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# Classification and Dynamics of Equivalence Classes in $SU(N)$ Gauge Theory on the Orbifold $S^1/Z_2$

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The dynamical determination of the boundary conditions in $SU(N)$ gauge theory on the orbifold $S^1/Z_2$ is investigated. We classify the equivalence classes of the boundary conditions, and then the vacuum energy density of the theory in each equivalence class is evaluated at one loop order. Unambiguous comparison of the vacuum energy densities in the two theories in different equivalence classes becomes possible in supersymmetric theories. It is found that in the supersymmetric $SU(5)$ models with the Scherk-Schwarz supersymmetry breaking, the theory with the boundary conditions yielding the standard model symmetry is in the equivalence class with the lowest energy density, though the low energy theory is not identically the minimal supersymmetric standard model. We also study how particular boundary conditions are chosen in the cosmological evolution of the universe.

## §1. Introduction

In higher-dimensional grand unified theory (GUT), the gauge fields and the Higgs fields in the adjoint representation in the lower four-dimensions are unified.[1),2)] Previously Manton attempted to unify the doublet Higgs fields with the gauge fields in the electroweak theory by adopting a monopole-type gauge field configuration in the extra-dimensional space in a larger gauge group.[3)] However, such a configuration with nonvanishing field strengths has a higher energy density, and the dynamical stability of the field configuration remains to be justified. It since has been recognized that in higher-dimensional GUT defined on a multiply connected manifold, the dynamics of Wilson line phases can induce dynamical gauge symmetry breaking by developing nonvanishing expectation values through radiative corrections. This provides real unification of the gauge fields and Higgs fields. Here the symmetry is broken by dynamics, not by hand. This Hosotani mechanism has been extensively investigated since that time,[4)−7)] but the incorporation of chiral fermions has been a major obstacle to constructing a realistic model.[6),8),9)] This idea has been recently revived under the name of 'gauge-Higgs unification'.[10)] Further, when the extra dimensions are deconstructed, some of the extra-dimensional components of the gauge fields acquire masses through radiative corrections, and pseudo-Nambu-Goldstone bosons in the lower four dimensions become 'little Higgs'.[11)]

Significant progress has been made in this direction by considering GUT on an orbifold. This provides a new way to solve such problems as the chiral fermion problem of how four-dimensional chiral fermions are generated from a higher-dimensional space-time and the Higgs mass splitting problem[12] (the 'triplet-doublet mass splitting problem' in $SU(5)$ GUT[13),14)]). In formulating gauge theory on an orbifold,[14)−24)] however, there arise many possibilities for boundary conditions (BCs) to be imposed on the fields in the extra-dimensional space. This leads to the problem of which type of BCs should be imposed without relying on phenomenological information. We refer to this problem as the 'arbitrariness problem'.[25)]

The arbitrariness problem is partially solved at the quantum level by the Hosotani mechanism as shown in our previous paper.[24),*)] The rearrangement of gauge symmetry takes place through the dynamics of the Wilson line phases. The physical symmetry of the theory, in general, differs from the symmetry of the BCs. Several sets of BCs with distinct symmetry can be related by large gauge transformations, belonging to the same equivalence class. This implies the reduction of the number of independent theories.

The remaining problem is to determine how a realistic theory is selected among various equivalence classes. We have not obtained a definite solution to this problem, partially because of a lack of understanding of a more fundamental theory that contains dynamics which select various BCs. Nevertheless, one can expect that the classification and characterization of equivalence classes will provide information as to how one of them is selected dynamically. In particular, the evaluation of the vacuum energy density in each equivalence class would provide critical information needed to solve the arbitrariness problem.[25)] This motivates the present work.

In the present paper, we classify the equivalence classes of the BCs in $SU(N)$ gauge theory on $S^1/Z_2$, and evaluate the vacuum energy density in each equivalence class. Some discussion is given with regard to the question of how particular BCs are selected in the cosmological evolution of the universe.

In §2 general arguments are given for BCs in gauge theories on the orbifold $S^1/Z_2$, and we classify those BCs with the aid of equivalence relations originating from the gauge invariance. In §3 generic formulas for the one-loop effective potential at vanishing Wilson line phases are derived. These can be applied to every equivalence class. The arguments are generalized to supersymmetric theories in §4. There it is recognized that unambiguous comparison of the vacuum energy densities in theories on different equivalence classes is possible. In §5, we discuss how particular boundary conditions are chosen in the cosmological evolution of the universe. Section 6 is devoted to conclusions and discussion.

## §2. Orbifold conditions and classification of equivalence classes

In this article, we focus on $SU(N)$ gauge theory defined on a five-dimensional space-time $M^4 \times (S^1/Z_2)$ where $M^4$ is the four-dimensional Minkowski spacetime. The fifth dimension, $S^1/Z_2$, is obtained by identifying two points on $S^1$ by parity. Let $x$ and $y$ be coordinates of $M^4$ and $S^1$, respectively. $S^1$ has a radius $R$, and therefore a point $y + 2\pi R$ is identified with a point $y$. The orbifold $S^1/Z_2$ is obtained by further

*) See Refs. 26) and 27) for SUSY breaking and Ref. 18) for gauge symmetry breaking on $S^1/Z_2$ by the Hosotani mechanism.

identifying $-y$ and $y$. The resultant fifth dimension is the interval $0 \leq y \leq \pi R$, which contains the information on $S^1$.

### 2.1. *Boundary conditions*

For $Z_2$ transformations around $y = 0$ and $y = \pi R$ and a loop translation along $S^1$, each defined by

$$Z_0: \ y \to -y, \quad Z_1: \ \pi R + y \to \pi R - y, \quad S: \ y \to y + 2\pi R \ , \tag{2·1}$$

the following relations hold:

$$Z_0^2 = Z_1^2 = I, \quad S = Z_1 Z_0, \quad S Z_0 S = Z_0 \ . \tag{2·2}$$

Here $I$ is the identity operation. Although we have the identification $y \sim y + 2\pi R \sim -y$ on $S^1/Z_2$, fields do not necessarily take identical values at $(x, y)$, $(x, y + 2\pi R)$ and $(x, -y)$ as long as the Lagrangian density is single-valued. The general BCs for a field $\varphi(x, y)$ are given by

$$\begin{aligned} &\varphi(x, -y) = T_\varphi[P_0]\varphi(x, y), \quad \varphi(x, \pi R - y) = T_\varphi[P_1]\varphi(x, \pi R + y), \\ &\varphi(x, y + 2\pi R) = T_\varphi[U]\varphi(x, y), \end{aligned} \tag{2·3}$$

where $T_\varphi[P_0]$, $T_\varphi[P_1]$ and $T_\varphi[U]$ represent appropriate representation matrices, including an arbitrary sign factor. The counterparts of (2·2) are given by

$$T_\varphi[P_0]^2 = T_\varphi[P_1]^2 = I, \ \ T_\varphi[U] = T_\varphi[P_0]T_\varphi[P_1], \ \ T_\varphi[U]T_\varphi[P_0]T_\varphi[U] = T_\varphi[P_0]. \tag{2·4}$$

In (2·4) $I$ represents an unit matrix. For instance, if $\varphi$ belongs to the fundamental representation of the $SU(N)$ gauge group, then $T_\varphi[P_0]\varphi$ is $\pm P_0\varphi$ where $P_0$ is a hermitian $U(N)$ matrix, i.e., $P_0^\dagger = P_0 = P_0^{-1}$. The same property applies to $P_1$.

The BCs imposed on a gauge field $A_M$ are

$$A_\mu(x, -y) = P_0 A_\mu(x, y) P_0^\dagger, \ A_y(x, -y) = -P_0 A_y(x, y) P_0^\dagger, \tag{2·5}$$
$$A_\mu(x, \pi R - y) = P_1 A_\mu(x, \pi R + y) P_1^\dagger, \ A_y(x, \pi R - y) = -P_1 A_y(x, \pi R + y) P_1^\dagger, \tag{2·6}$$
$$A_M(x, y + 2\pi R) = U A_M(x, y) U^\dagger. \tag{2·7}$$

The BCs of scalar fields $\phi^A$ are given by

$$\begin{aligned} &\phi^A(x, -y) = T_{\phi^A}[P_0]\phi^A(x, y) \ , \\ &\phi^A(x, \pi R - y) = \sum_B \left(e^{i\pi\beta M}\right)^A{}_B T_{\phi^B}[P_1]\phi^B(x, \pi R + y) \ , \\ &\phi^A(x, y + 2\pi R) = \sum_B \left(e^{i\pi\beta M}\right)^A{}_B T_{\phi^B}[U]\phi^B(x, y) \ , \end{aligned} \tag{2·8}$$

where $A$ and $B$ are indices and $M$ is a matrix in the flavor space. If nontrivial $Z_2$ parity is assigned in the flavor space and $M$ anti-commutes with the $Z_2$ parity, then $\beta$ can take an arbitrary value. For Dirac fields $\psi^A$ defined in the bulk, the gauge invariance of the kinetic energy term requires

$$\psi^A(x, -y) = T_{\psi^A}[P_0]\gamma^5\psi^A(x, y) \ ,$$

$$\psi^A(x,\pi R-y) = \sum_B \left(e^{i\pi\beta M}\right)^A{}_B T_{\psi^B}[P_1]\gamma^5\psi^B(x,\pi R+y) \quad ,$$
$$\psi(x,y+2\pi R) = \sum_B \left(e^{i\pi\beta M}\right)^A{}_B T_{\psi^B}[U]\psi^B(x,y) \quad . \tag{2·9}$$

To summarize, the BCs in gauge theories on $S^1/Z_2$ are specified with $(P_0, P_1, U, \beta)$ and additional sign factors. The matrices $P_0$ and $P_1$ need not be diagonal in general. One can always diagonalize one of them, say $P_0$, through a global gauge transformation, but $P_1$ might not be diagonal. For the reasons described in the subsequent sections, we consider BCs with diagonal $P_0$ and $P_1$.

The diagonal $P_0$ and $P_1$ are specified by three non-negative integers $(p, q, r)$ such that

$$\text{diag } P_0 = (\overbrace{+1,\cdots,+1,+1,\cdots,+1,-1,\cdots,-1,-1,\cdots,-1}^{N}) \ ,$$
$$\text{diag } P_1 = (\underbrace{+1,\cdots,+1}_{p},\underbrace{-1,\cdots,-1}_{q},\underbrace{+1,\cdots,+1}_{r},\underbrace{-1,\cdots,-1}_{s=N-p-q-r}) \ , \tag{2·10}$$

where $N \geq p, q, r, s \geq 0$. We denote each BCs specified by $(p, q, r)$ (or a theory with such BCs) as $[p; q, r; s]$. The matrix $P_0$ is interchanged with $P_1$ by the interchange of $q$ and $r$ such that

$$[p; q, r; s] \leftrightarrow [p; r, q; s]. \tag{2·11}$$

### 2.2. *Residual gauge invariance and equivalence classes*

Given the BCs $(P_0, P_1, U, \beta)$, there still remains residual gauge invariance. Recall that under a gauge transformation $\Omega(x, y)$, we have

$$A_M \to A'_M = \Omega A_M \Omega^\dagger - \frac{i}{g}\Omega\partial_M\Omega^\dagger \quad ,$$
$$\phi^A \to \phi'^A = T_{\phi^A}[\Omega]\phi^A \quad , \quad \psi^A \to \psi'^A = T_{\psi^A}[\Omega]\psi^A \quad , \tag{2·12}$$

where $g$ is a gauge coupling constant. The new fields $A'_M$ satisfy, instead of (2·6) – (2·7),

$$\begin{pmatrix} A'_\mu(x,-y) \\ A'_y(x,-y) \end{pmatrix} = P'_0 \begin{pmatrix} A'_\mu(x,y) \\ -A'_y(x,y) \end{pmatrix} P'^\dagger_0 - \frac{i}{g} P'_0 \begin{pmatrix} \partial_\mu \\ -\partial_y \end{pmatrix} P'^\dagger_0 \quad ,$$
$$\begin{pmatrix} A'_\mu(x,\pi R-y) \\ A'_y(x,\pi R-y) \end{pmatrix} = P'_1 \begin{pmatrix} A'_\mu(x,\pi R+y) \\ -A'_y(x,\pi R+y) \end{pmatrix} P'^\dagger_1 - \frac{i}{g} P'_1 \begin{pmatrix} \partial_\mu \\ -\partial_y \end{pmatrix} P'^\dagger_1 \quad ,$$
$$A'_M(x,y+2\pi R) = U' A'_M(x,y) U'^\dagger - \frac{i}{g} U'\partial_M U'^\dagger \quad , \tag{2·13}$$

where

$$P'_0 = \Omega(x,-y)\, P_0\, \Omega^\dagger(x,y) \quad , \quad P'_1 = \Omega(x,\pi R-y)\, P_1\, \Omega^\dagger(x,\pi R+y) \quad ,$$
$$U' = \Omega(x,y+2\pi R)\, U\, \Omega^\dagger(x,y) \quad . \tag{2·14}$$

Other fields $\phi'^A$ and $\psi'^A$ satisfy relations similar to (2·8) and (2·9) where $(P_0, P_1, U)$ are replaced by $(P_0', P_1', U')$.

The residual gauge invariance of the BCs is given by gauge transformations which preserve the given BCs, namely those transformations which satisfy $U' = U$, $P_0' = P_0$ and $P_1' = P_1$:

$$\begin{aligned} &\Omega(x,-y)\,P_0 = P_0\,\Omega(x,y) \quad , \quad \Omega(x,\pi R - y)\,P_1 = P_1\Omega(x,\pi R + y) \quad , \\ &\Omega(x, y + 2\pi R)\,U = U\,\Omega(x,y). \end{aligned} \tag{2·15}$$

We call the residual gauge invariance of the BCs the symmetry of the BCs.

The low energy symmetry of the BCs which is defined independently of the $y$-coordinate, is given by

$$\Omega(x)\,P_0 = P_0\,\Omega(x) \quad , \quad \Omega(x)\,P_1 = P_1\Omega(x) \quad , \quad \Omega(x)\,U = U\,\Omega(x) \quad , \tag{2·16}$$

that is, the symmetry is generated by generators that commute with $P_0$, $P_1$ and $U$.

Theories with different BCs can be equivalent with regard to physical content. The key observation is that in gauge theory, physics should not depend on the gauge chosen so that one is always free to choose the gauge. If $(P_0', P_1', U')$ satisfies the conditions

$$\partial_M P_0' = 0 \quad , \quad \partial_M P_1' = 0 \quad , \quad \partial_M U' = 0 \quad , \quad P_0'^{\dagger} = P_0' \quad , \quad P_1'^{\dagger} = P_1' \quad , \tag{2·17}$$

then the two sets of the BCs are equivalent:

$$(P_0', P_1', U') \sim (P_0, P_1, U) \quad . \tag{2·18}$$

It is easy to show that $(P_0', P_1', U')$ satisfy the relations (2·4). The equivalence relation (2·18) defines equivalence classes of the BCs. Here we stress that the BCs indeed change under general gauge transformations. To illustrate this, let us consider an $SU(2)$ gauge theory with $(P_0, P_1, U) = (\tau_3, \tau_3, I)$. If we carry out the gauge transformation $\Omega = \exp\{i(\alpha y/2\pi R)\tau_2\}$, we find the equivalence

$$(\tau_3, \tau_3, I) \sim (\tau_3, e^{i\alpha\tau_2}\tau_3, e^{i\alpha\tau_2}) \quad . \tag{2·19}$$

In particular, for $\alpha = \pi$ we have the equivalence

$$(\tau_3, \tau_3, I) \sim (\tau_3, -\tau_3, -I) \quad . \tag{2·20}$$

Using this equivalence, we can derive the following equivalence relations in $SU(N)$ gauge theory:

$$\begin{aligned} [p;q,r;s] &\sim [p-1;q+1,r+1;s-1], \quad \text{for} \quad p,s \geq 1 \quad , \\ &\sim [p+1;q-1,r-1;s+1], \quad \text{for} \quad q,r \geq 1. \end{aligned} \tag{2·21}$$

The symmetry of the BCs in one theory differs from that in the other, but the two theories are connected by the BCs-changing gauge transformation and are equivalent. This equivalence is guaranteed by the Hosotani mechanism as explained in the next subsection.

### 2.3. *The Hosotani mechanism and physical symmetry*

The two theories with distinct symmetry of BCs are equivalent to each other in physics content. This statement is verified by the dynamics of the Wilson line phases as a part of the Hosotani mechanism. The Hosotani mechanism in gauge theories defined on multiply connected manifolds consists of several parts.[1),2)]

(i) Wilson line phases are phases of $WU$ defined by

$$WU \equiv P \exp\left\{ ig \int_C dy A_y \right\} U, \tag{2·22}$$

where $C$ is a non-contractible loop. The eigenvalues of $WU$ are gauge invariant and become physical degrees of freedom. Wilson line phases cannot be gauged away and parametrize degenerate vacua at the classical level.

(ii) The degeneracy is lifted by quantum effects in general. The physical vacuum is given by the configuration of Wilson line phases that minimizes the effective potential $V_{\text{eff}}$.

(iii) If the configuration of the Wilson line phases is non-trivial, the gauge symmetry is spontaneously broken or restored by radiative corrections. Nonvanishing expectation values of the Wilson line phases give masses to those gauge fields in lower dimensions whose gauge symmetry is broken. Some of matter fields also acquire masses.

(iv) A nontrivial $V_{\text{eff}}$ also implies that all extra-dimensional components of gauge fields become massive.

(v) Two sets of BCs for fields can be related to each other by a BCs-changing gauge transformation. They are physically equivalent, even if the two sets have distinct symmetry of the BCs. This defines equivalence classes of the BCs. $V_{\text{eff}}$ depends on the BCs so that the expectation values of the Wilson line phases depend on the BCs. The physical symmetry of the theory is determined by the combination of the BCs and the expectation values of the Wilson line phases. Theories in the same equivalence class of the BCs have the same physical symmetry and physical content.

(vi) The physical symmetry of the theory is mostly dictated by the matter content of the theory.

(vii) The mechanism provides unification of gauge fields and Higgs scalar fields in the adjoint representation, namely the gauge-Higgs unification.

Let us spell out the part (v) of the mechanism in gauge theory defined on $M^4 \times (S^1/Z_2)$. Dynamical Wilson line phases are given by $\{g\pi R A_y^a \ , \ \frac{1}{2}\lambda^a \in \mathcal{H}_W\}$, where $\mathcal{H}_W$ is a set of generators which anti-commute with $P_0$ and $P_1$:

$$\mathcal{H}_W = \left\{ \ \frac{\lambda^a}{2} \ ; \ \{\lambda^a, P_0\} = \{\lambda^a, P_1\} = 0 \ \right\} \ . \tag{2·23}$$

Suppose that with $(P_0, P_1, U, \beta)$, $V_{\text{eff}}$ is minimized at $\langle A_y \rangle$ such that $W \equiv \exp(i2\pi g R \langle A_y \rangle) \neq I$. Perform a BCs-changing gauge transformation given by $\omega = \exp\{i\pi g(y+\alpha)\langle A_y \rangle\}$. This transforms $\langle A_y \rangle$ into $\langle A'{}_y \rangle = 0$. Under this transformation, the BCs change to

$$(P_0^{\rm sym}, P_1^{\rm sym}, U^{\rm sym}, \beta) \equiv (P_0', P_1', U', \beta) = (e^{2ig\alpha\langle A_y\rangle} P_0, e^{2ig(\alpha+\pi R)\langle A_y\rangle} P_1, WU, \beta). \tag{2·24}$$

Because the expectation values of $A_y'$ vanish in the new gauge, the physical symmetry is spanned by the generators that commute with $(P_0^{\rm sym}, P_1^{\rm sym}, U^{\rm sym})$:

$$\mathcal{H}^{\rm sym} = \left\{ \frac{\lambda^a}{2} \ ; \ [\lambda^a, P_0^{\rm sym}] = [\lambda^a, P_1^{\rm sym}] = 0 \right\} \ . \tag{2·25}$$

The group generated by $\mathcal{H}^{\rm sym}$, $H^{\rm sym}$, is the unbroken physical symmetry of the theory.

### 2.4. *Classification of equivalence classes*

The classification of equivalence classes of the BCs is reduced to the classification of $(P_0, P_1)$. As briefly mentioned in §2.1, $P_0$ can be made diagonal through a suitable global gauge transformation. Then $P_1$ is not diagonal in general. As explained in §2.3 two BCs, $(P_0, P_1)$ and $(P_0', P_1')$, can be in the same equivalence class. In each equivalence class the vacuum with the lowest energy density is chosen by the dynamics of Wilson line phases. Each equivalence class is characterized by $(P_0^{\rm sym}, P_1^{\rm sym})$ in (2·24).

Let $(P_0, P_1)$ be said to be diagonal if both $P_0$ and $P_1$ are diagonal. There are three questions of physical relevance;
(Q1) Does each equivalence class have a diagonal representative $(P_0, P_1)$?
(Q2) Can we choose $(P_0^{\rm sym}, P_1^{\rm sym})$ to be diagonal?
(Q3) Which $(P_0^{\rm sym}, P_1^{\rm sym})$, among all equivalence classes, has the lowest energy density?

The answer to (Q1) is affirmative. The proof is given in Appendix A. It is shown there that nontrivial BCs are reduced to BCs in $SU(2)$ subspaces. As for (Q2), we do not have a satisfactory answer at the moment. In a previous paper,[24] it is found that in many of the BCs of physical interest, $(P_0^{\rm sym}, P_1^{\rm sym})$ is diagonal. Even if one starts with a non-diagonal $(P_0, P_1)$, the Hosotani mechanism yields a diagonal $(P_0^{\rm sym}, P_1^{\rm sym})$. If the answer to (Q2) is affirmative, the investigation of (Q3) becomes feasible. What is to be done is (1) to list all diagonal pairs $(P_0, P_1)$ and (2) to evaluate the energy density or the effective potential at the vanishing Wilson line phases in each diagonal $(P_0, P_1)$.

One can show that the number of equivalence classes of BCs is $(N+1)^2$ in the $SU(N)$ model. The proof goes as follows. We count the number $n_1$ of diagonal pairs $(P_0, P_1)$ and the number $n_2$ of equivalence relations among those diagonal $(P_0, P_1)$. As proved in Appendix A, every equivalence class has a diagonal representative $(P_0, P_1)$. Hence, the number of equivalence classes is given by $n_1 - n_2$.

The number $n_1$ is found by counting all the different $[p; q, r; s]$ $(p+q+r+s = N)$ defined in (2·10). We write $[p; q, r; s] = [N-k; q, r; k-j]$, where $j = q + r$. The value $k$ runs from 0 to $N$, while $j$ runs from 0 to $k$. Given $(k, j)$, there are $(j+1)$

combinations for $(q, r)$. Hence,

$$n_1 = \sum_{k=0}^{N} \sum_{j=0}^{k} (j+1) = \frac{1}{6}(N+1)(N+2)(N+3) \quad . \tag{2·26}$$

The equivalence relation (2·21) is written as $[N-k; q, r; k-j] = [N-k-1; q+1, r+1; k-j-1]$. In this case, $k$ runs from 1 to $N-1$, while $j$ runs from 0 to $k-1$. Therefore

$$n_2 = \sum_{k=1}^{N-1} \sum_{j=0}^{k-1} (j+1) = \frac{1}{6}(N-1)N(N+1) \quad . \tag{2·27}$$

Thus the number of equivalence classes is $n_1 - n_2 = (N+1)^2$.

## §3. Effective potential

There exist $(N+1)^2$ equivalence classes in $SU(N)$ gauge theory on $M^4 \times (S^1/Z_2)$. We examine the question of which equivalence class has the lowest energy density. It may well be that such an equivalence class is preferentially chosen by the dynamics governing BCs, the arbitrariness problem thus being solved. Other scenarios are possible, however, in the cosmological evolution of the universe. We come back to this point in §5.

We evaluate the one-loop effective potential $V_{\rm eff}$ for each theory with diagonal $(P_0, P_1)$. The effective potential $V_{\rm eff}$ depends not only on Wilson line phases but also on BCs, i.e., $V_{\rm eff} = V_{\rm eff}[A_M^0; P_0, P_1, \beta]$ where $A_M^0$ is a background configuration of the gauge field $A_M$, in $SU(N)$ gauge theory. In a more fundamental theory, the BCs $(P_0, P_1)$ would not be parameters at our disposal, but would be determined by dynamics. The resultant effective theory belongs to a specific equivalence class of BCs.

Our goal is to find the global minimum of $V_{\rm eff}$. This is not an easy task, as it is difficult to write down a generic formula for $V_{\rm eff}$ including $A_M^0$ explicitly. We consider the case of vanishing VEV's of $A_M$ with diagonal $(P_0, P_1)$. As remarked in the previous section, the global minimum of $V_{\rm eff}$ in many cases of physical interest has been found at diagonal $(P_0^{\rm sym}, P_1^{\rm sym})$.

The effective potential for $A_M^0$ is derived by writing $A_M = A_M^0 + A_M^q$, taking a suitable gauge fixing and integrating over the quantum part $A_M^q$. If the gauge fixing term is also invariant under the gauge transformation, i.e.,

$$D^M(A^0)A_M = 0 \to D^M(A'^0)A'_M = \Omega D^M(A^0)A_M\Omega^\dagger = 0 \quad , \tag{3·1}$$

it is shown that $V_{\rm eff}$ on $M^4 \times (S^1/Z_2)$ satisfies the relation

$$V_{\rm eff}[A_M^0; P_0, P_1, U, \beta] = V_{\rm eff}[A'^0_M; P'_0, P'_1, U', \beta] \quad . \tag{3·2}$$

This property implies that the minimum of $V_{\rm eff}$ corresponds to the same symmetry as that of $(P_0^{\rm sym}, P_1^{\rm sym}, U^{\rm sym})$.

The one-loop effective potential for $A^0_M$ on $M^4 \times (S^1/Z_2)$ is given by

$$V_{\rm eff}[A^0_M; P_0, P_1, U, \beta] = \sum \mp \frac{i}{2} \,{\rm Tr}\, \ln D_M(A^0) D^M(A^0) \quad , \tag{3·3}$$

$$= \sum \mp \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \, \frac{1}{\pi R} \, \sum_n \ln(-p^2 + M_n^2 - i\varepsilon) \quad , \tag{3·4}$$

where we have supposed that $F^0_{MN} = 0$ and every scalar field is also massless. The sums extend over all degrees of freedom of fields in the bulk in (3·3) and all degrees of freedom of 4-dimensional fields whose masses are $M_n$ in (3·4). The sign is negative (positive) for bosons (FP ghosts and fermions). $D_M(A^0)$ denotes an appropriate covariant derivative with respect to $A^0_M$. The quantity $V_{\rm eff}$ depends on $A^0_M$ and the BCs, $(p, q, r; \beta)$. Hereafter we take $A^0_M = 0$ on the basis of the assumption that $V_{\rm eff}$ has a minimum there when both $P_0$ and $P_1$ are taken in an appropriate diagonal form.

On the orbifold $S^1/Z_2$ all fields are classified as either $Z_2$ singlets or $Z_2$ doublets. The mode expansion of $Z_2$ singlet fields $\phi^{(P_0 P_1)}(x, y)$ is given by

$$\phi^{(++)}(x,y) = \frac{1}{\sqrt{\pi R}} \, \phi_0(x) + \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \phi_n(x) \, \cos \frac{ny}{R} \quad ,$$

$$\phi^{(--)}(x,y) = \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \phi_n(x) \, \sin \frac{ny}{R} \quad ,$$

$$\phi^{(+-)}(x,y) = \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \phi_n(x) \, \cos \frac{(n+\frac{1}{2})y}{R} \quad ,$$

$$\phi^{(-+)}(x,y) = \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \phi_n(x) \, \sin \frac{(n+\frac{1}{2})y}{R} \quad , \tag{3·5}$$

where $\pm$ indicates the eigenvalue $\pm 1$ of $Z_2$ parity. The mass terms in four-dimensional space-time are derived from the kinetic $y$-derivative terms after compactification. They are

$$\left(\frac{n}{R}\right)^2 \quad (n \geq 0) \quad , \quad \left(\frac{n}{R}\right)^2 \quad (n \geq 1) \quad , \quad \left(\frac{n+\frac{1}{2}}{R}\right)^2 \quad (n \geq 0) \tag{3·6}$$

for $\phi^{(++)}$, $\phi^{(--)}$ and $\phi^{(+-)}$ / $\phi^{(-+)}$, respectively. Let us define $N^{(P_0P_1)} = N_B^{(P_0P_1)} - N_F^{(P_0P_1)}$, where $N_B^{(P_0P_1)}$ ($N_F^{(P_0P_1)}$) is the number of bosonic (fermionic) fields whose $Z_2$ parities are $P_0$ and $P_1$. The value $N^{(P_0P_1)}$ depends on $(p, q, r)$ of the BCs. For $Z_2$ singlet fields, the formula for $V_{\rm eff}$ becomes

$$V_{\rm eff}|_{Z_2 \text{ singlet}} = N_0 \epsilon_0 + N_\Delta \Delta\epsilon + N_v v(\tfrac{1}{2}) \tag{3·7}$$

where

$$N_0 = N^{(++)} + N^{(--)} + N^{(+-)} + N^{(-+)} \quad , \tag{3·8}$$

$$N_\Delta = N^{(++)} - N^{(--)} \quad , \tag{3·9}$$
$$N_v = N^{(+-)} + N^{(-+)} \quad , \tag{3·10}$$
$$\epsilon_0 \equiv -\frac{1}{4}\int \frac{d^4p_E}{(2\pi)^4}\,\frac{1}{\pi R}\sum_{n=-\infty}^{\infty}\ln\left[p_E^2+\left(\frac{n}{R}\right)^2\right] \quad , \tag{3·11}$$
$$\Delta\epsilon \equiv -\frac{1}{4}\int \frac{d^4p_E}{(2\pi)^4}\,\frac{1}{\pi R}\ln\left[p_E^2\right] \quad , \tag{3·12}$$
$$\begin{aligned} v(\beta) &\equiv -\frac{1}{4}\int \frac{d^4p_E}{(2\pi)^4}\,\frac{1}{\pi R}\sum_{n=-\infty}^{\infty}\left(\ln\left[p_E^2+\left(\frac{n+\beta}{R}\right)^2\right]-\ln\left[p_E^2+\left(\frac{n}{R}\right)^2\right]\right) \\ &= \frac{3}{256\pi^7R^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\left(1-\cos 2\pi n\beta\right) . \end{aligned} \tag{3·13}$$

Here, $p_E$ is a four-dimensional Euclidean momentum. Also the Wick rotation has been applied. The quantities $\epsilon_0$ and $\Delta\epsilon$ are divergent, whereas $v(\beta)$ is finite.

In gauge theory on the orbifold $S^1/Z_2$, there appears another important representation, a $Z_2$ doublet. A $Z_2$ doublet field $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ satisfies

$$\begin{aligned} &\phi(x,-y) = \tau_3\phi(x,y) \quad , \\ &\phi(x,y+2\pi R) = e^{-2\pi i\beta\tau_2}\,\phi(x,y) = \begin{pmatrix}\cos 2\pi\beta & -\sin 2\pi\beta \\ \sin 2\pi\beta & \cos 2\pi\beta\end{pmatrix}\phi(x,y) \quad . \end{aligned} \tag{3·14}$$

Its expansion is given by

$$\begin{pmatrix}\phi_1(x,y)\\ \phi_2(x,y)\end{pmatrix} = \frac{1}{\sqrt{\pi R}}\sum_{n=-\infty}^{\infty}\phi_n(x)\begin{pmatrix}\cos\dfrac{(n+\beta)y}{R}\\ \sin\dfrac{(n+\beta)y}{R}\end{pmatrix} \quad , \tag{3·15}$$

or, more concisely, we write

$$\begin{pmatrix}\phi_1(x,y)\\ \phi_2(x,y)\end{pmatrix} = \Big\{\phi_n(x)\ ;\ \beta\Big\} \quad . \tag{3·16}$$

This type of BCs frequently appears in the $SU(2)_R$ space in supersymmetric theories. As the mass for $\phi_n(x)$ is given by $[(n+\beta)/R]^2$, $V_{\rm eff}$ for a $Z_2$ doublet bosonic field is given by

$$\begin{aligned} V_{\rm eff}\big|_{Z_2\ \rm doublet} &= -\frac{1}{2}\int \frac{d^4p_E}{(2\pi)^4}\,\frac{1}{\pi R}\sum_{n=-\infty}^{\infty}\ln\left[p_E^2+\left(\frac{n+\beta}{R}\right)^2\right] \\ &= \epsilon_0 + v(\beta), \end{aligned} \tag{3·17}$$

where $\epsilon_0$ and $v(\beta)$ are defined in (3·11) and (3·13).

Let us apply the above result to non-supersymmetric $SU(N)$ gauge theories. To be definite, the matter content in the bulk is assumed to consist of $n_s$ species of complex scalar fields in the fundamental representation, $n_{fF}$ species of Dirac fermions in the fundamental representation, and $n_{fA}$ species of Dirac fermions in the 2nd rank antisymmetric representation. We further suppose no additional flavor symmetry that can generate $Z_2$ doublets whose BCs are given in (3·14).

The values $n_s^{(\eta_0\eta_1)}$, $n_{fF}^{(\eta_0\eta_1)}$ and $n_{fA}^{(\eta_0\eta_1)}$ are the numbers of scalar fields and Dirac fermions, whose BCs are given by

$$\begin{aligned}
&\phi(x,-y) = \eta_0 P_0 \phi(x,y) \quad , \quad \phi(x,\pi R - y) = \eta_1 P_1 \phi(x, \pi R + y) \quad , \\
&\psi_F(x,-y) = \eta_0 P_0 \gamma^5 \psi_F(x,y) \quad , \quad \psi_F(x,\pi R - y) = \eta_1 P_1 \gamma^5 \psi_F(x,\pi R + y) \quad , \\
&\psi_A(x,-y) = \eta_0 P_0 \gamma^5 \psi_A(x,y) P_0^t \quad , \quad \psi_A(x,\pi R - y) = \eta_1 P_1 \gamma^5 \psi_A(x,\pi R + y) P_1^t \quad ,
\end{aligned} \tag{3·18}$$

respectively. Here, $\eta_0$ and $\eta_1$ take the value 1 or $-1$. The sums of the numbers $n_s^{(\eta_0\eta_1)}$, $n_{fF}^{(\eta_0\eta_1)}$ and $n_{fA}^{(\eta_0\eta_1)}$ are denoted $n_s$, $n_{fF}$ and $n_{fA}$:

$$n_s = \sum_{(\eta_0\eta_1)} n_s^{(\eta_0\eta_1)} \quad , \quad n_{fF} = \sum_{(\eta_0\eta_1)} n_{fF}^{(\eta_0\eta_1)} \quad , \quad n_{fA} = \sum_{(\eta_0\eta_1)} n_{fA}^{(\eta_0\eta_1)} \quad . \tag{3·19}$$

Let us introduce $N_{\rm rep}^{(P_0,P_1)}$ for each representation by

$$\begin{aligned}
N_{Ad}^{(++)} &= p^2 + q^2 + r^2 + s^2 - 1 \quad , \\
N_{Ad}^{(--)} &= 2(ps + qr) \quad , \\
N_{Ad}^{(+-)} &= 2(pq + rs) \quad , \\
N_{Ad}^{(-+)} &= 2(pr + qs) \quad , \\
N_F^{(++)} &= p \quad , \quad N_F^{(--)} = s \quad , \quad N_F^{(+-)} = q \quad , \quad N_F^{(-+)} = r \quad , \\
N_A^{(++)} &= \tfrac{1}{2}\Big\{ p(p-1) + q(q-1) + r(r-1) + s(s-1) \Big\} \quad , \\
N_A^{(--)} &= ps + qr \quad , \\
N_A^{(+-)} &= pq + rs \quad , \\
N_A^{(-+)} &= pr + qs \quad .
\end{aligned} \tag{3·20}$$

Under the $Z_2$ parity assignment (2·10), the quantities $N^{(P_0P_1)}$ are found to be

$$N^{(P_0P_1)} = N_g^{(P_0P_1)} + N_s^{(P_0P_1)} - N_f^{(P_0P_1)} \quad , \tag{3·21}$$

$$N_g^{(P_0P_1)} = 2N_{Ad}^{(P_0,P_1)} + N_{Ad}^{(-P_0,-P_1)} \quad , \tag{3·22}$$

$$N_s^{(P_0,P_1)} = \sum_{\eta_0,\eta_1} 2 n_s^{(\eta_0,\eta_1)} N_F^{(\eta_0 P_0, \eta_1 P_1)} \quad , \tag{3·23}$$

$$\begin{aligned}
N_f^{(P_0P_1)} = &\sum_{\eta_0,\eta_1} 2 n_{fF}^{(\eta_0,\eta_1)} \big(N_F^{(\eta_0 P_0,\eta_1 P_1)} + N_F^{(-\eta_0 P_0,-\eta_1 P_1)}\big) \\
&+ \sum_{\eta_0,\eta_1} 2 n_{fA}^{(\eta_0,\eta_1)} \big(N_A^{(\eta_0 P_0,\eta_1 P_1)} + N_A^{(-\eta_0 P_0,-\eta_1 P_1)}\big),
\end{aligned} \tag{3·24}$$

where $N_g^{(P_0P_1)}$, $N_s^{(P_0P_1)}$ and $N_f^{(P_0P_1)}$ are the contributions from the gauge and FP ghost fields, complex scalar fields and Dirac fermions. The $A_y$ components of the gauge fields are opposite in parity to the four-dimensional components $A_\mu$. This leads to the expression (3·22). For fermions, the factor $\gamma^5$ in (3·18) implies that $(P_0, P_1)$ states are always accompanied by $(-P_0, -P_1)$ states. This leads to (3·24).

By use of (3·21), the formula for one-loop effective potential at $A_M^0 = 0$ is given by

$$
\begin{aligned}
V_{\rm eff} &= N_0\epsilon_0 + N_\Delta \Delta\epsilon + N_v v(\tfrac{1}{2}) \quad , \\
N_0 &= 3(N^2-1) + 2n_s N - 4n_{fF}N - 2n_{fA}N(N-1) \quad , \\
N_\Delta &= (p-s)^2 + (q-r)^2 - 1 + 2(n_s^{(++)} - n_s^{(--)})(p-s) \\
&\qquad + 2(n_s^{(+-)} - n_s^{(-+)})(q-r) \quad , \\
N_v &= (6-4n_{fA})(p+s)(q+r) \\
&\qquad + 2(n_s^{(++)} + n_s^{(--)} - 2n_{fF}^{(++)} - 2n_{fF}^{(--)})(q+r) \\
&\qquad + 2(n_s^{(+-)} + n_s^{(-+)} - 2n_{fF}^{(+-)} - 2n_{fF}^{(+-)})(p+s) \quad .
\end{aligned} \tag{3·25}
$$

At this stage, we recognize that $N_0$ is independent of the BCs, and therefore independent of $[p;q,r;s]$. The $N_0$ term does not distinguish BCs. By contrast, $N_\Delta$ and $N_v$ do depend on $[p;q,r;s]$. There appears a difference in the energy density among theories in different equivalence classes. It is tempting to seek a theory with the lowest energy density which may be most preferred, provided there exists dynamics connecting different equivalence classes. However, there arises fundamental ambiguity in the $N_\Delta$ term: $\Delta\epsilon$ is divergent. There is no symmetry principle that dictates unique regularization. One cannot compare the energy densities in two theories in different equivalence classes.

In a previous paper,[24)] we evaluated $V_{\rm eff}$ in one equivalence class as a function of the Wilson line phases. The value of $N_\Delta$ is the same in all theories in a given equivalence class, and therefore there appeared no ambiguity there. [See (2·21) and (3·25).] For instance, in the $SU(5)$ model, we have the equivalence relations

$$
[p;q,r;s] = [2;0,0;3] \sim [1;1,1;2] \sim [0;2,2;1] \quad . \tag{3·26}
$$

Suppose that all matter fields satisfy $\eta_0 = \eta_1 = 1$ and set $n_s^{(++)} = N_h$, $n_{fF}^{(++)} = N_f^5$, $n_{fA}^{(++)} = N_f^{10}$. It then follows from (3·25) for the effective potential $V_{\rm eff}[p,q,r,s] \equiv V_{\rm eff}[A^0 = 0; p,q,r,s]$ that

$$
\begin{aligned}
V_{\rm eff}[1,1,1,2] - V_{\rm eff}[2,0,0,3] &= 4(9 + N_h - 2N_f^5 - 6N_f^{10})v(\tfrac{1}{2}) \quad , \\
V_{\rm eff}[0,2,2,1] - V_{\rm eff}[2,0,0,3] &= 8(3 + N_h - 2N_f^5 - 2N_f^{10})v(\tfrac{1}{2}) \quad .
\end{aligned} \tag{3·27}
$$

These agree with the result in Ref. 24).[*)] When $N_f^5 = N_f^{10} = 0$, for instance, the theory with $[p;q,r;s] = [2;0,0;3]$, which has the $SU(3)\times SU(2)\times U(1)$ symmetry, has the lowest energy density.

*) $v(\beta)$ is related to $f_5(\beta)$ in Ref. 24) by $v(\beta) = \frac{1}{4}C[f_5(0) - f_5(2\beta)]$ and $C = 3/64\pi^7 R^5$. There was a factor 2 error in the normalization of the effective potential in Ref. 24).

However, if one tries to compare theories in different equivalence classes, the ambiguity in the $N_\Delta$ term cannot be avoided. This ambiguity naturally disappears in supersymmetric theories, as we spell out below.

## §4. Supersymmetric gauge theory

In this section, we derive generic formulas for the one-loop effective potential at vanishing Wilson line phases in supersymmetric (SUSY) $SU(N)$ gauge theories and compare the vacuum energy density in the theories belonging to various equivalence classes of the BCs.

If the theory has unbroken supersymmetry, then the effective potential for Wilson lines remains flat due to the cancellation among contributions from bosonic fields and fermionic fields. A nontrivial dependence of $V_{\rm eff}$ appears if SUSY is softly broken as the nature demands. There is a natural way to introduce soft SUSY breaking on an orbifold. $N=1$ SUSY in five-dimensional space-time corresponds to $N=2$ SUSY in four-dimensional spacetime. A five-dimensional gauge multiplet $\mathcal{V}=(A_M,\lambda,\lambda',\sigma)\equiv(A_M,\lambda_L^1,\lambda_L^2,\sigma)$ is decomposed into a vector superfield $V=(A_\mu,\lambda)$ and a chiral superfield $\Sigma=(\Phi\equiv\sigma+iA_y,\lambda')$ in four dimensions. Similarly, a hypermultiplet $\mathcal{H}=(h,h^{c\dagger},\tilde{h},\tilde{h}^{c\dagger})\equiv(h_1,h_2,\tilde{h}_L,\tilde{h}_R)$ is decomposed into two chiral superfields in four dimensions as $H=(h,\tilde{h})$ and $H^c=(h^c,\tilde{h}^c)$ where $H$ and $H^c$ undergo conjugated transformation under $SU(N)$. After a translation along a non-contractible loop, these fields and their superpartners may have different twist, depending on their $SU(2)_R$ charges.[15),17)] This is called the Scherk-Schwarz breaking mechanism.[28)] We adopt this mechanism for the SUSY breaking, which makes the evaluation of the effective potential easy.

The theory can contain several kinds of hypermultiplets in various kinds of representations in the bulk, some of which play the role of the Higgs multiplets or of the quark-lepton multiplets. Further there may exist $N=1$ supermultiplets on the boundary branes.[*)] Here we write down the bulk part of the typical Lagrangian density $\mathcal{L}_{\rm bulk}$ for $\mathcal{V}$ and $\mathcal{H}$ to discuss their BCs:

$$\begin{aligned}\mathcal{L}_{\rm bulk}=&\frac{1}{g^2}\left(-\frac{1}{2}{\rm Tr}F_{MN}^2+{\rm Tr}|D_M\Phi|^2+{\rm Tr}(i\bar{\lambda}^i\gamma^M D_M\lambda^i)-{\rm Tr}(\bar{\lambda}^i[\Phi,\lambda^i])\right)\\&+|D_M h_i|^2+\bar{\tilde{H}}(i\gamma^M D_M-\Phi)\tilde{H}-(i\sqrt{2}h_i^\dagger\bar{\lambda}^i\bar{\tilde{H}}+{\rm h.c.})\\&-h_i^\dagger\Phi^2 h_i-\frac{g^2}{2}\sum_{m,\alpha}\left(h_i^\dagger(\tau^m)_{ij}T_h^\alpha h_j\right)^2\quad.\end{aligned}\tag{4·1}$$

Here the quantities $\lambda^i$ are the symplectic Majorana spinors defined in Ref. 31): $\lambda^j=\begin{pmatrix}\lambda_L^j\\ i\epsilon^{jk}\sigma^2\lambda_L^{k\,*}\end{pmatrix}$. $\tilde{H}$ is a Dirac spinor, $\tilde{H}=\begin{pmatrix}\tilde{h}_L\\ \tilde{h}_R\end{pmatrix}$, and $T_h^\alpha$'s are representation

*) It is known that anomalies may arise at the boundaries with chiral fermions.[29)] These anomalies must be cancelled in the four-dimensional effective theory, for instance, by such local counter terms as the Chern-Simons term.[29),30)] We assume that the four-dimensional effective theory is anomaly free.

matrices of $SU(N)$ gauge generators for $h$.

The requirement that the Lagrangian density be single valued allows the following nontrivial BCs. For the gauge multiplet, we have

$$\begin{aligned}
&\begin{pmatrix} V \\ \Sigma \end{pmatrix}(x,-y) = P_0 \begin{pmatrix} V \\ -\Sigma \end{pmatrix}(x,y)\, P_0^\dagger \quad , \\[2ex]
&A_\mu(x,\pi R - y) = P_1\, A_\mu(x,\pi R + y)\, P_1^\dagger \quad , \\
&A_y(x,\pi R - y) = -P_1\, A_y(x,\pi R + y)\, P_1^\dagger \quad , \\
&\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}(x,\pi R - y) = e^{-2\pi i\beta\tau_2}\, P_1 \begin{pmatrix} \lambda \\ -\lambda' \end{pmatrix}(x,\pi R + y)\, P_1^\dagger \quad , \\
&\sigma(x,\pi R - y) = -P_1\, \sigma(x,\pi R + y)\, P_1^\dagger \quad , \\[2ex]
&A_M(x,y+2\pi R) = U\, A_M(x,y)\, U^\dagger \quad , \\
&\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}(x,y+2\pi R) = e^{-2\pi i\beta\tau_2}\, U \begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}(x,y)\, U^\dagger \quad , \\
&\sigma(x,y+2\pi R) = U\, \sigma(x,y)\, U^\dagger \quad .
\end{aligned} \tag{4·2}$$

For a hypermultiplet $\mathcal{H}$, the BCs are given by

$$\begin{aligned}
&\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,-y) = \eta_0 T_{\mathcal{H}}[P_0] \begin{pmatrix} h \\ -h^{c\dagger} \end{pmatrix}(x,y) \quad , \\
&\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,\pi R - y) = e^{-2\pi i\beta\tau_2}\, \eta_1 T_{\mathcal{H}}[P_1] \begin{pmatrix} h \\ -h^{c\dagger} \end{pmatrix}(x,\pi R + y) \quad , \\
&\begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,y+2\pi R) = e^{-2\pi i\beta\tau_2}\, \eta_0\eta_1 T_{\mathcal{H}}[U] \begin{pmatrix} h \\ h^{c\dagger} \end{pmatrix}(x,y) \quad , \\[2ex]
&\begin{pmatrix} \tilde h \\ \tilde h^{c\dagger} \end{pmatrix}(x,-y) = \eta_0 T_{\mathcal{H}}[P_0] \begin{pmatrix} \tilde h \\ -\tilde h^{c\dagger} \end{pmatrix}(x,y) \quad , \\
&\begin{pmatrix} \tilde h \\ \tilde h^{c\dagger} \end{pmatrix}(x,\pi R - y) = \eta_1 T_{\mathcal{H}}[P_1] \begin{pmatrix} \tilde h \\ -\tilde h^{c\dagger} \end{pmatrix}(x,\pi R + y) \quad , \\
&\begin{pmatrix} \tilde h \\ \tilde h^{c\dagger} \end{pmatrix}(x,y+2\pi R) = \eta_0\eta_1 T_{\mathcal{H}}[U] \begin{pmatrix} \tilde h \\ \tilde h^{c\dagger} \end{pmatrix}(x,y) \quad .
\end{aligned} \tag{4·3}$$

Here $T_{\mathcal{H}}[P_0]h$ represents $P_0 h$ or $P_0 h P_0^\dagger$ for $h$ in the fundamental or adjoint representation, respectively. A hypermultiplet with parity $(\eta_0,\eta_1)$ gives the same contribution to the vacuum energy density in the bulk as a hypermultiplet with parity $(-\eta_0,-\eta_1)$.

With a nonvanishing $\beta$, there appear soft SUSY breaking mass terms for gauginos $\lambda^i$ and scalar fields $h_i$ in the four-dimensional theory.*) From the above BCs (4·2) and (4·3), mode expansions of each field are obtained. Let $(P_0,P_1)_\lambda$ be the parity assignment of each component of $\lambda(x,y)$ defined by $P_0\lambda P_0$ and $P_1\lambda P_1$. Depending

*) $\beta$ should be of order $10^{-14}$, on the basis of phenomenological consideration, for the soft SUSY breaking masses to be $O(1)$ TeV if $1/R \sim 10^{16}$ GeV.

on $(P_0, P_1)_\lambda$, the mode expansion for each component of gauginos is given by

$$
\begin{aligned}
(P_0,P_1)_\lambda = (+1,+1):\quad & \begin{pmatrix}\lambda(x,y)\\ \lambda'(x,y)\end{pmatrix} = \Big\{\lambda_n(x)\ ;\ \beta\Big\}\ ,\\
(-1,-1):\quad & \begin{pmatrix}\lambda'(x,y)\\ \lambda(x,y)\end{pmatrix} = \Big\{\lambda_n(x)\ ;\ -\beta\Big\}\ ,\\
(+1,-1):\quad & \begin{pmatrix}\lambda(x,y)\\ \lambda'(x,y)\end{pmatrix} = \Big\{\lambda_n(x)\ ;\ \beta+\tfrac{1}{2}\Big\}\ ,\\
(-1,+1):\quad & \begin{pmatrix}\lambda'(x,y)\\ \lambda(x,y)\end{pmatrix} = \Big\{\lambda_n(x)\ ;\ -\beta-\tfrac{1}{2}\Big\}\ .
\end{aligned}
\tag{4·4}
$$

Hence the contribution to $V_{\rm eff}$ from the gauginos is given by

$$
\begin{aligned}
V_{\rm eff}|_{\rm gauginos} = -4\Big\{ & N_{Ad}^{(++)}(\epsilon_0 + v(\beta)) + N_{Ad}^{(--)}(\epsilon_0 + v(-\beta))\\
& + N_{Ad}^{(+-)}(\epsilon_0 + v(\beta+\tfrac{1}{2})) + N_{Ad}^{(-+)}(\epsilon_0 + v(-\beta-\tfrac{1}{2}))\Big\}\ ,
\end{aligned}
\tag{4·5}
$$

where the factor 4 comes from the number of the degrees of freedom for each Majorana fermion. The values $N_{Ad}^{(P_0P_1)}$ are defined in (3·20).

The boson part of a gauge multiplet contains an additional scalar field $\sigma$, which has the same parity assignment as $A_y$. Hence the contributions from the boson part are

$$
\begin{aligned}
V_{\rm eff}|_{\rm gauge,ghost,\sigma} = & (2N_{Ad}^{(++)} + 2N_{Ad}^{(--)})(\epsilon_0 + \varDelta\epsilon) + (2N_{Ad}^{(--)} + 2N_{Ad}^{(++)})(\epsilon_0 - \varDelta\epsilon)\\
& + (2N_{Ad}^{(+-)} + 2N_{Ad}^{(-+)})(\epsilon_0 + v(\tfrac{1}{2})) + (2N_{Ad}^{(-+)} + 2N_{Ad}^{(+-)})(\epsilon_0 + v(\tfrac{1}{2}))\ .
\end{aligned}
\tag{4·6}
$$

Notice that the $\varDelta\epsilon$ terms cancel among the contributions from the bosons in the supersymmetric theory.

The gauge multiplet part of $V_{\rm eff}$ is obtained by adding (4·5) and (4·6):

$$
\begin{aligned}
V_{\rm eff}|_{\rm gauge\ multiplet} &= -4(N^2-1)v(\beta) + 4(N_{Ad}^{(+-)} + N_{Ad}^{(-+)})w(\beta)\\
&= -4(N^2-1)v(\beta) + 4(p+s)(q+r)w(\beta)\ ,\\
w(\beta) &\equiv v(\tfrac{1}{2}) + v(\beta) - v(\beta+\tfrac{1}{2}) \ge 0\ ,
\end{aligned}
\tag{4·7}
$$

where the relation $v(-\beta) = v(\beta)$ has been used. $V_{\rm eff}|_{\rm gauge\ multiplet}$ takes its minimum value, $-4(N^2-1)v(\beta)$, at $p = s = 0$, $q + r = N$ or $p + s = N$, $q = r = 0$. Each case corresponds to

$$
\begin{aligned}
{\rm diag}\ P_0 &= (\overbrace{+1,\cdots\cdots,+1,-1,\cdots\cdots,-1}^{N})\ ,\\
{\rm diag}\ P_1 &= (\underbrace{-1,\cdots\cdots,-1}_{q},\underbrace{+1,\cdots\cdots,+1}_{r})
\end{aligned}
\tag{4·8}
$$

or

$$\begin{aligned}\text{diag } P_0 &= (\overbrace{+1,\cdots\cdots,+1,-1,\cdots\cdots,-1}^{N}) \ , \\ \text{diag } P_1 &= (\underbrace{+1,\cdots\cdots,+1}_{p},\underbrace{-1,\cdots\cdots,-1}_{s}) \ ,\end{aligned} \tag{4·9}$$

respectively. The gauge symmetry is broken to $SU(q)\times SU(N-q)\times U(1)$ or $SU(p)\times SU(N-p)\times U(1)$, respectively.

It is interesting to examine what types of breaking patterns are induced by the introduction of hypermultiplets. The mode expansion for the hypermultiplet $\mathcal{H}=(h,h^{c\dagger},\tilde{h},\tilde{h}^{c\dagger})$ is found in a similar manner. Let $(P_0,P_1)_h=(a,b)_h$ be the parity assignment of each component of $h(x,y)$ defined by $\eta_0(T_{\mathcal{H}}[P_0]h)^j=ah^j$ and $\eta_1(T_{\mathcal{H}}[P_1]h)^j=bh^j$. Then, depending on $(P_0,P_1)_h$, one finds

$$\begin{aligned}(P_0,P_1)_h=(+1,+1):\begin{pmatrix} h \\ h^{c\dagger}\end{pmatrix} &= \Big\{ h_n(x)\ ;\ \beta \Big\} \ , \\ (-1,-1):\begin{pmatrix} h^{c\dagger} \\ h\end{pmatrix} &= \Big\{ h_n(x)\ ;\ -\beta \Big\} \ , \\ (+1,-1):\begin{pmatrix} h \\ h^{c\dagger}\end{pmatrix} &= \Big\{ h_n(x)\ ;\ \beta+\tfrac{1}{2} \Big\} \ , \\ (-1,+1):\begin{pmatrix} h^{c\dagger} \\ h\end{pmatrix} &= \Big\{ h_n(x)\ ;\ -\beta-\tfrac{1}{2} \Big\} \ .\end{aligned} \tag{4·10}$$

For their fermionic superpartners, the mode expansions are obtained by setting $\beta=0$ in (4·10). Hence the contribution to $V_{\rm eff}$ from $\mathcal{H}$ is given by

$$\begin{aligned}V_{\rm eff}|_{\mathcal{H}} &= 4\Big(N_h^{(++)}+N_h^{(--)}\Big)v(\beta)+4\Big(N_h^{(+-)}+N_h^{(-+)}\Big)\Big(v(\beta+\tfrac{1}{2})-v(\tfrac{1}{2})\Big) \\ &= 4N_h^{\rm total}v(\beta)-4\Big(N_h^{(+-)}+N_h^{(-+)}\Big)w(\beta) \ .\end{aligned} \tag{4·11}$$

Here $N_h^{(P_0,P_1)}$ is the number of components of $h(x,y)$ with parity $(P_0,P_1)$. The value $N_h^{\rm total}=\sum_{a,b}N_h^{(ab)}$ is independent of $(p,q,r,s)$. We note that

$$N_h^{(P_0,P_1)}=\sum_{{\rm rep}=Ad,F,A} n_{\rm rep}^{(\eta_0,\eta_1)}N_{\rm rep}^{(\eta_0 P_0,\eta_1 P_1)} \ , \tag{4·12}$$

where the quantities $N_{\rm rep}^{(a,b)}$ are given in (3·20).

Suppose that the matter content in the bulk is given by $n_{Ad}^{(\pm)}$, $n_F^{(\pm)}$ and $n_A^{(\pm)}$ species of hypermultiplets with $\eta_0\eta_1=\pm1$ in the adjoint, fundamental, and 2nd rank antisymmetric representations, respectively. (Note that $n_{\rm rep}^{(+)}=n_{\rm rep}^{(++)}+n_{\rm rep}^{(--)}$ and $n_{\rm rep}^{(-)}=n_{\rm rep}^{(+-)}+n_{\rm rep}^{(-+)}$.) It is straightforward to extend our analysis including hypermultiplets in bigger representations. The total effective potential is given by

$$\begin{aligned}
V_{\text{eff}} = & \\
& 4\left\{(-1+n_{Ad}^{(+)}+n_{Ad}^{(-)})(N^2-1)+(n_F^{(+)}+n_F^{(-)})N+(n_A^{(+)}+n_A^{(-)})\frac{N(N-1)}{2}\right\}v(\beta) \\
& +4\left\{(p+s)(q+r)(2-2n_{Ad}^{(+)}+2n_{Ad}^{(-)}-n_A^{(+)}+n_A^{(-)})\right. \\
& \qquad \left.-n_{Ad}^{(-)}(N^2-1)-n_A^{(-)}\frac{N(N-1)}{2}-n_F^{(+)}(q+r)-n_F^{(-)}(p+s)\right\}w(\beta) \\
= & \ (p,q,r,s\text{-independent terms})+4w(\beta)\ h(q+r) \ . 
\end{aligned} \tag{4·13}$$

Here the function $h(x)$ $(0 \le x = q+r = N-(p+s) \le N)$ is defined by

$$\begin{aligned}
& h(x) = ax(N-x) - bx \ , \\
& \begin{cases} a = 2-2n_{Ad}^{(+)}+2n_{Ad}^{(-)}-n_A^{(+)}+n_A^{(-)} \ , \\ b = n_F^{(+)}-n_F^{(-)} \ . \end{cases}
\end{aligned} \tag{4·14}$$

As $w(\beta) > 0$ for nonintegral $\beta$, the minimum of the energy density is given by that of $h(x)$, which is determined by $a$ and $b$.

Let us classify various cases.

(i) The case with $a = 0$

In this case the BCs which give the minimum energy density are

$$\begin{cases} n_F^{(+)} > n_F^{(-)} & \Rightarrow \quad q+r = N \ , \\ n_F^{(+)} = n_F^{(-)} & \Rightarrow \quad \text{completely degenerate} \ , \\ n_F^{(+)} < n_F^{(-)} & \Rightarrow \quad q+r = 0 \ . \end{cases} \tag{4·15}$$

(ii) The case with $a > 0$

In this case, $h(x)$ is minimized either at $x = 0$ or at $x = N$. As $h(N) - h(0) = -Nb$, we have

$$\begin{cases} n_F^{(+)} > n_F^{(-)} & \Rightarrow \quad q+r = N \ , \\ n_F^{(+)} = n_F^{(-)} & \Rightarrow \quad q+r = 0, N \ , \\ n_F^{(+)} < n_F^{(-)} & \Rightarrow \quad q+r = 0 \ . \end{cases} \tag{4·16}$$

(iii) The case with $a < 0$

In this case, we have

$$h(x) = |a|\left\{(x-x_0)^2 - x_0^2\right\} \quad , \quad x_0 = \frac{N}{2} + \frac{b}{2|a|} \ , \tag{4·17}$$

so that $h(q+r)$ can have a minimum at $q+r$ between 1 and $N-1$. Let $[x_0]_{\rm nearest}$ be the integer nearest to $x_0$. Then

$$\begin{cases} x_0 \leq 0 & \Rightarrow \quad q+r=0 \ , \\ 0 < x_0 < N & \Rightarrow \quad q+r=[x_0]_{\rm nearest} \ , \\ x_0 \geq N & \Rightarrow \quad q+r=N \ . \end{cases} \tag{4·18}$$

Let us examine a couple of examples. In the $SU(5)$ GUT proposed in Ref. 13), only the gauge multiplet and the fundamental Higgs multiplets exist in the bulk, whereas the quark and lepton multiplets are confined on one of the boundary branes. The parity of the Higgs hypermultiplets is assigned such that $n_F^{(++)} = n_F^{(--)} = 1$. This corresponds to the case with $n_F^{(+)} = 2$ and $n_F^{(-)} = n_A^{(\pm)} = n_{Ad}^{(\pm)} = 0$. In one of the models considered in Ref. 17), the quarks and leptons also reside in the bulk. The three generations of quarks and leptons add $n_F^{(++)} = n_F^{(+-)} = n_A^{(++)} = n_A^{(+-)} = 3$. In all, $n_F^{(+)} = 5$, $n_F^{(-)} = 3$, $n_A^{(\pm)} = 3$ and $n_{Ad}^{(\pm)} = 0$. The same matter content was examined in Ref. 24). In all of those models, $a=b=2$, and thus the equivalence classes with $q+r=N$ have the lowest energy density. In Refs. 13) and 17), the BC $[p;q,r;s]=[2;3,0;0]$ has been adopted to reproduce MSSM at low energies. In Ref. 24), the BC $[p;q,r;s]=[2;0,0;3]$ is examined. The result in the present paper shows that with the given matter content, the equivalence classes to which these BCs belong are not those with the lowest energy density. They may not be selected, provided that there exist dynamics connecting different equivalence classes.

This, however, does not preclude the possibility of having these BCs. As pointed out in Refs. 13) and 14), $[p;q,r;s]=[2;3,0;0]$ has the nice feature of reproducing MSSM at low energies with the natural triplet-doublet splitting. To obtain exactly three families of matter chiral multiplets and two weak Higgs chiral multiplets as zero modes (namely, massless particles or light particles with masses of $O(\beta/R)$ in four dimensions) of hypermultiplets in the bulk and of chiral multiplets on the brane, the relations

$$\begin{aligned} & n_5^{(++)} + n_{\overline{5}}^{(--)} = 1 \ , \\ & n_{\overline{5}}^{(++)} + n_5^{(--)} + n_{\overline{5}}^{(\rm Brane)} = 4 \ , \\ & n_{\overline{5}}^{(+-)} + n_5^{(-+)} + n_{\overline{5}}^{(\rm Brane)} = 3 \ , \\ & n_{10}^{(++)} + n_{\overline{10}}^{(--)} + n_{10}^{(\rm Brane)} = 3 \ , \\ & n_{10}^{(+-)} + n_{\overline{10}}^{(-+)} + n_{10}^{(\rm Brane)} = 3 \ , \\ & n_{24}^{(+)} = n_{24}^{(-)} = 0 \ , \end{aligned} \tag{4·19}$$

must hold, as can be inferred from Table I. Here $n_{\overline{5}}^{(\rm Brane)}$ and $n_{10}^{(\rm Brane)}$ are the numbers of chiral multiplets on the brane whose representations are $\overline{\mathbf{5}}$ and $\mathbf{10}$, respectively. In order for the class $[p;q,r;s]=[2;3,0;0]$ to have the lowest energy density, we need $a<0$ and $[x_0]_{\rm nearest}=3$. In other words we need

$$2\Big\{2n_{Ad}^{(+)} - 2n_{Ad}^{(-)} + n_A^{(+)} - n_A^{(-)} - 2\Big\} \geq n_F^{(+)} - n_F^{(-)} \geq 0 \ . \tag{4·20}$$

Table I. Standard model gauge quantum numbers of zero modes in $SU(5)$ hypermultiplets with $(\eta_0, \eta_1)$.

| | $(\eta_0,\eta_1)=(++)$ | $(--)$ | $(+-)$ | $(-+)$ |
|---|---|---|---|---|
| $\mathbf{5}^{(\eta_0,\eta_1)}$ | $(\mathbf{1},\mathbf{2})_{1/2}$ | $(\mathbf{1},\mathbf{2})_{-1/2}$ | $(\mathbf{3},\mathbf{1})_{-1/3}$ | $(\overline{\mathbf{3}},\mathbf{1})_{1/3}$ |
| $\overline{\mathbf{5}}^{(\eta_0,\eta_1)}$ | $(\mathbf{1},\mathbf{2})_{-1/2}$ | $(\mathbf{1},\mathbf{2})_{1/2}$ | $(\overline{\mathbf{3}},\mathbf{1})_{1/3}$ | $(\mathbf{3},\mathbf{1})_{-1/3}$ |
| $\mathbf{10}^{(\eta_0,\eta_1)}$ | $(\overline{\mathbf{3}},\mathbf{1})_{-2/3}+(\mathbf{1},\mathbf{1})_{1}$ | $(\mathbf{3},\mathbf{1})_{2/3}+(\mathbf{1},\mathbf{1})_{-1}$ | $(\mathbf{3},\mathbf{2})_{1/6}$ | $(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$ |
| $\overline{\mathbf{10}}^{(\eta_0,\eta_1)}$ | $(\mathbf{3},\mathbf{1})_{2/3}+(\mathbf{1},\mathbf{1})_{-1}$ | $(\overline{\mathbf{3}},\mathbf{1})_{-2/3}+(\mathbf{1},\mathbf{1})_{1}$ | $(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$ | $(\mathbf{3},\mathbf{2})_{1/6}$ |
| $\mathbf{24}^{(\eta_0,\eta_1)}$ | $(\mathbf{8},\mathbf{1})_0+(\mathbf{3},\mathbf{1})_0$ $+(\mathbf{1},\mathbf{1})_0$ | $(\mathbf{8},\mathbf{1})_0+(\mathbf{3},\mathbf{1})_0$ $+(\mathbf{1},\mathbf{1})_0$ | $(\mathbf{3},\mathbf{2})_{-5/6}$ $+(\overline{\mathbf{3}},\mathbf{2})_{5/6}$ | $(\mathbf{3},\mathbf{2})_{-5/6}$ $+(\overline{\mathbf{3}},\mathbf{2})_{5/6}$ |

The set of relations in (4·19) is incompatible with the first inequality in (4·20). In order to reconcile these relations, we need an extension of the model with extra hypermultiplets (and brane fields). Then there appear additional zero modes, which might threaten the stability of protons and the successful gauge coupling unification based on MSSM. If these zero modes acquire large masses, say, through coupling with extra singlets on the brane, the phenomenological disaster can be avoided.

Let us give an example with $a<0$ and $[x_0]_{\text{nearest}}=3$. It is realized with $a=-2$ and $b=2$, that is, $(n_{Ad}^{(+)}-n_{Ad}^{(-)}, n_A^{(+)}-n_A^{(-)})=(0,4)$, $(1,2)$, $(2,0)$ and $n_F^{(+)}-n_F^{(-)}=2$. The latter equality holds in (4·19). Consider $(n_{Ad}^{(-)}, n_A^{(-)})=(0,0)$ for simplicity. To have $(n_{Ad}^{(+)}, n_A^{(+)})=(0,4)$, we need to pick two extra sets of hypermultiplet pairs in $(\mathbf{10}^{(++)},\mathbf{10}^{(--)})$, $(\mathbf{10}^{(++)},\overline{\mathbf{10}}^{(++)})$, $(\overline{\mathbf{10}}^{(++)},\overline{\mathbf{10}}^{(--)})$ or $(\mathbf{10}^{(--)},\overline{\mathbf{10}}^{(--)})$. There appear two pairs of zero modes with representations $(\overline{\mathbf{3}},\mathbf{1})_{-2/3}+(\mathbf{1},\mathbf{1})_1$ and $(\mathbf{3},\mathbf{1})_{2/3}+(\mathbf{1},\mathbf{1})_{-1}$ with the standard model gauge group. They can form SUSY mass terms through the interactions $(\overline{\mathbf{3}},\mathbf{1})_{-2/3}\cdot(\mathbf{3},\mathbf{1})_{2/3}\cdot(\text{singlet})$ and $(\mathbf{1},\mathbf{1})_1\cdot(\mathbf{1},\mathbf{1})_{-1}\cdot(\text{singlet})$ and decouple from the low energy theory if the magnitudes of the VEVs of singlets on the brane are sufficiently large. In other cases with $(n_{Ad}^{(+)}, n_A^{(+)})=(1,2)$ and $(n_{Ad}^{(+)}, n_A^{(+)})=(2,0)$ as well, phenomenologically interesting low energy theories can be derived in a similar manner. A careful analysis is necessary to check whether or not this scenario is realistic.

Another possibility is to have $[p;q,r;s]=[2;0,0;3]$ as the preferred equivalence class. For this end we need $a\geq 0$ and $n_F^{(+)}\leq n_F^{(-)}$. This is realized if, for instance, $n_F^{(\pm)}=5$, $n_A^{(\pm)}=3$ and $n_{Ad}^{(\pm)}=0$. In this scenario, however, there appear additional light particles which may threaten the stability of protons without further implementing such symmetry as $U(1)_R$ to forbid it and also may ruin the gauge coupling unification in MSSM.

Further, irrespective of the matter content in the bulk, there remains the degeneracy. The effective potential (4·13) is a function of $q+r$ only. It does not select unique $[p;q,r;s]$. We need to find a mechanism to lift the degeneracy.

## §5. Cosmological evolution

In the present paper we have evaluated the vacuum energy density in each equivalence class of BCs in SUSY $SU(N)$ gauge theory on the orbifold $S^1/Z_2$. With the soft SUSY breaking there arise energy differences among different vacua.

An imminent question of great concern is how the BCs are selected or determined. Although it is natural to expect that the BCs yielding the lowest energy density would be selected, the problem is not so simple, as the mechanism for transitions among different BCs is not well understood.

Here we have to distinguish two kinds of transitions. In each equivalence class there are, in general, infinitely many theories with different BCs. There are Wilson line phases whose dynamics yield and guarantee the same physics in the equivalence class. The effective potential for the Wilson line phases has, in general, more than one minimum. These minima are separated by barriers whose heights are about $\beta^2/R^4$ in the four-dimensional energy density for $|\beta| \ll 1$. In the GUT picture, $M_{\rm GUT} \sim 1/R$ and $M_{\rm SUSY} \sim \beta/R$. The energy scale characterizing the barrier is $V_{\rm B}^{(1)} \sim \sqrt{\beta}/R \sim \sqrt{M_{\rm GUT} M_{\rm SUSY}}$. Transitions among different minima occur either thermally or quantum mechanically. The quantum tunneling transition rate at zero temperature, however, is negligibly small.[24)]

Nothing definite can be said about transitions among different equivalence classes without an understanding of the dynamics connecting these different equivalence classes. We are supposing that there exist such dynamics. This must certainly be true if the structure of the spacetime is determined dynamically as in string theory. One needs to know the height, $V_{\rm B}^{(2)}$, of the effective barrier separating the different equivalence classes. It would be below $M_{\rm GUT}$ where the spacetime structure $S^1/Z_2$ is selected. As the typical energy difference among different equivalence classes is $\beta^2/R^4$ for $|\beta| \ll 1$, it should be above $V_{\rm B}^{(1)}$. Hence we have

$$V_{\rm B}^{(1)} \sim \sqrt{M_{\rm GUT} M_{\rm SUSY}} \le V_{\rm B}^{(2)} \le M_{\rm GUT} \quad . \tag{5·1}$$

The quantum tunneling rate at zero temperature from one equivalence class to another is probably negligibly small.

To understand how the low energy symmetry is determined, one needs to trace the cosmological evolution of the universe. At the very early stage around the scale $M_{\rm GUT} \sim 1/R$, it is suspected that the effective five-dimensional orbifold $M^4 \times (S^1/Z_2)$ emerges. The universe then can be either cold or hot.

If the universe is cold, the selection of the equivalence class is due solely to the dynamics connecting different classes. Without understanding the dynamics, nothing can be said for sure about the selection.

If the universe is hot with temperature $T \sim M_{\rm GUT}$, there is an ample amount of thermal transitions among theories with different BCs in various equivalence classes. When the temperature drops below $V_{\rm B}^{(2)}$, thermal transitions among different equivalence classes would cease to exist. In each region of the space one of the equivalence classes would be selected. It is possible that the universe forms a domain structure such that each domain is in its own equivalence class. Then, at the edge of each domain, there forms a domain wall that connects two distinct equivalence classes. Such domain walls would be something exotic which could not be described in the language of gauge theory alone.

The universe continues to expand and the temperature drops further. When

$V_{\rm B}^{(2)} > T > V_{\rm B}^{(1)}$, the entire universe or each domain in the universe remains in one equivalence class. Wilson line phases are thermally excited. When $T$ drops below $V_{\rm B}^{(1)}$, the Wilson line phases settle into one of the minima of the effective potential, which determines the low energy symmetry. The Wilson line phases may happen to be trapped in a 'false' vacuum (a local minimum) instead of the 'true' vacuum (the global minimum).

The inflation may take place somewhere between the scales $M_{\rm GUT}$ and $M_{\rm SUSY}$. In this connection we recognize that $V_{\rm B}^{(1)} \sim \sqrt{M_{\rm GUT} M_{\rm SUSY}}$ defines the intermediate scale. The Wilson line phases themselves may serve as inflatons at the scale $\sqrt{M_{\rm GUT} M_{\rm SUSY}}$. As shown in Ref. 24), the potential for the Wilson line phases (or the extra-dimensional components of the gauge fields) takes a special form which may be suited for natural inflation.[32] Such a scenario has been employed in Ref. 33) to realize the quintessence scenario. In our case, the effective cosmological constant obtained is of $O(\sqrt{M_{\rm GUT} M_{\rm SUSY}})$, not of the order of the cosmological constant observed recently.

## §6. Conclusions

We have tackled the arbitrariness problem of the boundary conditions (BCs) in $SU(N)$ gauge theory on the orbifold $S^1/Z_2$; i.e., we have attempted to explain how one particular set of the boundary conditions is dynamically selected over many other possibilities. Two theories are equivalent if they are related by a BC-changing gauge transformation. According to the Hosotani mechanism, the physical symmetry of each equivalence class is uniquely determined by the dynamics of the Wilson line phases. Hence the number of inequivalent theories is equal to the number of the equivalence classes of the BCs. We have classified the equivalence classes of the BCs. It is found that each equivalence class always has a diagonal representative $(P_0, P_1)$ and that the number of equivalence classes is $(N+1)^2$ in $SU(N)$ gauge theory.

Next, we have derived generic formulas for the one-loop effective potential at the vanishing Wilson line phases. These formulas can be applied to any equivalence class. It has been presumed that $(P_0^{\rm sym}, P_1^{\rm sym})$ is diagonal, as has been confirmed in many examples investigated to this date. When applied to non-supersymmetric theory, there arises an intrinsic ambiguity in comparing the energy densities in two theories in different equivalence classes; the difference between the vacuum energy densities is, in general, infinite, and therefore one cannot compare them.

The unambiguous comparison of the vacuum energy densities in two theories in different equivalence classes becomes possible in supersymmetric theories. We have found that in the supersymmetric $SU(5)$ models with the Scherk-Schwarz supersymmetry breaking, the theory with the BCs yielding the standard model symmetry can be in the equivalence class with the lowest energy density, though the low energy theory may not reproduce the minimal supersymmetric standard model. Possibilities to derive a low energy theory with the standard model gauge group have been studied particularly for the equivalence classes $[p; q, r; s] = [2; 3, 0; 0]$ and $[p; q, r; s] = [2; 0, 0; 3]$.

Further, we have discussed how particular BCs are selected in the cosmological evolution of the universe. It is believed that the quantum tunneling transition rate at zero temperature from one equivalence class to another is probably negligibly small, though the dynamics connecting such different equivalence classes are not understood at all. In one scenario, the thermal transitions among different equivalence classes cease to exist below a temperature whose magnitude is roughly that of the energy barrier separating them. The universe may or may not form a domain structure and settle into one of the minima of the effective potential as the universe cools further.

We would like to stress that the arbitrariness problem has not been completely solved yet as there remains a degeneracy among the theories of the lowest energy density. A new mechanism must be found to lift this degeneracy. It is certainly necessary to understand the dynamics in a more fundamental theory in order to determine the boundary conditions.

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## Appendix A
### —— *Diagonal Representative of* $(P_0, P_1)$ ——

The boundary condition matrices $P_0$ and $P_1$ are $N \times N$ hermitian, unitary matrices. As discussed in §2, two distinct sets, $(P_0, P_1)$ and $(P_0', P_1')$, can be related by a large gauge transformation. The two theories are said to be in the same equivalence class when (2·17) and (2·18) hold. In this appendix we prove that in every equivalence class there is at least one diagonal $(P_0, P_1)$.

Through a global $SU(N)$ transformation, $P_0$ can be diagonalized as

$$P_0 = \begin{pmatrix} I_m & \\ & -I_n \end{pmatrix} \quad , \quad P_1 = \begin{pmatrix} A & C^\dagger \\ C & B \end{pmatrix} \quad , \tag{A·1}$$

where $I_m$ is an $m \times m$ unit matrix and $m + n = N$. $P_0$ still has $SU(m) \times SU(n)$ invariance. Utilizing this invariance, one can diagonalize the hermitian matrices $A$ and $B$. Let us write

$$P_1 = \begin{pmatrix} a_1 & & & & \vec{c}_1{}^\dagger & \\ & \ddots & & & \vdots & \\ & & a_m & & \vec{c}_m{}^\dagger & \\ & & & b_1 & & \\ \vec{c}_1 & \cdots & \vec{c}_m & & \ddots & \\ & & & & & b_n \end{pmatrix} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & \vec{d}_1 & \cdots & \vec{d}_n \\ & & a_m & & & \\ & \vec{d}_1{}^\dagger & & b_1 & & \\ & \vdots & & & \ddots & \\ & \vec{d}_n{}^\dagger & & & & b_n \end{pmatrix} . \quad (\text{A}\cdot 2)$$

As $P_1$ is unitary, we have

$$\begin{aligned} &a_j^2 + \vec{c}_j{}^\dagger \vec{c}_j = 1 \quad , \quad b_j^2 + \vec{d}_j{}^\dagger \vec{d}_j = 1 \quad , \\ &\vec{c}_j{}^\dagger \vec{c}_k = \vec{d}_j{}^\dagger \vec{d}_k = 0 \quad \text{for } j \neq k \ . \end{aligned} \quad (\text{A}\cdot 3)$$

Let the rank of $C$ be $r$. Only $r$ of the vectors $\vec{c}_j$ are linearly independent. (A·3) implies that the other $(m-r)$ of the vectors $\vec{c}_j$ identically vanish. Similarly, only $r$ of the vectors $\vec{d}_j$ are nonvanishing. Through a rearrangement of the rows and columns, one can bring $P_1$ into the form

$$P_1 = \begin{pmatrix} \widetilde{P}_1 & \\ & \hat{I}_{N-2r} \end{pmatrix} \quad ,$$

$$\widetilde{P}_1 = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & \widetilde{C}^\dagger & \\ & & a_r & & & \\ & & & b_1 & & \\ & \widetilde{C} & & & \ddots & \\ & & & & & b_r \end{pmatrix} . \quad (\text{A}\cdot 4)$$

Here, $\widetilde{C}$ is a square $r \times r$ matrix, and $\hat{I}_{N-2r}$ is a diagonal matrix whose diagonal elements are either $+1$ or $-1$. Notice that $(\widetilde{P}_1)^2 = I_{2r}$ implies

$$(a_j + b_k)\, \widetilde{C}_{jk} = 0 \quad . \quad (1 \leq j, k \leq r) \quad (\text{A}\cdot 5)$$

Making use of (A·3) and (A·5), one can reshuffle rows and columns such that

$$\widetilde{P}_1 = \begin{pmatrix} \widetilde{P}_1^{(1)} & & \\ & \ddots & \\ & & \widetilde{P}_1^{(t)} \end{pmatrix} \quad ,$$

$$\widetilde{P}_1^{(l)} = \begin{pmatrix} a_l I_{s_l} & C_l^\dagger \\ C_l & -a_l I_{s_l} \end{pmatrix} \quad ,$$

$$(s_1 + \cdots + s_t = r) \quad . \quad (\text{A}\cdot 6)$$

Now, consider the submatrix $\widetilde{P}_1^{(l)}$. It is also hermitian and unitary, which in particular implies that $U^{(l)} = (1-a_l^2)^{-1/2} C_l^\dagger$ is unitary; $U^{(l)\dagger} U^{(l)} = I_{s_l}$. Through another global unitary transformation, $\widetilde{P}_1^{(l)}$ is brought into the canonical form:

$$\begin{pmatrix} I_{s_l} & \\ & U^{(l)} \end{pmatrix} \widetilde{P}_1^{(l)} \begin{pmatrix} I_{s_l} & \\ & U^{(l)\dagger} \end{pmatrix} = \begin{pmatrix} a_l I_{s_l} & \sqrt{1-a_l^2}\ I_{s_l} \\ \sqrt{1-a_l^2}\ I_{s_l} & -a_l I_{s_l} \end{pmatrix} . \tag{A·7}$$

We note that $P_0$ remains invariant under this transformation. The new $\widetilde{P}_1^{(l)}$ decomposes into $SU(2)$ submatrices. In each subspace, we have

$$\hat{P}_0 = \tau_3 \quad , \quad \hat{P}_1 = \cos\theta \tau_3 + \sin\theta \tau_1 = e^{-i\theta\tau_2}\tau_3 \quad , \quad \cos\theta = a_l . \tag{A·8}$$

Finally, the $y$-dependent transformation $\Omega(y) = e^{i(\theta y/2\pi R)\tau_2}$ in the subspace, with (2·14), transforms $(\hat{P}_0, \hat{P}_1)$ into $(\tau_3, \tau_3)$ in the subspace. Hence, the original general $(P_0, P_1)$ has been transformed into a diagonal one under a series of transformations. This completes the proof.

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