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Generalized Heisenberg Dynamics

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We formulate a dynamical system based on many-index objects. These objects yield a generalization of the Heisenberg equation. Systems describing harmonic oscillators are presented.

§1. Introduction

Recently, a new type of mechanics has been proposed that is based on three-index objects,[1] and its basic structure has been studied from an algebraic point of view.[2],[3] This mechanics has a counterpart in the canonical structure of classical mechanics, or Nambu mechanics,[4] and it can be interpreted as its 'quantum' or 'discretized' version. It can also be regarded as a generalization of Heisenberg's matrix mechanics, because generalizations of the Ritz rule and the Bohr frequency condition are employed as guiding principles.

The same type of mechanics in the case that physical variables are n-index objects ($n \geq 4$) was also proposed in Ref. 1), but its formulation has not yet been completed. The purpose of the present paper is to construct a mechanics for multi-index objects that models the dynamical structure of Heisenberg's matrix mechanics.

This paper is organized as follows. In the next section, we define n-index objects. We formulate a dynamical system based on n-index objects in §3. Section 4 is devoted to conclusions.

§2. Generalized matrices

2.1. *Definitions*

Here, we define n-index objects (referred to as "n-th power matrices")**) and define related terminology. An n-th power matrix is an object with n indices written $A_{l_1 l_2 \cdots l_n}$. This is a generalization of a usual matrix, written analogously as $B_{l_1 l_2}$. We treat n-th power 'square' matrices, i.e., $N \times N \times \cdots \times N$ matrices, and treat the elements of these matrices as c-numbers throughout this paper.

First, we define the hermiticity of an n-th power matrix by the relation $A_{l'_1 l'_2 \cdots l'_n} = A^*_{l_1 l_2 \cdots l_n}$ for odd permutations among indices and refer to an n-th power matrix possessing the property of hermiticity as a hermitian n-th power matrix. Here, the asterisk indicates complex conjugation. A hermitian n-th power matrix satisfies the

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**) Many-index objects have been introduced to construct a quantum version of the Nambu bracket.[5],[6] The definition of the n-fold product we use is the same as that used by Xiong.

relation $A_{l'_1 l'_2 \cdots l'_n} = A_{l_1 l_2 \cdots l_n}$ for even permutations among indices. The components for which at least two indices are identical, e.g., $A_{l_1 \cdots l_i \cdots l_i \cdots l_n}$, which is the counterpart of the diagonal part of a hermitian matrix, are real-valued and symmetric with respect to permutations among indices $\{l_1, \cdots, l_i, \cdots, l_i, \cdots, l_n\}$. We refer to a special type of hermitian matrix whose components with all different indices are vanishing as a "real normal form" or a "real normal n-th power matrix". A normal n-th power matrix is written

$$A^{(N)}_{l_1 l_2 \cdots l_n} = \sum_{i<j} \delta_{l_i l_j} a_{l_j l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_n}, \tag{2·1}$$

where the summation is over all pairs among $\{l_1, \cdots, l_n\}$, the hatted indices are omitted, and $a_{l_j l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_n}$ is symmetric under the exchange of $n-2$ indices $\{l_1, \cdots, \hat{l}_i, \cdots, \hat{l}_j, \cdots, l_n\}$.

We define the n-fold product of n-th power matrices $(A_i)_{l_1 l_2 \cdots l_n}$, $(i = 1, 2, \cdots, n)$ by

$$(A_1 A_2 \cdots A_n)_{l_1 l_2 \cdots l_n} \equiv \sum_k (A_1)_{l_1 \cdots l_{n-1} k} (A_2)_{l_1 \cdots l_{n-2} k l_n} \cdots (A_n)_{k l_2 \cdots l_n}. \tag{2·2}$$

The resultant n-index object, $(A_1 A_2 \cdots A_n)_{l_1 l_2 \cdots l_n}$, is not necessarily hermitian, even if the n-th power matrices $(A_i)_{l_1 l_2 \cdots l_n}$ are hermitian. Note that the above product is, in general, neither commutative nor associative; for example, we have

$$\begin{aligned} &(A_1 A_2 \cdots A_n)_{l_1 l_2 \cdots l_n} \neq (A_2 A_1 \cdots A_n)_{l_1 l_2 \cdots l_n}, \\ &(A_1 \cdots A_{n-1}(A_n A_{n+1} \cdots A_{2n-1}))_{l_1 l_2 \cdots l_n} \\ &\qquad \neq ((A_1 \cdots A_{n-1} A_n) A_{n+1} \cdots A_{2n-1})_{l_1 l_2 \cdots l_n}. \end{aligned} \tag{2·3}$$

The n-fold commutator is defined by

$$\begin{aligned} &[A_1, A_2, \cdots, A_n]_{l_1 l_2 \cdots l_n} \\ &\quad \equiv \sum_{(i_1, \cdots, i_n)} \sum_k \mathrm{sgn}(P) (A_{i_1})_{l_1 \cdots l_{n-1} k} (A_{i_2})_{l_1 \cdots l_{n-2} k l_n} \cdots (A_{i_n})_{k l_2 \cdots l_n}, \end{aligned} \tag{2·4}$$

where the first summation is over all permutations among the subscripts $\{i_1, \cdots, i_n\}$. Here, $\mathrm{sgn}(P)$ is $+1$ and -1 for even and odd permutations among the subscripts $\{i_1, \cdots, i_n\}$, respectively. If the n-th power matrices $(A_i)_{l_1 l_2 \cdots l_n}$ are hermitian, then $i[A_1, A_2, \cdots, A_n]_{l_1 l_2 \cdots l_n}$ is also hermitian.

2.2. *Properties*

Here, we study some properties of the n-fold commutator $[A_1, A_2, \cdots, A_n]_{l_1 l_2 \cdots l_n}$. This commutator is written

$$\begin{aligned} [A_1, A_2, \cdots, A_n]_{l_1 l_2 \cdots l_n} = &(A_1)_{l_1 l_2 \cdots l_n} (A_2 \widetilde{A_3 \cdots} A_n)_{l_1 l_2 \cdots l_n} \\ &+ (-1)^{n-1} (A_2)_{l_1 l_2 \cdots l_n} (A_3 \widetilde{\cdots A_n} A_1)_{l_1 l_2 \cdots l_n} \\ &+ \cdots + (-1)^{n-1} (A_n)_{l_1 l_2 \cdots l_n} (A_1 \widetilde{A_2 \cdots} A_{n-1})_{l_1 l_2 \cdots l_n} \\ &+ ([A_1, A_2, \cdots, A_n])^0_{l_1 l_2 \cdots l_n}, \end{aligned} \tag{2·5}$$

where $(A_2\widetilde{A_3\cdots} A_n)_{l_1l_2\cdots l_n}$ and $([A_1,A_2,\cdots,A_n])^0_{l_1l_2\cdots l_n}$ are defined by

$$\begin{aligned}&(A_2\widetilde{A_3\cdots} A_n)_{l_1l_2\cdots l_n}\\&\quad\equiv\sum_{(i_2,\cdots,i_n)}\mathrm{sgn}(P)\Big((A_{i_2})_{l_1\cdots l_{n-2}l_nl_n}(A_{i_3})_{l_1\cdots l_{n-3}l_nl_{n-1}l_n}\cdots(A_{i_n})_{l_nl_2\cdots l_{n-1}l_n}\\&\qquad+(-1)^{n-1}(A_{i_2})_{l_1\cdots l_{n-3}l_{n-1}l_{n-1}l_n}(A_{i_3})_{l_1\cdots l_{n-4}l_{n-1}l_{n-2}l_{n-1}l_n}\cdots(A_{i_n})_{l_1\cdots l_{n-2}l_{n-1}l_{n-1}}\\&\qquad+\cdots+(-1)^{n-1}(A_{i_2})_{l_1\cdots l_{n-1}l_1}(A_{i_3})_{l_1\cdots l_{n-2}l_1l_n}\cdots(A_{i_n})_{l_1l_1l_3\cdots l_n}\Big)\end{aligned}\tag{2·6}$$

and

$$\begin{aligned}&([A_1,A_2,\cdots,A_n])^0_{l_1l_2\cdots l_n}\\&\quad\equiv\sum_{(i_1,\cdots i_n)}\sum_{k\neq l_1,l_2,\cdots,l_n}\mathrm{sgn}(P)(A_{i_1})_{l_1\cdots l_{n-1}k}(A_{i_2})_{l_1\cdots l_{n-2}kl_n}\cdots(A_{i_n})_{kl_2\cdots l_n},\end{aligned}\tag{2·7}$$

respectively.

We now discuss features of $(A_1\widetilde{A_2\cdots} A_{n-1})_{l_1l_2\cdots l_n}$. It is skew-symmetric with respect to permutations among indices, i.e.,

$$(A_1\widetilde{A_2\cdots} A_{n-1})_{l_1\cdots l_i\cdots l_j\cdots l_n}=-(A_1\widetilde{A_2\cdots} A_{n-1})_{l_1\cdots l_j\cdots l_i\cdots l_n},\tag{2·8}$$

if each $(A_k)_{l_jl_1\cdots\hat{l}_i\cdots l_n}$, $(k=1,\cdots,n-1)$ is symmetric with respect to permutations among the n-indices $\{l_j,l_1,\cdots,\hat{l}_i,\cdots,l_n\}$, as are hermitian n-th power matrices. We define the following operation on k-th antisymmetric objects $\omega_{l_1l_2\cdots l_k}$:

$$(\delta\omega)_{l_0l_1l_2\cdots l_k}\equiv\sum_{i=0}^{k}(-1)^i\omega_{l_0l_1\cdots\hat{l}_i\cdots l_k}.\tag{2·9}$$

Here, the operator δ is regarded as a coboundary operator that changes k-th antisymmetric objects into $(k+1)$-th objects, and it is nilpotent, i.e. $\delta^2(*)=0$.*) If $\omega_{l_1l_2\cdots l_k}$ satisfies the cocycle condition $(\delta\omega)_{l_0l_1l_2\cdots l_k}=0$, it is called a k-cocycle. We now give an example of a solution for the cocycle condition $(\delta(A_1\widetilde{A_2\cdots} A_{n-1}))_{l_0l_1l_2\cdots l_n}=0$. When one A_l is an arbitrary hermitian n-th power matrix and all the rest have components in the form $(A_l)_{l_jl_1\cdots\hat{l}_i\cdots l_n}=\sum_{l_k\neq\hat{l}_i}(a_l)_{l_k}$, then $(A_1\widetilde{A_2\cdots} A_{n-1})_{l_1l_2\cdots l_n}$ can be written

$$(A_1\widetilde{A_2\cdots} A_{n-1})_{l_1l_2\cdots l_n}=\sum_{i=1}^{n}(-1)^{i-1}\gamma_{l_1\cdots\hat{l}_i\cdots l_n}\equiv(\delta\gamma)_{l_1l_2\cdots l_n},\tag{2·10}$$

where each $\gamma_{l_1l_2\cdots l_{n-1}}$ is an $(n-1)$-th power antisymmetric object. Then, the n-th antisymmetric object $(A_1\widetilde{A_2\cdots} A_{n-1})_{l_1l_2\cdots l_n}$ automatically satisfies the cocycle condition

$$(\delta(A_1\widetilde{A_2\cdots} A_{n-1}))_{l_0l_1l_2\cdots l_n}=(\delta^2\gamma)_{l_0l_1\cdots\hat{l}_i\cdots l_n}=0\tag{2·11}$$

*) See Ref. 7) for treatments of cohomology.

due to the nilpotency of coboundary operations. This type of solution is called an n-coboundary.

We can demonstrate the following relations regarding the n-fold commutator from the above expressions and properties.

1. For arbitrary n-th power hermitian matrices A_j, we have $[A_1, \cdots, A_{n-1}, \Delta]_{l_1 l_2 \cdots l_n} = 0$ with $\Delta_{l_1 l_2 \cdots l_n} = \prod_{i<j} \delta_{l_i l_j}$. Here, the product is over all pairs among indices.
2. For arbitrary normal n-th power matrices $A_j^{(N)}$, the n-fold commutator among A and $A_j^{(N)}$ is given by
$$[A, A_1^{(N)}, \cdots, A_{n-1}^{(N)}]_{l_1 l_2 \cdots l_n} = A_{l_1 l_2 \cdots l_n} (\widetilde{A_1^{(N)} \cdots A_{n-1}^{(N)}})_{l_1 l_2 \cdots l_n}. \tag{2·12}$$
3. The n-fold commutator among arbitrary normal n-th power matrices $A_i^{(N)}$ is vanishing:
$$[A_1^{(N)}, A_2^{(N)}, \cdots, A_n^{(N)}]_{l_1 l_2 \cdots l_n} = 0. \tag{2·13}$$
4. If $(\widetilde{B_1^{(N)} B_2^{(N)}} \cdots B_{n-1}^{(N)})_{l_1 l_2 \cdots l_n}$ is an n-cocycle for normal n-th power matrices $B_j^{(N)}$, the following fundamental identity holds:
$$[[A_1, \cdots, A_n], B_1^{(N)}, \cdots, B_{n-1}^{(N)}]_{l_1 l_2 \cdots l_n} = \sum_{i=1}^{n} [A_1, \cdots, [A_i, B_1^{(N)}, \cdots, B_{n-1}^{(N)}], \cdots, A_n]_{l_1 l_2 \cdots l_n}. \tag{2·14}$$

§3. Dynamical system

In this section, we employ generalizations of the Ritz rule and the Bohr frequency condition as guiding principles and construct a generalization of Heisenberg's matrix mechanics based on hermitian n-th power matrices.

3.1. *Framework*

The time-dependent variables are hermitian n-th power matrices given by
$$(V_\alpha(t))_{l_1 l_2 \cdots l_n} = (V_\alpha)_{l_1 l_2 \cdots l_n} e^{i\Omega_{l_1 l_2 \cdots l_n} t}, \tag{3·1}$$
where the angular frequency $\Omega_{l_1 l_2 \cdots l_n}$ has the properties
$$\Omega_{l'_1 l'_2 \cdots l'_n} = \mathrm{sgn}(P) \Omega_{l_1 l_2 \cdots l_n}, \tag{3·2}$$
$$(\delta\Omega)_{l_0 l_1 l_2 \cdots l_n} \equiv \sum_{i=0}^{n} (-1)^i \Omega_{l_0 l_1 \cdots \hat{l}_i \cdots l_n} = 0. \tag{3·3}$$
The frequency $\Omega_{l_1 l_2 \cdots l_n}$ is regarded as an n-cocycle, from Eq. (3·3). This equation is a generalization of the Ritz rule,*) and it is required for the consistency

*) The Ritz rule is given by $\Omega_{l_1 l_3} = \Omega_{l_1 l_2} + \Omega_{l_2 l_3}$ in quantum mechanics, where $\Omega_{l_i l_j}$ is the angular frequency of the radiation from an atom.

of the time evolution of a system, as shown below. Note that the n-fold product, $(V_{\alpha_1} V_{\alpha_2} \cdots V_{\alpha_n})_{l_1 l_2 \cdots l_n} e^{i\Omega_{l_1 l_2 \cdots l_n} t}$, takes the same form as (3·1), with the relation (3·3).

The time-independent variables are real normal n-th power matrices given by

$$(U_a)_{l_1 l_2 \cdots l_n} = \sum_{i<j} \delta_{l_i l_j} (u_a)_{l_j l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_n}. \tag{3·4}$$

These variables are conserved quantities, and they are regarded as generators of a symmetry transformation.

Next we discuss the time evolution of physical variables. It is expressed as a symmetry transformation if the physical system is closed. In our mechanics, we conjecture that real normal n-th power matrices generate such a transformation. We refer to these as 'Hamiltonians' H_A $(A = 1, \cdots, n-1)$. Hamiltonians are functions of physical variables: $H_A = H_A(V_\alpha(t), U_a)$. By analogy with Heisenberg's matrix mechanics, we require that the n-th power matrices $(V_\alpha(t))_{l_1 l_2 \cdots l_n}$ yield the following generalization of the Heisenberg equation:

$$\frac{d}{dt}(V_\alpha(t))_{l_1 l_2 \cdots l_n} = \frac{1}{i\hbar^{(n)}}[V_\alpha(t), H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n}, \tag{3·5}$$

where $\hbar^{(n)}$ is a new physical constant. The left-hand side of (3·5) is written

$$\frac{d}{dt}(V_\alpha(t))_{l_1 l_2 \cdots l_n} = i\Omega_{l_1 l_2 \cdots l_n}(V_\alpha(t))_{l_1 l_2 \cdots l_n}, \tag{3·6}$$

by definition (3·1), while the right-hand side is written

$$\begin{aligned} &\frac{1}{i\hbar^{(n)}}[V_\alpha(t), H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} \\ &\qquad = \frac{1}{i\hbar^{(n)}}(\widetilde{H_1 \cdots H_{n-1}})_{l_1 l_2 \cdots l_n}(V_\alpha(t))_{l_1 l_2 \cdots l_n}, \end{aligned} \tag{3·7}$$

by use of the formula (2·12). From Eqs. (3·6) and (3·7), we obtain the relation

$$\hbar^{(n)}\Omega_{l_1 l_2 \cdots l_n} = -(\widetilde{H_1 \cdots H_{n-1}})_{l_1 l_2 \cdots l_n}. \tag{3·8}$$

This relation is a generalization of Bohr's frequency condition.*)

Let us make a consistency check for the time evolution of a system. By definition, an arbitrary normal n-th power matrix $A^{(N)}$ [and the time-independent part of $V_\alpha(t)$] should be a constant of motion, and this is verified by use of the equation of motion,

$$\frac{d}{dt}(A^{(N)})_{l_1 l_2 \cdots l_n} = \frac{1}{i\hbar^{(n)}}[A^{(N)}, H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} = 0, \tag{3·9}$$

where the formula (2·13) has been used. Because the Hamiltonians are real normal n-th power matrices, they are conserved quantities. The n-fold commutator,

*) Bohr's frequency condition is given by $\hbar\Omega_{l_1 l_2} = -\widetilde{H}_{l_1 l_2} = E_{l_1} - E_{l_2}$ in quantum mechanics, where E_l is the energy eigenvalue of an atom.

$i[V_1(t), \cdots, V_n(t)]$ should satisfy the fundamental identity including the Hamiltonians,

$$[i[V_1(t), \cdots, V_n(t)], H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} \\ = \sum_{i=1}^{n} i[V_1(t), \cdots, [V_i(t), H_1, \cdots, H_{n-1}], \cdots, V_n(t)]_{l_1 l_2 \cdots l_n} \quad (3\cdot10)$$

from the requirement that the derivation rule for the time hold as follows:

$$\frac{d}{dt} i[V_1(t), \cdots, V_n(t)]_{l_1 l_2 \cdots l_n} = \sum_{i=1}^{n} i[V_1(t), \cdots, \frac{d}{dt} V_i(t), \cdots, V_n(t)]_{l_1 l_2 \cdots l_n}. \quad (3\cdot11)$$

The fundamental identity (3·10) holds in the case that the frequency $\Omega_{l_1 l_2 \cdots l_n}$ is an n-cocycle from the 4-th relation regarding the n-fold commutator.

3.2. *Examples*

Now, we study the simple example of a harmonic oscillator whose variables are two kinds of hermitian n-th power matrices given by $\xi(t)_{l_1 l_2 \cdots l_n} = \xi_{l_1 l_2 \cdots l_n} e^{i\Omega_{l_1 l_2 \cdots l_n} t}$ and $\eta(t)_{l_1 l_2 \cdots l_n} = \eta_{l_1 l_2 \cdots l_n} e^{i\Omega_{l_1 l_2 \cdots l_n} t}$. Here, each of the indices l_i runs from 1 to n. The coefficients $\xi_{l_1 l_2 \cdots l_n}$ and $\eta_{l_1 l_2 \cdots l_n}$ are given by

$$\xi_{l_1 l_2 \cdots l_n} = \sqrt{\frac{\hbar^{(n)}}{2m\Omega}} |\varepsilon_{l_1 l_2 \cdots l_n}|, \quad \eta_{l_1 l_2 \cdots l_n} = \frac{1}{i} \sqrt{\frac{m\Omega\hbar^{(n)}}{2}} \varepsilon_{l_1 l_2 \cdots l_n}, \quad (3\cdot12)$$

where the quantity m in the square root represents a mass, $\Omega = |\Omega_{12\cdots n}|$, and $\varepsilon_{l_1 l_2 \cdots l_n}$ is the n-dimensional Levi-Civita symbol.

If $\Omega_{l_1 l_2 \cdots l_n} = -\Omega \varepsilon_{l_1 l_2 \cdots l_n}$, we obtain the equations of motion describing the harmonic oscillator:

$$\frac{d}{dt} \xi(t)_{l_1 l_2 \cdots l_n} = \frac{1}{m} \eta(t)_{l_1 l_2 \cdots l_n}, \quad (3\cdot13)$$

$$\frac{d}{dt} \eta(t)_{l_1 l_2 \cdots l_n} = -m\Omega^2 \xi(t)_{l_1 l_2 \cdots l_n}. \quad (3\cdot14)$$

The Hamiltonians H_A satisfy the relation

$$\hbar^{(n)} \Omega \varepsilon_{l_1 l_2 \cdots l_n} = (H_1 \widetilde{\cdots H}_{n-1})_{l_1 l_2 \cdots l_n} \quad (3\cdot15)$$

from the requirement that physical variables yield the generalized Heisenberg equation (3·5). As an example of H_A, we have

$$(H_1)_{n\ n\ 1 \cdots n-2} = \hbar^{(n)} \Omega, \quad (H_B)_{n\ n\ 1 \cdots \hat{i} \cdots n-1} = \delta_{n-i\ B}, \quad (3\cdot16)$$

where each H_A is symmetric with respect to permutations among indices, other components of H_A are vanishing, and B runs from 2 to $n-1$. The following algebraic relation holds among $\xi(t)$, $\eta(t)$ and the Hamiltonians H_A:

$$(H_1)_{l_1 l_2 \cdots l_n} = i\Omega[\xi(t), \eta(t), H_2, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n}. \quad (3\cdot17)$$

Finally, we give an example of an 'n-plet', $(V_\alpha(t))_{l_1 l_2 \cdots l_n}$ ($\alpha = 1, \cdots, n$), where each of the indices l_i runs from 1 to n^2. The components of $V_\alpha(t)$ are defined by

$$(V_\alpha(t))_{l_1 l_2 \cdots l_n} \equiv \begin{cases} \eta(t)_{l_1 l_2 \cdots l_n} & \text{for } \; l_i = (\alpha-1)n+1, (\alpha-1)n+2, \cdots, \alpha n, \\ \xi(t)_{l_1 l_2 \cdots l_n} & \text{for } \; l_i = \alpha n+1, \alpha n+2, \\ & \qquad \cdots, (\alpha+1)n \pmod{n^2}, \\ (\zeta_{n-j})_{l_1 l_2 \cdots l_n} & \text{for } \; l_i = (\alpha+j-1)n+1, (\alpha+j-1)n+2, \\ & \qquad \cdots, (\alpha+j)n \pmod{n^2}, \end{cases} \tag{3·18}$$

where $(\zeta_{n-j})_{l_1 l_2 \cdots l_n}$ represents the $n \times n \times \cdots \times n$ matrices whose non-vanishing components are given by

$$(\zeta_{n-j})_{kn\ kn\ (k-1)n+1 \cdots \widehat{(k-1)n+i} \cdots kn-1} = \delta_{n-i\ n-j+1}. \tag{3·19}$$

Here, i and j run from 2 to $n-1$ and each ζ_{n-j} is symmetric with respect to permutations among indices $\{kn\ kn\ (k-1)n+1 \cdots \widehat{(k-1)n+i} \cdots kn-1\}$ for $k = 1, \cdots, n$. We find that the time-dependent components in $V_\alpha(t)$ yield the equation of motion of harmonic oscillators for the Hamiltonians whose non-vanishing components are given by

$$(H_1)_{kn\ kn\ (k-1)n+1 \cdots kn-2} = (-1)^{k(n-1)} \hbar^{(n)} \Omega, \tag{3·20}$$

$$(H_B)_{kn\ kn\ (k-1)n+1 \cdots \widehat{(k-1)n+i} \cdots kn-1} = \delta_{n-i\ B}, \tag{3·21}$$

where B runs from 2 to $n-1$ and k runs from 1 to n. The Hamiltonians H_A are symmetric with respect to permutations among indices. The n-th power matrices $(V_\alpha(t))_{l_1 l_2 \cdots l_n}$ form an algebra characterized as follows:

$$[V_1(t), V_2(t), \cdots, V_n(t)]_{l_1 l_2 \cdots l_n} = -i\hbar^{(n)} (J^{(N)})_{l_1 l_2 \cdots l_n}, \tag{3·22}$$

where $(J^{(N)})_{l_1 l_2 \cdots l_n}$ is the real normal $n^2 \times n^2 \times \cdots \times n^2$ matrices whose non-vanishing components are given by

$$(J^{(N)})_{kn\ kn\ (k-1)n+1 \cdots kn-2} = (-1)^{k(n-1)}. \tag{3·23}$$

In both cases, the $n+1$ variables, $(\xi(t), \eta(t), H_A)$ or $(V_\alpha(t), J^{(N)})$, form a closed algebra for the n-fold commutator, which is regarded as a generalization of the spin algebra.[9),8),6),10)]

§4. Conclusions

We have defined n-index objects (n-th power matrices) and their algebraic properties and formulated a dynamical system based on hermitian n-th power matrices. The basic structure of our mechanics is summarized as follows. For hermitian n-th power matrices $(V_\alpha(t))_{l_1 l_2 \cdots l_n} = (V_\alpha)_{l_1 l_2 \cdots l_n} e^{i\Omega_{l_1 l_2 \cdots l_n} t}$, their time evolution is regarded as the symmetry transformation generated by the Hamiltonians $(H_A)_{l_1 l_2 \cdots l_n}$ given by $i\hbar^{(n)} \delta(V_\alpha(t))_{l_1 l_2 \cdots l_n} = [V_\alpha(t), H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} \delta t$, which is a generalization of the Heisenberg equation. The Hamiltonians $(H_A)_{l_1 l_2 \cdots l_n}$ are real normal forms, where $(H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \cdots l_n}$ satisfies the n-cocycle condition. Among the $\Omega_{l_1 l_2 \cdots l_n}$ and H_A, we have the relation $\hbar^{(n)} \Omega_{l_1 l_2 \cdots l_n} = -(H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \cdots l_n}$. An arbitrary normal n-th power matrix $A^{(N)}_{l_1 l_2 \cdots l_n}$ is a constant of motion; i.e., $i\hbar^{(n)} dA^{(N)}_{l_1 l_2 \cdots l_n}/dt = [A^{(N)}, H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} = 0$. There are simple systems of harmonic oscillators described by hermitian n-th power matrices.

Our mechanics can be regarded as a generalization of Heisenberg's matrix mechanics because they are equivalent in case $n = 2$. In quantum mechanics, a matrix element $A_{l_1 l_2}$ is interpreted as a probability amplitude between the state labeled by l_1 and that labeled by l_2. A similar interpretation for an n-th power matrix element $(n \geq 3)$, however, is not yet known, and it is not clear whether many-index objects are applicable to real physical systems.*) It would be worthwhile to explore the physical meaning of many-index objects based on generalized spin variables.

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References

1) Y. Kawamura, Prog. Theor. Phys. **107** (2002), 1105; hep-th/0203007.
2) Y. Kawamura, Prog. Theor. Phys. **109** (2003), 1; hep-th/0206184.
3) Y. Kawamura, Prog. Theor. Phys. **109** (2003), 153; hep-th/0207054.
4) Y. Nambu, Phys. Rev. D **7** (1973), 2405.
5) H. Awata, M. Li, D. Minic and T. Yoneya, J. High Energy Phys. **02** (2001), 013.
6) C.-S. Xiong, Phys. Lett. B **486** (2000), 228.
7) R. Bott and L. Tu, *Differential forms in algebraic topology* (Springer, 1982).
C. Nath and S. Sen, *Topology and Geometry for Physicists* (Academic Press, London, 1983).
8) J. Hoppe, Helv. Phys. Acta. **70** (1997), 302.
9) V. A. Filippov, Sib. Math. J. **26** (1985), 126.
10) Y. Kawamura, Prog. Theor. Phys. **110** (2003), 579; hep-th/0304149.

*) In Ref. 10), a generalization of spin algebra using three-index objects is proposed, and the connection between triple commutation relations and uncertainty relations is discussed.