here, $\sigma(m)$ stands for the permutation among indices.

We have studied the transformation properties of the system described by the Lagrangian (3·1). Note that there is a similarity between generalizations of the unitary transformation in the dynamical system of generalized matrices and the volume preserving diffeomorphism in the classical system of p-branes. It is important to explore the relationship between these two systems and make clear whether our theory describes the microscopic physics of p-brane-like extended objects.*)

Finally, we comment on several other similar systems.

(i) Supersymmetric theory:
The supersymmetric version of the Lagrangian (3·1) with $\alpha = 1$ and $\beta = \gamma = 0$ is given by the following:

$$\begin{aligned} L = & \frac{1}{2} \sum_i \sum_{l_1, l_2, \cdots, l_n} (D_0 X^i)_{l_1 l_2 \cdots l_n} (D_0 X^i)_{l_2 l_1 \cdots l_n} \\ & + \frac{1}{n \cdot n!} \sum_{i_1, i_2, \cdots, i_n} \sum_{l_1, l_2, \cdots, l_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_1 l_2 \cdots l_n} [X^{i_1}, X^{i_2}, \cdots, X^{i_n}]_{l_2 l_1 \cdots l_n} \\ & + \frac{i}{2} \sum_{l_1, l_2, \cdots, l_n} (\bar{S})_{l_1 l_2 \cdots l_n} (D_0 S)_{l_2 l_1 \cdots l_n} \\ & + \frac{i}{2(n-1)!} \sum_{i_1, \cdots, i_{n-1}} \sum_{l_1, l_2, \cdots, l_n} (\bar{S})_{l_1 l_2 \cdots l_n} \gamma^{i_1 \cdots i_{n-1}} [X^{i_1}, \cdots, X^{i_{n-1}}, S]_{l_2 l_1 \cdots l_n}, \end{aligned} \tag{3·37}$$

where S is a Grassmann-valued n-th power matrix, and $\gamma^{i_1 \cdots i_{n-1}}$ is a product of Dirac γ matrices. This Lagrangian is the counterpart of the super p-brane given by (D·21), and the system possesses supersymmetry between X^i and S for specific values of n and N.

(ii) Generalization of the matrix model:
The action of the 0-dimensional system analogous to matrix model is

$$\begin{aligned} S = & \frac{\alpha}{n \cdot n!} \sum_{\mu_1, \mu_2, \cdots, \mu_n} \sum_{l_1, l_2, \cdots, l_n} [X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_n}]_{l_1 l_2 \cdots l_n} [X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_n}]_{l_2 l_1 \cdots l_n} \\ & - \beta \sum_\mu \sum_{l_1, l_2, \cdots, l_n} (X^\mu)_{l_1 l_2 \cdots l_n} (X^\mu)_{l_2 l_1 \cdots l_n} \end{aligned}$$

*) Matsuo and Shibusa have given a representation of the volume preserving diffeomorphism using the non-commutative branes.[26] It is interesting to establish a link between their results and our realization.

$$-\gamma\frac{2i}{n+1}\sum_{\mu,\mu_1,\mu_2,\cdots,\mu_n}\sum_{l_1,l_2,\cdots,l_n} f^{\mu\mu_1\mu_2\cdots\mu_n}(X^\mu)_{l_1l_2\cdots l_n}(X^{\mu_1}X^{\mu_2}\cdots X^{\mu_n})_{l_2l_1\cdots l_n}, \quad (3\cdot38)$$

where X^μ are hermitian n-th power matrices, and α, β and γ are real parameters. This action with $\beta=\gamma=0$ is interpreted as the n-th power matrix analog of the action (D·24). In the case with $n=2$, $\alpha=1/g^2$ and $\beta=\gamma=0$, the action is equivalent to the bosonic part of the type IIB matrix model.[25]

§4. Conclusions

We have proposed a generalization of spin algebra using multi-index objects called n-th power matrices and studied a dynamical system analogous to matrix theory. We have found that this system has a solution described by generalized spin representation matrices and possesses a symmetry similar to the volume preserving diffeomorphism in the classical p-brane action.

Our system is interpreted as a generalization of the bosonic part of the BFSS matrix theory. The BFSS matrix theory has several interesting physical implications. For example, it is regarded as a regularized theory of a supermembrane, and it describes a system of D0-branes and can offer a microscopic description of M-theory. This theory also has a special position with regard to symmetry properties. Our system for $n=2$ has a larger symmetry, that is, invariance under an arbitrary time dependent unitary transformation, but it seems to possess a restricted type of local symmetry for $n\geq 3$. We have treated the abelian local transformations (3·31) and (3·32) as an example. We have also considered the case in which the transformations form a group whose elements are factorized into a product of matrices, as an extension of unitary transformations. It is important to explore the physical implications and transformation properties beyond the group theoretical analysis in our system for $n\geq 3$.

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Appendix A
—— *Definition of n-th Power Matrices* ——

In this appendix, we define n-index objects, which we refer to as 'n-th power matrices',*) and define related terminology.[14] An n-th power matrix is an object with n indices written $B_{l_1l_2\cdots l_n}$. This is a generalization of an ordinary matrix, written analogously as $B_{l_1l_2}$. We treat n-th power "square" matrices, i.e., $N\times N\times\cdots\times N$ matrices, and in many cases treat the elements of these matrices as c-numbers throughout this paper.

*) Many-index objects have been introduced to construct a quantum version of the Nambu bracket.[20),23)] The definition of the n-fold product we use is the same as that used by Xiong.

First, we define the hermiticity of an n-th power matrix by the relation $B_{l'_1 l'_2 \cdots l'_n} = B^*_{l_1 l_2 \cdots l_n}$ for odd permutations among indices and refer to an n-th power matrix possessing the property of hermiticity as a 'hermitian n-th power matrix'. Here, the asterisk indicates complex conjugation. A hermitian n-th power matrix satisfies the relation $B_{l'_1 l'_2 \cdots l'_n} = B_{l_1 l_2 \cdots l_n}$ for even permutations among indices. The components for which at least two indices are identical, e.g., $B_{l_1 \cdots l_i \cdots l_i \cdots l_n}$, which is the counterpart of the diagonal part of a hermitian matrix, are real-valued and symmetric with respect to permutations among indices $\{l_1, \cdots, l_i, \cdots, l_i, \cdots, l_n\}$. We refer to a special type of hermitian matrix whose components possessing all distinct indices vanish as a 'real normal form' or a 'real normal n-th power matrix'. A normal n-th power matrix is written

$$B^{(N)}_{l_1 l_2 \cdots l_n} = \sum_{i<j} \delta_{l_i l_j} b_{l_j l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_n}, \qquad (\mathrm{A}\cdot 1)$$

where the summation is over all pairs among $\{l_1, \cdots, l_n\}$, the hatted indices are omitted, and $b_{l_j l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_n}$ is symmetric under the exchange of any $(n-2)$ indices, excluding l_j.

We define the n-fold product of n-th power matrices $(B_i)_{l_1 l_2 \cdots l_n}$ $(i = 1, 2, \cdots, n)$ by

$$(B_1 B_2 \cdots B_n)_{l_1 l_2 \cdots l_n} \equiv \sum_k (B_1)_{l_1 \cdots l_{n-1} k} (B_2)_{l_1 \cdots l_{n-2} k l_n} \cdots (B_n)_{k l_2 \cdots l_n}. \qquad (\mathrm{A}\cdot 2)$$

The resultant n-index object, $(B_1 B_2 \cdots B_n)_{l_1 l_2 \cdots l_n}$, is not necessarily hermitian, even if the n-th power matrices $(B_i)_{l_1 l_2 \cdots l_n}$ are all hermitian. Note that the above product is, in general, neither commutative nor associative; for example, we have

$$\begin{gathered}(B_1 B_2 \cdots B_n)_{l_1 l_2 \cdots l_n} \neq (B_2 B_1 \cdots B_n)_{l_1 l_2 \cdots l_n}, \\ (B_1 \cdots B_{n-1}(B_n B_{n+1} \cdots B_{2n-1}))_{l_1 l_2 \cdots l_n} \neq ((B_1 \cdots B_{n-1} B_n) B_{n+1} \cdots B_{2n-1})_{l_1 l_2 \cdots l_n}.\end{gathered} \qquad (\mathrm{A}\cdot 3)$$

The n-fold commutator is defined by

$$\begin{aligned}&[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n} \\ &\quad \equiv \sum_{(i_1, i_2, \cdots, i_n)} \sum_k \mathrm{sgn}(P) (B_{i_1})_{l_1 \cdots l_{n-1} k} (B_{i_2})_{l_1 \cdots l_{n-2} k l_n} \cdots (B_{i_n})_{k l_2 \cdots l_n},\end{aligned} \qquad (\mathrm{A}\cdot 4)$$

where the first summation is over all permutations among the subscripts $\{i_1, i_2, \cdots, i_n\}$. Here, $\mathrm{sgn}(P)$ is $+1$ and -1 for even and odd permutations among the subscripts $\{i_1, i_2, \cdots, i_n\}$, respectively. If the n-th power matrices $(B_i)_{l_1 l_2 \cdots l_n}$ are hermitian, then $i[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n}$ is also hermitian.

We now study some properties of the n-fold commutator $[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n}$. This commutator is written

$$\begin{aligned}[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n} &= (B_1)_{l_1 l_2 \cdots l_n} (B_2 \widetilde{B_3 \cdots B_n})_{l_1 l_2 \cdots l_n} \\ &\quad + (-1)^{n-1} (B_2)_{l_1 l_2 \cdots l_n} (\widetilde{B_3 \cdots B_n} B_1)_{l_1 l_2 \cdots l_n}\end{aligned}$$

$$+\cdots+(-1)^{n-1}(B_n)_{l_1l_2\cdots l_n}(B_1\widetilde{B_2\cdots B_{n-1}})_{l_1l_2\cdots l_n}$$
$$+([B_1,B_2,\cdots,B_n])^0_{l_1l_2\cdots l_n}, \tag{A·5}$$

where $(B_2\widetilde{B_3\cdots B_n})_{l_1l_2\cdots l_n}$ and $([B_1,B_2,\cdots,B_n])^0_{l_1l_2\cdots l_n}$ are defined by

$$\begin{aligned}
&(B_2\widetilde{B_3\cdots B_n})_{l_1l_2\cdots l_n}\\
&\equiv\sum_{(i_2,i_3,\cdots,i_n)}\mathrm{sgn}(P)\Big((B_{i_2})_{l_1\cdots l_{n-2}l_nl_n}(B_{i_3})_{l_1\cdots l_{n-3}l_nl_{n-1}l_n}\cdots(B_{i_n})_{l_nl_2\cdots l_{n-1}l_n}\\
&+(-1)^{n-1}(B_{i_2})_{l_1\cdots l_{n-3}l_{n-1}l_{n-1}l_n}(B_{i_3})_{l_1\cdots l_{n-4}l_{n-1}l_{n-2}l_{n-1}l_n}\cdots(B_{i_n})_{l_1\cdots l_{n-2}l_{n-1}l_{n-1}}\\
&+\cdots+(-1)^{n-1}(B_{i_2})_{l_1\cdots l_{n-1}l_1}(B_{i_3})_{l_1\cdots l_{n-2}l_1l_n}\cdots(B_{i_n})_{l_1l_1l_3\cdots l_n}\Big)
\end{aligned}\tag{A·6}$$

and

$$\begin{aligned}
&([B_1,B_2,\cdots,B_n])^0_{l_1l_2\cdots l_n}\\
&\equiv\sum_{(i_1,i_2,\cdots i_n)}\sum_{k\neq l_1,l_2,\cdots,l_n}\mathrm{sgn}(P)(B_{i_1})_{l_1\cdots l_{n-1}k}(B_{i_2})_{l_1\cdots l_{n-2}kl_n}\cdots(B_{i_n})_{kl_2\cdots l_n},
\end{aligned}\tag{A·7}$$

respectively.

We now discuss features of $(B_1\widetilde{B_2\cdots B_{n-1}})_{l_1l_2\cdots l_n}$. It is skew-symmetric with respect to permutations among indices; i.e., we have

$$(B_1\widetilde{B_2\cdots B_{n-1}})_{l_1\cdots l_i\cdots l_j\cdots l_n}=-(B_1\widetilde{B_2\cdots B_{n-1}})_{l_1\cdots l_j\cdots l_i\cdots l_n} \tag{A·8}$$

if each $(B_k)_{l_jl_1\cdots\hat{l}_i\cdots l_n}$ $(k=1,\cdots,n-1)$ is symmetric with respect to permutations among the n-indices $\{l_j,l_1,\cdots,\hat{l}_i,\cdots,l_n\}$, as are hermitian n-th power matrices. Here we define the following operation on an n-th antisymmetric object $\omega_{m_1m_2\cdots m_n}$:

$$(\delta\omega)_{m_0m_1\cdots m_n}\equiv\sum_{i=0}^{n}(-1)^i\omega_{m_0m_1\cdots\hat{m}_i\cdots m_n}, \tag{A·9}$$

where the operator δ is regarded as a coboundary operator that changes n-th antisymmetric objects into $(n+1)$-th objects. This operator is nilpotent, i.e. $\delta^2(*)=0$.*) If $\omega_{m_1m_2\cdots m_k}$ satisfies the cocycle condition $(\delta\omega)_{m_0m_1\cdots m_n}=0$, it is called a cocycle.

For arbitrary normal n-th power matrices $B_j^{(N)}$, the n-fold commutator among B and $B_j^{(N)}$ is given by

$$[B_1^{(N)},\cdots,B_{n-1}^{(N)},B]_{l_1l_2\cdots l_n}=(-1)^{n-1}(B_1^{(N)}\widetilde{\cdots B_{n-1}^{(N)}})_{l_1l_2\cdots l_n}B_{l_1l_2\cdots l_n}. \tag{A·10}$$

If $(B_1^{(N)}\widetilde{\cdots B_{n-1}^{(N)}})_{l_1l_2\cdots l_n}$ is a cocycle for normal n-th power matrices $B_j^{(N)}$, the following fundamental identity holds:

$$[[C_1,\cdots,C_n],B_1^{(N)},\cdots,B_{n-1}^{(N)}]_{l_1l_2\cdots l_n}$$

*) See Ref. 27) for treatments of cohomology.

$$= \sum_{i=1}^{n} [C_1, \cdots, [C_i, B_1^{(N)}, \cdots, B_{n-1}^{(N)}], \cdots, C_n]_{l_1 l_2 \cdots l_n}. \tag{A·11}$$

Next, we give two kinds of trace operations on n-th power matrices. The first one is a generalization of the trace of $B_{l_1 l_2}$, defined by

$$\mathrm{Tr} B \equiv \sum_{l} B_{ll} = \sum_{l_1, l_2} B_{l_1 l_2} \delta_{l_1 l_2}. \tag{A·12}$$

We define the trace operation on the n-th power matrix $B_{l_1 l_2 \cdots l_n}$ by

$$\mathrm{Tr}_{(1)} B \equiv \sum_{l} B_{ll\cdots l} = \sum_{l_1, l_2, \cdots, l_n} B_{l_1 l_2 \cdots l_n} \delta_{l_1 l_2 \cdots l_n}, \tag{A·13}$$

where $\delta_{l_1 l_2 \cdots l_n} \equiv \delta_{l_1 l_2} \delta_{l_2 l_3} \cdots \delta_{l_{n-1} l_n}$. The second one is a generalization of the trace of $(B_1 B_2)_{l_1 l_2}$, which is written

$$\mathrm{Tr}(B_1 B_2) \equiv \sum_{l_1} \sum_{k} (B_1)_{l_1 k} (B_2)_{k l_1} = \sum_{l_1, l_2} (B_1 B_2)_{l_1 l_2} \delta_{l_1 l_2}. \tag{A·14}$$

Here we define the product of the n-th power matrices B_1 and B_2 by

$$(B_1 B_2)_{l_1 l_2 \cdots l_n} \equiv \sum_{k} (B_1)_{l_1 \cdots l_{n-1} k} (B_2)_{l_1 \cdots l_{n-2} k l_n}. \tag{A·15}$$

This product is also obtained by setting $B_3 = \cdots = B_n = T$ in the n-fold product (A·2), where T is the n-th power matrix in which every component has the value of 1, i.e., $T_{l_1 l_2 \cdots l_n} = 1$. Note that this product is not commutative, but it is associative:

$$(B_1 B_2)_{l_1 l_2 \cdots l_n} \neq (B_2 B_1)_{l_1 l_2 \cdots l_n}, \quad (B_1 (B_2 B_3))_{l_1 l_2 \cdots l_n} = ((B_1 B_2) B_3)_{l_1 l_2 \cdots l_n}. \tag{A·16}$$

Now we define the trace operation on the n-th power matrix $(B_1 B_2)_{l_1 l_2 \cdots l_n}$ by

$$\begin{aligned} \mathrm{Tr}_{(2)}(B_1 B_2) &\equiv \sum_{l_1, l_2 \cdots, l_n} (B_1 B_2)_{l_1 l_2 \cdots l_n} \delta_{l_{n-1} l_n} \\ &= \sum_{l_1, l_2 \cdots, l_n} \sum_{k} (B_1)_{l_1 \cdots l_{n-1} k} (B_2)_{l_1 \cdots l_{n-2} k l_n} \delta_{l_{n-1} l_n} \\ &= \sum_{l_1, \cdots, l_{n-1}, l_n} (B_1)_{l_1 \cdots l_{n-1} l_n} (B_2)_{l_1 \cdots l_n l_{n-1}}. \end{aligned} \tag{A·17}$$

For a hermitian n-th power matrix $B_{l_1 \cdots l_n}$, the second kind of trace for $(B^2)_{l_1 l_2 \cdots l_n}$ is positive semi-definite:

$$\mathrm{Tr}_{(2)} B^2 = \sum_{l_1, \cdots, l_n} |B_{l_1 \cdots l_n}|^2 \geq 0. \tag{A·18}$$

For hermitian n-th power matrices B_1 and B_2, the following formula holds:

$$\begin{aligned} \mathrm{Tr}_{(2)}(B_1 B_2) &= \sum_{l_1, \cdots, l_{n-1}, l_n} (B_1)_{l_1 \cdots l_{n-1} l_n} (B_2)_{l_1 \cdots l_n l_{n-1}} \\ &= \sum_{l_1, l_2 \cdots, l_n} (B_1)_{l_1 l_2 \cdots l_n} (B_2)_{l_2 l_1 \cdots l_n}. \end{aligned} \tag{A·19}$$

Appendix B
—— *Transformation Properties of n-th Power Matrices* ——

In this appendix, we study the analog of unitary transformations for n-th power matrices. First, we review unitary transformations for ordinary matrices. A unitary transformation for $B_{l_1 l_2}$ is defined by

$$\begin{aligned} B_{l_1 l_2} \to B'_{l_1 l_2} &= \sum_{m_1, m_2} U_{l_1 m_1} B_{m_1 m_2} U^\dagger_{m_2 l_2} = \sum_{m_1, m_2} U_{l_1 m_1} U^*_{l_2 m_2} B_{m_1 m_2} \\ &\equiv \sum_{m_1, m_2} R^{m_1 m_2}_{l_1 l_2} B_{m_1 m_2}, \end{aligned} \tag{B·1}$$

where U_{lm} is a unitary matrix ($\sum_m U_{lm} U^\dagger_{mn} = \sum_m U^\dagger_{lm} U_{mn} = \delta_{ln}$) and $R^{m_1 m_2}_{l_1 l_2}$ is a "transformation matrix" defined by $R^{m_1 m_2}_{l_1 l_2} \equiv U_{l_1 m_1} U^*_{l_2 m_2}$. By the definition of $R^{m_1 m_2}_{l_1 l_2}$, we have the relation

$$(R^{m_1 m_2}_{l_1 l_2})^* = R^{m_2 m_1}_{l_2 l_1}. \tag{B·2}$$

Then, from the unitarity of U_{lm}, we obtain the relations

$$\sum_m R^{mm}_{l_1 l_2} = \delta_{l_1 l_2}, \qquad \sum_l R^{m_1 m_2}_{ll} = \delta_{m_1 m_2}, \tag{B·3}$$

$$\sum_k R^{m_1 m_2}_{l_1 k} R^{n_2 n_1}_{k l_2} = R^{m_1 n_1}_{l_1 l_2} \delta_{m_2 n_2}, \tag{B·4}$$

$$\sum_{k,l} R^{m_1 m_2}_{lk} R^{n_2 n_1}_{kl} = \delta_{m_1 n_1} \delta_{m_2 n_2}. \tag{B·5}$$

In terms of $R^{m_1 m_2}_{l_1 l_2}$, the quantities $\delta_{l_1 l_2}$ and $\mathrm{Tr}B \equiv \sum_l B_{ll}$ are shown to be invariant under the unitary transformation from the first and second relations in (B·3), respectively. The relation (B·4) is related to the covariance of $C_{l_1 l_2} = (B_1 B_2)_{l_1 l_2}$ under the unitary transformation

$$C_{l_1 l_2} \to C'_{l_1 l_2} \equiv (B'_1 B'_2)_{l_1 l_2} = \sum_{m_1, m_2} R^{m_1 m_2}_{l_1 l_2} C_{m_1 m_2}. \tag{B·6}$$

The relation (B·5) is related to the invariance of $(B_1 B_2)_{ll}$ under the unitary transformation.

An infinitesimal unitary transformation is given by

$$\delta B_{l_1 l_2} = i[\Lambda, B]_{l_1 l_2} = i \sum_{m_1, m_2} r^{(\Lambda)}{}^{m_1 m_2}_{l_1 l_2} B_{m_1 m_2}, \tag{B·7}$$

where the "transformation matrix" $r^{(\Lambda)}{}^{m_1 m_2}_{l_1 l_2}$ is given by $r^{(\Lambda)}{}^{m_1 m_2}_{l_1 l_2} = \Lambda_{l_1 m_1} \delta_{l_2 m_2} - \Lambda_{m_2 l_2} \delta_{l_1 m_1}$. We find that the identity

$$\sum_{n_1, n_2} (r^{(\Lambda)}{}^{n_1 n_2}_{l_1 l_2} r^{(\Lambda')}{}^{m_1 m_2}_{n_1 n_2} - r^{(\Lambda')}{}^{n_1 n_2}_{l_1 l_2} r^{(\Lambda)}{}^{m_1 m_2}_{n_1 n_2}) = r^{([\Lambda, \Lambda'])}{}^{m_1 m_2}_{l_1 l_2} \tag{B·8}$$

holds among transformation matrices, from the Jacobi identity. We can also show that the commutator $[B_1, B_2]$ transforms as

$$\begin{aligned}\delta[B_1, B_2]_{l_1 l_2} &= [\delta B_1, B_2]_{l_1 l_2} + [B_1, \delta B_2]_{l_1 l_2} \\ &= i[\Lambda, [B_1, B_2]]_{l_1 l_2} = i \sum_{m_1, m_2} r^{(\Lambda)}{}^{m_1 m_2}_{l_1 l_2} B_{m_1 m_2}.\end{aligned} \tag{B·9}$$

Next, we study the case of n-th power matrices with $n \geq 3$. We define an extension of the unitary transformation (B·1) for $B_{l_1 l_2 \cdots l_n}$ by

$$B_{l_1 l_2 \cdots l_n} \to B'_{l_1 l_2 \cdots l_n} = \sum_{m_1, m_2, \cdots, m_n} R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} B_{m_1 m_2 \cdots m_n}, \tag{B·10}$$

where $R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n}$ is a "transformation matrix". From the transformation (B·10) and the hermiticity of $B_{l_1 l_2 \cdots l_n}$, we obtain the relations

$$(R^{m'_1 m'_2 \cdots m'_n}_{l'_1 l'_2 \cdots l'_n})^* = R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} \tag{B·11}$$

for odd permutations among the pairs of indices (l_k, m_k) $(k = 1, \cdots, n)$, and

$$(R^{m'_1 m'_2 \cdots m'_n}_{l'_1 l'_2 \cdots l'_n}) = R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} \tag{B·12}$$

for even permutations among the pairs of indices (l_k, m_k) $(k = 1, \cdots, n)$. When the transformation (B·10) is given by an n-fold product, the transformations do not necessarily form a group, because the n-fold product is, in general, not associative. However, for simplicity, here we treat the case in which the transformations form a group and the transformation matrix $R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n}$ can be factorized into a product of matrices as

$$R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} = V^{m_1}_{l_1} V^{m_2}_{l_2} \cdots V^{m_n}_{l_n}, \tag{B·13}$$

where V^m_l should be a real matrix, from the relations (B·11) and (B·12). The form of the matrix V^m_l is restricted by suitable requirements. We now give examples of such requirements. The first requirement is that the second kind of trace, $\mathrm{Tr}_{(2)}(BC) = \sum_{l_1, l_2, \cdots, l_n} B_{l_1 l_2 \cdots l_n} C_{l_2 l_1 \cdots l_n}$ be invariant under the transformation

$$\begin{aligned}B_{l_1 l_2 \cdots l_n} \to B'_{l_1 l_2 \cdots l_n} &= \sum_{m_1, m_2, \cdots, m_n} R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} B_{m_1 m_2 \cdots m_n}, \\ C_{l_1 l_2 \cdots l_n} \to C'_{l_1 l_2 \cdots l_n} &= \sum_{m_1, m_2, \cdots, m_n} R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} C_{m_1 m_2 \cdots m_n}.\end{aligned} \tag{B·14}$$

The necessary condition to satisfy this requirement is

$$\sum_{l_1, l_2, \cdots, l_n} R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} R^{m'_2 m'_1 \cdots m'_n}_{l_2 l_1 \cdots l_n} = \delta_{m_1 m'_1} \delta_{m_2 m'_2} \cdots \delta_{m_n m'_n}, \tag{B·15}$$

in which case $R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n}$ is given by

$$R^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} = O^{m_1}_{l_1} O^{m_2}_{l_2} \cdots O^{m_n}_{l_n}, \tag{B·16}$$

where O_l^m is an orthogonal matrix ($\sum_l O_l^m O_l^n = \delta_{mn}$, $\sum_m O_l^m O_n^m = \delta_{ln}$).

Next, we require that the trace $\mathrm{Tr}_{(2)} C^2 = \sum_{l_1,l_2,\cdots,l_n} C_{l_1 l_2 \cdots l_n} C_{l_2 l_1 \cdots l_n}$ be invariant under the transformation

$$(B_k)_{l_1 l_2 \cdots l_n} \to (B_k)'_{l_1 l_2 \cdots l_n} = \sum_{m_1,m_2,\cdots,m_n} R_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} (B_k)_{m_1 m_2 \cdots m_n}, \quad \text{(B·17)}$$

where $C_{l_1 l_2 \cdots l_n}$ is the n-fold product given by $C_{l_1 l_2 \cdots l_n} = (B_1 B_2 \cdots B_n)_{l_1 l_2 \cdots l_n}$. We find that the following $R_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n}$ satisfies the above requirement:

$$R_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} = D_{l_1}^{m_1} D_{l_2}^{m_2} \cdots D_{l_n}^{m_n}, \quad \text{(B·18)}$$

where $D_l^m = \delta_l^{\sigma(m)}$, and $\sigma(m)$ stands for a permutation of the index m. Note that D_l^i satisfies the relations

$$\sum_i D_{m_1}^i D_{m_2}^i \cdots D_{m_k}^i = \delta_{m_1 m_2 \cdots m_k}, \quad \sum_m D_m^{i_1} D_m^{i_2} \cdots D_m^{i_k} = \delta_{i_1 i_2 \cdots i_k} \quad \text{(B·19)}$$

for an arbitrary integer k. Using the relations (B·19) with $k = n$, we find that $\delta_{l_1 l_2 \cdots l_n}$ and $\mathrm{Tr}_{(1)} B$ are invariant under the extended transformation (B·10). Further, we find that the n-fold product $C_{l_1 l_2 \cdots l_n} = (B_1 B_2 \cdots B_n)_{l_1 l_2 \cdots l_n}$ transforms covariantly, i.e.,

$$C_{l_1 l_2 \cdots l_n} \to C'_{l_1 l_2 \cdots l_n} = \sum_{m_1,m_2,\cdots,m_n} R_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} C_{m_1 m_2 \cdots m_n}, \quad \text{(B·20)}$$

under the transformation (B·17) with the transformation matrix (B·18).

Finally, we study the generalization of the infinitesimal transformation (B·7) for n-th power matrices with $n \geq 3$. We consider the following transformation by use of n-fold commutator:

$$\begin{aligned} \delta B_{l_1 l_2 \cdots l_n} &= i[\Lambda_1, \cdots, \Lambda_{n-1}, B]_{l_1 l_2 \cdots l_n} \\ &= i \sum_{m_1,m_2,\cdots,m_n} r^{(\Lambda)}{}_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} B_{m_1 m_2 \cdots m_n}. \end{aligned} \quad \text{(B·21)}$$

Here, $r^{(\Lambda)}{}_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n}$ is a "transformation matrix", and Λ_k, $(k = 1, \cdots, n)$ is a set of "generators". The n-fold commutator $[B_1, B_2, \cdots, B_n]$ transforms as

$$\begin{aligned} \delta[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n} &= \sum_{k=1}^{n} [B_1, \cdots, \delta(B_k), \cdots, B_n]_{l_1 l_2 \cdots l_n} \\ &= i \sum_p \sum_{(i_1,\cdots,i_n)} \sum_k \mathrm{sgn}(P) (B_{i_1})_{l_1 \cdots l_{n-1} k} \\ \cdots \sum_{m_1,m_2,\cdots,m_n} & r^{(\Lambda)}{}_{l_1 \cdots l_{n-p} k l_{n+2-p} \cdots l_n}^{m_1 m_2 \cdots m_n} (B_{i_p})_{m_1 m_2 \cdots m_n} \cdots (B_{i_n})_{k l_2 \cdots l_n} \end{aligned} \quad \text{(B·22)}$$

under the transformation

$$\delta(B_k)_{l_1 l_2 \cdots l_n} = i \sum_{m_1,m_2,\cdots,m_n} r^{(\Lambda)}{}_{l_1 l_2 \cdots l_n}^{m_1 m_2 \cdots m_n} (B_k)_{m_1 m_2 \cdots m_n}. \quad \text{(B·23)}$$

Note that the transformation law

$$\delta[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n} = i \sum_{m_1, m_2, \cdots, m_n} r^{(\Lambda)}{}^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} [B_1, B_2, \cdots, B_n]_{m_1 m_2 \cdots m_n} \qquad (\mathrm{B{\cdot}24})$$

does not necessarily hold for n-th power matrices with $n \geq 3$. This means that the fundamental identity does not always hold among n-th power matrices with $n \geq 3$. Here we treat the case that Λ_k $(k = 1, 2, \cdots, n)$ are normal n-th power matrices as an example that the law (B·24) holds. In this case, the n-th power matrices B_k transform as

$$\delta(B_k)_{l_1 l_2 \cdots l_n} = i[\Lambda_1, \cdots, \Lambda_{n-1}, B_k]_{l_1 l_2 \cdots l_n} = i\Lambda_{l_1 l_2 \cdots l_n}(B_k)_{l_1 l_2 \cdots l_n}, \qquad (\mathrm{B{\cdot}25})$$

where $\Lambda_{l_1 l_2 \cdots l_n} = (-1)^{n-1}(\Lambda_1 \widetilde{\cdots \Lambda_{n-1}})_{l_1 l_2 \cdots l_n}$. In terms of $\Lambda_{l_1 l_2 \cdots l_n}$, the transformation matrix $r^{(\Lambda)}$ is written

$$r^{(\Lambda)}{}^{m_1 m_2 \cdots m_n}_{l_1 l_2 \cdots l_n} = \Lambda_{m_1 m_2 \cdots m_n} \delta^{m_1}_{l_1} \delta^{m_2}_{l_2} \cdots \delta^{m_n}_{l_n}. \qquad (\mathrm{B{\cdot}26})$$

If $\Lambda_{l_1 l_2 \cdots l_n}$ satisfy the cocycle condition,

$$(\delta\Lambda)_{l_0 l_1 \cdots l_n} \equiv \sum_{i=0}^{n} (-1)^i \Lambda_{l_0 l_1 \cdots \hat{l}_i \cdots l_n} = 0, \qquad (\mathrm{B{\cdot}27})$$

we find that the n-fold commutator $[B_1, B_2, \cdots, B_n]$ transforms covariantly:

$$\delta[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n} = i\Lambda_{l_1 l_2 \cdots l_n}[B_1, B_2, \cdots, B_n]_{l_1 l_2 \cdots l_n}. \qquad (\mathrm{B{\cdot}28})$$

Appendix C
—— *Classical Analog of Generalized Spin Algebra* ——

In this appendix, we study the classical analog of generalized spin algebra. First, we consider the following action integral, whose variables are $\phi^i(t)$:[28]

$$S = \int \left(\sum_i A_i(\phi) \frac{d\phi^i}{dt} - H(\phi) \right) dt. \qquad (\mathrm{C{\cdot}1})$$

The change in S under an infinitesimal variation of $\phi^i(t)$ is given by

$$\delta S = \int \sum_i \left(\sum_j F_{ij}(\phi) \frac{d\phi^j}{dt} - \frac{\partial H}{\partial \phi^i} \right) \delta\phi^i dt, \qquad (\mathrm{C{\cdot}2})$$

where $F_{ij}(\phi)$ is defined by

$$F_{ij}(\phi) \equiv \frac{\partial A_j}{\partial \phi^i} - \frac{\partial A_i}{\partial \phi^j}. \qquad (\mathrm{C{\cdot}3})$$

From the least action principle, we obtain the equation of motion

$$\frac{d\phi^i}{dt} = \sum_j F^{ij} \frac{\partial H}{\partial \phi^j}, \tag{C·4}$$

where F^{ij} is the inverse of F_{ij}, i.e., $\sum_j F^{ij} F_{jk} = \delta^i_k$. Then, the Poisson bracket is defined by

$$\{f, g\}_{\mathrm{PB}} \equiv \sum_{i,j} F^{ij} \frac{\partial f}{\partial \phi^i} \frac{\partial g}{\partial \phi^j}, \tag{C·5}$$

and using it, the equation of motion (C·4) is rewritten

$$\frac{d\phi^i}{dt} = \{\phi^i, H\}_{\mathrm{PB}}. \tag{C·6}$$

When the variables compose a triplet X^i ($i = 1, 2, 3$) and are such that we have $F^{ij} = \sum_k \varepsilon^{ijk} X^k$, then these variables form the algebra described by

$$\{X^i, X^j\}_{\mathrm{PB}} = \sum_k \varepsilon^{ijk} X^k. \tag{C·7}$$

This is the classical analog of the spin algebra $su(2)$. In this case, the first term of the action integral (C·1) is rewritten as

$$\int \sum_i A_i(X) dX^i = \frac{1}{2} \int\int \sum_{i,j} F_{ij}(X) dX^i \wedge dX^j = \int\int R \sin\theta d\theta \wedge d\varphi, \tag{C·8}$$

where $\wedge$ represents Cartan's wedge product, and the variables X^i are coordinates on S^2 with radius R, which can therefore be written in polar coordinates as

$$X^1 = R\sin\theta\cos\varphi, \quad X^2 = R\sin\theta\sin\varphi, \quad X^3 = R\cos\theta. \tag{C·9}$$

The action integral (C·8) is regarded as an area on S^2.

Next, we consider a generalization of the action integral (C·1) whose variables are $\phi^i = \phi^i(t, \sigma_1, \cdots, \sigma_{n-1})$,

$$S = \int \cdots \int \Bigg(\sum_{i_1, \cdots, i_{n-1}, i_n} A_{i_1 \cdots i_{n-1} i_n}(\phi) \frac{\partial \phi^{i_1}}{\partial \sigma_1} \cdots \frac{\partial \phi^{i_{n-1}}}{\partial \sigma_{n-1}} \frac{\partial \phi^{i_n}}{\partial t} - H_1 \frac{\partial(H_2, \cdots, H_n)}{\partial(\sigma_1, \cdots, \sigma_{n-1})} \Bigg) d\sigma_1 \cdots d\sigma_{n-1} dt, \tag{C·10}$$

where $A_{i_1 \cdots i_n}(\phi)$ is antisymmetric under the exchange of indices and the quantities H_i are "Hamiltonians". The change in S under an infinitesimal variation of ϕ^i is given by

$$\delta S = \int \cdots \int \sum_{j, i_1, \cdots, i_{n-1}} \Bigg(\sum_{i_n} F_{j i_1 \cdots i_{n-1} i_n} \frac{\partial \phi^{i_n}}{\partial t} - \frac{\partial(H_1, H_2, \cdots, H_n)}{\partial(\phi^j, \phi^{i_1} \cdots, \phi^{i_{n-1}})} \Bigg) \delta\phi^j \frac{\partial \phi^{i_1}}{\partial \sigma_1} \cdots \frac{\partial \phi^{i_{n-1}}}{\partial \sigma_{n-1}} d\sigma_1 \cdots d\sigma_{n-1} dt, \tag{C·11}$$

where $F_{ji_1\cdots i_n}$ is defined by

$$F_{ji_1\cdots i_n} \equiv \frac{\partial A_{i_1\cdots i_n}}{\partial \phi^j} + (-1)^n \frac{\partial A_{i_2\cdots i_n j}}{\partial \phi^{i_1}} + \cdots + (-1)^n \frac{\partial A_{ji_1\cdots i_{n-1}}}{\partial \phi^{i_n}}. \tag{C·12}$$

From the least action principle, we obtain the equation of motion

$$\frac{d\phi^i}{dt} = \sum_{i_1,\cdots,i_n} F^{ii_1\cdots i_n} \frac{\partial(H_1, H_2, \cdots, H_n)}{\partial(\phi^{i_1}\cdots, \phi^{i_n})}, \tag{C·13}$$

where $F^{ii_1\cdots i_n}$ is the inverse of $F_{ii_1\cdots i_n}$, i.e., $\sum_{i_1,\cdots,i_n} F^{ii_1\cdots i_n} F_{i_1\cdots i_n j} = \delta^i_j$. Then the Nambu bracket is defined by

$$\{f_1, \cdots, f_{n+1}\}_{\rm NB} \equiv \sum_{i_1,\cdots,i_{n+1}} F^{i_1\cdots i_{n+1}} \frac{\partial f_1}{\partial \phi^{i_1}} \cdots \frac{\partial f_{n+1}}{\partial \phi^{i_{n+1}}}, \tag{C·14}$$

and using it, the equation of motion (C·13) is rewritten

$$\frac{d\phi^i}{dt} = \{\phi^i, H_1, \cdots, H_n\}_{\rm NB}. \tag{C·15}$$

The equation of motion (C·15) is equivalent to the Hamilton-Nambu equation.[16)] When the variables compose an $(n+2)$-let X^i $(i = 1, \cdots, n+2)$ and are such that we have $F^{i_1\cdots i_{n+1}} = \sum_{i_{n+2}} \varepsilon^{i_1\cdots i_{n+1} i_{n+2}} X^{i_{n+2}}$, then these X^i form the algebra described by

$$\{X^{i_1}, \cdots, X^{i_{n+1}}\}_{\rm NB} = \sum_{i_{n+2}} \varepsilon^{i_1\cdots i_{n+1} i_{n+2}} X^{i_{n+2}}. \tag{C·16}$$

This is the classical analog of the generalized spin algebra. In this case, the first term of the action integral (C·10) is rewritten as

$$\begin{aligned}
&\frac{1}{n!} \int \cdots \int \sum_{i_1,\cdots,i_n} A_{i_1\cdots i_n}(X) dX^{i_1} \wedge \cdots \wedge dX^{i_n} \\
&= \frac{1}{(n+1)!} \int \cdots \int \sum_{i_1,\cdots,i_{n+1}} F_{i_1\cdots i_{n+1}}(X) dX^{i_1} \wedge \cdots \wedge dX^{i_{n+1}} \\
&= \int \cdots \int R^n \sin\theta_2 \sin^2\theta_3 \cdots \sin^n \theta_{n+1} d\theta_2 \wedge d\theta_1 \wedge d\theta_3 \wedge \cdots \wedge d\theta_{n+1},
\end{aligned} \tag{C·17}$$

where the variables X^i are coordinates on S^{n+1} with radius R, which therefore can be written in polar coordinates as

$$\begin{aligned}
X^1 &= R \sin\theta_{n+1} \sin\theta_n \cdots \sin\theta_3 \sin\theta_2 \cos\theta_1, \\
X^2 &= R \sin\theta_{n+1} \sin\theta_n \cdots \sin\theta_3 \sin\theta_2 \sin\theta_1, \\
X^3 &= R \sin\theta_{n+1} \sin\theta_n \cdots \sin\theta_3 \cos\theta_2, \quad \cdots, X^{n+1} = R \sin\theta_{n+1} \cos\theta_n, \\
X^{n+2} &= R \cos\theta_{n+1}.
\end{aligned} \tag{C·18}$$

The action integral (C·17) is regarded as an "area" on S^{n+1}.

Appendix D
—— *Classical Analog of Generalized Matrix Systems* ——

In this appendix, we explain the framework of classical p-branes. The bosonic p-brane action is given by[17)]

$$S = -\int d^{p+1}\sigma \sqrt{-g}, \tag{D·1}$$

where $d^{p+1}\sigma$ represents the $(p+1)$-dimensional world-volume element, and $g = \det g_{\alpha\beta}$. Here, $g_{\alpha\beta}$ is the induced world-volume metric given by

$$g_{\alpha\beta} = \sum_{\mu,\nu} \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}, \tag{D·2}$$

where X^μ $(\mu = 0, 1, \cdots, D-1)$ are the target space coordinates of the p-brane and σ^α $(\alpha = 0, 1, \cdots, p)$ are the $(p+1)$-dimensional world-volume coordinates. We assume that the target space is the D-dimensional Minkowski space. Then, the action integral (D·1) is invariant under the reparametrization

$$\delta X^\mu = \sum_\alpha \epsilon^\alpha \partial_\alpha X^\mu, \tag{D·3}$$

where ϵ^α is an arbitrary function of σ^α.

Let us next introduce the light-cone coordinates in space-time:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \tag{D·4}$$

The transverse coordinates are denoted by X^i $(i = 1, \cdots, D-2)$. By using the reparametrization invariance, we can choose the light-cone gauge,

$$X^+ = x^+ + p^+ \tau, \tag{D·5}$$

where x^+ and p^+ are the center of mass position and momemtum, respectively, and $\tau = \sigma^0$. In the light-cone gauge, the action is written (up to a zero mode term)

$$S = \frac{1}{2}\int d^{p+1}\sigma \left(\sum_i (D_0 X^i)^2 - \det g_{ab} \right), \tag{D·6}$$

where g_{ab} is the induced p-dimensional metric given by

$$g_{ab} = \sum_{i,j} \eta_{ij} \frac{\partial X^i}{\partial \sigma^a} \frac{\partial X^j}{\partial \sigma^b}. \tag{D·7}$$

Here, σ^a $(a = 1, \cdots, p)$ are the p-dimensional volume coordinates. The covariant time derivative D_0 is defined by

$$D_0 X^i \equiv \left(\frac{\partial}{\partial \tau} + \sum_a u^a \frac{\partial}{\partial \sigma^a} \right) X^i, \tag{D·8}$$

where u^a is regarded as the "gauge field" of time, which should satisfy the equation

$$\sum_a \frac{\partial u^a}{\partial \sigma^a} = 0. \tag{D·9}$$

The action (D·6) is rewritten

$$S = \frac{1}{2}\int d^{p+1}\sigma \left(\sum_i (D_0 X^i)^2 - \frac{1}{p!}\sum_{i_1,\cdots,i_p} \{X^{i_1},\cdots,X^{i_p}\}^2 \right), \tag{D·10}$$

where the symbol $\{f_1,\cdots,f_p\}$ is defined by

$$\{f_1,\cdots,f_p\} \equiv \sum_{a_1,\cdots,a_p} \varepsilon^{a_1\cdots a_p} \frac{\partial f_1}{\partial \sigma^{a_1}} \cdots \frac{\partial f_p}{\partial \sigma^{a_p}}. \tag{D·11}$$

(If the coordinates σ^i form a canonical p-let, the symbol $\{f_1,\cdots,f_p\}$ is regarded as the Nambu bracket.) In the case that u^a is written in terms of functions A_k ($k = 1,\cdots,p-1$) as

$$u^a = \sum_{a_1,\cdots,a_{p-1}} \varepsilon^{a_1\cdots a_{p-1}a} \frac{\partial A_1}{\partial \sigma^{a_1}} \cdots \frac{\partial A_{p-1}}{\partial \sigma^{a_{p-1}}}, \tag{D·12}$$

the covariant time derivative (D·8) can be written as

$$D_0 X^i = \frac{\partial X^i}{\partial \tau} + \{A_1,\cdots,A_{p-1},X^i\}. \tag{D·13}$$

The action (D·10) is invariant under the p-dimensional volume preserving diffeomorphism:

$$\delta X^i = \sum_a \lambda^a \partial_a X^i, \tag{D·14}$$

$$\delta u^a = -\frac{\partial \lambda^a}{\partial \tau} - \sum_b u^b \partial_b \lambda^a + \sum_b \lambda^b \partial_b u^a, \tag{D·15}$$

where λ^a satisfies the condition

$$\sum_a \frac{\partial \lambda^a}{\partial \sigma^a} = 0. \tag{D·16}$$

In the case that λ^a can be written in terms of functions Λ_k ($k = 1,\cdots,p-1$) as

$$\lambda^a = \sum_{a_1,\cdots,a_{p-1}} \varepsilon^{a_1\cdots a_{p-1}a} \frac{\partial \Lambda_1}{\partial \sigma^{a_1}} \cdots \frac{\partial \Lambda_{p-1}}{\partial \sigma^{a_{p-1}}}, \tag{D·17}$$

the transformation laws of the p-dimensional volume preserving diffeomorphism (D·14) and (D·15) are rewritten as

$$\delta X^i = \{\Lambda_1,\cdots,\Lambda_{p-1},X^i\}, \tag{D·18}$$

$$\delta u^a = -\frac{\partial \lambda^a}{\partial \tau} - \{A_1,\cdots,A_{p-1},\lambda^a\} + \{\Lambda_1,\cdots,\Lambda_{p-1},u^a\}, \tag{D·19}$$

respectively. This system has the extra symmetry that $u^a = \sum_b \partial_b W^{ab}$ is invariant under the transformation

$$W^{ab} \left(= \sum_{a_1, \cdots, a_{p-2}} \varepsilon^{a_1 \cdots a_{p-2} ba} \frac{\partial A_1}{\partial \sigma^{a_1}} \cdots \frac{\partial A_{p-2}}{\partial \sigma^{a_{p-2}}} A_{p-1} \right)$$
$$\to W'^{ab} = W^{ab} + \sum_c \partial_c \Theta^{abc}, \tag{D·20}$$

where Θ^{abc} is an arbitrary antisymmetric function of σ^a.

Next, we write down the action integral of the super p-brane:

$$S = \frac{1}{2} \int d^{p+1}\sigma \left(\sum_i (D_0 X^i)^2 - \frac{1}{p!} \sum_{i_1, \cdots, i_p} \{X^{i_1}, \cdots, X^{i_p}\}^2 \right.$$
$$\left. + \frac{i}{2} \bar{S} D_0 S + \frac{1}{(p-1)!} \sum_{i_1, \cdots, i_{p-1}} \bar{S} \gamma^{i_1 \cdots i_{p-1}} \{X^{i_1}, \cdots, X^{i_{p-1}}, S\}^2 \right). \tag{D·21}$$

Here, S on the right-hand side is a spinor of $SO(d-2)$, and $\gamma^{i_1 \cdots i_{n-1}}$ is a product of Dirac γ matrices. It is well known that super p-branes exist in space-times possessing certain particular numbers of dimensions.[29]

Finally, we discuss an alternative formulation of bosonic p-branes. First, the action (D·1) is rewritten as

$$S = -\int d^{p+1}\sigma \sqrt{\frac{1}{(p+1)!} \sum_{\mu_1, \cdots, \mu_{p+1}} \{X^{\mu_1}, \cdots, X^{\mu_{p+1}}\}^2}, \tag{D·22}$$

where the symbol $\{f_1, \cdots, f_{p+1}\}$ is defined by

$$\{f_1, \cdots, f_{p+1}\} \equiv \sum_{\alpha_1, \cdots, \alpha_{p+1}} \varepsilon^{\alpha_1 \cdots \alpha_{p+1}} \frac{\partial f_1}{\partial \sigma^{\alpha_1}} \cdots \frac{\partial f_{p+1}}{\partial \sigma^{\alpha_{p+1}}}. \tag{D·23}$$

Then by introducing the auxiliary field $e = e(\sigma)$, we can write down the following action, which is classically equivalent to the above action (D·22):

$$S = \frac{1}{2} \int d^{p+1}\sigma \left(\frac{1}{(p+1)!e} \sum_{\mu_1, \cdots, \mu_{p+1}} \{X^{\mu_1}, \cdots, X^{\mu_{p+1}}\}^2 - e \right). \tag{D·24}$$

The action (D·24) is a p-brane generalization of the so-called Schild action.[30]

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