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Generalized Matrix Mechanics

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We propose a generalization of Heisenberg's matrix mechanics based on many-index objects. It is shown that there exists a solution describing a harmonic oscillator and that the many-index objects lead to a generalization of spin algebra.

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§1. Introduction

Until the end of 19th century, it was generally believed that any experimental results could be explained with classical mechanics (CM). The phenomenon of black body radiation destroyed this belief, and the concept of energy quanta was introduced by Planck in 1900 to overcome the difficulty presented. Since that time, quantum mechanics (QM) has been applied to very broad areas of physics with indisputable success. Considering its success, it is natural to ask the following questions:

1. Why does QM describe the microscopic world so successfully?
2. Does QM hold without limit?
3. If there are limitations, how is QM modified beyond them?

Unfortunately, we presently have no definite answers to these questions, although there are some conjectures. We expect that a generalization of CM and/or QM will provide information that can help to answer the above questions. From this point of view, it is meaningful to construct a new, generalized mechanics based on CM and/or QM.

Nambu proposed a generalization of Hamiltonian dynamics through the extension of phase space based on the Liouville theorem and gave a suggestion for its quantization.[1] The structure of this mechanics has been studied in the framework of constrained systems[2] and in geometric and algebraic formulations.[3] There are several works in which the quantization of Nambu mechanics is investigated.[3–8] This approach is quite interesting, but it is not the unique way to explore new mechanics. There is also the possibility of examining the generalization of QM directly, and here we consider this possibility.

In this paper, we propose a generalization of Heisenberg's matrix mechanics based on many-index objects (which we refer to as the M-matrix).**) It is shown that there exists a solution describing a harmonic oscillator and that the many-index

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**) Recently, Awata, Li, Minic and Yoneya introduced many-index objects to quantize Nambu mechanics.[6] We find that our definition of the triple product among cubic matrices is different from theirs, because we require a generalization of the Ritz rule in the phase factor, but not necessarily the associativity of the products.

objects lead to a generalization of spin algebra. A conjecture concerning operator formalism is also given.

This paper is organized as follows. In the next section, we review Heisenberg's matrix mechanics and explore its generalization. We formulate (cubic) matrix mechanics based on three-index objects in §3. Section 4 is devoted to conclusions and discussion.

§2. Matrix mechanics and generalization

2.1. *Heisenberg's matrix mechanics*

Here we review Heisenberg's matrix mechanics. For a closed physical system, physical quantities are represented by hermitian square matrices that can be written as

$$F_{mn}(t) = F_{mn} e^{i\Omega_{mn} t} = F_{mn} e^{\frac{i}{\hbar}(E_m - E_n)t}, \tag{2·1}$$

where the phase factor implies that a change in energy $E_m - E_n$ appears as radiation with angular frequency Ω_{mn}, and the hermiticity of $F_{mn}(t)$ is expressed by $F^*_{nm}(t) = F_{mn}(t)$. By the usual definition of the product of two square matrices $A_{mn}(t) = A_{mn} e^{i\Omega_{mn} t}$ and $B_{mn}(t) = B_{mn} e^{i\Omega_{mn} t}$,

$$(AB)_{mn}(t) \equiv \sum_k A_{mk}(t) B_{kn}(t) = \sum_k A_{mk} B_{kn} e^{i\Omega_{mn} t}, \tag{2·2}$$

it is seen that the product $(AB)_{mn}(t)$ has the same form as (2·1), with the Ritz rule $\Omega_{mn} = \Omega_{mk} + \Omega_{kn}$. The time development of $F_{mn}(t)$ is expressed by the Heisenberg equation

$$\begin{aligned} \frac{d}{dt} F_{mn}(t) &= i\Omega_{mn} F_{mn} = \frac{i}{\hbar}(E_m - E_n) F_{mn}(t) \\ &= \frac{1}{i\hbar}((F(t)H)_{mn} - (HF(t))_{mn}) \equiv \frac{1}{i\hbar}[F(t), H]_{mn}, \end{aligned} \tag{2·3}$$

where the Hamiltonian H is a diagonal matrix written $H_{mn} \equiv E_m \delta_{mn}$.

Here we give a simple example of a harmonic oscillator whose variables are two hermitian matrices, $\xi_{mn}(t) = \xi_{mn} e^{i\Omega_{mn} t}$ and $\eta_{mn}(t) = \eta_{mn} e^{i\Omega_{mn} t}$. The coefficients ξ_{mn} and η_{mn} are given by

$$\xi_{mn} = \sqrt{\frac{\hbar}{2m\Omega}} (\sigma^1)_{mn} \quad \text{and} \quad \eta_{mn} = \sqrt{\frac{m\Omega\hbar}{2}} (\sigma^2)_{mn}, \tag{2·4}$$

respectively. Here the quantity m in the square root represents a mass, the $(\sigma^a)_{mn}$ are Pauli matrices, and $\Omega = \Omega_{21} (> 0)$. The variables $\xi_{mn}(t)$ and $\eta_{mn}(t)$ satisfy the following anticommutation relations:

$$\{\xi(t), \xi(t)\}_{mn} = \frac{\hbar}{m\Omega} \delta_{mn}, \quad \{\eta(t), \eta(t)\}_{mn} = m\Omega\hbar\delta_{mn}, \tag{2·5}$$

$$\{\xi(t), \eta(t)\}_{mn} = 0. \tag{2·6}$$

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With the above, we obtain the equations of motion describing the harmonic oscillator,

$$\frac{d}{dt}\xi_{mn}(t) = \frac{1}{i\hbar}[\xi, H]_{mn} = \frac{1}{m}\eta_{mn}(t), \tag{2·7}$$

$$\frac{d}{dt}\eta_{mn}(t) = \frac{1}{i\hbar}[\eta, H]_{mn} = -m\Omega^2\xi_{mn}(t), \tag{2·8}$$

where the Hamiltonian H_{mn} is written

$$H_{mn} = i\Omega\sum_k \xi_{mk}(t)\eta_{kn}(t) = -\frac{1}{2}\hbar\Omega(\sigma^3)_{mn}. \tag{2·9}$$

2.2. *Conjecture on M-matrix mechanics*

Let us extend the formulation described in the previous subsection to a system with M-matrix valued quantities, whose variables are given by

$$F_{m_1m_2\cdots m_n}(t) = F_{m_1m_2\cdots m_n}e^{i\Omega_{m_1m_2\cdots m_n}t}, \tag{2·10}$$

where the angular frequency $\Omega_{m_1m_2\cdots m_n}$ is written in terms of antisymmetric quantities $\omega_{m_1m_2\cdots m_{n-1}}$ as

$$\Omega_{m_1m_2\cdots m_n} = \sum_{j=1}^{n}(-1)^{n-j}\omega_{m_1\cdots m_{j-1}m_{j+1}\cdots m_n} \equiv (\partial\omega)_{m_1m_2\cdots m_n}. \tag{2·11}$$

Here we assume the generalization of Bohrs' frequency condition[*)]

$$\Omega_{m_1m_2\cdots m_n} = \frac{1}{\hbar}\sum_{j=1}^{n}(-1)^{n-j}E_{m_1\cdots m_{j-1}m_{j+1}\cdots m_n}. \tag{2·12}$$

The antisymmetric property is expressed by

$$\Omega_{m'_1m'_2\cdots m'_n} = \mathrm{sgn}(P)\Omega_{m_1m_2\cdots m_n},\ \omega_{m'_1m'_2\cdots m'_{n-1}} = \mathrm{sgn}(P)\omega_{m_1m_2\cdots m_{n-1}}, \tag{2·13}$$

where $\mathrm{sgn}(P)$ is $+1$ and -1 for even and odd permutation among indices, respectively. The operator ∂ is regarded as a boundary operator that changes k-th antisymmetric objects into $(k+1)$-th objects, and this operation is nilpotent, i.e. $\partial^2(*) = 0$.[9)] Hence a homology group can be constructed from a set of phase factors of M-matrices. The $\Omega_{m_1m_2\cdots m_n}$ are regarded as $(n-1)$-boundaries. We define the hermiticity of an n-index object by $F_{m'_1m'_2\cdots m'_n}(t) = F^*_{m_1m_2\cdots m_n}(t)$ for odd permutations of the subscripts. If we define an n-fold product among $F^{(a)}_{m_1m_2\cdots m_n}(t)$ $(a = 1, 2, ...)$ by

$$\begin{aligned}(F^{(1)}\cdots F^{(n)})_{m_1m_2\cdots m_n}(t) &\equiv \sum_k F^{(1)}_{m_1\cdots m_{n-1}k}(t)F^{(2)}_{m_1\cdots m_{n-2}km_n}(t)\cdots F^{(n)}_{km_2\cdots m_n}(t)\\ &= \sum_k (F^{(1)}\cdots F^{(n)})_{m_1m_2\cdots m_n}e^{i\Omega_{m_1m_2\cdots m_n}t},\end{aligned} \tag{2·14}$$

*) Here and hereafter we use the reduced Planck constant $\hbar = \frac{h}{2\pi}$ as the unit of action, with the expectation that M-matrix mechanics reduces to QM in a particular limit and is characterized by the same physical constant.

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the outcome has the same form as (2·10) with the relation $(\partial\Omega)_{m_1m_2\cdots m_{n+1}} = 0$, which is a generalization of the Ritz rule.

Next, we discuss the time evolution of M-matrices, $F^{(a)}_{m_1m_2\cdots m_n}(t)$. It is natural to have conjecture that the equation of motion is given by

$$\begin{aligned}\frac{d}{dt}F^{(a)}_{m_1m_2\cdots m_n}(t) &= i\Omega_{m_1m_2\cdots m_n}F^{(a)}_{m_1m_2\cdots m_n}(t) \\ &= \frac{i}{\hbar}\sum_{j=1}^{n}(-1)^{n-j}E_{m_1\cdots m_{j-1}m_{j+1}\cdots m_n}F^{(a)}_{m_1m_2\cdots m_n}(t) \\ &= \frac{1}{i\hbar}(F^{(a)}(t), K^{(1)}, \cdots, K^{(n-2)}, H)_{m_1m_2\cdots m_n}, \end{aligned} \tag{2·15}$$

where the quantities $K^{(1)}$, $\cdots$, $K^{(n-2)}$ and H are time-independent n-index objects called *Hamiltonians*, and $(*,*,\cdots,*)$ is a linear combination of n-fold products among variables. Equation (2·15) is regarded as a generalization of the Heisenberg equation. An ansatz for Hamiltonians and $(*,*,\cdots,*)$ is given by

$$\begin{aligned}K^{(1)} = \cdots = K^{(n-2)} &= I_{m_1m_2\cdots m_n} \\ &\equiv \sum_{(i,j)} I^{m_1\cdots m_{i-1}}{}_{m_i}{}^{m_{i+1}\cdots m_{j-1}}{}_{m_j}{}^{m_{j+1}\cdots m_n}, \end{aligned} \tag{2·16}$$

$$H_{m_1m_2\cdots m_n} = -\frac{1}{2}\sum_{j=1}^{n}(-1)^{n-j}E_{m_1\cdots m_{j-1}m_{j+1}\cdots m_n}\delta_{m_{j-1}m_j} \tag{2·17}$$

and

$$\begin{aligned}(F^{(1)}, F^{(2)}, \cdots, F^{(n)})_{m_1m_2\cdots m_n} = &\sum_{\text{cyclic}}(F^{(1)}F^{(2)}\cdots F^{(n)})_{m_1m_2\cdots m_n} \\ &- \sum_{\text{cyclic}}(F^{(n)}\cdots F^{(2)}F^{(1)})_{m_1m_2\cdots m_n}, \end{aligned} \tag{2·18}$$

where $I^{m_1\cdots m_{i-1}}{}_{m_i}{}^{m_{i+1}\cdots m_{j-1}}{}_{m_j}{}^{m_{j+1}\cdots m_n} = \delta_{m_im_j}\prod_{(k,l)\neq(i,j)}(1-\delta_{m_km_l})$ and $\delta_{m_0m_1} = \delta_{m_nm_1}$ in (2·17), and the summation in (2·18) is over all cyclic permutations. The quantity $I_{m_1m_2\cdots m_n}$ plays the role of a unit matrix.

We now give a comment on a set of n-index objects. We find that the $(n+1)\times(n+1)\times\cdots\times(n+1)$ matrices defined by $J^{(a)}_{m_1m_2\cdots m_n} \equiv -i\hbar\varepsilon_{am_1m_2\cdots m_n}$ satisfy the following interesting algebra:

$$\begin{aligned}&[J^{(a_1)}, J^{(a_2)}, \cdots, J^{(a_n)}]_{m_1m_2\cdots m_n} \\ &\qquad = -(-1)^{\frac{n(n+1)(n-1)}{2}}(i\hbar)^{n-1}\varepsilon_{a_1a_2\cdots a_na_{n+1}}J^{(a_{n+1})}_{m_1m_2\cdots m_n}. \end{aligned} \tag{2·19}$$

In this equation, the n-fold commutator is defined by

$$\begin{aligned}&[F^{(a_1)}, F^{(a_2)}, \cdots, F^{(a_n)}]_{m_1m_2\cdots m_n} \\ &\qquad \equiv \sum \text{sgn}(P)(F^{(a'_1)}F^{(a'_2)}\cdots F^{(a'_n)})_{m_1m_2\cdots m_n}, \end{aligned} \tag{2·20}$$

where the summation is over all permutations among the superscripts. The algebra (2·19) is a generalization of ordinary spin algebra [$su(2)$ algebra] and is equivalent to a special case of M-algebra discussed in Ref. 5).

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2.3. *Relation to classical dynamics*

Before we study a cubic matrix, we discuss the structure of classical dynamics from the viewpoint of matrix mechanics. First we review the relation between CM and QM. A physical variable $F(t)$ in CM is regarded as a linear combination of one-index objects (a 1×1 matrix) such that

$$F(t) = \sum_n F_n e^{i\Omega_n t}, \tag{2·21}$$

where $F_n^* = F_{-n}$, because $F(t)$ should be a real quantity, and the angular frequency Ω_n is an integer multiple of the basic frequency ω, i.e. $\Omega_n = n\omega$. Under the guidance of Bohrs' correspondence principle and the frequency condition, we obtain a relation between ω and the Hamiltonian H,

$$\omega = \frac{\Omega_{\Delta n}}{\Delta n} = \lim_{\frac{\hbar \Delta n}{n} \to 0} \frac{\Omega_{n+\Delta n n}}{\Delta n} = \lim_{\frac{\hbar \Delta n}{n} \to 0} \frac{E_{n+\Delta n} - E_n}{\hbar \Delta n} = \frac{dE}{dJ} = \frac{\partial H}{\partial J}, \tag{2·22}$$

where J is the action variable and we use $J = \oint p dq = hn$ (Bohr-Sommerfeld quantization condition). The equation of motion for $F(t)$ is written

$$\frac{d}{dt} F(t) = \sum_n in\omega F_n e^{i\Omega_n t} = \frac{\partial F(t)}{\partial(\omega t)} \frac{\partial H}{\partial J} = \{F(t), H\}_{\mathrm{PB}}, \tag{2·23}$$

where $\{*, *\}_{\mathrm{PB}}$ is the Poisson bracket and we use the fact that J is the canonical conjugate of the angle variable ωt. Equation (2·23) is Hamilton's canonical equation.

Next, we study the 'classical' limit of M-matrix mechanics based on an n-index object, whose frequency condition is given by (2·12). We require that there are generalizations of Bohr's correspondence principle and the Bohr-Sommerfeld quantization condition and that the 'classical' counterpart of the n-index object satisfies Hamilton's canonical equation. A system which satisfies these requirements is obtained under the assumption that the variables depend on intrinsic $(n-2)$ parameters $\vec{\sigma} = (\sigma_1, \cdots, \sigma_{n-2})$; that is, a physical variable $F(t, \vec{\sigma})$ is given by

$$F(t, \vec{\sigma}) = \sum_n F_n(\vec{\sigma}) e^{i\Omega_n t}, \tag{2·24}$$

where $F_n^*(\vec{\sigma}) = F_{-n}(\vec{\sigma})$ and $\Omega_n = n\omega$. The energy $E(J(\vec{\sigma}))$ is given by the functional integral

$$E(J(\vec{\sigma})) = \int_\Sigma \mathcal{E}(J(\vec{\sigma})) d^{n-2}\sigma, \tag{2·25}$$

where Σ is a closed $(n-2)$-dimensional surface and $J(\vec{\sigma})$ is an action variable. The correspondence of $E(J(\vec{\sigma}))$ to $E_{m_1 m_2 \cdots m_{n-1}}$ is obtained by the replacement of Σ with an oriented $(n-1)$-simplex $\partial(m_1 m_2 \cdots m_n) = \sum_{j=1}^n (-1)^{n-j} (m_1 \cdots m_{j-1} m_{j+1} \cdots m_n)$,

$$\int_{\partial(m_1 m_2 \cdots m_n)} \mathcal{E}(J(\vec{\sigma})) d^{n-2}\sigma = \sum_{j=1}^n (-1)^{n-j} \int_{(m_1 \cdots m_{j-1} m_{j+1} \cdots m_n)} \mathcal{E}(J(\vec{\sigma})) d^{n-2}\sigma$$
$$\Longleftrightarrow \sum_{j=1}^n (-1)^{n-j} E_{m_1 \cdots m_{j-1} m_{j+1} \cdots m_n}, \tag{2·26}$$

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that is,

$$\int_{(m_1\cdots m_{j-1}m_{j+1}\cdots m_n)} \mathcal{E}(J(\vec{\sigma}))d^{n-2}\sigma \Longleftrightarrow E_{m_1\cdots m_{j-1}m_{j+1}\cdots m_n}, \qquad (2\cdot 27)$$

where $\Longleftrightarrow$ indicates the correspondence. Generalizations of Bohr's correspondence principle and the Bohr-Sommerfeld quantization condition are given by

$$\omega = \frac{\Omega_{\Delta N}}{\Delta N} = \lim_{\Delta N \to 0} \frac{\Omega_{n+\Delta n_{n-1}\cdots n+\Delta n_1 n}}{\Delta N} \qquad (2\cdot 28)$$

and

$$\int_{\Delta\Sigma} J(\vec{\sigma})d^{n-2}\sigma = \hbar\Delta N, \qquad (2\cdot 29)$$

respectively. Here ΔN is a function of the quantities Δn_i and goes to zero as the Δn_i do, and $\Delta\Sigma$ is an infinitesimal closed $(n-2)$-dimensional surface attached to a point $\vec{\sigma}$. By use of (2·12), (2·28) and (2·29), we derive the relation

$$\begin{aligned} \omega &= \lim_{\Delta N \to 0} \frac{\sum_{j=1}^{n}(-1)^{n-j} E_{n+\Delta n_{n-1}\cdots n+\Delta n_{n-j+1}n+\Delta n_{n-j-1}\cdots n+\Delta n_1 n}}{\hbar\Delta N} \\ &\Longleftrightarrow \frac{\int_{\Delta\Sigma} \mathcal{E}(J(\vec{\sigma}))d^{n-2}\sigma}{\int_{\Delta\Sigma} J(\vec{\sigma})d^{n-2}\sigma} \\ &= \frac{\int_{\Delta\Sigma}(\mathcal{E}(J(\vec{\sigma})+\Delta J(\vec{\sigma})) - \mathcal{E}(J(\vec{\sigma})))d^{n-2}\sigma}{\int_{\Delta\Sigma} \Delta J(\vec{\sigma})d^{n-2}\sigma} = \frac{\delta E(J(\vec{\sigma}))}{\delta J(\vec{\sigma})} = \frac{\delta H}{\delta J}, \end{aligned} \qquad (2\cdot 30)$$

where we have used a special type of infinitesimal deformation of J such that $\tilde{J} = J + \Delta J = 0$ on $\Delta\Sigma$ with $\mathcal{E}(\tilde{J} = 0) = 0$, and $\delta/\delta J(\vec{\sigma})$ represents the functional derivative with respect to $J(\vec{\sigma})$. Hence, we find that $F(t, \vec{\sigma})$ satisfies Hamilton's canonical equation $dF/dt = \{F, H\}_{\mathrm{PB}}$.

Finally, we discuss the physical meaning of the $(n-2)$ parameters $\vec{\sigma}$. The position of an object is represented by $x^i(t, \vec{\sigma})$, or $x^\mu(\tau, \vec{\sigma})$ in a system with relativistic invariance. Here, τ is a parameter that corresponds to time development and the $\vec{\sigma}$ are interpreted as spatial coordinates that describe an extended object. In this way, we have arrived at the interesting conjecture that *the 'classical' counterpart of an n-index object is an $(n-2)$-dimensional object and that M-matrix mechanics can describe the 'quantum' physics of extended objects.*

§3. Cubic matrix mechanics

3.1. *Cubic matrix*

We now consider a three-index object (cubic matrix) given by

$$C_{lmn}(t) = C_{lmn}e^{i\Omega_{lmn}t}, \qquad (3\cdot 1)$$

where the C_{lmn} possesses cyclic symmetry, i.e., $C_{lmn} = C_{mnl} = C_{nlm}$, and the angular frequency Ω_{lmn} has the form

$$\Omega_{lmn} = \omega_{lm} - \omega_{ln} + \omega_{mn} \equiv (\partial\omega)_{lmn}, \quad \omega_{ml} = -\omega_{lm}. \qquad (3\cdot 2)$$

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The angular frequencies Ω_{lmn} have the following properties:

$$\Omega_{l'm'n'} = \mathrm{sgn}(P)\Omega_{lmn}, \tag{3·3}$$

$$(\partial\Omega)_{lmnk} \equiv \Omega_{lmn} - \Omega_{lmk} + \Omega_{lnk} - \Omega_{mnk} = 0. \tag{3·4}$$

The relations (3·2) and (3·4) show that the Ω_{lmn} are 2-boundaries when ∂ is regarded as a boundary operator. We define the hermiticity of a cubic matrix by $C_{l'm'n'}(t) = C^*_{lmn}(t)$ for odd permutations among indices. For a hermitian cubic matrix, there are relations

$$C_{lmn}(t) = C_{mnl}(t) = C_{nlm}(t) = C^*_{mln}(t) = C^*_{lnm}(t) = C^*_{nml}(t). \tag{3·5}$$

If we define the triple product among cubic matrices $C_{lmn}(t) = C_{lmn}e^{i\Omega_{lmn}t}$, $D_{lmn}(t) = D_{lmn}e^{i\Omega_{lmn}t}$ and $E_{lmn}(t) = E_{lmn}e^{i\Omega_{lmn}t}$ by

$$(C(t)D(t)E(t))_{lmn} \equiv \sum_k C_{lmk}(t)D_{lkn}(t)E_{kmn}(t) = (CDE)_{lmn}e^{i\Omega_{lmn}t}, \tag{3·6}$$

the product takes the same form as (3·1) with the relation (3·4). Note that this product is, in general, neither commutative nor associative, that is, $(CDE)_{lmn} \neq (DCE)_{lmn}$ and $(AB(CDE))_{lmn} \neq (A(BCD)E)_{lmn} \neq ((ABC)DE)_{lmn}$. Taking the hermitian conjugate of products for hermitian cubic matrices, we obtain the relations

$$\begin{aligned}(C(t)D(t)E(t))_{lmn} &= (E(t)D(t)C(t))^*_{nml} = (C(t)E(t)D(t))^*_{mln}\\ &= (D(t)C(t)E(t))^*_{lnm} = (D(t)E(t)C(t))_{nlm} = (E(t)C(t)D(t))_{mnl}.\end{aligned} \tag{3·7}$$

The triple-commutator and anticommutator are defined by

$$\begin{aligned}[C(t), D(t), E(t)]_{lmn} \equiv (&C(t)D(t)E(t) + D(t)E(t)C(t) + E(t)C(t)D(t)\\ &- D(t)C(t)E(t) - C(t)E(t)D(t) - E(t)D(t)C(t))_{lmn}\end{aligned} \tag{3·8}$$

and

$$\begin{aligned}\{C(t), D(t), E(t)\}_{lmn} \equiv (&C(t)D(t)E(t) + D(t)E(t)C(t) + E(t)C(t)D(t)\\ &+ D(t)C(t)E(t) + C(t)E(t)D(t) + E(t)D(t)C(t))_{lmn},\end{aligned} \tag{3·9}$$

respectively. With the above definitions, we have the relation

$$[A^{(a')}(t), A^{(b')}(t), A^{(c')}(t)]_{lmn} = \mathrm{sgn}(P)[A^{(a)}(t), A^{(b)}(t), A^{(c)}(t)]_{lmn}. \tag{3·10}$$

If $C_{lmn}(t)$, $D_{lmn}(t)$ and $E_{lmn}(t)$ are hermitian matrices, $[C(t), D(t), E(t)]_{lmn}$ and $\{C(t), D(t), E(t)\}_{lmn}$ are also hermitian cubic matrices.

3.2. *Dynamics*

The cyclically symmetric cubic matrices $C^{(a)}_{lmn}(t)$ yield the generalization of the Heisenberg equation

$$\frac{d}{dt}C^{(a)}_{lmn}(t) = i\Omega_{lmn}C^{(a)}_{lmn}(t) = \frac{1}{i\hbar}[C^{(a)}(t), K, H]_{lmn}, \tag{3·11}$$

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where K and H are time independent 3-index objects. A possible form of K and H is given by

$$K_{lmn} = I_{lm}{}^{n} + I^{l}{}_{mn} + I_{l}{}^{m}{}_{n} \equiv I_{lmn}, \tag{3·12}$$

$$H_{lmn} = \frac{1}{2}\hbar\omega_{mn} I_{lm}{}^{n} + \frac{1}{2}\hbar\omega_{nl} I^{l}{}_{mn} + \frac{1}{2}\hbar\omega_{lm} I_{l}{}^{m}{}_{n}, \tag{3·13}$$

where $I_{lm}{}^{n}$, $I_{l}{}^{m}{}_{n}$ and $I^{l}{}_{mn}$ are defined by

$$I_{lm}{}^{n} \equiv \delta_{lm}(1-\delta_{nl}), \quad I_{l}{}^{m}{}_{n} \equiv \delta_{ln}(1-\delta_{mn}), \quad I^{l}{}_{mn} \equiv \delta_{mn}(1-\delta_{lm}). \tag{3·14}$$

Our triple-commutator in general, does not satisfy conditions such as the derivation rule (which is a counterpart of the Leibniz rule in differential calculus) and a generalization of the Jacobi identity called *a fundamental identity*, both of which are possessed by the Nambu-Poisson bracket. As an exceptional case, the derivation rule and the fundamental identity hold for the triple-commutator including the Hamiltonians K and H:

$$\begin{aligned}\frac{d}{dt}(C(t)D(t)E(t))_{lmn} &= \left(\frac{dC(t)}{dt}D(t)E(t)\right)_{lmn} + \left(C(t)\frac{dD(t)}{dt}E(t)\right)_{lmn} \\ &\quad + \left(C(t)D(t)\frac{dE(t)}{dt}\right)_{lmn} \\ &= i\Omega_{lmn}(C(t)D(t)E(t))_{lmn} \\ &= ([C(t),K,H]D(t)E(t))_{lmn} + (C(t)[D(t),K,H]E(t))_{lmn} \\ &\quad + (C(t)D(t)[E(t),K,H])_{lmn} \\ &= [C(t)D(t)E(t),K,H]_{lmn}\end{aligned} \tag{3·15}$$

for $(CDE)_{llm} = (CDE)_{lml} = (CDE)_{mll}$ and

$$\begin{aligned}[[C(t),D(t),E(t)],K,H]_{lmn} &= [[C(t),K,H],D(t),E(t)]_{lmn} \\ &\quad + [C(t),[D(t),K,H],E(t)]_{lmn} + [C(t),D(t),[E(t),K,H]]_{lmn}.\end{aligned} \tag{3·16}$$

It is thus seen that our description of the time development is consistent for cyclically symmetric matrices.

3.3. *Example*

We now study the simple example of a harmonic oscillator whose variables are two hermitian $3 \times 3 \times 3$ matrices $\xi_{lmn}(t) = \xi_{lmn} e^{i\Omega_{lmn}t}$ and $\eta_{lmn}(t) = \eta_{lmn} e^{i\Omega_{lmn}t}$. The coefficients ξ_{lmn} and η_{lmn} are given by

$$\xi_{lmn} = -\sqrt{\frac{\hbar}{2m\Omega}}\frac{\Omega_{lmn}}{\Omega}\varepsilon_{lmn}, \quad \eta_{lmn} = \frac{1}{i}\sqrt{\frac{m\Omega\hbar}{2}}\varepsilon_{lmn}, \tag{3·17}$$

where the quantity m in the square root represents a mass, and $\Omega = \Omega_{321}(>0)$. The variables $\xi_{lmn}(t)$ and $\eta_{lmn}(t)$ satisfy the relations

$$(I\xi^2)_{lmn} = \frac{\hbar}{2m\Omega} I_{lm}{}^{n}, \ (\xi^2 I)_{lmn} = \frac{\hbar}{2m\Omega} I^{l}{}_{mn}, \ (\xi I \xi)_{lmn} = \frac{\hbar}{2m\Omega} I_{l}{}^{m}{}_{n}, \tag{3·18}$$

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$$(I\eta^2)_{lmn} = \frac{\hbar m\Omega}{2} I_{lm}{}^{n}, \ (\eta^2 I)_{lmn} = \frac{\hbar m\Omega}{2} I^{l}{}_{mn}, \ (\eta I\eta)_{lmn} = \frac{\hbar m\Omega}{2} I_{l}{}^{m}{}_{n}, \quad (3{\cdot}19)$$

$$(I\xi\eta)_{lmn} + (I\eta\xi)_{lmn} = (\xi\eta I)_{lmn} + (\eta\xi I)_{lmn} = (\xi I\eta)_{lmn} + (\eta I\xi)_{lmn} = 0, \quad (3{\cdot}20)$$

$$(I\eta\xi)_{lmn} = \frac{i\hbar}{2} I^{(3)}_{lm}{}^{n}, \ (\eta\xi I)_{lmn} = \frac{i\hbar}{2} I^{(3)l}{}_{mn}, \ (\xi I\eta)_{lmn} = \frac{i\hbar}{2} I^{(3)}_{l}{}^{m}{}_{n}, \quad (3{\cdot}21)$$

$$(I^{(3)}\eta\xi)_{lmn} = \frac{i\hbar}{2} I_{lm}{}^{n}, \ (\eta\xi I^{(3)})_{lmn} = \frac{i\hbar}{2} I^{l}{}_{mn}, \ (\xi I^{(3)}\eta)_{lmn} = \frac{i\hbar}{2} I_{l}{}^{m}{}_{n}, \quad (3{\cdot}22)$$

$$(\xi^3)_{lmn} = (\xi^2\eta)_{lmn} = \cdots = (\eta^2\xi)_{lmn} = (\eta^3)_{lmn} = 0, \quad (3{\cdot}23)$$

where $I = I_{lmn}$ and $I^{(3)} = I^{(3)}_{lmn} \equiv I^{(3)}_{lm}{}^{n} + I^{(3)l}{}_{mn} + I^{(3)}_{l}{}^{m}{}_{n}$. Here $I^{(3)}_{lm}{}^{n}$, $I^{(3)l}{}_{mn}$ and $I^{(3)}_{l}{}^{m}{}_{n}$ are defined by

$$I^{(3)}_{lm}{}^{n} \equiv \delta_{lm}\varepsilon_{mn}, \quad I^{(3)l}{}_{mn} \equiv \delta_{mn}\varepsilon_{nl}, \quad I^{(3)}_{l}{}^{m}{}_{n} \equiv \delta_{ln}\varepsilon_{lm}, \quad (3{\cdot}24)$$

where $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = -\varepsilon_{21} = -\varepsilon_{32} = -\varepsilon_{13} = 1$. With the above, we obtain the equations of motion describing the harmonic oscillator

$$\frac{d}{dt}\xi_{lmn}(t) = \frac{1}{i\hbar}[\xi, K, H]_{lmn} = \frac{1}{m}\eta_{lmn}(t), \quad (3{\cdot}25)$$

$$\frac{d}{dt}\eta_{lmn}(t) = \frac{1}{i\hbar}[\eta, K, H]_{lmn} = -m\Omega^2\xi_{lmn}(t), \quad (3{\cdot}26)$$

where K and H are given by

$$K_{lmn} = \frac{1}{i\hbar}[\xi, I^{(3)}, \eta]_{lmn} = I_{lmn}, \quad H_{lmn} = \frac{i}{6}\Omega[\xi, I, \eta]_{lmn} = -\frac{1}{6}\hbar\Omega I^{(3)}_{lmn}. \quad (3{\cdot}27)$$

3.4. *Operator formalism*

In the preceding sections, we have studied a generalization of QM using the M-matrix formalism. The mechanics we obtain has an interesting algebraic structure, but the formalism is not practical, because it is only applicable to stationary systems. From experience, it is known that in order to be of practical use operator formalism must be capable of handling problems in a wider class of physical systems. By analogy to QM, we now study the operator formalism of cubic matrix mechanics. First, we make the following basic assumptions.

1. For a given physical system, there exist triplet of state vectors $|m_1; P_{m_1m_2m_3}\rangle$, $|m_2; P_{m_1m_2m_3}\rangle$ and $|m_3; P_{m_1m_2m_3}\rangle$ that depend on both the quantum numbers m_i, (e.g., these m_i represent l, m or n) and their ordering. Here, the ordering is represented by a permutation (denoted by $P_{m_1m_2m_3}$) for a standard ordering, (e.g., $m_1 = l, m_2 = m, m_3 = n$).
2. For every physical observable, there is a one-to-one correspondence to a linear operator $\hat{C}$.

Under the above assumptions, it is natural to identify the cubic matrix element C_{lmn} with $\hat{C}|l; P_{lmn}\rangle|m; P_{lmn}\rangle|n; P_{lmn}\rangle$. In general, the quantity $C_{m_1m_2m_3}$ is identified with $\hat{C}|m_1; P_{m_1m_2m_3}\rangle|m_2; P_{m_1m_2m_3}\rangle|m_3; P_{m_1m_2m_3}\rangle$. By use of (3·11), the following equations of motion for the states are derived:

$$i\hbar\frac{d}{dt}|l; P_{lmn}\rangle = [\hat{K}, \hat{H}]|l; P_{lmn}\rangle, \quad i\hbar\frac{d}{dt}|m; P_{lmn}\rangle = [\hat{K}, \hat{H}]|m; P_{lmn}\rangle,$$

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$$i\hbar \frac{d}{dt}|n; P_{lmn}\rangle = [\hat{K}, \hat{H}]|n; P_{lmn}\rangle. \quad (3\cdot28)$$

Here, $[\hat{K}, \hat{H}]$ is the commutator of the operators $\hat{K}$ and $\hat{H}$. (Note that $[\hat{K}, \hat{H}]$ in the third equation corresponds to $\sum_k (K_{lkn}H_{kmn} - H_{lkn}K_{kmn})$ in cubic matrix mechanics.) The above equations (3·28) are regarded as a generalization of the Schrödinger equation. The commutator $\hat{\mathcal{H}} \equiv [\hat{K}, \hat{H}]$ is interpreted as the generalized Hamiltonian operator. By use of (3·12) and (3·13), the time evolution of state vectors is given by

$$|l; P_{lmn}\rangle = \exp\left(\frac{i}{2}(\omega_{nl} + \omega_{lm})t\right)|l; P_{lmn}\rangle_0,$$
$$|m; P_{lmn}\rangle = \exp\left(\frac{i}{2}(\omega_{lm} + \omega_{mn})t\right)|m; P_{lmn}\rangle_0,$$
$$|n; P_{lmn}\rangle = \exp\left(\frac{i}{2}(\omega_{mn} + \omega_{nl})t\right)|n; P_{lmn}\rangle_0, \quad (3\cdot29)$$

where the subscript 0 indicates that the state is that at an initial time. In the same way, the time development of state vectors for the matrix element C_{mln} is given by

$$|l; P_{mln}\rangle = \exp\left(\frac{i}{2}(\omega_{ml} + \omega_{ln})t\right)|l; P_{mln}\rangle_0,$$
$$|m; P_{mln}\rangle = \exp\left(\frac{i}{2}(\omega_{nm} + \omega_{ml})t\right)|m; P_{mln}\rangle_0,$$
$$|n; P_{mln}\rangle = \exp\left(\frac{i}{2}(\omega_{ln} + \omega_{nm})t\right)|n; P_{mln}\rangle_0. \quad (3\cdot30)$$

We can identify $|l; P_{mln}\rangle$ with the complex conjugate of $|l; P_{lmn}\rangle$ from (3·29) and (3·30). It is seen that this identification is consistent with the relations (3·5).

§4. Conclusions and discussion

We have proposed a generalization of Heisenberg's matrix mechanics based on many-index objects. It has been shown that there exists a solution describing a harmonic oscillator [the three-index objects $\xi_{lmn}(t)$ and $\eta_{lmn}(t)$ defined by (3·17) satisfy Eqs. (3·25) and (3·26)] and that many-index objects lead to a generalization of spin algebra [the $4 \times 4 \times 4$ matrices defined by $J^{(a)}_{lmn} \equiv -i\hbar\varepsilon_{almn}$ satisfy the algebra $[J^{(a)}, J^{(b)}, J^{(c)}]_{lmn} = \hbar^2 \varepsilon_{abcd} J^{(d)}_{lmn}$, where a, b, c, d, l, m, n are integers from 1 to 4.] We have studied the 'classical' limit of generalized matrix mechanics and obtained evidence that M-matrix mechanics can be regarded as a 'quantum' theory of extended objects. We have also made a conjecture on the operator formalism of cubic matrix mechanics. The basic equations are given by (3·28).

Finally we give comments regarding the questions raised in the Introduction.

With regard to the question of why QM describes the microscopic world so successfully, *the simplicity or variety of structure* in mechanics could be the key.

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Quantum mechanics might represent a special case in the entirely M-matrix mechanics. For example, matrix mechanics with many-index objects could be reduced to Heisenberg's matrix mechanics or to a physically meaningless system by a change of variables. It is important to make clear the entire structure of M-matrix mechanics and find relations between its various limiting forms.

With regard to the question of whether QM holds without limit, there is the proposal that QM should be modified near the Planck scale, on the basis of the problem of information loss at a black hole.[10)] This problem is deeply related to the difficulty involved in the quantization of gravity. Superstring theory and/or M-theory are the most promising theories that include quantum gravity. In fact, the problem of the counting of entropy has been solved for a class of (near) extremal black holes in superstring theory.[11)]

With regard to the question of how QM is modified if it has limitations, if elementary objects in nature are not point particles but, rather, extended objects, the correct way to arrive at a final theory must be to construct a theory based on a (new) mechanics appropriate for these fundamental constituents. The study of generalized matrix mechanics might shed new light on this subject. Or, there is the possibility that superstring theory and/or M-theory can be used to build a new mechanics. It would be worthwhile to explore the generalization of QM in order to approach the construction of a fundamental theory of nature from every possible direction.[*)]

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[*)] Several different ideas have been proposed for the generalization of QM.[12)]

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