

Cubic Matrix, Nambu Mechanics and Beyond

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We propose a generalization of cubic matrix mechanics by introducing a canonical triplet and study its relation to Nambu mechanics. The generalized cubic matrix mechanics we consider can be interpreted as a ‘quantum’ generalization of Nambu mechanics.

§1. Introduction

The study of a new, generalized mechanics beyond classical mechanics (CM) and quantum mechanics (QM) is often regarded as ambitious, because QM has been applied to very broad areas of physics with indisputable success. There is, however, no strong reason to believe that QM is the unique mechanics to describe nature at a fundamental level (around and beyond the gravitational scale). In fact, M-theory¹⁾ is a promising candidate of a fundamental theory of nature, and there is an intriguing proposal for a formulation of M-theory based on the infinite momentum frame,²⁾ deeply related to the quantum mechanics of supermembranes,³⁾ but no complete such formulation has yet been made. There is a possibility that an ultimate theory requires a new mechanics combined with a configuration of fundamental objects. Therefore it is still meaningful to construct a new, generalized mechanics and study its properties.

Nambu proposed a generalization of Hamiltonian dynamics through the extension of phase space based on the Liouville theorem and gave a suggestion for its quantization.⁴⁾ The structure of this mechanics has been studied in the framework of constrained systems⁵⁾ and in geometric and algebraic formulations.⁶⁾ There are several works in which the quantization of Nambu mechanics (NM) is investigated.^{6)–11)} As an interesting approach, Awata, Li, Minic and Yoneya introduced many-index objects to realize the quantum version of Nambu bracket.⁹⁾

Recently, a new mechanics has been proposed based on many-index objects,¹²⁾ which is a generalization of Heisenberg’s matrix mechanics, and its basic structure has been studied from the algebraic point of view.¹³⁾ The definition of the triple product among three-index objects called ‘cubic matrices’ given in Ref. 12) is different from that given in Ref. 9) in the respect that a generalization of the Ritz rule in the phase factor is required, but the associativity of the products is not necessary. This mechanics possesses a counterpart to the canonical structure in CM and can be generalized through the extension of phase space modeling, following NM. It is quite interesting to investigate this type of generalization and its relation to NM.

In this paper, we propose a generalization of cubic matrix mechanics, which we

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refer to as ‘generalized cubic matrix mechanics’, by introducing a canonical triplet and study the correspondence to NM. A conjecture concerning operator formalism is also given.

Our strategy is almost the same as that in Ref. 13). In the next section, we review the canonical structure of NM and discuss the basic structure that a mechanics beyond NM should possess. We formulate a generalized cubic matrix mechanics and study its structure from an algebraic viewpoint and its relation to NM in §3. Section 4 is devoted to conclusions and discussion.

§2. Nambu mechanics and beyond

2.1. Canonical structure of Nambu mechanics

Here we review the canonical structure of Nambu mechanics.⁴⁾ For simplicity, we treat a system with a 3-dimensional phase space whose variables are $x = x(t)$, $y = y(t)$ and $z = z(t)$. They satisfy the “Hamilton’s equations”

$$\frac{dx}{dt} = \frac{\partial(K, H)}{\partial(y, z)}, \quad \frac{dy}{dt} = \frac{\partial(K, H)}{\partial(z, x)}, \quad \frac{dz}{dt} = \frac{\partial(K, H)}{\partial(x, y)}, \quad (2.1)$$

where the right-hand sides represent 2-dimensional Jacobians, and K and H are the “Hamiltonians”. Physical variables are given by functions of the canonical variables and the time variable t ; e.g., $A = A(x, y, z, t)$, $B = B(x, y, z, t)$, and $C = C(x, y, z, t)$. Hereafter we consider systems such that physical variables do not contain t explicitly, that is, closed physical systems. The Nambu bracket of three variables A , B and C with respect to x , y and z is defined by

$$\{A, B, C\}_{\text{NB}} \equiv \frac{\partial(A, B, C)}{\partial(x, y, z)}, \quad (2.2)$$

where the right-hand side represents a 3-dimensional Jacobian. Hence, the Nambu brackets of the canonical variables are given by

$$\{x, y, z\}_{\text{NB}} = 1, \quad \{x, x, z\}_{\text{NB}} = \{x, y, y\}_{\text{NB}} = \cdots = \{z, z, z\}_{\text{NB}} = 0. \quad (2.3)$$

The basic features of the Nambu bracket are as follows:

$$\begin{aligned} \{A, B, C\}_{\text{NB}} &= \{B, C, A\}_{\text{NB}} = \{C, A, B\}_{\text{NB}} \\ &= -\{C, B, A\}_{\text{NB}} = -\{B, A, C\}_{\text{NB}} = -\{A, C, B\}_{\text{NB}}, \quad (\text{skew-symmetry}) \end{aligned} \quad (2.4)$$

$$\{A + B, C, D\}_{\text{NB}} = \{A, C, D\}_{\text{NB}} + \{B, C, D\}_{\text{NB}}, \quad (\text{linearity}) \quad (2.5)$$

$$\begin{aligned} \{\{A, B, C\}_{\text{NB}}, D, E\}_{\text{NB}} &= \{\{A, D, E\}_{\text{NB}}, B, C\}_{\text{NB}} + \{A, \{B, D, E\}_{\text{NB}}, C\}_{\text{NB}} \\ &+ \{A, B, \{C, D, E\}_{\text{NB}}\}_{\text{NB}} = 0, \quad (\text{fundamental identity}) \end{aligned} \quad (2.6)$$

$$\{AB, C, D\}_{\text{NB}} = A\{B, C, D\}_{\text{NB}} + \{A, C, D\}_{\text{NB}}B. \quad (\text{derivation rule}) \quad (2.7)$$

By use of (2.1) and (2.2), the physical variable A is shown to satisfy the equation

$$\frac{dA}{dt} = \{A, K, H\}_{\text{NB}}. \quad (2.8)$$

We call a transformation $A \rightarrow A' = \mathcal{C}(A)$ that preserves the bracket structure ‘canonical’:

$$\{A, B, C\}_{\text{NB}} \longrightarrow \mathcal{C}(\{A, B, C\}_{\text{NB}}) = \{\mathcal{C}(A), \mathcal{C}(B), \mathcal{C}(C)\}_{\text{NB}}. \quad (2.9)$$

The infinitesimal version $A \rightarrow A' = A + \delta A$ is given by

$$\delta A = \{A, G_1, G_2\}_{\text{NB}} \delta s, \quad (2.10)$$

where G_1 and G_2 are generators of the transformation, and δs is an infinitesimal parameter. We can show that the bracket structure is preserved under the transformation (2.10) by using the identity (2.6).

2.2. Beyond Nambu mechanics

The structure of Nambu mechanics is so elegant that it is natural to expect the existence of a ‘quantum’ counterpart of NM. In this subsection, we present a conjecture for the basic structure of a new mechanics beyond NM based on a requirement that *the algebraic structure of equations of motion and symmetry transformations be preserved (up to anomalous breakings)*.

The above requirement is expressed as the following properties:

1. There are counterparts of the canonical variables in NM, which are denoted $X = X(t)$, $Y = Y(t)$ and $Z = Z(t)$, and physical variables are functions of X , Y and Z in a closed system. There exists a counterpart of the Nambu bracket, which we call the ‘generalized bracket’, and the bracket relations for X , Y and Z are conditions that place restrictions on the phase space (like quantization conditions in QM). The generalized bracket does not necessarily possess all the algebraic properties of the Nambu bracket. However, at least it possesses properties of skew-symmetry and linearity.
2. The equations of motion for physical variables are of the same type as those in NM. More specifically, an equation of motion is obtained from the corresponding equation in NM by replacing the Nambu bracket with the generalized bracket.
3. There is a transformation that preserves the generalized bracket structure that we call a ‘generalized canonical transformation’. The ‘fundamental identity’ analogous to (2.6) for the generalized bracket holds, including generators, which are conserved quantities. Continuous symmetry transformations are realized as generalized canonical transformations of conserved quantities.

Now we formulate the basic structure of a new mechanics based on the above properties.

1. Let us denote the generalized bracket by $\mathcal{B}(*, *, *)$ and impose the following conditions on X , Y and Z :

$$\mathcal{B}(X, Y, Z) = \Theta, \mathcal{B}(X, X, Z) = \mathcal{B}(X, Y, Y) = \dots = \mathcal{B}(Z, Z, Z) = 0. \quad (2.11)$$

Here Θ is a constant of motion, and the bracket of Θ and any conserved quantities A_i vanishes; i.e. $\mathcal{B}(\Theta, A_i, A_j) = 0$. The skew-symmetry and linearity

conditions are expressed by

$$\begin{aligned}\mathcal{B}(A, B, C) &= \mathcal{B}(B, C, A) = \mathcal{B}(C, A, B) \\ &= -\mathcal{B}(C, B, A) = -\mathcal{B}(B, A, C) = -\mathcal{B}(A, C, B),\end{aligned}\quad (2.12)$$

$$\mathcal{B}(A + B, C, D) = \mathcal{B}(A, C, D) + \mathcal{B}(B, C, D).\quad (2.13)$$

We do not necessarily require correspondences between the fundamental identity nor the derivation rule in NM as properties of $\mathcal{B}(*, *, *)$ for generic variables.

2. The equation of motion for a physical quantity A is given by

$$\frac{dA}{dt} = \mathcal{B}(A, K, H),\quad (2.14)$$

where K and H are the ‘‘Hamiltonians’’.

3. A generalized canonical transformation is defined by the transformation $A \rightarrow A' = \mathcal{G}(A)$, which preserves the structure of $\mathcal{B}(*, *, *)$:

$$\mathcal{B}(A, B, C) \longrightarrow \mathcal{G}(\mathcal{B}(A, B, C)) = \mathcal{B}(\mathcal{G}(A), \mathcal{G}(B), \mathcal{G}(C)).\quad (2.15)$$

The infinitesimal version of (2.15) is written

$$\delta\mathcal{B}(A, B, C) = \mathcal{B}(\delta A, B, C) + \mathcal{B}(A, \delta B, C) + \mathcal{B}(A, B, \delta C),\quad (2.16)$$

under the infinitesimal generalized canonical transformation $A \rightarrow A' = A + \delta A$. For conserved quantities G_1 and G_2 [i.e., $dG_i/dt = \mathcal{B}(G_i, K, H) = 0$], the fundamental identity holds:

$$\begin{aligned}\mathcal{B}(\mathcal{B}(A, B, C), G_1, G_2) &= \mathcal{B}(\mathcal{B}(A, G_1, G_2), B, C) \\ &+ \mathcal{B}(A, \mathcal{B}(B, G_1, G_2), C) + \mathcal{B}(A, B, \mathcal{B}(C, G_1, G_2)).\end{aligned}\quad (2.17)$$

Then, a symmetry transformation is given by the infinitesimal generalized canonical transformation,^{*)}

$$\delta A = \mathcal{B}(A, G_1, G_2)\delta s.\quad (2.18)$$

§3. Generalized cubic matrix mechanics

We have discussed the basic structure that a new mechanics beyond Nambu mechanics should possess. It is expected that the study of NM will be helpful to understand the structure of M-theory,^{1),2)} through the quantum theory of supermembranes.³⁾ For this reason, it is important to construct a ‘quantum’ version of NM and study its features. In this section, we propose a new mechanics based on cubic matrices, which is a generalization of the cubic matrix mechanics examined in Refs. 12) and 13), and study its structure and the correspondence to NM.

^{*)} It is not certain whether every continuous generalized canonical transformation $A \rightarrow A' = \mathcal{G}(A)$ can be constructed from the infinitesimal one given by $\delta A = \mathcal{B}(A, F_1, F_2)\delta s$, where F_i are generators. Here, we require the algebraic structure of symmetry transformations to be identical to that in NM.

3.1. Cubic matrix

Here we state our definition of a cubic matrix and its related terminology. A cubic matrix is an object with three indices, A_{lmn} , which is a generalization of a usual matrix, such as B_{mn} . We refer to a cubic matrix whose elements possess cyclic symmetry, i.e., $A_{lmn} = A_{mnl} = A_{nlm}$, as a cyclic cubic matrix. We define the hermiticity of a cubic matrix by $A_{l'm'n'}(t) = A_{lmn}^*(t)$ for odd permutations among indices and refer to a cubic matrix possessing hermiticity as a hermitian cubic matrix. Here, the asterisk indicates complex conjugation. A hermitian cubic matrix is a special type of cyclic cubic matrix, because it obeys the relations $A_{lmn} = A_{mnl}^* = A_{nml}^* = A_{nlm}^* = A_{lmn}^*$. We refer to the following form of a cubic matrix as a normal form or a normal cubic matrix:

$$A_{lmn}^{(N)} = \delta_{lm}a_{mn} + \delta_{mn}a_{nl} + \delta_{nl}a_{lm}. \quad (3.1)$$

A normal cubic matrix is also a special type of cyclic cubic matrix. The elements of a cubic matrix are treated as c -numbers throughout this paper.

3.2. Generalized cubic matrix mechanics and its structure

The physical variables are cyclic cubic matrices given by

$$A_{lmn}(t) = A_{lmn}e^{i\Omega_{lmn}t}, \quad (3.2)$$

where the angular frequency Ω_{lmn} has the properties

$$\Omega_{l'm'n'} = \text{sgn}(P)\Omega_{lmn}, \quad (\delta\Omega)_{lmnk} \equiv \Omega_{lmn} - \Omega_{lmk} + \Omega_{lnk} - \Omega_{mnk} = 0. \quad (3.3)$$

Here, $\text{sgn}(P)$ is $+1$ and -1 for even and odd permutations among indices, respectively. The operator δ is regarded as a coboundary operator that changes k -th antisymmetric objects into $(k + 1)$ -th objects, and this operation is nilpotent, i.e. $\delta^2(*) = 0$.¹⁴⁾ The frequency Ω_{lmn} is regarded as a 3-cocycle, from the second equation in (3.3).

If we define the triple product among cubic matrices $A_{lmn}(t) = A_{lmn}e^{i\Omega_{lmn}t}$, $B_{lmn}(t) = B_{lmn}e^{i\Omega_{lmn}t}$ and $C_{lmn}(t) = C_{lmn}e^{i\Omega_{lmn}t}$ by

$$(A(t)B(t)C(t))_{lmn} \equiv \sum_k A_{lmk}(t)B_{lkn}(t)C_{kmn}(t) = (ABC)_{lmn}e^{i\Omega_{lmn}t}, \quad (3.4)$$

this product takes the same form as (3.2), with the relation (3.3), which is a generalization of the Ritz rule.^{*)} We comment that the resultant three-index object, $(ABC)_{lmn}e^{i\Omega_{lmn}t}$, does not always have cyclic symmetry, even if $A_{lmn}(t)$, $B_{lmn}(t)$ and $C_{lmn}(t)$ are cyclic cubic matrices. Note that this product is, in general, neither commutative nor associative; that is, $(ABC)_{lmn} \neq (BAC)_{lmn}$ and $(AB(CDE))_{lmn} \neq (A(BCD)E)_{lmn} \neq ((ABC)DE)_{lmn}$. The triple-commutator is defined by

$$[A(t), B(t), C(t)]_{lmn} \equiv (A(t)B(t)C(t) + B(t)C(t)A(t) + C(t)A(t)B(t) - B(t)A(t)C(t) - A(t)C(t)B(t) - C(t)B(t)A(t))_{lmn}. \quad (3.5)$$

^{*)} The Ritz rule is given by $\Omega_{ln} = \Omega_{lm} + \Omega_{mn}$ in QM, where Ω_{mn} is the angular frequency of radiation from an atom.

The triple-anticommutator is defined by

$$\begin{aligned} \{A(t), B(t), C(t)\}_{lmn} \equiv & (A(t)B(t)C(t) + B(t)C(t)A(t) + C(t)A(t)B(t) \\ & + B(t)A(t)C(t) + A(t)C(t)B(t) + C(t)B(t)A(t))_{lmn}. \end{aligned} \quad (3.6)$$

If $A_{lmn}(t)$, $B_{lmn}(t)$ and $C_{lmn}(t)$ are hermitian matrices, $i[A(t), B(t), C(t)]_{lmn}$ and $\{A(t), B(t), C(t)\}_{lmn}$ are also hermitian cubic matrices.

The generalized bracket is defined by use of the triple-commutator (3.5) as

$$\mathcal{B}(A, B, C)_{lmn} \equiv \frac{1}{i\hbar_C} [A(t), B(t), C(t)]_{lmn}, \quad (3.7)$$

where \hbar_C is a new physical constant. By definition, we find that the generalized bracket (3.7) has the properties of skew-symmetry and linearity, as seen from the relations

$$\begin{aligned} [A(t), B(t), C(t)]_{lmn} &= [B(t), C(t), A(t)]_{lmn} = [C(t), A(t), B(t)]_{lmn} \\ &= -[C(t), B(t), A(t)]_{lmn} = -[B(t), A(t), C(t)]_{lmn} = -[A(t), C(t), B(t)]_{lmn}, \end{aligned} \quad (3.8)$$

$$[A(t) + B(t), C(t), D(t)]_{lmn} = [A(t), C(t), D(t)]_{lmn} + [B(t), C(t), D(t)]_{lmn}. \quad (3.9)$$

Note that neither the fundamental identity nor the derivation rule necessarily holds for generic variables. (See Appendix A for properties of the triple-commutator $[A, B, C]$.)

We impose the following conditions on the canonical triplet $X_{lmn}(t)$, $Y_{lmn}(t)$ and $Z_{lmn}(t)$:

$$\begin{aligned} [X(t), Y(t), Z(t)]_{lmn} &= i\hbar_C \Theta_{lmn}, \\ [X(t), X(t), Z(t)]_{lmn} &= \cdots = [Z(t), Z(t), Z(t)]_{lmn} = 0. \end{aligned} \quad (3.10)$$

Here, Θ_{lmn} can be a normal cubic matrix, because the conditions should be imposed time independently, and an arbitrary normal cubic matrix is a constant of motion, as seen below.

The cyclic cubic matrix $A_{lmn}(t)$ yields the generalization of the Heisenberg equation

$$\frac{d}{dt} A_{lmn}(t) = i\Omega_{lmn} A_{lmn}(t) = \frac{1}{i\hbar_C} [A(t), K, H]_{lmn}, \quad (3.11)$$

where K and H are the Hamiltonians given by

$$K_{lmn} = \delta_{lm} k_{mn} + \delta_{mn} k_{nl} + \delta_{nl} k_{lm} \quad (3.12)$$

and

$$H_{lmn} = \delta_{lm} h_{mn} + \delta_{mn} h_{nl} + \delta_{nl} h_{lm}, \quad (3.13)$$

respectively. By use of (3.12) and (3.13), Ω_{lmn} can be written

$$\begin{aligned} \Omega_{lmn} = \frac{1}{\hbar_C} \left(k_{ml} h_{mn} + k_{nm} h_{nl} + k_{ln} h_{lm} \right. \\ \left. - h_{ml} k_{mn} - h_{nm} k_{nl} - h_{ln} k_{lm} \right). \end{aligned} \quad (3.14)$$

Because the Hamiltonians K_{lmn} and H_{lmn} are normal forms, we find that an arbitrary normal cubic matrix $A^{(N)}$ is a constant of motion: $i\hbar_C dA^{(N)}/dt = [A^{(N)}, K, H]_{lmn} = 0$. The Hamiltonians are conserved quantities, and the time evolution of $A_{lmn}(t)$ is regarded as the symmetry transformation generated by them. The fundamental identity with K_{lmn} and H_{lmn} must hold in order to preserve the bracket structure under the transformation. This requirement is equivalent to the requirement that Ω_{lmn} be a 3-cocycle, i.e., $(\delta\Omega)_{lmnk} = 0$. An extreme case in which Ω_{lmn} is a 3-cocycle is that in which $k_{lm} = 1$ or $h_{lm} = 1$. When $k_{lm} = 1$, K_{lmn} and Ω_{lmn} can be written

$$K_{lmn} = \delta_{lm} + \delta_{mn} + \delta_{nl}, \tag{3.15}$$

$$\Omega_{lmn} = \frac{2}{\hbar_C} \left(h_{lm}^{(-)} + h_{mn}^{(-)} + h_{nl}^{(-)} \right), \tag{3.16}$$

respectively. Here $h_{lm}^{(-)} = \frac{1}{2}(h_{lm} - h_{ml})$. In this case, our generalization of cubic matrix mechanics is equivalent to ordinary cubic matrix mechanics*⁾ discussed in Refs. 12) and 13).

Next, we consider the case that $k_{lm} = -k_{ml}$ and $h_{lm} = -h_{ml}$ and both k_{lm} and h_{lm} are 2-cocycles, i.e., $(\delta k)_{lmn} = 0$ and $(\delta h)_{lmn} = 0$. In this case, we can show that Ω_{lmn} is a 3-cocycle, and then Ω_{lmn} can be rewritten as

$$\Omega_{lmn} = \frac{3}{\hbar_C} \left(k_{ml}h_{mn} - h_{ml}k_{mn} \right). \tag{3.17}$$

Rewriting this further, we have

$$\begin{aligned} \Omega_{lmn} = & -\frac{3}{\hbar_C} \left((k_{pl}h_{pm} - h_{pl}k_{pm}) + (k_{pm}h_{pn} - h_{pm}k_{pn}) \right. \\ & \left. + (k_{pn}h_{pl} - h_{pn}k_{pl}) \right), \end{aligned} \tag{3.18}$$

where p is arbitrary. The relation (3.18) shows that Ω_{lmn} is a 3-coboundary, and it leads to the conjecture that generalized cubic matrix mechanics can be reduced to cubic matrix mechanics by a suitable change of Hamiltonians. We discuss a simple example for variables that yield the generalized Heisenberg equation (3.11) in Appendix B.

The generalized bracket structure (3.7) is preserved by the infinitesimal transformation

$$\delta A_{lmn}(t) = \frac{1}{i\hbar_C} [A(t), G_1^{(N)}, G_2^{(N)}]_{lmn} \delta s, \tag{3.19}$$

where $G_1^{(N)}$ and $G_2^{(N)}$ are normal cubic matrices and $(\widetilde{G_1^{(N)} G_2^{(N)}})_{lmn}$ is a 3-cocycle. Here, we use the fact that the fundamental identity holds for such normal cubic

*⁾ In cubic matrix mechanics, K_{lmn} is given by $I_{lmn} = \delta_{lm}(1 - \delta_{mn}) + \delta_{mn}(1 - \delta_{nl}) + \delta_{nl}(1 - \delta_{lm})$. The difference between K_{lmn} in (3.15) and I_{lmn} has no effect on the equations of motion, because there is the identity $[A, B, \Delta]_{lmn} = 0$ for arbitrary cyclic cubic matrices A and B and $\Delta_{lmn} = \delta_{lm}\delta_{mn}$.

matrices $G_1^{(N)}$ and $G_2^{(N)}$, so that

$$\begin{aligned} [[A, B, C], G_1^{(N)}, G_2^{(N)}]_{lmn} &= [[A, G_1^{(N)}, G_2^{(N)}], B, C]_{lmn} \\ &+ [A, [B, G_1^{(N)}, G_2^{(N)}], C]_{lmn} + [A, B, [C, G_1^{(N)}, G_2^{(N)}]]_{lmn}. \end{aligned} \quad (3.20)$$

Further, we find that the derivation rule

$$\begin{aligned} [ABC, G_1^{(N)}, G_2^{(N)}]_{lmn} &= ([A, G_1^{(N)}, G_2^{(N)}]BC)_{lmn} + (A[B, G_1^{(N)}, G_2^{(N)}]C)_{lmn} \\ &+ (AB[C, G_1^{(N)}, G_2^{(N)}])_{lmn} \end{aligned} \quad (3.21)$$

holds for $G_1^{(N)}$ and $G_2^{(N)}$ if $(ABC)_{lmn}$ is a cyclic cubic matrix.

3.3. Correspondence to Nambu mechanics

We now discuss the relation between Nambu mechanics and generalized cubic matrix mechanics from the viewpoint of the correspondence principle. First we review the relation between classical mechanics and quantum mechanics. A physical variable $F(t)$ in CM is regarded as a linear combination of one-index objects (a 1×1 matrix) in the form

$$F(t) = \sum_n F_n e^{i\Omega_n t}, \quad (3.22)$$

where $F_n^* = F_{-n}$, because $F(t)$ should be a real quantity, and the angular frequency Ω_n is an integer multiple of the basic frequency ω , i.e. $\Omega_n = n\omega$. By use of the fact that the action variable $J = \frac{1}{2\pi} \oint pdq$ is the canonical conjugate of the angle variable ωt , the equation of motion for $F(t)$ can be written

$$\frac{d}{dt}F(t) = \sum_n in\omega F_n e^{i\Omega_n t} = \{F(t), H\}_{\text{PB}}, \quad (3.23)$$

where $\{*, *\}_{\text{PB}}$ is the Poisson bracket with respect to the canonical pair ωt and J , and we use Hamilton's canonical equation for the angle variable,

$$\frac{d}{dt}(\omega t) = \{\omega t, H\}_{\text{PB}} = \frac{\partial H}{\partial J}. \quad (3.24)$$

Under the guidance of Bohr's correspondence principle, there is the following correspondence between ω and Ω_{mn} :

$$\omega = \frac{\Omega_{\Delta n}}{\Delta n} \iff \lim_{\frac{\Delta n}{n} \rightarrow 0} \frac{\Omega_{n+\Delta n}}{\Delta n} = \lim_{\frac{\Delta n}{n} \rightarrow 0} \frac{E_{n+\Delta n} - E_n}{\hbar \Delta n}. \quad (3.25)$$

Here, \iff indicates the correspondence, and we use the Bohr frequency condition $\hbar\Omega_{mn} = E_m - E_n$. We find that the equation on the right-hand side in (3.25) corresponds to (3.24) with the Bohr-Sommerfeld quantization condition, $J = \hbar n$.

Next, we study the 'classical' limit of generalized matrix mechanics based on 3-index objects, whose frequency condition is given by (3.17). We consider the case

that a physical variable $A(t)$ in NM is expanded as a linear combination of one-index objects, so that

$$A(t) = \sum_n A_n e^{i\Omega_n t}, \tag{3.26}$$

where $A_n^* = A_{-n}$ and the angular frequency Ω_n is an integer multiple of the basic frequency ω , i.e. $\Omega_n = n\omega$. The equation of motion for $A(t)$ is written

$$\frac{d}{dt}A(t) = \sum_n in\omega A_n e^{i\Omega_n t} = \{A(t), K, H\}_{\text{NB}}, \tag{3.27}$$

where $\{*, *, *\}_{\text{NB}}$ is the Nambu bracket with respect to the canonical triplet \mathcal{J}_1 , ωt and \mathcal{J}_2 , and we use ‘‘Hamilton’s equation’’ for the angle variable ωt ,

$$\frac{d}{dt}(\omega t) = \{\omega t, K, H\}_{\text{NB}} = \frac{\partial(K, H)}{\partial(\mathcal{J}_2, \mathcal{J}_1)}. \tag{3.28}$$

Here \mathcal{J}_1 and \mathcal{J}_2 are conserved quantities. (See Appendix C for the ‘‘Hamilton-Jacobi formalism’’ of NM.) It is natural to assume the existence of the following correspondence between ω and Ω_{lmn} :

$$\begin{aligned} \omega = \frac{\Omega_{\Delta N}}{\Delta N} &\iff \lim_{\frac{\Delta l}{l}, \frac{\Delta n}{n} \rightarrow 0} \frac{\Omega_{l+\Delta l, n}}{\Delta l \Delta n} \\ &= \lim_{\frac{\Delta l}{l}, \frac{\Delta n}{n} \rightarrow 0} \frac{3(k_{l+\Delta l} h_{n+\Delta n} - h_{l+\Delta l} k_{n+\Delta n})}{\hbar_C \Delta l \Delta n}, \\ &= -\frac{3}{\hbar_C} \lim_{\frac{\Delta l}{l}, \frac{\Delta n}{n} \rightarrow 0} \left(\frac{k_{l+\Delta l} - k_{ln}}{\Delta l} \frac{h_{ln+\Delta n} - h_{ln}}{\Delta n} \right. \\ &\quad \left. - \frac{h_{l+\Delta l} - h_{ln}}{\Delta l} \frac{k_{ln+\Delta n} - k_{ln}}{\Delta n} \right). \end{aligned} \tag{3.29}$$

Here, $\Delta N = \Delta l \Delta n$, $\Delta l = m - l$, $\Delta n = m - n$, \iff indicates the correspondence, and we use the frequency condition (3.17) with the property that k_{lm} and h_{lm} are 2-cocycles. We find that the equation on the right-hand side in (3.29) corresponds to (3.28) if \mathcal{J}_1 and \mathcal{J}_2 are quantized in analogy to the Bohr-Sommerfeld quantization condition. In this way, the generalized cubic matrix mechanics can be interpreted as a ‘quantum’ generalization of NM.

3.4. Conjecture on operator formalism

We have studied the structure of generalized cubic matrix mechanics using a matrix formalism. This mechanics has an interesting algebraic structure, but the formalism is not practical, because it is only applicable to stationary systems. From experience, it is known that in order to be of practical use, operator formalism must be capable of handling problems in a wider class of physical systems. By analogy to quantum mechanics, we now give a conjecture on the operator formalism of generalized cubic matrix mechanics. First, we make the following basic assumptions.

1. For a given physical system, there exists a triplet of state vectors $|m_1; P_{m_1 m_2 m_3}\rangle$, $|m_2; P_{m_1 m_2 m_3}\rangle$ and $|m_3; P_{m_1 m_2 m_3}\rangle$ that depend on both the quantum numbers m_i (e.g., these m_i represent l, m or n) and their ordering. Here, the ordering is represented by a permutation (denoted by $P_{m_1 m_2 m_3}$) for a standard ordering (e.g., $m_1 = l, m_2 = m, m_3 = n$).
2. For every physical observable, there is a one-to-one correspondence to a linear operator \hat{A} .

Under the above assumptions, it is natural to identify the cubic matrix element A_{lmn} with $\hat{A}|l; P_{lmn}\rangle|m; P_{lmn}\rangle|n; P_{lmn}\rangle$. In general, the quantity $A_{m_1 m_2 m_3}$ is identified with $\hat{A}|m_1; P_{m_1 m_2 m_3}\rangle|m_2; P_{m_1 m_2 m_3}\rangle|m_3; P_{m_1 m_2 m_3}\rangle$. By use of (3·11), the following equations of motion for the states are derived:

$$\begin{aligned} i\hbar_C \frac{d}{dt}|l; P_{lmn}\rangle &= \sum_{l'} |l'; P_{l'mn}\rangle [K, H]_{l'l}^{(mn)}, \\ i\hbar_C \frac{d}{dt}|m; P_{lmn}\rangle &= \sum_{m'} |m'; P_{lm'n}\rangle [K, H]_{m'm}^{(nl)}, \\ i\hbar_C \frac{d}{dt}|n; P_{lmn}\rangle &= \sum_{n'} |n'; P_{lmn'}\rangle [K, H]_{n'n}^{(lm)}, \end{aligned} \quad (3\cdot30)$$

where $[K, H]_{l'l}^{(mn)} \equiv K_{ml'l} H_{l'l} - H_{ml'l} K_{l'l}$, and we employ the Schrödinger picture. By use of relations (3·12) and (3·13), $[K, H]_{l'l}^{(mn)}$ can be written as

$$[K, H]_{l'l}^{(mn)} = (k_{lm} h_{l'n} - h_{lm} k_{l'n}) \delta_{ll'}. \quad (3\cdot31)$$

The equations (3·30) are regarded as a generalization of the Schrödinger equation. The time evolution of state vectors is given by

$$\begin{aligned} |l; P_{lmn}\rangle &= \exp\left(\frac{i}{\hbar_C}(k_{ln} h_{lm} - h_{ln} k_{lm})t\right) |l; P_{lmn}\rangle_0, \\ |m; P_{lmn}\rangle &= \exp\left(\frac{i}{\hbar_C}(k_{ml} h_{mn} - h_{ml} k_{mn})t\right) |m; P_{lmn}\rangle_0, \\ |n; P_{lmn}\rangle &= \exp\left(\frac{i}{\hbar_C}(k_{nm} h_{nl} - h_{nm} k_{nl})t\right) |n; P_{lmn}\rangle_0, \end{aligned} \quad (3\cdot32)$$

where the subscript 0 indicates that the state is that at an initial time. In the same way, the time development of state vectors for the matrix element A_{mln} is given by

$$\begin{aligned} |l; P_{mln}\rangle &= \exp\left(\frac{i}{\hbar_C}(k_{nl} h_{lm} - h_{nl} k_{lm})t\right) |l; P_{mln}\rangle_0, \\ |m; P_{mln}\rangle &= \exp\left(\frac{i}{\hbar_C}(k_{mn} h_{ml} - h_{mn} k_{ml})t\right) |m; P_{mln}\rangle_0, \\ |n; P_{mln}\rangle &= \exp\left(\frac{i}{\hbar_C}(k_{nl} h_{nm} - h_{nl} k_{nm})t\right) |n; P_{mln}\rangle_0. \end{aligned} \quad (3\cdot33)$$

From (3·32) and (3·33), we can identify $|l; P_{mln}\rangle$ with the complex conjugate of $|l; P_{lmn}\rangle$. It is seen that this identification is consistent with the skew-symmetric property of the phase factor in (3·2).

§4. Conclusions and discussion

We have proposed a generalization of cubic matrix mechanics by introducing a canonical triplet and studied the structure of the mechanics and the relation to Nambu mechanics. The basic structure of generalized cubic matrix mechanics is summarized as follows. The infinitesimal symmetry transformation of a physical quantity $A_{lmn}(t)$, which is a cyclic cubic matrix, is given by $\delta A_{lmn}(t) = \frac{1}{i\hbar_C} [A(t), G_1, G_2]_{lmn} \delta s$. Here the triple-commutator is the counterpart of the Nambu bracket in NM, and G_1 and G_2 are generators of the transformation, which are normal cubic matrices. The time evolution of $A_{lmn}(t)$ is regarded as the symmetry transformation generated by the Hamiltonians K_{lmn} and H_{lmn} , such that $i\hbar_C \delta A_{lmn}(t) = [A(t), K, H]_{lmn} \delta t$, which is a generalization of the Heisenberg equation. A normal cubic matrix, $G_{lmn}^{(N)}$, is a constant of motion; i.e., $i\hbar_C dG_{lmn}^{(N)}/dt = [G^{(N)}, K, H]_{lmn} = 0$. The fundamental identity and the derivation rule hold in the case that they contain a special type of conserved quantities, G_1 and G_2 , such as (3·20) and (3·21), and the bracket structure is preserved under the symmetry transformation, as seen from the fundamental identity. There is a correspondence between generalized cubic matrix mechanics and NM, and hence our matrix mechanics can be interpreted as a ‘quantum’ version of NM. There is a simple system of harmonic oscillators described by $3 \times 3 \times 3$ matrices, which yield the generalization of the Heisenberg equation (3·11), but this is not a non-trivial example entirely. The dynamical variables in this system are essentially $X_{lmn}(t)$ and $Y_{lmn}(t)$, and the introduction of a special type of normal cubic matrices F_{lmn}^i seems tricky. Moreover, the system can also be described in terms of cubic matrix mechanics. It would be interesting to find a non-trivial system, where all members of a canonical triplet are time-dependent and satisfy (3·11), and study its dynamics and relation to reality. For this purpose, it would be useful to explore a counterpart to the rigid rotator that yields the Euler equation.

There still exist several obstacles that must be overcome before we can arrive at a final formulation. For example, there is the conjecture that no global symmetry exists in a quantum theory including gravity.¹⁵⁾ If this conjecture holds, then we would need a formulation including local symmetries. Another modification would be necessary if we incorporate a gravitational interaction. The theory should be formulated in a background-independent way, as the theory of general relativity. Therefore, the scheme discussed in this paper can be interpreted as an effective description of an underlying mechanics after fixing the background geometry and ignoring dynamical degrees of freedom for the graviton.

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Appendix A

— Features of Triple-Commutator —

In this appendix, we study the properties of the triple-commutator $[A, B, C]$ for cyclic cubic matrices A_{lmn} , B_{lmn} and C_{lmn} . This commutator is written

$$[A, B, C]_{lmn} = A_{lmn} \widetilde{(BC)}_{lmn} + B_{lmn} \widetilde{(CA)}_{lmn} + C_{lmn} \widetilde{(AB)}_{lmn} + ([A, B, C])_{lmn}^0, \quad (\text{A}\cdot 1)$$

where $\widetilde{(BC)}_{lmn}$ and $([A, B, C])_{lmn}^0$ are defined by

$$\begin{aligned} \widetilde{(BC)}_{lmn} &\equiv B_{lnn} C_{nmn} + B_{lml} C_{lln} + B_{mnn} C_{lmm} \\ &\quad - B_{nmn} C_{lnn} - B_{lln} C_{lml} - B_{lmm} C_{mnn} \end{aligned} \quad (\text{A}\cdot 2)$$

and

$$\begin{aligned} ([A, B, C])_{lmn}^0 &\equiv \sum_{k \neq l, m, n} \left(A_{lmk} (B_{lkn} C_{kmn} - C_{lkn} B_{kmn}) \right. \\ &\quad \left. + B_{lmk} (C_{lkn} A_{kmn} - A_{lkn} C_{kmn}) + C_{lmk} (A_{lkn} B_{kmn} - B_{lkn} A_{kmn}) \right), \end{aligned} \quad (\text{A}\cdot 3)$$

respectively. The features of $\widetilde{(BC)}_{lmn}$ are as follows:

1. $\widetilde{(BC)}_{lmn}$ possesses skew-symmetry with respect to permutations among indices:

$$\begin{aligned} \widetilde{(BC)}_{lmn} &= \widetilde{(BC)}_{mnl} = \widetilde{(BC)}_{nlm} \\ &= -\widetilde{(BC)}_{nml} = -\widetilde{(BC)}_{mln} = -\widetilde{(BC)}_{lnm}. \end{aligned} \quad (\text{A}\cdot 4)$$

2. If $b_{lm} \equiv B_{llm}$ and $c_{lm} \equiv C_{llm}$ are 2-cocycles, i.e., $b_{lm} (= -b_{ml}) = b_{ln} + b_{nm}$ and $c_{lm} (= -c_{ml}) = c_{ln} + c_{nm}$, then $\widetilde{(BC)}_{lmn}$ is a 3-cocycle:

$$(\delta \widetilde{(BC)})_{lmnk} \equiv \widetilde{(BC)}_{lmn} - \widetilde{(BC)}_{lmk} + \widetilde{(BC)}_{lnk} - \widetilde{(BC)}_{mnk} = 0. \quad (\text{A}\cdot 5)$$

We can show the following relations from the above expressions and properties.

1. For arbitrary cyclic cubic matrices A and B , $[A, B, \Delta]_{lmn} = 0$ with $\Delta_{lmn} = \delta_{lm} \delta_{mn}$.
2. For arbitrary normal cubic matrices $B_{lmn}^{(N)}$ and $C_{lmn}^{(N)}$, the triple-commutator among A , $B_{lmn}^{(N)}$ and $C_{lmn}^{(N)}$ is given by $[A, B^{(N)}, C^{(N)}]_{lmn} = A_{lmn} \widetilde{(B^{(N)} C^{(N)})}_{lmn}$.
3. The triple-commutator among arbitrary normal cubic matrices $A_{lmn}^{(N)}$, $B_{lmn}^{(N)}$ and $C_{lmn}^{(N)}$ is vanishing; that is, $[A^{(N)}, B^{(N)}, C^{(N)}]_{lmn} = 0$.
4. The fundamental identity holds if any two of A, B, C, D and E are normal forms (e.g., $D = D_{lmn}^{(N)}$ and $E = E_{lmn}^{(N)}$), and $(D^{(N)} E^{(N)})_{lmn}$ is a 3-cocycle:

$$\begin{aligned} [[A, B, C], D^{(N)}, E^{(N)}]_{lmn} &= [[A, D^{(N)}, E^{(N)}], B, C]_{lmn} \\ &\quad + [A, [B, D^{(N)}, E^{(N)}], C]_{lmn} + [A, B, [C, D^{(N)}, E^{(N)}]]_{lmn}. \end{aligned} \quad (\text{A}\cdot 6)$$

5. The derivation rule holds for normal cubic matrices $D_{lmn}^{(N)}$ and $E_{lmn}^{(N)}$, so that

$$[ABC, D^{(N)}, E^{(N)}]_{lmn} = ([A, D^{(N)}, E^{(N)}]BC)_{lmn} + (A[B, D^{(N)}, E^{(N)}]C)_{lmn} + (AB[C, D^{(N)}, E^{(N)}])_{lmn}, \tag{A.7}$$

if $(\widetilde{D^{(N)}E^{(N)}})_{lmn}$ is a 3-cocycle and $(ABC)_{lmn}$ is a cyclic cubic matrix.

Appendix B

— Example —

Here we study a simple example for variables that yield the generalization of the Heisenberg equation (3.11). The variables are three kinds of cyclic $3 \times 3 \times 3$ matrices defined by

$$\begin{aligned} X_{lmn}(t) &\equiv \frac{\hbar_C}{\sqrt{2}} |\varepsilon_{lmn}| e^{i\Omega_{lmn}t}, \\ Y_{lmn}(t) &\equiv \frac{\hbar_C}{i\sqrt{2}} \varepsilon_{lmn} e^{i\Omega_{lmn}t}, \\ Z_{lmn} &\equiv -\frac{\hbar_C}{\sqrt{6}} (\delta_{lm}\varepsilon_{mn} + \delta_{mn}\varepsilon_{nl} + \delta_{nl}\varepsilon_{lm}), \end{aligned} \tag{B.1}$$

where each of the indices l, m and n runs from 1 to 3, ε_{lmn} is the Levi-Civita symbol, and $\varepsilon_{lm} = \sum_k \varepsilon_{lmk}$. The above variables satisfy the relations

$$\begin{aligned} [X(t), Y(t), Z]_{lmn} &= -\hbar_C^2 W_{lmn}, \quad [Y(t), Z, W]_{lmn} = -\hbar_C^2 X_{lmn}(t), \\ [Z, W, X(t)]_{lmn} &= \hbar_C^2 Y_{lmn}(t), \quad [W, X(t), Y(t)]_{lmn} = -\hbar_C^2 Z_{lmn}, \end{aligned} \tag{B.2}$$

where W_{lmn} is proportional to I_{lmn} and defined as

$$W_{lmn} \equiv i \frac{\hbar_C}{\sqrt{6}} (\delta_{lm}(1 - \delta_{mn}) + \delta_{mn}(1 - \delta_{nl}) + \delta_{nl}(1 - \delta_{lm})). \tag{B.3}$$

When we consider $X_{lmn}(t), Y_{lmn}(t)$ and Z_{lmn} as a canonical triplet, the first relation in (B.2) is regarded as the first condition in (3.10).

Next, we introduce normal cubic matrices defined by

$$E_{lmn}^i \equiv \delta_{lm}(\delta_m^i - \delta_n^i) + \delta_{mn}(\delta_n^i - \delta_l^i) + \delta_{nl}(\delta_l^i - \delta_m^i), \tag{B.4}$$

where $i = 1, 2, 3$. The quantity $e_{lm}^i \equiv \delta_l^i - \delta_m^i$ satisfies $e_{lm}^i + e_{mn}^i + e_{nl}^i = 0$, because e_{lm}^i is a 2-coboundary. Further, there is the relation

$$e_{ml}^i e_{mn}^j - e_{ml}^j e_{mn}^i = -\varepsilon_{ij} \varepsilon_{lmn}. \tag{B.5}$$

The variables $X_{lmn}(t), Y_{lmn}(t)$ and E_{lmn}^i satisfy the relations

$$\begin{aligned} \{X(t), E^i, X(t)\}_{lmn} &= \{Y(t), E^i, Y(t)\}_{lmn} = \hbar_C^2 F_{lmn}^i, \\ \{X(t), F^i, X(t)\}_{lmn} &= \{Y(t), F^i, Y(t)\}_{lmn} = \hbar_C^2 E_{lmn}^i, \end{aligned} \tag{B.6}$$

where F_{lmn}^i are normal cubic matrices given by

$$F_{lmn}^i = \delta_{lm}(|\varepsilon_{mn}|\delta_m^i - |\varepsilon_{mni}|) + \delta_{mn}(|\varepsilon_{nl}|\delta_n^i - |\varepsilon_{nli}|) + \delta_{nl}(|\varepsilon_{lm}|\delta_l^i - |\varepsilon_{lmi}|). \quad (\text{B}\cdot 7)$$

Here, we use the formula of triple-anticommutator

$$\begin{aligned} \{A, B, C^{(N)}\}_{lmn} &= A_{lmn}(\widehat{BC^{(N)}})_{lmn} + B_{lmn}(\widehat{AC^{(N)}})_{lmn} \\ &+ \delta_{lm} \left(\sum_k (A_{mnk}B_{nmk} + B_{mnk}A_{nmk})c_{mk} \right) \\ &+ \delta_{mn} \left(\sum_k (A_{nlk}B_{lnk} + B_{nlk}A_{lnk})c_{nk} \right) \\ &+ \delta_{nl} \left(\sum_k (A_{lmk}B_{mlk} + B_{lmk}A_{mlk})c_{lk} \right), \end{aligned} \quad (\text{B}\cdot 8)$$

where $C^{(N)} = \delta_{lm}c_{mn} + \delta_{mn}c_{nl} + \delta_{nl}c_{lm}$ and $(\widehat{BC^{(N)}})_{lmn}$ is defined by

$$\begin{aligned} (\widehat{BC^{(N)}})_{lmn} &\equiv B_{lmn}c_{nm} + B_{lml}c_{ln} + B_{mnn}c_{ml} \\ &+ B_{lmm}c_{mn} + B_{nmn}c_{nl} + B_{lln}c_{lm}. \end{aligned} \quad (\text{B}\cdot 9)$$

When the Hamiltonians are given by

$$K_{lmn} = E_{lmn}^i, \quad H_{lmn} = \frac{1}{3}\hbar_C \Omega E_{lmn}^j, \quad (\text{B}\cdot 10)$$

the frequency Ω_{lmn} can be written

$$\Omega_{lmn} = -\Omega \varepsilon_{ij} \varepsilon_{lmn}, \quad (\text{B}\cdot 11)$$

by use of (3.14). Then, the time-development of the variables $X_{lmn}(t)$, $Y_{lmn}(t)$ and Z_{lmn} are given by

$$\frac{d}{dt} X_{lmn}(t) = \frac{1}{i\hbar_C} [X(t), K, H]_{lmn} = \Omega Y_{lmn}(t), \quad (\text{B}\cdot 12)$$

$$\frac{d}{dt} Y_{lmn}(t) = \frac{1}{i\hbar_C} [Y(t), K, H]_{lmn} = -\Omega X_{lmn}(t), \quad (\text{B}\cdot 13)$$

$$\frac{d}{dt} Z_{lmn} = \frac{1}{i\hbar_C} [Z, K, H]_{lmn} = 0, \quad (\text{B}\cdot 14)$$

where we take $i = 1(2, 3)$ and $j = 2(3, 1)$. We find that $X_{lmn}(t)$ and $Y_{lmn}(t)$ describe a harmonic oscillator from the above equations. The quantities K and H are expressed in terms of $X_{lmn}(t)$ and $Y_{lmn}(t)$ as^{*)}

$$\begin{aligned} K_{lmn} &= \frac{1}{2\hbar_C^2} \left(\{X(t), F^i, X(t)\}_{lmn} + \{Y(t), F^i, Y(t)\}_{lmn} \right), \\ H_{lmn} &= \frac{\Omega}{6\hbar_C} \left(\{X(t), F^j, X(t)\}_{lmn} + \{Y(t), F^j, Y(t)\}_{lmn} \right). \end{aligned} \quad (\text{B}\cdot 15)$$

^{*)} These expressions for K and H are not unique.

It is known that equations of the same forms as (B·12) and (B·13) are derived in cubic matrix mechanics.¹²⁾ We have

$$\frac{d}{dt}X_{lmn}(t) = \frac{1}{i\hbar_C}[X(t), I, H]_{lmn} = \Omega Y_{lmn}(t), \tag{B·16}$$

$$\frac{d}{dt}Y_{lmn}(t) = \frac{1}{i\hbar_C}[Y(t), I, H]_{lmn} = -\Omega X_{lmn}(t), \tag{B·17}$$

where I and H are given by

$$\begin{aligned} I_{lmn} &= \delta_{lm}(1 - \delta_{mn}) + \delta_{mn}(1 - \delta_{nl}) + \delta_{nl}(1 - \delta_{lm}), \\ H_{lmn} &= \frac{i\Omega}{6\hbar_C^2}[X(t), I, Y(t)]_{lmn}, \\ &= -\frac{1}{6}\hbar_C\Omega(\delta_{lm}\varepsilon_{mn} + \delta_{mn}\varepsilon_{nl} + \delta_{ln}\varepsilon_{lm}). \end{aligned} \tag{B·18}$$

Appendix C

— “Hamilton-Jacobi Formalism” for Nambu Mechanics —

In this appendix, we study “Hamilton-Jacobi formalism” for Nambu mechanics. The basic ingredient is the differential 2-form relation^{*)}

$$d\mathcal{S} = xdy \wedge dz - KdH \wedge dt, \tag{C·1}$$

where \wedge represents Cartan’s wedge product and $\mathcal{S} = \mathcal{S}(y, z, t)$ is a differential 1-form. Hamilton’s equations (2·1) are derived by taking the exterior derivatives of the above equation (C·1):

$$0 = dx \wedge dy \wedge dz - dK \wedge dH \wedge dt. \tag{C·2}$$

By use of the skew-symmetric property of the Nambu bracket and the equation of motion (2·8), we find that the Hamiltonians K and H are constants of motion. The trajectory of the physical system in the phase space (x, y, z) is determined by the intersection of two surfaces, $K(x, y, z) = k = \text{const}$ and $H(x, y, z) = h = \text{const}$. Hereafter, we consider periodic motion on the intersection given by $C(x, y) = \text{const}$ and $z = \text{const}$ for simplicity.

Next, we consider the canonical transformation from the canonical triplet (x, y, z) to $(\mathcal{J}_1, \omega t, \mathcal{J}_2)$,

$$xdy \wedge dz - KdH \wedge dt = \mathcal{J}_1d\theta \wedge d\mathcal{J}_2 - K'dH' \wedge dt + d\mathcal{W}, \tag{C·3}$$

where $\theta \equiv \omega t$ is the angle variable, $\mathcal{J}_2 = z$, and $\mathcal{W} = \mathcal{W}(y, z, \mathcal{J}_1, t)$ is a differential 1-form called the ‘generating function’. With the relation $\mathcal{W} = \alpha d\beta$ and the equation (C·3), we obtain the equations

$$x = \frac{\partial(\alpha, \beta)}{\partial(y, z)}, \quad \theta = \frac{\partial(\alpha, \beta)}{\partial(\mathcal{J}_1, \mathcal{J}_2)}, \quad \frac{\partial(\alpha, \beta)}{\partial(y, \mathcal{J}_1)} = 0, \tag{C·4}$$

$$-K \frac{\partial H}{\partial y} = \frac{\partial(\alpha, \beta)}{\partial(y, t)}, \quad -K \frac{\partial H}{\partial z} + K' \frac{\partial H'}{\partial \mathcal{J}_2} = \frac{\partial(\alpha, \beta)}{\partial(z, t)}, \quad K' \frac{\partial H'}{\partial \mathcal{J}_1} = \frac{\partial(\alpha, \beta)}{\partial(\mathcal{J}_1, t)}. \tag{C·5}$$

^{*)} It is known that the dynamics of relativistic strings are described by Hamilton-Jacobi formalism based on a slightly different 2-form including two evolution parameters from (C·1).¹⁶⁾

For a conserved system, we find that \mathcal{J}_1 and \mathcal{J}_2 are constants of motion from Hamilton's equations for \mathcal{J}_1 and \mathcal{J}_2 :

$$\frac{d\mathcal{J}_1}{dt} = \frac{\partial(K', H')}{\partial(\theta, \mathcal{J}_2)} = 0, \quad \frac{d\mathcal{J}_2}{dt} = \frac{\partial(K', H')}{\partial(\mathcal{J}_1, \theta)} = 0. \quad (\text{C}\cdot 6)$$

By solving the equations (C·6), we obtain the relations $\mathcal{J}_1 = \mathcal{J}_1(k, h)$ and $\mathcal{J}_2 = \mathcal{J}_2(k, h)$.

Finally, we study the change in θ over a complete cycle of y , given by

$$\Delta\theta = \oint \frac{\partial\theta}{\partial y} dy = \oint \frac{\partial}{\partial y} \frac{\partial(\alpha, \beta)}{\partial(\mathcal{J}_1, \mathcal{J}_2)} dy = \frac{d}{d\mathcal{J}_1} \oint \frac{\partial(\alpha, \beta)}{\partial(y, \mathcal{J}_2)} dy = \frac{d}{d\mathcal{J}_1} \oint x dy, \quad (\text{C}\cdot 7)$$

where we have used the equations (C·4). Because $\Delta\theta = 2\pi$, \mathcal{J}_1 is given by

$$\mathcal{J}_1 = \frac{1}{2\pi} \oint x dy. \quad (\text{C}\cdot 8)$$

Hence, \mathcal{J}_1 corresponds to the action variable in CM. The period T of a complete cycle of rotation is given by

$$T = 2\pi \frac{\partial(\mathcal{J}_2, \mathcal{J}_1)}{\partial(k, h)}, \quad (\text{C}\cdot 9)$$

as seen from Hamilton's equation for $\theta(= \frac{2\pi}{T}t)$,

$$\frac{d\theta}{dt} = \frac{\partial(K', H')}{\partial(\mathcal{J}_2, \mathcal{J}_1)}. \quad (\text{C}\cdot 10)$$

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