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Orbifold family unification on six dimensions

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We study the possibility of family unification on the basis of $SU(N)$ gauge theory on the six-dimensional space-time, $M^4 \times T^2/Z_N$. We obtain enormous numbers of models with three families of $SU(5)$ matter multiplets and those with three families of the standard model multiplets, from a single massless Dirac fermion with a higher-dimensional representation of $SU(N)$, through the orbifold breaking mechanism.

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I. INTRODUCTION

The origin of the family replication has been a big riddle. The family unification based on a large symmetry group can provide a possible solution. The studies have been carried out intensively, and they are classified into two categories. One is the investigation based on the four-dimensional Minkowski space-time [1–7], and the other is that based on higher-dimensional space-times [8–16].

In the family unification based on a gauge group on the four dimensions, we encounter the following difficulties relating the chiralness of fermions and its anomalies. The one is that chiral fermions do not, in general, come from fermions with an anomaly free representation, e.g., $\mathbf{2}^{n-1}$ for $SO(2n)$ $(n \neq 1, 3)$ or a nonchiral set of representations, e.g., $\boldsymbol{N} + \bar{\boldsymbol{N}}$ for $SU(N)$. There appear extra fermions including mirror particles. Here, the mirror particles are particles with opposite quantum numbers under the standard model (SM) gauge group. If we adopt the "survival hypothesis" to get rid of the unwelcomed particles, our family members would also disappear from the low-energy spectrum. Here, the survival hypothesis is the assumption that if a symmetry is broken down into a smaller symmetry at a scale $M_{\rm SB}$, then any fermion mass terms invariant under the smaller group induce fermion masses of $O(M_{\rm SB})$ [3,17]. The other is that we need fermions with several representations to produce only the SM family members using the survival hypothesis. Georgi found that three families are derived from the anomaly free chiral set $[11,4] + [11,8] + [11,9] + [11,10]$ in the $SU(11)$ model [3]. In any case, it is impossible to generate only the three families up to SM singlets from a single anomaly free representation by the help of the survival hypothesis on the four dimensions.[1]

The advantage of higher-dimensional theories is that substances including mirror particles can be reduced using the symmetry breaking mechanism concerning extra dimensions, as originally discussed in superstring theory [18–20]. Hence, a candidate realizing the family unification is grand unified theories (GUTs) on a higher-dimensional space-time including an orbifold as an extra space.[2] Through several preceding studies, three replicas in the GUT group such as $SU(5)$ and E_6 are derived from a single multiplet of a larger gauge group, but models to derive three families via the direct orbifold breaking down to the SM gauge group have not yet been found. For example, in $SU(N)$ gauge theory on five-dimensional space-time including S^1/Z_2, three replicas in $SU(5)$ have been derived from a single bulk field of $SU(N)$ gauge group $(N \geq 9)$, but there are no models to derive the three families of the SM group multiplets [14].

In this paper, we study the possibility of family unification on the basis of $SU(N)$ gauge theory on $M^4 \times T^2/Z_N$ using the method in Ref. [14]. We investigate whether or not three families are derived from a single massless Dirac fermion of $SU(N)$ for two patterns of symmetry breaking.

The contents of this paper are as follows. In Sec. II, we provide general arguments on the orbifold breaking based on T^2/Z_N and formulas for numbers of species. In Sec. III, we investigate the family unification for each T^2/Z_N $(N = 2, 3, 4, 6)$, in the framework of six-dimensional $SU(N)$ GUTs. Section IV is devoted to conclusions and discussions.

II. Z_N ORBIFOLD BREAKING AND FORMULAS FOR NUMBERS OF SPECIES

We explain the orbifold T^2/Z_N and give formulas for numbers of species, in the case with diagonal embeddings for representation matrices of Z_N transformations.

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[1]There is a possibility that extra particles are confined at a high-energy scale by some strong dynamics [2,6].

[2]Five-dimensional supersymmetric GUTs on $M^4 \times S^1/Z_2$ possess the attractive feature that the triplet-doublet splitting of Higgs multiplets is elegantly realized [21,22].

A. Z_N orbifold breaking

Let z be the complex coordinate of T^2/Z_N. Here, T^2 is constructed from a two-dimensional lattice. On T^2, the points $z + e_1$ and $z + e_2$ are identified with the point z, where e_1 and e_2 are basis vectors. The orbifold T^2/Z_N is obtained by dividing T^2 by the Z_N transformation Z_N: $z \to \xi z$ ($\xi^N = 1$) so that the point z is identified with ξz, or z is generally identified with $\xi^k z + ae_1 + be_2$, where k, a, and b are integers.

Let us explain the orbifold breaking using T^2/Z_2. Accompanied by the identification of points on T^2/Z_2, the following boundary conditions (BCs) for a field $\Phi(x, z)$ can be imposed on

$$\begin{aligned}\Phi(x, -z) &= T_\Phi[P_0]\Phi(x, z),\\ \Phi(x, e_1 - z) &= T_\Phi[P_1]\Phi(x, z),\\ \Phi(x, e_2 - z) &= T_\Phi[P_2]\Phi(x, z),\end{aligned} \tag{1}$$

where $e_1 = 1$, $e_2 = i$, and $T_\Phi[P_0]$, $T_\Phi[P_1]$, and $T_\Phi[P_2]$ represent appropriate representation matrices. The P_0, P_1, and P_2 stand for the representation matrices of the Z_2 transformations $z \to -z$, $z \to e_1 - z$, and $z \to e_2 - z$ for fields with the fundamental representation.

The eigenvalues of $T_\Phi[P_0]$, $T_\Phi[P_1]$, and $T_\Phi[P_2]$ are interpreted as the Z_2 parities for the extra space. The fields with even Z_2 parities have zero modes, but those including an odd Z_2 parity do not have zero modes. Here, zero modes mean four-dimensional massless fields surviving after compactification. Kaluza-Klein modes do not appear in our low-energy world, because they have heavy masses of $O(1/R)$, with the same magnitude as the unification scale. Unless all components of nonsinglet field have a common Z_2 parity, a symmetry reduction occurs upon compactification because zero modes are absent in fields with an odd parity. This type of symmetry breaking mechanism is called the "orbifold breaking mechanism."[3]

Basis vectors, representation matrices, and their transformation properties of T^2/Z_N are summarized in Table I [31,32].[4] Note that there is a choice in representation matrices, and P_1 concerning the Z_2 transformation $z \to e_1 - z$ is also used in T^2/Z_4 and T^2/Z_6.

Fields possess discrete charges relating the eigenvalues of the representation matrices for the Z_M transformation. Here, $M = N$ for $N = 2$, 3 and $M = N$, 2 for $N = 4$, 6. The discrete charges are assigned as numbers n/M ($n = 0, 1, \ldots, M-1$) and $e^{2\pi i n/M}$ are the elements of the Z_M transformation. We refer to them as Z_M elements.

TABLE I. The characters of T^2/Z_N.

N	Basis vectors	Rep. matrices	Transformation properties
2	$1, i$	P_0, P_1, P_2	$z \to -z,\ z \to e_1 - z,\ z \to e_2 - z$
3	$1, e^{2\pi i/3}$	Θ_0, Θ_1	$z \to e^{2\pi i/3}z,\ z \to e^{2\pi i/3}z + e_1$
4	$1, i$	Q_0, P_1	$z \to iz,\ z \to e_1 - z$
6	$1, (-3 + i\sqrt{3})/2$	Ξ_0, P_1	$z \to e^{\pi i/3}z,\ z \to e_1 - z$

A fermion with spin 1/2 in six dimensions is regarded as a Dirac fermion or a pair of Weyl fermions with opposite chiralities in four dimensions. There are two choices in a six-dimensional Weyl fermion, i.e.,

$$\Psi_+ = \frac{1+\Gamma_7}{2}\Psi = \begin{pmatrix} \frac{1-\gamma_5}{2} & 0 \\ 0 & \frac{1+\gamma_5}{2} \end{pmatrix}\begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} = \begin{pmatrix} \Psi^1_L \\ \Psi^2_R \end{pmatrix}, \tag{2}$$

$$\Psi_- = \frac{1-\Gamma_7}{2}\Psi = \begin{pmatrix} \frac{1+\gamma_5}{2} & 0 \\ 0 & \frac{1-\gamma_5}{2} \end{pmatrix}\begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} = \begin{pmatrix} \Psi^1_R \\ \Psi^2_L \end{pmatrix}, \tag{3}$$

where Ψ_+ and Ψ_- are fermions with positive and negative chirality, respectively, and Γ_7 and γ_5 are the chirality operators for six-dimensional fermions and four-dimensional ones, respectively.[5] Here and hereafter, the subscript $\pm$ stands for the chiralities on six dimensions.

From the Z_M invariance of the kinetic term and the transformation property of the covariant derivatives Z_M: $D_z \to \bar{\rho} D_z$ and $D_{\bar{z}} \to \rho D_{\bar{z}}$ with $\bar{\rho} = e^{-2\pi i/M}$ and $\rho = e^{2\pi i/M}$, the following relations hold between the Z_M element of $\Psi^1_{L(R)}$ and $\Psi^2_{R(L)}$:

$$\mathcal{P}_{\Psi^2_R} = \rho \mathcal{P}_{\Psi^1_L}, \qquad \mathcal{P}_{\Psi^1_R} = \bar{\rho} \mathcal{P}_{\Psi^2_L}, \tag{4}$$

where $z \equiv x^5 + ix^6$ and $\bar{z} \equiv x^5 - ix^6$.

Chiral gauge theories including Weyl fermions on even-dimensional space-time become, in general, anomalous in the presence of gauge anomalies, gravitational anomalies, mixed anomalies, and/or global anomaly [34,35]. In $SU(N)$ GUTs on six-dimensional space-time, the global anomaly is absent because of $\Pi_6(SU(N)) = 0$ for $N \geq 4$. Here, $\Pi_6(SU(N))$ is the sixth homotopy group of $SU(N)$. In our analysis, we consider a massless Dirac fermion (Ψ_+, Ψ_-) under the $SU(N)$ gauge group ($N \geq 8$) on six-dimensional space-time. In this case, anomalies are canceled out by the contributions from fermions with different chiralities.

[3] The Z_2 orbifolding was used in superstring theory [23] and heterotic M theory [24,25]. In field theoretical models, it was applied to the reduction of global supersymmetry (SUSY) [26,27], which is an orbifold version of the Scherk-Schwarz mechanism [28,29], and then to the reduction of gauge symmetry [30].

[4] Though the number of independent representation matrices for T^2/Z_6 is stated to be three in Ref. [15], it should be two because other operations are generated using s_0: $z \to e^{\pi i/3}z$ and r_1: $z \to e_1 - z$. For example, t_1: $z \to z + e_1$ and t_2: $z \to z + e_2$ are generated as $t_1 = r_1(s_0)^3$ and $t_2 = (s_0)^2 r_1 (s_0)^4 r_1$, respectively.

[5] For more detailed explanations for six-dimensional fermions, see Ref. [33].

B. Formulas for numbers of species

With suitable diagonal representation matrices R_a ($a = 0, 1, 2$ for T^2/Z_2 and $a = 0, 1$ for T^2/Z_3, T^2/Z_4, and T^2/Z_6), the $SU(N)$ gauge group is broken down into its subgroup such that

$$SU(N) \to SU(p_1) \times SU(p_2) \times \cdots \times SU(p_n) \times U(1)^{n-m-1}, \tag{5}$$

where $N = \sum_{i=1}^{n} p_i$. Here and hereafter, $SU(1)$ unconventionally stands for $U(1)$, $SU(0)$ means nothing, and m is a sum of the number of $SU(0)$ and $SU(1)$. The concrete form of R_a will be given in the next section.

After the breakdown of $SU(N)$, the rank k totally antisymmetric tensor representation $[N, k]$, whose dimension is ${}_NC_k$, is decomposed into a sum of multiplets of the subgroup $SU(p_1) \times \cdots \times SU(p_n)$ as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} ({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_n}C_{l_n}), \tag{6}$$

where $l_n = k - l_1 - \cdots - l_{n-1}$ and our notation is that ${}_nC_l = 0$ for $l > n$ and $l < 0$. Here and hereafter, we use ${}_nC_l$ instead of $[n, l]$ in many cases. We sometimes use the ordinary notation for representations, too, e.g., $\mathbf{5}$ and $\bar{\mathbf{5}}$ in place of ${}_5C_1$ and ${}_5C_4$.

The $[N, k]$ is constructed by the antisymmetrization of the *k-ple* product of the fundamental representation $\boldsymbol{N} = [N, 1]$:

$$[N, k] = (\boldsymbol{N} \times \cdots \times \boldsymbol{N})_{\mathrm{A}}. \tag{7}$$

We define the intrinsic Z_M elements η_k^a such that

$$(\boldsymbol{N} \times \cdots \times \boldsymbol{N})_{\mathrm{A}} \to \eta_k^a (R_a\boldsymbol{N} \times \cdots \times R_a\boldsymbol{N})_{\mathrm{A}}. \tag{8}$$

By definition, η_k^a take a value of Z_M elements, i.e., $e^{2\pi i n/M}$ ($n = 0, 1, \ldots, M-1$). Note that η_k^a for Ψ_+ are not necessarily the same as those of Ψ_-, and the chiral symmetry is still respected.

Let us investigate the family unification in two cases. Each breaking pattern is given by

$$SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_n) \times U(1)^{n-m-1}, \tag{9}$$

$$SU(N) \to SU(3) \times SU(2) \times SU(p_3) \times \cdots \times SU(p_n) \times U(1)^{n-m-1}, \tag{10}$$

where $SU(3)$ and $SU(2)$ are identified with $SU(3)_C$ and $SU(2)_L$ in the SM gauge group.

1. Formulas for $SU(5)$ *multiplets*

We study the breaking pattern (9). After the breakdown of $SU(N)$, $[N, k]$ is decomposed as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} ({}_5C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_n}C_{l_n}). \tag{11}$$

As mentioned before, ${}_5C_0$, ${}_5C_1$, ${}_5C_2$, ${}_5C_3$, ${}_5C_4$, and ${}_5C_5$ stand for representations $\mathbf{1}$, $\mathbf{5}$, $\mathbf{10}$, $\overline{\mathbf{10}}$, $\bar{\mathbf{5}}$, and $\bar{\mathbf{1}}$.[6]

Utilizing the survival hypothesis and the equivalence of $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ with $\bar{\mathbf{5}}_L$ and $\mathbf{10}_L$, respectively,[7] we write the numbers of $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations for left-handed Weyl fermions as

$$n_{\bar{5}} \equiv \sharp\bar{\mathbf{5}}_L - \sharp\mathbf{5}_L + \sharp\mathbf{5}_R - \sharp\bar{\mathbf{5}}_R, \tag{12}$$

$$n_{10} \equiv \sharp\mathbf{10}_L - \sharp\overline{\mathbf{10}}_L + \sharp\overline{\mathbf{10}}_R - \sharp\mathbf{10}_R, \tag{13}$$

where $\sharp$ represents the number of each multiplet.

The $SU(5)$ singlets are regarded as the right-handed neutrinos, which can obtain heavy Majorana masses among themselves as well as the Dirac masses with left-handed neutrinos. Some of them can be involved in a seesaw mechanism [2,36,37]. The total number of $SU(5)$ singlets (with heavy masses) is given by

$$n_1 \equiv \sharp\mathbf{1}_L + \sharp\bar{\mathbf{1}}_L + \sharp\bar{\mathbf{1}}_R + \sharp\mathbf{1}_R. \tag{14}$$

Formulas for $n_{\bar{5}}$, n_{10}, and n_1 from a Dirac fermion (Ψ_+, Ψ_-) whose intrinsic Z_M elements are $(\eta_{k+}^a, \eta_{k-}^a)$ are given by

$$n_{\bar{5}} = \sum_{\pm} \sum_{l_1=1,4} (-1)^{l_1} \times \left(\sum_{\{l_2,\ldots,l_{n-1}\}_{\eta^a_{l_1L\pm}}} - \sum_{\{l_2,\ldots,l_{n-1}\}_{\eta^a_{l_1R\pm}}} \right) {}_{p_2}C_{l_2} \cdots {}_{p_n}C_{l_n}, \tag{15}$$

$$n_{10} = \sum_{\pm} \sum_{l_1=2,3} (-1)^{l_1} \times \left(\sum_{\{l_2,\ldots,l_{n-1}\}_{\eta^a_{l_1L\pm}}} - \sum_{\{l_2,\ldots,l_{n-1}\}_{\eta^a_{l_1R\pm}}} \right) {}_{p_2}C_{l_2} \cdots {}_{p_n}C_{l_n}, \tag{16}$$

$$n_1 = \sum_{\pm} \sum_{l_1=0,5} \left(\sum_{\{l_2,\ldots,l_{n-1}\}_{\eta^a_{l_1L\pm}}} + \sum_{\{l_2,\ldots,l_{n-1}\}_{\eta^a_{l_1R\pm}}} \right) {}_{p_2}C_{l_2} \cdots {}_{p_n}C_{l_n}, \tag{17}$$

where $p_n = N - \sum_{i=1}^{n-1} p_i$ and $l_n = N - \sum_{i=1}^{n-1} l_i$. $\sum_{\pm}$ represents the summation of contributions from Ψ_+ and Ψ_-.

[6] We denote the $SU(5)$ singlet relating to ${}_5C_5$ as $\bar{\mathbf{1}}$, for convenience sake, to avoid the confusion over singlets.

[7] As usual, $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ represent the charge conjugate of $\mathbf{5}_R$ and $\overline{\mathbf{10}}_R$, respectively. Note that $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ transform as the left-handed Weyl fermions under the four-dimensional Lorentz transformations.

TABLE II. The specific relations for l_j.

Orbifolds	$\bar{\rho}^k \eta^a_{k\pm}$	Specific relations
T^2/Z_2	$(-1)^k \eta^0_{k\pm} = (-1)^{\alpha_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + l_3 + l_4 = 2 - l_1 - \alpha_\pm \pmod 2$
	$(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^1_{l_1 L\pm} \equiv l_2 + l_5 + l_6 = 2 - l_1 - \beta_\pm \pmod 2$
	$(-1)^k \eta^2_{k\pm} = (-1)^{\gamma_\pm}$	$n^2_{l_1 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \gamma_\pm \pmod 2$
T^2/Z_3	$(e^{-2\pi i/3})^k \eta^0_{k\pm} = (e^{2\pi i/3})^{\alpha_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + l_3 + 2(l_4 + l_5 + l_6) = 3 - l_1 - \alpha_\pm \pmod 3$
	$(e^{-2\pi i/3})^k \eta^1_{k\pm} = (e^{2\pi i/3})^{\beta_\pm}$	$n^1_{l_1 L\pm} \equiv l_4 + l_7 + 2(l_2 + l_5 + l_8) = 3 - l_1 - \beta_\pm \pmod 3$
T^2/Z_4	$(-i)^k \eta^0_{k\pm} = i^{\alpha_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + 2(l_3 + l_4) + 3(l_5 + l_6) = 4 - l_1 - \alpha_\pm \pmod 4$
	$(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^1_{l_1 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \beta_\pm \pmod 2$
T^2/Z_6	$(e^{-\pi i/3})^k \eta^0_{k\pm} = (e^{\pi i/3})^{\alpha_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + 2(l_3 + l_4) + 3(l_5 + l_6) + 4(l_7 + l_8) + 5(l_9 + l_{10}) = 6 - l_1 - \alpha_\pm \pmod 6$
	$(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^1_{l_1 L\pm} \equiv l_3 + l_5 + l_7 + l_9 + l_{11} = 2 - l_1 - \beta_\pm \pmod 2$

Furthermore, $\sum_{\{l_2,\ldots,l_{n-1}\}_{n^a_{l_1 L\pm}}}$ means that the summations over $l_j = 0, \ldots, k - l_1 - \cdots - l_{j-1}$ $(j = 2, \ldots, n-1)$ are carried out under the condition that l_j should satisfy specific relations on T^2/Z_N given in Table II. The relations will be confirmed in the next section. In the same way, $\sum_{\{l_2,\ldots,l_{n-1}\}_{n^a_{l_1 R\pm}}}$ means that the summations over $l_j = 0, \ldots, k - l_1 - \cdots - l_{j-1}$ $(j = 2, \ldots, n-1)$ are carried out under the condition that l_j should satisfy specific relations $n^a_{l_1 R\pm} = n^a_{l_1 L\pm} \mp 1 \pmod M$ for $\Psi_\pm$. In the next section, the formulas (15)–(17) will be rewritten in more concrete form for each T^2/Z_N $(N = 2, 3, 4, 6)$ by the use of projection operators.

2. Formulas for the SM multiplets

We study the breaking pattern (10). After the breakdown of $SU(N)$, $[N, k]$ is decomposed as

$$[N,k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \times \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} ({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3}, \ldots, {}_{p_n}C_{l_n}). \quad (18)$$

The flavor numbers of downtype antiquark singlets $(d_R)^c$, lepton doublets l_L, uptype antiquark singlets $(u_R)^c$, positrontype lepton singlets $(e_R)^c$, and quark doublets q_L are denoted as $n_{\bar{d}}$, n_l, $n_{\bar{u}}$, $n_{\bar{e}}$, and n_q. Using the survival hypothesis and the equivalence on charge conjugation, we define the flavor number of each chiral fermion as

$$n_{\bar{d}} \equiv \sharp({}_3C_2, {}_2C_2)_L - \sharp({}_3C_1, {}_2C_0)_L + \sharp({}_3C_1, {}_2C_0)_R - \sharp({}_3C_2, {}_2C_2)_R, \quad (19)$$

$$n_l \equiv \sharp({}_3C_3, {}_2C_1)_L - \sharp({}_3C_0, {}_2C_1)_L + \sharp({}_3C_0, {}_2C_1)_R - \sharp({}_3C_3, {}_2C_1)_R, \quad (20)$$

$$n_{\bar{u}} \equiv \sharp({}_3C_2, {}_2C_0)_L - \sharp({}_3C_1, {}_2C_2)_L + \sharp({}_3C_1, {}_2C_2)_R - \sharp({}_3C_2, {}_2C_0)_R, \quad (21)$$

$$n_{\bar{e}} \equiv \sharp({}_3C_0, {}_2C_2)_L - \sharp({}_3C_3, {}_2C_0)_L + \sharp({}_3C_3, {}_2C_0)_R - \sharp({}_3C_0, {}_2C_2)_R, \quad (22)$$

$$n_q \equiv \sharp({}_3C_1, {}_2C_1)_L - \sharp({}_3C_2, {}_2C_1)_L + \sharp({}_3C_2, {}_2C_1)_R - \sharp({}_3C_1, {}_2C_1)_R, \quad (23)$$

where $\sharp$ again represents the number of each multiplet. The total number of (heavy) neutrino singlets $(\nu_R)^c$ is denoted $n_{\bar{\nu}}$ and defined as

$$n_{\bar{\nu}} \equiv \sharp({}_3C_0, {}_2C_0)_L + \sharp({}_3C_3, {}_2C_2)_L + \sharp({}_3C_3, {}_2C_2)_R + \sharp({}_3C_0, {}_2C_0)_R. \quad (24)$$

Formulas for the SM species including neutrino singlets are given by

$$n_{\bar{d}} = \sum_{\pm} \sum_{(l_1,l_2)=(2,2),(1,0)} (-1)^{l_1+l_2} \times \left(\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n}, \quad (25)$$

$$n_l = \sum_{\pm} \sum_{(l_1,l_2)=(3,1),(0,1)} (-1)^{l_1+l_2} \times \left(\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n}, \quad (26)$$

$$n_{\bar{u}} = \sum_{\pm} \sum_{(l_1,l_2)=(2,0),(1,2)} (-1)^{l_1+l_2} \times \left(\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n}, \quad (27)$$

$$n_{\bar{e}} = \sum_{\pm} \sum_{(l_1,l_2)=(0,2),(3,0)} (-1)^{l_1+l_2} \times \left(\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n}, \quad (28)$$

$$n_{q} = \sum_{\pm} \sum_{(l_1,l_2)=(1,1),(2,1)} (-1)^{l_1+l_2} \times \left(\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n}, \quad (29)$$

$$n_{\bar{\nu}} = \sum_{\pm} \sum_{(l_1,l_2)=(0,0),(3,2)} \times \left(\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} + \sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n}, \quad (30)$$

where $\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}}$ means that the summations over $l_j = 0, \ldots, k - l_1 - \cdots - l_{j-1}$ ($j = 3, \ldots, n-1$) are carried out under the condition that l_j should satisfy specific relations on T^2/Z_N given in Table III. The relations will be confirmed in the next section. In the same way, $\sum_{\{l_3,\ldots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}}$ means that the summations over $l_j = 0, \ldots, k - l_1 - \cdots - l_{j-1}$ ($j = 3, \ldots, n-1$) are carried out under the condition that l_j should satisfy specific relations $n^a_{l_1 l_2 R\pm} = n^a_{l_1 l_2 L\pm} \mp 1$ (mod M) for $\Psi_\pm$. In the next section, Eqs. (25)–(30) will be also rewritten in more concrete forms for each T^2/Z_N by the use of projection operators.

C. Generic features of flavor numbers

We list generic features of flavor numbers.

(i) Each flavor number from $[N, k]$ with intrinsic Z_M elements $\eta^a_{k\pm}$ is equal to that from $[N, N-k]$ with appropriate ones $\eta^a_{N-k\pm}$.

Let us explain this feature using the $SU(5)$ multiplets. From Eq. (11) and the decomposition of $[N, N-k]$ such that

$$[N, N-k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} \times \left({}_5C_{5-l_1}, {}_{p_2}C_{p_2-l_2}, \ldots, {}_{p_n}C_{p_n-l_n}\right), \quad (31)$$

there is a one-to-one correspondence between $({}_5C_{5-l_1}, {}_{p_2}C_{p_2-l_2}, \ldots, {}_{p_n}C_{p_n-l_n})$ in $[N, N-k]$ and $({}_5C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_n}C_{l_n})$ in $[N, k]$. The right-handed Weyl fermion whose representation is $({}_5C_{5-l_1}, {}_{p_2}C_{p_2-l_2}, \ldots, {}_{p_n}C_{p_n-l_n})$ is regarded as the left-handed one whose representation is the conjugate representation $({}_5C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_n}C_{l_n})$, and hence, we obtain the same numbers for Eqs. (15)–(17) with a suitable assignment of intrinsic Z_M elements for $[N, N-k]$.

Here, we give an example for T^2/Z_2. Each flavor number obtained from $[N, k]$ with $(-1)^k \eta^0_{k\pm} = (-1)^{\alpha_\pm}$, $(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$, and $(-1)^k \eta^2_{k\pm} = (-1)^{\gamma_\pm}$ agrees with that from $[N, N-k]$ with $(-1)^{N-k} \eta^0_{N-k\pm} = (-1)^{\alpha'_\pm}$, $(-1)^{N-k} \eta^1_{N-k\pm} = (-1)^{\beta'_\pm}$, and $(-1)^{N-k} \eta^2_{N-k\pm} = (-1)^{\gamma'_\pm}$, where $\alpha'_\pm$, $\beta'_\pm$, and $\gamma'_\pm$ satisfy the relations $\alpha'_\pm = \alpha_\pm + p_2 + p_3 + p_4 (\mathrm{mod}\, 2)$, $\beta'_\pm = \beta_\pm + p_2 + p_5 + p_6 (\mathrm{mod}\, 2)$, and $\gamma'_\pm = \gamma_\pm + p_3 + p_5 + p_7 (\mathrm{mod}\, 2)$, respectively.

(ii) Each flavor number from $[N, k]$ with intrinsic Z_2 elements $(-1)^k \eta^a_{k\pm} = (-1)^{\delta^a_\pm}$ is equal to that from $[N, k]$ with the exchanged ones ($\delta^a_+ \leftrightarrow \delta^a_-$), i.e., $(-1)^k \eta^a_{k\pm} = (-1)^{\delta^a_\mp}$.

This feature is understood from the fact that specific relations on l_j for Ψ_+ change into those of Ψ_- and vice

TABLE III. The specific relations for l_j.

Orbifolds	$\bar{\rho}^k \eta^a_{k\pm}$	Specific relations
T^2/Z_2	$(-1)^k \eta^0_{k\pm} = (-1)^{\alpha_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv l_3 + l_4 = 2 - l_1 - l_2 - \alpha_\pm$ (mod 2)
	$(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^1_{l_1 l_2 L\pm} \equiv l_5 + l_6 = 2 - l_1 - l_2 - \beta_\pm$ (mod 2)
	$(-1)^k \eta^2_{k\pm} = (-1)^{\gamma_\pm}$	$n^2_{l_1 l_2 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \gamma_\pm$ (mod 2)
T^2/Z_3	$(e^{-2\pi i/3})^k \eta^0_{k\pm} = (e^{2\pi i/3})^{\alpha_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv l_3 + 2(l_4 + l_5 + l_6) = 3 - l_1 - l_2 - \alpha_\pm$ (mod 3)
	$(e^{-2\pi i/3})^k \eta^1_{k\pm} = (e^{2\pi i/3})^{\beta_\pm}$	$n^1_{l_1 l_2 L\pm} \equiv l_4 + l_7 + 2(l_5 + l_8) = 3 - l_1 - 2l_2 - \beta_\pm$ (mod 3)
T^2/Z_4	$(-i)^k \eta^0_{k\pm} = i^{\alpha_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv 2(l_3 + l_4) + 3(l_5 + l_6) = 4 - l_1 - l_2 - \alpha_\pm$ (mod 4)
	$(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^1_{l_1 l_2 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \beta_\pm$ (mod 2)
T^2/Z_6	$(e^{-\pi i/3})^k \eta^0_{k\pm} = (e^{\pi i/3})^{\alpha_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv 2(l_3 + l_4) + 3(l_5 + l_6) + 4(l_7 + l_8) + 5(l_9 + l_{10}) = 6 - l_1 - l_2 - \alpha_\pm$ (mod 6)
	$(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^1_{l_1 l_2 L\pm} \equiv l_3 + l_5 + l_7 + l_9 + l_{11} = 2 - l_1 - \beta_\pm$ (mod 2)

versa, under the exchange of Z_2 parity of Ψ_+ and that of Ψ_-.

Here, we give an example for T^2/Z_2. Under the exchange of α_+ and α_-, $n^0_{l_1L+}$ and $n^0_{l_1R+}$ change into $n^0_{l_1L-}$ and $n^0_{l_1R-}$ (mod 2), respectively. Each flavor number remains the same because the summation is taken for Ψ_+ and Ψ_-.

(iii) Each flavor number from $[N, k]$ is invariant under several types of exchange among p_j and intrinsic Z_M elements.

From specific relations in Table II, we find that the same number for each $SU(5)$ multiplet is obtained under the exchange,

$$\begin{aligned}&(p_3, p_4, \alpha_\pm) \Leftrightarrow (p_5, p_6, \beta_\pm),\\&(p_2, p_6, \beta_\pm) \Leftrightarrow (p_3, p_7, \gamma_\pm),\\&(p_2, p_4, \alpha_\pm) \Leftrightarrow (p_5, p_7, \gamma_\pm) \quad \text{for } T^2/Z_2,\end{aligned} \tag{32}$$

$$(p_2, p_3, p_6, \alpha_\pm) \Leftrightarrow (p_4, p_7, p_8, \beta_\pm) \quad \text{for } T^2/Z_3, \tag{33}$$

where the exchange is done independently.

In the same way, from specific relations in Table III, we find that the same number for each SM multiplet is obtained under the exchange,

$$(p_3, p_4, \alpha_\pm) \Leftrightarrow (p_5, p_6, \beta_\pm), \quad \text{for } T^2/Z_2. \tag{34}$$

Under the above exchanges, although the unbroken gauge symmetry remains, the numbers of zero modes for extradimensional components of gauge bosons are, in general, different, and hence, a model is transformed into a different one.

(iv) Each flavor number obtained from $[N, k]$ is invariant in the introduction of Wilson line phases.

Let us give some examples.

On T^2/Z_2, the numbers $n_{\bar{5}}$ and n_{10} obtained from the breaking pattern $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_8) \times U(1)^{7-m}$ are the same as those from $SU(N) \to SU(5) \times SU(p'_2) \times \cdots \times SU(p'_8) \times U(1)^{7-m}$ if the following relations are satisfied:

$$\begin{aligned}&p'_2 - p_2 = p'_7 - p_7 = p_3 - p'_3 = p_6 - p'_6,\\&\quad p'_4 = p_4, \qquad p'_5 = p_5, \qquad p'_8 = p_8,\end{aligned} \tag{35}$$

or

$$\begin{aligned}&p'_2 - p_2 = p'_7 - p_7 = p_4 - p'_4 = p_5 - p'_5,\\&\quad p'_3 = p_3, \qquad p'_6 = p_6, \qquad p'_8 = p_8,\end{aligned} \tag{36}$$

or

$$\begin{aligned}&p'_3 - p_3 = p'_6 - p_6 = p_4 - p'_4 = p_5 - p'_5,\\&\quad p'_2 = p_2, \qquad p'_7 = p_7, \qquad p'_8 = p_8.\end{aligned} \tag{37}$$

The above BCs are connected by a singular gauge transformation, and they are regarded as equivalent in the presence of Wilson line phases. This equivalence originates from the Hosotani mechanism [38–41] and is shown by the following relations among the diagonal representatives for 2×2 submatrices of (P_0, P_1, P_2) [32]:

$$\begin{aligned}(\tau_3, \tau_3, \tau_3) &\sim (\tau_3, \tau_3, -\tau_3) \sim (\tau_3, -\tau_3, \tau_3)\\&\sim (\tau_3, -\tau_3, -\tau_3),\end{aligned} \tag{38}$$

where τ_3 is the third component of Pauli matrices.

In our present case, we assume that the BC is chosen as a physical one; i.e., the system with the physical vacuum is realized with the vanishing Wilson line phases after a suitable gauge transformation is performed. Hence, it is understood that each net flavor number obtained from $[N, k]$ does not change even though the vacuum changes different ones in the presence of Wilson line phases.

In the same way, the numbers $n_{\bar{d}}$, n_l, $n_{\bar{u}}$, $n_{\bar{e}}$, and n_q obtained from the breaking pattern $SU(N) \to SU(3) \times SU(2) \times SU(p_3) \times \cdots \times SU(p_8) \times U(1)^{7-m}$ are the same as those from $SU(N) \to SU(3) \times SU(2) \times SU(p'_3) \times \cdots \times SU(p'_8) \times U(1)^{7-m}$, if the following relations are satisfied:

$$\begin{aligned}&p'_3 - p_3 = p'_6 - p_6 = p_4 - p'_4 = p_5 - p'_5,\\&\quad p'_7 = p_7, \qquad p'_8 = p_8.\end{aligned} \tag{39}$$

On T^2/Z_3, the numbers $n_{\bar{5}}$ and n_{10} obtained from the breaking pattern $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_9) \times U(1)^{8-m}$ are the same as those from $SU(N) \to SU(5) \times SU(p'_2) \times \cdots \times SU(p'_9) \times U(1)^{8-m}$ if the following relations are satisfied:

$$\begin{aligned}p'_2 - p_2 &= p'_6 - p_6 = p'_7 - p_7 = p_3 - p'_3\\&= p_4 - p'_4 = p_8 - p'_8,\\p'_5 &= p_5, \qquad p'_9 = p_9.\end{aligned} \tag{40}$$

The above BCs are also connected by a singular gauge transformation, and they are regarded as equivalent in the presence of Wilson line phases. The equivalence is shown using the following relations among the diagonal representatives for 3×3 submatrices of (Θ_0, Θ_1) on T^2/Z_3 [32]:

$$(X, X) \sim (X, \bar{\omega} X) \sim (X, \omega X), \tag{41}$$

where $\omega = e^{2\pi i/3}$, $\bar{\omega} = e^{4\pi i/3}$, and $X = \mathrm{diag}(1, \omega, \bar{\omega})$.

For these cases, it is also understood that each net flavor number does not change even though the vacuum changes different ones in the presence of Wilson line phases.

Although this feature holds for models on T^2/Z_4 and T^2/Z_6, there are no examples in our setting because of the absence of Wilson line phases changing BCs but keeping $SU(5)$ or the SM gauge group for T^2/Z_4 and because of the absence of equivalence relations between diagonal representatives for T^2/Z_6 [32].

III. ORBIFOLD FAMILY UNIFICATION ON $M^4 \times T^2/Z_N$

We investigate the family unification in $SU(N)$ GUTs for each T^2/Z_N ($N = 2, 3, 4, 6$).

A. Total numbers of models with three families

Let us present the total numbers of models with the three families, for reference. The total numbers of models with the three families of $SU(5)$ multiplets and the SM multiplets, which originate from a Dirac fermion whose representation is $[N, k]$ ($k \le N/2$) of $SU(N)$, are summarized up to $SU(12)$ in Table IV and up to $SU(13)$ in Table V, respectively. In the tables, the three centered dots ($\cdots$) mean no models. We omit the total numbers of models from $[N, N-k]$ because they agree with those from $[N, k]$ reflecting the feature (i) in Sec. II C.

B. T^2/Z_2

For the representation matrices given by

$$\begin{aligned} P_0 &= \mathrm{diag}([+1]_{p_1}, [+1]_{p_2}, [+1]_{p_3}, [+1]_{p_4}, [-1]_{p_5}, [-1]_{p_6}, [-1]_{p_7}, [-1]_{p_8}), \\ P_1 &= \mathrm{diag}([+1]_{p_1}, [+1]_{p_2}, [-1]_{p_3}, [-1]_{p_4}, [+1]_{p_5}, [+1]_{p_6}, [-1]_{p_7}, [-1]_{p_8}), \\ P_2 &= \mathrm{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{p_3}, [-1]_{p_4}, [+1]_{p_5}, [-1]_{p_6}, [+1]_{p_7}, [-1]_{p_8}), \end{aligned} \quad (42)$$

the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N) \to SU(p_1) \times SU(p_2) \times \cdots \times SU(p_8) \times U(1)^{7-n}, \quad (43)$$

where $[\pm 1]_{p_i}$ represents ± 1 for all p_i elements.

After the breakdown of $SU(N)$, $[N, k]_\pm$ is decomposed as

$$[N, k]_\pm = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} ({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_8}C_{l_8})_\pm, \quad (44)$$

where $l_8 = k - l_1 - \cdots - l_7$.

Using the definition of the intrinsic Z_2 parities $\eta^a_{k\pm}$ ($a = 0, 1, 2$), such that

$$(\boldsymbol{N} \times \cdots \times \boldsymbol{N})_{\mathrm{A}\pm} \to \eta^a_{k\pm}(P_a \boldsymbol{N} \times \cdots \times P_a \boldsymbol{N})_{\mathrm{A}\pm}, \quad (45)$$

the Z_2 parities of the representation $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_8}C_{l_8})_\pm$ are given by

$$\begin{aligned} \mathcal{P}_{0\pm} &= (-1)^{l_5+l_6+l_7+l_8}\eta^0_{k\pm} = (-1)^{l_1+l_2+l_3+l_4}(-1)^k\eta^0_{k\pm} \\ &= (-1)^{l_1+l_2+l_3+l_4+\alpha_\pm}, \end{aligned} \quad (46)$$

$$\begin{aligned} \mathcal{P}_{1\pm} &= (-1)^{l_3+l_4+l_7+l_8}\eta^1_{k\pm} = (-1)^{l_1+l_2+l_5+l_6}(-1)^k\eta^1_{k\pm} \\ &= (-1)^{l_1+l_2+l_5+l_6+\beta_\pm}, \end{aligned} \quad (47)$$

$$\begin{aligned} \mathcal{P}_{2\pm} &= (-1)^{l_2+l_4+l_6+l_8}\eta^2_{k\pm} = (-1)^{l_1+l_3+l_5+l_7}(-1)^k\eta^2_{k\pm} \\ &= (-1)^{l_1+l_3+l_5+l_7+\gamma_\pm}, \end{aligned} \quad (48)$$

TABLE IV. Total numbers of models with the three families of $SU(5)$ multiplets.

	T^2/Z_2	T^2/Z_3	T^2/Z_4	T^2/Z_6
$SU(8)$	$\cdots$	[8,3]:24 [8,4]:12	[8,3]:14 [8,4]:16	[8,3]:28 [8,4]:20
$SU(9)$	[9,3]:192	[9,3]:182 [9,4]:348	[9,3]:142 [9,4]:32	[9,3]:512 [9,4]:800
$SU(10)$	$\cdots$	[10,3]:852 [10,4]:1308 [10,5]:48	[10,3]:160 [10,4]:92	[10,3]:2484 [10,4]:2654 [10,5]:1532
$SU(11)$	[11,3]:768 [11,4]:768	[11,3]:1608 [11,4]:1716 [11,5]:1794	[11,3]:456 [11,4]:436 [11,5]:186	[11,3]:6530 [11,4]:6768 [11,5]:5540
$SU(12)$	[12,3]:1104	[12,3]:2214 [12,4]:1020	[12,3]:748 [12,4]:676 [12,5]:534 [12,6]:632	[12,3]:17084 [12,4]:13692 [12,5]:10498 [12,6]:13188

TABLE V. Total numbers of models with the three families of SM multiplets.

	T^2/Z_2	T^2/Z_3	T^2/Z_4	T^2/Z_6
$SU(8)$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$SU(9)$	[9,3]:32	$\cdots$	[9,3]:8	[9,3]:8 [9,4]:32
$SU(10)$	$\cdots$	$\cdots$	$\cdots$	[10,3]:80 [10,4]:108
$SU(11)$	[11,3]:80 [11,4]:80	[11,4]:80	[11,3]:20 [11,4]:20	[11,3]:84 [11,4]:144 [11,5]:156
$SU(12)$	[12,3]:120	[12,3]:80	[12,4]:88 [12,6]:240	[12,3]:392 [12,4]:120 [12,5]:72 [12,6]:552
$SU(13)$	[13,3]:144	$\cdots$	[13,4]:40	[13,3]:712 [13,4]:88 [13,5]:140 [13,6]:200

where $\eta^a_{k\pm}$ take a value $+1$ or -1 by definition, and we parametrize them as $(-1)^k\eta^0_{k\pm}=(-1)^{\alpha_\pm}$, $(-1)^k\eta^1_{k\pm}=(-1)^{\beta_\pm}$, and $(-1)^k\eta^2_{k\pm}=(-1)^{\gamma_\pm}$.

1. Numbers of $SU(5)$ multiplets on T^2/Z_2

After the breakdown $SU(N)\to SU(5)\times SU(p_2)\times\cdots\times SU(p_8)\times U(1)^{7-m}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm=\sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}({}_5C_{l_1},{}_{p_2}C_{l_2},\ldots,{}_{p_8}C_{l_8})_\pm. \tag{49}$$

Using the assignment of the Z_2 parities (46)–(48), we find that zero modes appear if the following relations are satisfied:

$$\begin{aligned} n^0_{l_1L\pm}&\equiv l_2+l_3+l_4=2-l_1-\alpha_\pm \pmod 2,\\ n^1_{l_1L\pm}&\equiv l_2+l_5+l_6=2-l_1-\beta_\pm \pmod 2,\\ n^2_{l_1L\pm}&\equiv l_3+l_5+l_7=2-l_1-\gamma_\pm \pmod 2.\end{aligned} \tag{50}$$

Utilizing the survival hypothesis and the equivalence of charge conjugation, we obtain the formulas (15)–(17) with $n=8$. Because the Z_2 projection operator $P_\pm$ that picks up $\mathcal{P}=\pm1$ is defined as $P_\pm\equiv(1\pm\mathcal{P})/2$, the Z_2 projection operator that picks up zero modes of left-handed ones, i.e., massless modes in fields with $(\mathcal{P}_{0\pm},\mathcal{P}_{1\pm},\mathcal{P}_{2\pm})=(1,1,1)$, is given by

$$P^{(1,1,1)}\equiv\frac{1}{8}(1+\mathcal{P}_{0\pm})(1+\mathcal{P}_{1\pm})(1+\mathcal{P}_{2\pm}), \tag{51}$$

and the Z_2 projection operator that picks up the zero modes of right-handed ones, i.e., massless modes in fields with $(\mathcal{P}_{0\pm},\mathcal{P}_{1\pm},\mathcal{P}_{2\pm})=(-1,-1,-1)$, is given by

$$P^{(-1,-1,-1)}\equiv\frac{1}{8}(1-\mathcal{P}_{0\pm})(1-\mathcal{P}_{1\pm})(1-\mathcal{P}_{2\pm}). \tag{52}$$

From Eqs. (51) and (52),

$$P^{(1,1,1)}-P^{(-1,-1,-1)}=\frac{1}{4}(\mathcal{P}_{0\pm}+\mathcal{P}_{1\pm}+\mathcal{P}_{2\pm}+\mathcal{P}_{0\pm}\mathcal{P}_{1\pm}\mathcal{P}_{2\pm}), \tag{53}$$

$$P^{(1,1,1)}+P^{(-1,-1,-1)}=\frac{1}{4}(1+\mathcal{P}_{0\pm}\mathcal{P}_{1\pm}+\mathcal{P}_{0\pm}\mathcal{P}_{2\pm}+\mathcal{P}_{1\pm}\mathcal{P}_{2\pm}). \tag{54}$$

Using Eqs. (46)–(48), (53), and (54), the formulas (15)–(17) are rewritten as

TABLE VI. Examples for the three families of $SU(5)$ from T^2/Z_2.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	$(\alpha_+,\beta_+,\gamma_+)$	$(\alpha_-,\beta_-,\gamma_-)$
[9,3]	(5,0,0,0,3,0,0,1)	(0,1,1)	(0,0,1)
[11,3]	(5,0,1,0,4,0,1,0)	(0,0,1)	(1,1,0)
[11,4]	(5,0,3,1,0,1,1,0)	(0,0,0)	(0,0,1)
[12,3]	(5,2,0,0,2,0,1,2)	(1,0,1)	(0,0,0)

$$\begin{aligned} n_{\bar 5}&=\sum_\pm\sum_{l_1=1,4}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(-1)^{l_1}(P^{(1,1,1)}-P^{(-1,-1,-1)})\,{}_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}\\ &=\sum_\pm\sum_{l_1=1,4}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}\frac{1}{4}((-1)^{l_2+l_3+l_4+\alpha_\pm}+(-1)^{l_2+l_5+l_6+\beta_\pm}+(-1)^{l_3+l_5+l_7+\gamma_\pm}\\ &\quad+(-1)^{l_4+l_6+l_7+\alpha_\pm+\beta_\pm+\gamma_\pm})\,{}_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8},\end{aligned} \tag{55}$$

$$\begin{aligned} n_{10}&=\sum_\pm\sum_{l_1=2,3}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(-1)^{l_1}(P^{(1,1,1)}-P^{(-1,-1,-1)})\,{}_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}\\ &=\sum_\pm\sum_{l_1=2,3}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}\frac{1}{4}((-1)^{l_2+l_3+l_4+\alpha_\pm}+(-1)^{l_2+l_5+l_6+\beta_\pm}+(-1)^{l_3+l_5+l_7+\gamma_\pm}\\ &\quad+(-1)^{l_4+l_6+l_7+\alpha_\pm+\beta_\pm+\gamma_\pm})\,{}_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8},\end{aligned} \tag{56}$$

$$\begin{aligned} n_{1}&=\sum_\pm\sum_{l_1=0,5}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(P^{(1,1,1)}+P^{(-1,-1,-1)})\,{}_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}\\ &=\sum_\pm\sum_{l_1=0,5}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}\frac{1}{4}(1+(-1)^{l_3+l_4+l_5+l_6+\alpha_\pm+\beta_\pm}+(-1)^{l_2+l_4+l_5+l_7+\alpha_\pm+\gamma_\pm}\\ &\quad+(-1)^{l_2+l_3+l_6+l_7+\beta_\pm+\gamma_\pm})\,{}_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}.\end{aligned} \tag{57}$$

In Table VI, we give some examples for representations and BCs to derive $n_{\bar 5}=n_{10}=3$.

2. Numbers of the SM multiplets on T^2/Z_2

After the breakdown, $SU(N)\to SU(3)\times SU(2)\times SU(p_2)\times\cdots\times SU(p_8)\times U(1)^{7-m}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_{\pm} = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\sum_{l_3=0}^{k-l_1-l_2}\cdots \times \sum_{l_7=0}^{k-l_1-\cdots-l_6} ({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3}, \ldots, {}_{p_8}C_{l_8})_{\pm}. \quad (58)$$

Using the assignment of the Z_2 parities (46)–(48), we find that zero modes appear if the following relations are satisfied:

$$\begin{aligned} n^0_{l_1 l_2 L\pm} &\equiv l_3 + l_4 = 2 - l_1 - l_2 - \alpha_{\pm} \pmod 2, \\ n^1_{l_1 l_2 L\pm} &\equiv l_5 + l_6 = 2 - l_1 - l_2 - \beta_{\pm} \pmod 2, \\ n^2_{l_1 l_2 L\pm} &\equiv l_3 + l_5 + l_7 = 2 - l_1 - \gamma_{\pm} \pmod 2, \end{aligned} \quad (59)$$

for $(-1)^k\eta^0_{k\pm} = (-1)^{\alpha_\pm}$, $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$, and $(-1)^k\eta^2_{k\pm} = (-1)^{\gamma_\pm}$.

TABLE VII. The three families of SM multiplets from [9,3] on T^2/Z_2.

$[N,k]$	$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)$	$(\alpha_+, \beta_+, \gamma_+)$	$(\alpha_-, \beta_-, \gamma_-)$
	(3,2,0,0,0,3,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,0,3,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,0,3,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,0,3,1,0)	(0,1,0)	(0,1,1)
	(3,2,0,0,3,0,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,3,0,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,3,0,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,3,0,1,0)	(0,1,0)	(0,1,1)
	(3,2,0,3,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,0,3,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,0,3,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,0,3,0,0,1,0)	(1,0,0)	(1,0,1)
	(3,2,3,0,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,3,0,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,3,0,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,3,0,0,0,1,0)	(1,0,0)	(1,0,1)
[9,3]	(3,2,0,0,1,2,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,1,2,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,1,2,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,1,2,1,0)	(0,1,0)	(0,1,1)
	(3,2,0,0,2,1,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,2,1,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,2,1,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,2,1,1,0)	(0,1,0)	(0,1,1)
	(3,2,1,2,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,1,2,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,1,2,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,1,2,0,0,1,0)	(1,0,0)	(1,0,1)
	(3,2,2,1,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,2,1,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,2,1,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,2,1,0,0,1,0)	(1,0,0)	(1,0,1)

Then, we obtain Eqs. (25)–(30) with $n = 8$. Using Eqs. (46)–(48), (53), and (54), the formulas for $(d_R)^c$ and $(\nu_R)^c$ are rewritten as

$$n_{\bar d} = \sum_{\pm}\sum_{(l_1,l_2)=(2,2),(1,0)}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}\frac{1}{4}((-1)^{l_3+l_4+\alpha_\pm} + (-1)^{l_5+l_6+\beta_\pm} + (-1)^{l_2+l_3+l_5+l_7+\gamma_\pm} + (-1)^{l_2+l_4+l_6+l_7+\alpha_\pm+\beta_\pm+\gamma_\pm})\, {}_{p_3}C_{l_3}\cdots{}_{p_8}C_{l_8}, \quad (60)$$

$$n_{\bar\nu} = \sum_{\pm}\sum_{(l_1,l_2)=(0,0),(3,2)}\sum_{l_2=0}^{k-l_1}\cdots \times\sum_{l_7=0}^{k-l_1-\cdots-l_6}\frac{1}{4}(1 + (-1)^{l_3+l_4+l_5+l_6+\alpha_\pm+\beta_\pm} + (-1)^{l_2+l_4+l_5+l_7+\alpha_\pm+\gamma_\pm} + (-1)^{l_2+l_3+l_6+l_7+\beta_\pm+\gamma_\pm})\, {}_{p_3}C_{l_3}\cdots{}_{p_8}C_{l_8}. \quad (61)$$

The formulas for l_L, $(u_R)^c$, $(e_R)^c$, and q_L are obtained by replacing the summation of (l_1, l_2) for $n_{\bar d}$ with $\{(3,1),(0,1)\}$, $\{(2,0),(1,2)\}$, $\{(0,2),(3,0)\}$, and $\{(1,1),(2,1)\}$.

In Table VII, we give a list of all BCs to derive three families of SM fermions from [9,3]. We find that the features (ii) and (iii) presented in Sec. II C hold on.

C. T^2/Z_3

For the representation matrices given by

$$\begin{aligned} \Theta_0 &= \mathrm{diag}([1]_{p_1}, [1]_{p_2}, [1]_{p_3}, [\omega]_{p_4}, [\omega]_{p_5}, [\omega]_{p_6}, \\ &\quad\times [\bar\omega]_{p_7}, [\bar\omega]_{p_8}, [\bar\omega]_{p_9}), \\ \Theta_1 &= \mathrm{diag}([1]_{p_1}, [\omega]_{p_2}, [\bar\omega]_{p_3}, [1]_{p_4}, [\omega]_{p_5}, [\bar\omega]_{p_6}, \\ &\quad\times [1]_{p_7}, [\omega]_{p_8}, [\bar\omega]_{p_9}), \end{aligned} \quad (62)$$

the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N) \to SU(p_1)\times SU(p_2)\times\cdots\times SU(p_9)\times U(1)^{7-n}, \quad (63)$$

where $[1]_{p_i}$, $[\omega]_{p_i}$, and $[\bar\omega]_{p_i}$ represent 1, $\omega(\equiv e^{2\pi i/3})$, and $\bar\omega(\equiv e^{4\pi i/3})$ for all p_i elements.

After the breakdown of $SU(N)$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_9}C_{l_9})_\pm, \quad (64)$$

where $l_9 = k - l_1 - \cdots - l_8$. The $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_9}C_{l_9})_\pm$ has the Z_3 elements

$$\mathcal{P}_{0\pm} = \omega^{l_4+l_5+l_6}\bar{\omega}^{l_7+l_8+l_9}\eta^0_{k\pm} = \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)}\bar{\omega}^k\eta^0_{k\pm} = \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)+\alpha_\pm}, \tag{65}$$

$$\mathcal{P}_{1\pm} = \omega^{l_2+l_5+l_8}\bar{\omega}^{l_3+l_6+l_9}\eta^1_{k\pm} = \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)}\bar{\omega}^k\eta^1_{k\pm} = \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)+\beta_\pm}, \tag{66}$$

where $\eta^a_{k\pm}$ take a value 1, ω, or $\bar{\omega}$, and we parametrize them as $\bar{\omega}^k\eta^0_{k\pm} = \omega^{\alpha_\pm}$ and $\bar{\omega}^k\eta^1_{k\pm} = \omega^{\beta_\pm}$.

1. Numbers of $SU(5)$ *multiplets on* T^2/Z_3

After the breakdown of $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_9) \times U(1)^{8-m}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}({}_5C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_9}C_{l_9})_\pm. \tag{67}$$

Using the assignment of the Z_3 elements (65) and (66), we find that zero modes appear if the following relations are satisfied:

$$n^0_{l_1L\pm} \equiv l_2 + l_3 + 2(l_4+l_5+l_6) = 3 - l_1 - \alpha_\pm \pmod 3,$$
$$n^1_{l_1L\pm} \equiv l_4 + l_7 + 2(l_2+l_5+l_8) = 3 - l_1 - \beta_\pm \pmod 3. \tag{68}$$

The relation $n^a_{l_1R\pm} = n^a_{l_1L\pm} \mp 1 \pmod 3$ holds from Eq. (4).

Then, we obtain the formulas (15)–(17) with $n = 9$, and they are rewritten as

$$n_{\bar{5}} = \sum_{l_1=1,4}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}(-1)^{l_1}(P^{(1,1)}_+ - P^{(\omega,\omega)}_+ + P^{(1,1)}_- - P^{(\bar{\omega},\bar{\omega})}_-)\,{}_{p_2}C_{l_2}\cdots{}_{p_9}C_{l_9}, \tag{69}$$

$$n_{10} = \sum_{l_1=2,3}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}(-1)^{l_1}(P^{(1,1)}_+ - P^{(\omega,\omega)}_+ + P^{(1,1)}_- - P^{(\bar{\omega},\bar{\omega})}_-)\,{}_{p_2}C_{l_2}\cdots{}_{p_9}C_{l_9}, \tag{70}$$

$$n_1 = \sum_{l_1=0,5}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}(P^{(1,1)}_+ + P^{(\omega,\omega)}_+ + P^{(1,1)}_- + P^{(\bar{\omega},\bar{\omega})}_-)\,{}_{p_2}C_{l_2}\cdots{}_{p_9}C_{l_9}, \tag{71}$$

where $P^{(\rho,\rho)}_\pm$ are projection operators that pick up the part relating $(\mathcal{P}_{0\pm}, \mathcal{P}_{1\pm}) = (\rho,\rho)$ and are written by

$$P^{(\rho,\rho)}_\pm = \frac{1}{9}(1 + \bar{\rho}\mathcal{P}_{0\pm} + \bar{\rho}^2\mathcal{P}^2_{0\pm})(1 + \bar{\rho}\mathcal{P}_{1\pm} + \bar{\rho}^2\mathcal{P}^2_{1\pm}). \tag{72}$$

In Table VIII, we give some examples for representations and BCs to derive $n_{\bar{5}} = n_{10} = 3$.

TABLE VIII. Examples for the three families of $SU(5)$ from T^2/Z_3.

$[N,k]$	$(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9)$	(α_+, β_+)	(α_-, β_-)
[8,3]	(5,0,0,0,3,0,0,0,0)	(2,0)	(2,2)
[8,4]	(5,1,1,0,1,0,0,0,0)	(0,0)	(2,2)
[9,3]	(5,0,0,2,0,1,0,0,1)	(2,0)	(2,1)
[9,4]	(5,0,2,0,0,0,0,2,0)	(2,2)	(0,2)
[10,3]	(5,0,0,0,3,2,0,0,0)	(2,0)	(2,2)
[10,4]	(5,0,0,1,0,1,1,1,1)	(2,2)	(2,2)
[10,5]	(5,1,0,0,1,0,2,0,1)	(0,0)	(0,0)
[11,3]	(5,1,0,0,1,4,0,0,0)	(0,0)	(2,1)
[11,4]	(5,2,2,0,0,1,0,1,0)	(1,2)	(2,1)
[11,5]	(5,1,1,1,1,0,0,0,2)	(0,1)	(1,1)
[12,3]	(5,0,0,3,3,0,0,0,1)	(2,0)	(0,2)
[12,4]	(5,0,3,1,0,1,0,2,0)	(1,2)	(0,1)

2. Numbers of the SM multiplets on T^2/Z_3

After the breakdown $SU(N) \to SU(3) \times SU(2) \times SU(p_2) \times \cdots \times SU(p_9) \times U(1)^{8-m}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\sum_{l_3=0}^{k-l_1-l_2}\cdots \times \sum_{l_8=0}^{k-l_1-\cdots-l_7}({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3}, \ldots, {}_{p_9}C_{l_9})_\pm. \tag{73}$$

Using the assignment of the Z_3 elements (65) and (66), we find that zero modes appear if the following relations are satisfied:

$$n^0_{l_1l_2L\pm} \equiv l_3 + 2(l_4+l_5+l_6) = 3 - l_1 - l_2 - \alpha_\pm \pmod 3,$$
$$n^1_{l_1l_2L\pm} \equiv l_4 + l_7 + 2(l_5+l_8) = 3 - l_1 - 2l_2 - \beta_\pm \pmod 3. \tag{74}$$

The relation $n^a_{l_1l_2R\pm} = n^a_{l_1l_2L\pm} \mp 1 \pmod 3$ holds from Eq. (4).

Then, we obtain Eqs. (25)–(30) with $n = 9$. Using the projection operators (72), the formulas for $(d_R)^c$ and $(\nu_R)^c$ are rewritten as

$$n_{\bar{d}} = \sum_{(l_1,l_2)=(2,2),(1,0)}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}(-1)^{l_1+l_2}(P^{(1,1)}_+ - P^{(\omega,\omega)}_+ + P^{(1,1)}_- - P^{(\bar{\omega},\bar{\omega})}_-)\,{}_{p_3}C_{l_3}\cdots{}_{p_9}C_{l_9}, \tag{75}$$

$$n_{\bar{\nu}} = \sum_{(l_1,l_2)=(0,0),(3,2)}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}(P^{(1,1)}_+ + P^{(\omega,\omega)}_+ + P^{(1,1)}_- + P^{(\bar{\omega},\bar{\omega})}_-)\,{}_{p_3}C_{l_3}\cdots{}_{p_9}C_{l_9}. \tag{76}$$

TABLE IX. Examples for the three families of SM multiplets from T^2/Z_3.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9)$	(α_+,β_+)	(α_-,β_-)
[11,4]	(3,2,0,0,1,2,3,0,0)	(0,1)	(0,1)
[12,3]	(3,2,0,1,1,0,1,2,2)	(1,0)	(0,1)

The formulas for l_L, $(u_R)^c$, $(e_R)^c$, and q_L are obtained by replacing the summation of (l_1, l_2) for $n_{\bar{d}}$ with $\{(3,1),(0,1)\}$, $\{(2,0),(1,2)\}$, $\{(0,2),(3,0)\}$, and $\{(1,1),(2,1)\}$.

In Table IX, we give some examples for representations and BCs to derive three families of SM fermions.

D. T^2/Z_4

For the representation matrices given by

$$\begin{aligned} Q_0 &= \mathrm{diag}([+1]_{p_1},[+1]_{p_2},[+i]_{p_3},[+i]_{p_4},[-1]_{p_5},[-1]_{p_6},\\ &\quad \times[-i]_{p_7},[-i]_{p_8}),\\ P_1 &= \mathrm{diag}([+1]_{p_1},[-1]_{p_2},[+1]_{p_3},[-1]_{p_4},[+1]_{p_5},[-1]_{p_6},\\ &\quad \times[+1]_{p_7},[-1]_{p_8}), \end{aligned} \tag{77}$$

the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N)\to SU(p_1)\times SU(p_2)\times\cdots\times SU(p_8)\times U(1)^{7-n}, \tag{78}$$

where $[\pm 1]_{p_i}$ and $[\pm i]_{p_i}$ represent ± 1 and $\pm i$ for all p_i elements.

After the breakdown of $SU(N)$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}({}_{p_1}C_{l_1},{}_{p_2}C_{l_2},\ldots,{}_{p_8}C_{l_8})_\pm, \tag{79}$$

where $l_8 = k - l_1 - \cdots - l_7$. The $({}_{p_1}C_{l_1},{}_{p_2}C_{l_2},\ldots,{}_{p_8}C_{l_8})_\pm$ has the Z_4 and Z_2 elements

$$\begin{aligned} \mathcal{P}_{0\pm} &= i^{l_3+l_4}(-1)^{l_5+l_6}(-i)^{l_7+l_8}\eta^0_{k\pm}\\ &= i^{l_1+l_2+2(l_3+l_4)+3(l_5+l_6)}(-i)^k\eta^0_{k\pm}\\ &= i^{l_1+l_2+2(l_3+l_4)+3(l_5+l_6)+\alpha_\pm}, \end{aligned} \tag{80}$$

$$\begin{aligned} \mathcal{P}_1 &= (-1)^{l_2+l_4+l_6+l_8}\eta^1_{k\pm} = (-1)^{l_1+l_3+l_5+l_7}(-1)^k\eta^1_{k\pm}\\ &= (-1)^{l_1+l_3+l_5+l_7+\beta_\pm}, \end{aligned} \tag{81}$$

where $\eta^0_{k\pm}$ takes a value 1, -1, i, or $-i$, and we parametrize the intrinsic Z_M elements ($M = 4, 2$) as $(-i)^k\eta^0_{k\pm} = i^{\alpha_\pm}$ and $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$.

1. Numbers of $SU(5)$ multiplets on T^2/Z_4

After the breakdown of $SU(N)\to SU(5)\times SU(p_2)\times\cdots\times SU(p_8)\times U(1)^{7-m}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}({}_5C_{l_1},{}_{p_2}C_{l_2},\ldots,{}_{p_8}C_{l_8})_\pm. \tag{82}$$

Using the assignment of the Z_4 and Z_2 elements (80) and (81), we find that zero modes appear if the following relations are satisfied:

$$\begin{aligned} n^0_{l_1L\pm} &\equiv l_2+2(l_3+l_4)+3(l_5+l_6) = 4-l_1-\alpha_\pm \pmod 4,\\ n^1_{l_1L\pm} &\equiv l_3+l_5+l_7 = 2-l_1-\beta_\pm \pmod 2. \end{aligned} \tag{83}$$

The relation $n^a_{l_1R\pm} = n^a_{l_1L\pm} \mp 1 \pmod 4$ holds from Eq. (4).

Then, we obtain the formulas (15)–(17) with $n = 8$, and they are rewritten as

$$\begin{aligned} n_{\bar{5}} = \sum_{l_1=1,4}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(-1)^{l_1}(P_+^{(1,1)}-P_+^{(i,-1)}\\ +P_-^{(1,1)}-P_-^{(-i,-1)})_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}, \end{aligned} \tag{84}$$

$$\begin{aligned} n_{10} = \sum_{l_1=2,3}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(-1)^{l_1}(P_+^{(1,1)}-P_+^{(i,-1)}\\ +P_-^{(1,1)}-P_-^{(-i,-1)})_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}, \end{aligned} \tag{85}$$

$$\begin{aligned} n_{1} = \sum_{l_1=0,5}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(P_+^{(1,1)}+P_+^{(i,-1)}+P_-^{(1,1)}\\ +P_-^{(-i,-1)})_{p_2}C_{l_2}\cdots{}_{p_8}C_{l_8}, \end{aligned} \tag{86}$$

where $P_\pm^{(\rho,\rho')}$ are projection operators that pick up the part relating $(\mathcal{P}_{0\pm},\mathcal{P}_{1\pm}) = (\rho,\rho')$ and are written by

TABLE X. Examples for the three families of $SU(5)$ from T^2/Z_4.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	(α_+,β_+)	(α_-,β_-)
[8,3]	(5,0,0,0,0,0,3,0)	(2,1)	(0,0)
[8,4]	(5,0,0,3,0,0,0,0)	(0,0)	(2,0)
[9,3]	(5,3,0,0,0,0,0,1)	(1,0)	(0,1)
[9,4]	(5,0,2,0,0,0,1,1)	(2,0)	(2,0)
[10,3]	(5,0,0,0,3,0,0,2)	(1,0)	(2,0)
[10,4]	(5,0,0,0,0,4,0,1)	(0,0)	(2,1)
[11,3]	(5,0,0,1,2,2,0,1)	(3,1)	(2,0)
[11,4]	(5,0,3,1,2,0,0,0)	(2.0)	(1,1)
[11,5]	(5,0,0,2,0,0,1,3)	(0,1)	(3,0)
[12,3]	(5,4,0,1,0,0,0,2)	(3,1)	(1,0)
[12,4]	(5,0,4,0,1,2,0,0)	(2,0)	(3,0)
[12,5]	(5,1,2,0,2,2,0,0)	(3,1)	(1,1)
[12,6]	(5,0,3,0,1,0,3,0)	(2,0)	(2,1)

$$P_\pm^{(\rho,\rho')} = \frac{1}{8}(1+\bar\rho\mathcal{P}_{0\pm}+\bar\rho^2\mathcal{P}_{0\pm}^2+\bar\rho^3\mathcal{P}_{0\pm}^3)(1+\bar\rho'\mathcal{P}_{1\pm}). \tag{87}$$

In Table X, we give some examples for representations and BCs to derive $n_{\bar 5} = n_{10} = 3$.

2. Numbers of the SM multiplets on T^2/Z_4

After the breakdown of $SU(N) \to SU(3)\times SU(2)\times SU(p_3)\times\cdots\times SU(p_8)\times U(1)^{7-m}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3},\ldots,{}_{p_8}C_{l_8})_\pm. \tag{88}$$

Using the assignment of the Z_4 and Z_2 elements (80) and (81), we find that zero modes appear if the following relations are satisfied:

$$n^0_{l_1l_2L\pm} \equiv 2(l_3+l_4)+3(l_5+l_6) = 4-l_1-l_2-\alpha_\pm \pmod 4,$$
$$n^1_{l_1l_2L\pm} \equiv l_3+l_5+l_7 = 2-l_1-\beta_\pm \pmod 2. \tag{89}$$

The relation $n^a_{l_1l_2R\pm} = n^a_{l_1l_2L\pm} \mp 1 \pmod 4$ holds from Eq. (4).

Then, we obtain the formulas (15)–(17) with $n=8$. Using the projection operators (87), the formulas for $(d_R)^c$ and $(\nu_R)^c$ are rewritten as

$$n_{\bar d} = \sum_{(l_1,l_2)=(2,2),(1,0)}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(-1)^{l_1+l_2} \times (P_+^{(1,1)}-P_+^{(i,-1)}+P_-^{(1,1)}-P_-^{(-i,-1)})_{p_3}C_{l_3}\cdots{}_{p_8}C_{l_8}, \tag{90}$$

$$n_{\bar\nu} = \sum_{(l_1,l_2)=(0,0),(3,2)}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}(P_+^{(1,1)}+P_+^{(i,-1)} + P_-^{(1,1)}+P_-^{(-i,-1)})_{p_3}C_{l_3}\cdots{}_{p_8}C_{l_8}. \tag{91}$$

The formulas for l_L, $(u_R)^c$, $(e_R)^c$, and q_L are obtained by replacing the summation of (l_1, l_2) for $n_{\bar d}$ with $\{(3,1),(0,1)\}$, $\{(2,0),(1,2)\}$, $\{(0,2),(3,0)\}$, and $\{(1,1),(2,1)\}$.

In Table XI, we give some examples of representations and BCs to derive three families of SM fermions.

E. T^2/Z_6

For the representation matrices given by

$$\Xi_0 = \mathrm{diag}([+1]_{p_1},[+1]_{p_2},[\varphi]_{p_3},[\varphi]_{p_4},[\varphi^2]_{p_5},[\varphi^2]_{p_6},[-1]_{p_7},[-1]_{p_8},[-\varphi]_{p_9},[-\varphi]_{p_{10}},[-\varphi^2]_{p_{11}},[-\varphi^2]_{p_{12}}),$$
$$P_1 = \mathrm{diag}([+1]_{p_1},[-1]_{p_2},[+1]_{p_3},[-1]_{p_4},[+1]_{p_5},[-1]_{p_6},[+1]_{p_7},[-1]_{p_8},[+1]_{p_9},[-1]_{p_{10}},[+1]_{p_{11}},[-1]_{p_{12}}), \tag{92}$$

the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N)\to SU(p_1)\times SU(p_2)\times\cdots\times SU(p_{12})\times U(1)^{11-m}, \tag{93}$$

where $\varphi = e^{\pi i/3}$ and $[c]_{p_i}$ represents the number c for all p_i elements.

After the breakdown of $SU(N)$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}}({}_{p_1}C_{l_1},{}_{p_2}C_{l_2},\ldots,{}_{p_{12}}C_{l_{12}})_\pm, \tag{94}$$

where $l_{12} = k-l_1-\cdots-l_{11}$. The $({}_{p_1}C_{l_1},{}_{p_2}C_{l_2},\ldots,{}_{p_{12}}C_{l_{12}})_\pm$ has the Z_6 and Z_2 elements

$$\begin{aligned}\mathcal{P}_0 &= \varphi^{l_3+l_4}(\varphi^2)^{l_5+l_6}(-1)^{l_7+l_8}(-\varphi)^{l_9+l_{10}}(-\varphi^2)^{l_{11}+l_{12}}\eta^0_{k\pm}\\ &= \varphi^{l_1+l_2+2(l_3+l_4)+3(l_5+l_6)+4(l_7+l_8)+5(l_9+l_{10})}\bar\varphi^k\eta^0_{k\pm}\\ &= \varphi^{l_1+l_2+2(l_3+l_4)+3(l_5+l_6)+4(l_7+l_8)+5(l_9+l_{10})+\alpha_\pm},\end{aligned} \tag{95}$$

$$\begin{aligned}\mathcal{P}_1 &= (-1)^{l_2+l_4+l_6+l_8+l_{10}+l_{12}}\eta^1_{k\pm}\\ &= (-1)^{l_1+l_3+l_5+l_7+l_9+l_{11}}(-1)^k\eta^1_{k\pm}\\ &= (-1)^{l_1+l_3+l_5+l_7+l_9+l_{11}+\beta_\pm},\end{aligned} \tag{96}$$

where $\eta^0_{k\pm}$ takes a value $e^{n\pi i/3}$ $(n=0,1,\ldots,5)$, and we parametrize the intrinsic Z_M elements $(M=6,\ 2)$ as $(e^{-\pi i/3})^k\eta^0_{k\pm} = (e^{\pi i/3})^{\alpha_\pm}$ and $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$.

TABLE XI. Examples for the three families of SM multiplets from T^2/Z_4.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	(α_+,β_+)	(α_-,β_-)
[9,3]	(3,2,1,0,0,0,2,1)	(0,1)	(0,0)
[11,3]	(3,2,1,1,0,4,0,0)	(1,0)	(1,1)
[11,4]	(3,2,0,0,3,1,1,1)	(0,1)	(0,0)
[12,4]	(3,2,1,0,2,1,3,0)	(0,1)	(0,0)
[12,6]	(3,2,1,2,0,0,0,4)	(0,1)	(1,1)
[13,4]	(3,2,1,2,2,2,0,1)	(0,1)	(0,0)

1. Numbers of $SU(5)$ multiplets on T^2/Z_6

After the breakdown of $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_{12}) \times U(1)^{11-m}$, $[N, k]_\pm$ is decomposed as

$$[N, k]_\pm = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}} ({}_5C_{l_1}, {}_{p_2}C_{l_2}, \ldots, {}_{p_{12}}C_{l_{12}})_\pm. \tag{97}$$

Using the assignment of the Z_6 and Z_2 elements (95) and (96), we find that zero modes appear if the following relations are satisfied:

$$\begin{aligned} n^0_{l_1 L\pm} &\equiv l_2 + 2(l_3 + l_4) + 3(l_5 + l_6) + 4(l_7 + l_8) \\ &\quad + 5(l_9 + l_{10}) \quad = 6 - l_1 - \alpha_\pm \quad (\text{mod}\, 6), \\ n^1_{l_1 L\pm} &\equiv l_3 + l_5 + l_7 + l_9 + l_{11} = 2 - l_1 - \beta_\pm \quad (\text{mod}\, 2). \end{aligned} \tag{98}$$

The relation $n^a_{l_1 R\pm} = n^a_{l_1 L\pm} \mp 1 \ (\text{mod}\, 6)$ holds from Eq. (4).

Then, we obtain the formulas (15)–(17) with $n = 12$, and they are rewritten as

$$\begin{aligned} n_{\bar{5}} = \sum_{l_1=1,4} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}} (-1)^{l_1} (P^{(1,1)}_+ - P^{(\varphi,-1)}_+ \\ + P^{(1,1)}_- - P^{(\bar{\varphi},-1)}_-)\, {}_{p_2}C_{l_2} \cdots {}_{p_{12}}C_{l_{12}}, \end{aligned} \tag{99}$$

$$\begin{aligned} n_{10} = \sum_{l_1=2,3} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}} (-1)^{l_1} (P^{(1,1)}_+ - P^{(\varphi,-1)}_+ \\ + P^{(1,1)}_- - P^{(\bar{\varphi},-1)}_-)\, {}_{p_2}C_{l_2} \cdots {}_{p_{12}}C_{l_{12}}, \end{aligned} \tag{100}$$

TABLE XII. Examples for the three families of $SU(5)$ from T^2/Z_6.

$[N, k]$	$(p_1, p_2, p_3, \ldots, p_{11}, p_{12})$	(α_+, β_+)	(α_-, β_-)
[8,3]	(5,0,0,3,0,0,0,0,0,0,0,0)	(0,1)	(2,0)
[8,4]	(5,0,0,1,0,0,0,2,0,0,0,0)	(0,0)	(2,0)
[9,3]	(5,0,0,0,0,0,3,0,0,0,0,1)	(0,1)	(5,0)
[9,4]	(5,2,0,1,0,0,1,0,0,0,0,0)	(2,0)	(2,0)
[10,3]	(5,0,0,1,1,0,0,0,0,0,3,0)	(0,1)	(4,1)
[10,4]	(5,0,1,0,1,1,0,0,0,1,1,0)	(5,0)	(2,0)
[10,5]	(5,0,0,0,0,0,1,2,0,2,0,0)	(4,1)	(1,0)
[11,3]	(5,0,0,1,0,0,0,0,0,1,4,0)	(3,1)	(4,1)
[11,4]	(5,0,0,0,0,2,0,0,2,1,0,1)	(5,0)	(2,0)
[11,5]	(5,3,0,0,0,0,0,0,0,0,3,0)	(1,1)	(1,1)
[12,3]	(5,3,0,1,0,0,0,0,0,0,0,3)	(0,1)	(3,0)
[12,4]	(5,0,0,0,0,0,0,1,0,4,1,1)	(5,0)	(2,0)
[12,5]	(5,0,0,0,0,0,2,1,2,1,1,0)	(1,1)	(1,1)
[12,6]	(5,0,0,0,0,3,1,1,2,0,0,0)	(3,0)	(0,0)

$$\begin{aligned} n_1 = \sum_{l_1=0,5} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}} (P^{(1,1)}_+ + P^{(\varphi,-1)}_+ + P^{(1,1)}_- \\ + P^{(\bar{\varphi},-1)}_-)\, {}_{p_2}C_{l_2} \cdots {}_{p_{12}}C_{l_{12}}, \end{aligned} \tag{101}$$

where $P^{(\rho,\rho')}_\pm$ are projection operators that pick up the part relating $(\mathcal{P}_{0\pm}, \mathcal{P}_{1\pm}) = (\rho, \rho')$ and are written by

$$\begin{aligned} P^{(\rho,\rho')}_\pm = \frac{1}{12}(1 + \bar{\rho}\mathcal{P}_{0\pm} + \bar{\rho}^2\mathcal{P}^2_{0\pm} + \bar{\rho}^3\mathcal{P}^3_{0\pm} + \bar{\rho}^4\mathcal{P}^4_{0\pm} \\ + \bar{\rho}^5\mathcal{P}^5_{0\pm})(1 + \bar{\rho}'\mathcal{P}_{1\pm}). \end{aligned} \tag{102}$$

In Table XII, we give some examples for representations and BCs to derive $n_{\bar{5}} = n_{10} = 3$.

2. Numbers of the SM multiplets on T^2/Z_6

After the breakdown of $SU(N) \to SU(3) \times SU(2) \times SU(p_2) \times \cdots \times SU(p_{12}) \times U(1)^{11-m}$, $[N, k]_\pm$ is decomposed as

$$[N, k]_\pm = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \times \sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}} ({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3}, \ldots, {}_{p_{12}}C_{l_{12}})_\pm. \tag{103}$$

Using the assignment of the Z_6 and Z_2 elements (95) and (96), we find that zero modes appear if the following relations are satisfied:

$$\begin{aligned} n^0_{l_1 l_2 L\pm} &\equiv 2(l_3 + l_4) + 3(l_5 + l_6) + 4(l_7 + l_8) \\ &\quad + 5(l_9 + l_{10}) = 6 - l_1 - l_2 - \alpha_\pm \quad (\text{mod}\, 6), \\ n^1_{l_1 l_2 L\pm} &\equiv l_3 + l_5 + l_7 + l_9 + l_{11} = 2 - l_1 - \beta_\pm \quad (\text{mod}\, 2). \end{aligned} \tag{104}$$

The relation $n^a_{l_1 l_2 R\pm} = n^a_{l_1 l_2 L\pm} \mp 1$ (mod 6) holds from Eq. (4).

Then, we obtain the formulas (15)–(17) with $n = 12$. Using the projection operators (102), the formulas for $(d_R)^c$ and $(\nu_R)^c$ are rewritten as

$$\begin{aligned} n_{\bar{d}} = \sum_{(l_1,l_2)=(2,2),(1,0)} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{11}=0}^{k-l_1-\cdots-l_{12}} (-1)^{l_1+l_2} (P^{(1,1)}_+ \\ - P^{(\varphi,-1)}_+ + P^{(1,1)}_- - P^{(\bar{\varphi},-1)}_-)\, {}_{p_3}C_{l_3} \cdots {}_{p_{12}}C_{l_{12}}, \end{aligned} \tag{105}$$

$$\begin{aligned} n_{\bar{\nu}} = \sum_{(l_1,l_2)=(0,0),(3,2)} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{11}=0}^{k-l_1-\cdots-l_{10}} (P^{(1,1)}_+ + P^{(\varphi,-1)}_+ \\ + P^{(1,1)}_- + P^{(\bar{\varphi},-1)}_-)\, {}_{p_3}C_{l_3} \cdots {}_{p_{12}}C_{l_{12}}. \end{aligned} \tag{106}$$

The formulas for l_L, $(u_R)^c$, $(e_R)^c$, and q_L are obtained by replacing the summation of (l_1, l_2) for $n_{\bar{d}}$ with

TABLE XIII. Examples for the three families of SM multiplets from T^2/Z_6.

$[N, k]$	$(p_1, p_2, p_3, \ldots, p_{11}, p_{12})$	(α_+, β_+)	(α_-, β_-)
[9,3]	(3,2,0,1,0,0,0,0,0,0,1,2)	(0,0)	(0,1)
[9,4]	(3,2,0,0,0,1,0,0,1,2,0,0)	(1,1)	(1,0)
[10,3]	(3,2,0,0,3,0,0,0,0,0,1,1)	(1,0)	(1,1)
[10,4]	(3,2,0,1,1,2,0,0,0,0,1,0)	(0,1)	(0,0)
[11,3]	(3,2,1,1,1,0,0,0,0,1,1,1)	(0,1)	(0,0)
[11,4]	(3,2,0,1,0,2,0,0,0,3,0,0)	(0,1)	(1,0)
[11,5]	(3,2,0,0,1,0,4,0,1,0,0,0)	(0,1)	(0,0)
[12,3]	(3,2,0,1,3,1,0,1,0,0,0,1)	(1,0)	(1,1)
[12,4]	(3,2,0,0,0,1,1,2,0,2,1,0)	(1,1)	(1,0)
[12,5]	(3,2,1,1,0,3,1,1,0,0,0,0)	(1,0)	(1,1)
[12,6]	(3,2,0,0,0,1,0,0,3,0,0,3)	(1,1)	(1,1)
[13,3]	(3,2,1,0,0,0,0,3,2,0,0,2)	(0,0)	(0,1)
[13,4]	(3,2,2,0,1,1,1,1,0,0,1,1)	(1,0)	(1,1)
[13,5]	(3,2,1,0,0,4,0,0,0,3,0,0)	(1,1)	(1,0)
[13,6]	(3,2,1,0,0,0,0,2,4,0,0,1)	(0,0)	(0,1)

{(3, 1), (0, 1)}, {(2, 0), (1, 2)}, {(0, 2), (3, 0)}, and {(1, 1), (2, 1)}.

In Table XIII, we give some examples for representations and BCs to derive three families of SM fermions.

IV. CONCLUSIONS

We have studied the possibility of family unification on the basis of $SU(N)$ gauge theory on the six-dimensional space-time, $M^4 \times T^2/Z_N$. We have obtained enormous numbers of models with three families of $SU(5)$ matter multiplets and those with three families of the SM multiplets, from a single massless Dirac fermion with a higher-dimensional representation of $SU(N)$, after the orbifold breaking. The total numbers of models with the three families of $SU(5)$ multiplets and the SM multiplets are summarized in Tables IV and V, respectively. Our results can give a starting point for the construction toward a more realistic model, because three families of chiral fermions in the SM are contained in our models.

Now, the following open questions should be tackled as a future work.

The unwanted matter degrees of freedom can be successfully made massive thanks to the orbifolding. However, some extra gauge fields remain massless, even after the symmetry breaking due to the Hosotani mechanism [38,39]. In most cases, this kind of non-Abelian gauge subgroup plays the role of family symmetry. These massless degrees of freedom must be made massive by further breaking of the family symmetry. Extra scalar fields can play the role of Higgs fields for the breakdown of extra gauge symmetries including non-Abelian gauge symmetries. As a result, extra massless fields including the family gauge bosons can be massive.

If fields localized around fixed points (brane fields) are introduced, there is a potential such that three families are generated after the survival hypothesis works between the bulk fields and brane fields. In such a case, models with families greater than three derived from a single bulk multiplet would be favorable.

In general, there appear D-term contributions to scalar masses in supersymmetric models after the breakdown of such extra gauge symmetries and the D-term contributions lift the mass degeneracy [42–46]. The mass degeneracy for each squark and slepton species in the first two families is favorable for suppressing flavor-changing neutral current processes. The dangerous flavor-changing neutral current processes can be avoided if the sfermion masses in the first two families are rather large or the fermion and its superpartner mass matrices are aligned. The requirement of degenerate masses would yield a constraint on the D-term condensations and/or SUSY breaking mechanism unless other mechanisms work. If we consider the Scherk-Schwarz mechanism [28,29] for $N = 1$ SUSY breaking, the D-term condensations can vanish for the gauge symmetries broken at the orbifold breaking scale because of a universal structure of the soft SUSY breaking parameters. The D-term contributions have been studied in the framework of $SU(N)$ orbifold GUTs [47,48].

Can the gauge coupling unification be successfully achieved? If the particle contents in the minimal supersymmetric standard model only remain in the low-energy spectrum around and below the TeV scale and a big desert exists after the breakdown of extra gauge symmetries, an ordinary grand unification scenario can be realized up to the threshold corrections due to the Kaluza-Klein modes and the brane contributions from nonunified gauge kinetic terms.

Another problem is whether or not the realistic fermion mass spectrum and the generation mixings are successfully achieved. Fermion mass hierarchy and generation mixings can also occur through the Froggatt-Nielsen mechanism [49] on the breakdown of extra gauge symmetries and the suppression of brane-localized Yukawa coupling constants among brane weak Higgs doublets and bulk matters with the volume suppression factor [50].

It would be interesting to reconsider or reconstruct our models in the framework of string theory. Various four-dimensional string models including three families have been constructed from several methods, see, e.g., Ref. [51] and references therein for useful articles.[8]

It has been pointed out that $SO(1, D-1)$ space-time symmetry can lead to family structure [53,54], and hence, it would offer a hint to explore the family structure in our models.

Furthermore, it would be interesting to study cosmological implications of the class of models presented in this paper, see, e.g., Ref. [55] and references therein for useful articles toward this direction.

[8]See also Ref. [52] and references therein for recent works.

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[1] P. Ramond, arXiv:hep-ph/9809459.
[2] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by P. van Nieuwenhuizen and D. Z. Freedman (North Holland, Amsterdam, 1979).
[3] H. Georgi, Nucl. Phys. **B156**, 126 (1979).
[4] P. H. Frampton, Phys. Lett. **88B**, 299 (1979).
[5] P. H. Frampton, Phys. Lett. **89B**, 352 (1980).
[6] F. Wilczek and A. Zee, Phys. Rev. D **25**, 553 (1982).
[7] T. Kugo and T. Yanagida, Phys. Lett. **134B**, 313 (1984).
[8] K. S. Babu, S. M. Barr, and B. Kyae, Phys. Rev. D **65**, 115008 (2002).
[9] T. Watari and T. Yanagida, Phys. Lett. B **532**, 252 (2002).
[10] I. Gogoladze, Y. Mimura, and S. Nandi, Phys. Lett. B **562**, 307 (2003).
[11] I. Gogoladze, T. Li, Y. Mimura, and S. Nandi, Phys. Rev. D **72**, 055006 (2005).
[12] I. Gogoladze, C. A. Lee, Y. Mimura, and Q. Shafi, Phys. Lett. B **649**, 212 (2007).
[13] I. Gogoladze, Y. Mimura, and S. Nandi, Phys. Rev. Lett. **91**, 141801 (2003).
[14] Y. Kawamura, T. Kinami, and K. Oda, Phys. Rev. D **76**, 035001 (2007).
[15] Y. Kawamura and T. Miura, Phys. Rev. D **81**, 075011 (2010).
[16] Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, Phys. Rev. D **87**, 086001 (2013).
[17] R. Barbieri, D. V. Nanopoulos, G. Morchio, and F. Strocchi, Phys. Lett. **90B**, 91 (1980).
[18] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258**, 46 (1985).
[19] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985).
[20] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B274**, 285 (1986).
[21] Y. Kawamura, Prog. Theor. Phys. **105**, 999 (2001).
[22] L. Hall and Y. Nomura, Phys. Rev. D **64**, 055003 (2001).
[23] L. Antoniadis, Phys. Lett. B **246**, 377 (1990).
[24] P. Hořava and E. Witten, Nucl. Phys. **B460**, 506 (1996).
[25] P. Hořava and E. Witten, Nucl. Phys. **B475**, 94 (1996).
[26] E. A. Mirabelli and M. E. Peskin, Phys. Rev. D **58**, 065002 (1998).
[27] A. Pomarol and M. Quirós, Phys. Lett. B **438**, 255 (1998).
[28] J. Scherk and J. H. Schwarz, Phys. Lett. **82B**, 60 (1979).
[29] J. Scherk and J. H. Schwarz, Nucl. Phys. **B153**, 61 (1979).
[30] Y. Kawamura, Prog. Theor. Phys. **103**, 613 (2000).
[31] Y. Kawamura, T. Kinami, and T. Miura, Prog. Theor. Phys. **120**, 815 (2008).
[32] Y. Kawamura and T. Miura, Prog. Theor. Phys. **122**, 847 (2009).
[33] Y. Kawamura and T. Miura, Int. J. Mod. Phys. A **28**, 1350055 (2013).
[34] B. A. Dobrescu and E. Poppitz, Phys. Rev. Lett. **87**, 031801 (2001).
[35] N. Borghini, Y. Gouverneur, and M. H. G. Tytgat, Phys. Rev. D **65**, 025017 (2001).
[36] P. Minkowski, Phys. Lett. **67B**, 421 (1977).
[37] T. Yanagida, in *Proceedings of Workshop on the Unified Theory and the Baryon Number in the Universe*, edited by O. Sawada and A. Sugamoto econf C7902131, 95 (1979).
[38] Y. Hosotani, Phys. Lett. **126B**, 309 (1983).
[39] Y. Hosotani, Ann. Phys. (N.Y.) **190**, 233 (1989).
[40] N. Haba, M. Harada, Y. Hosotani, and Y. Kawamura, Nucl. Phys. B **657**, 169 (2003); **B669**, 381(E) (2003).
[41] N. Haba, Y. Hosotani, and Y. Kawamura, Prog. Theor. Phys. **111**, 265 (2004).
[42] M. Drees, Phys. Lett. B **181**, 279 (1986).
[43] J. S. Hagelin and S. Kelley, Nucl. Phys. **B342**, 95 (1990).
[44] Y. Kawamura, H. Murayama, and M. Yamaguchi, Phys. Lett. B **324**, 52 (1994).
[45] Y. Kawamura, H. Murayama, and M. Yamaguchi, Phys. Rev. D **51**, 1337 (1995).
[46] Y. Kawamura and M. Tanaka, Prog. Theor. Phys. **91**, 949 (1994).
[47] Y. Kawamura and T. Kinami, Int. J. Mod. Phys. A **22**, 4617 (2007); Prog. Theor. Phys. **119**, 285 (2008).
[48] Y. Kawamura, T. Kinami, and T. Miura, J. High Energy Phys. 01 (2009) 064.
[49] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B147**, 277 (1979).
[50] K. Yoshioka, Mod. Phys. Lett. A **15**, 29 (2000).
[51] L. E. Ibáñez and A. M. Uranga, *String Theory and Particle Physics* (Cambridge University Press, Cambridge, England 2012).
[52] S. Mizoguchi and M. Yata, Prog. Theor. Exp. Phys. **2013**, 053B01 (2013).
[53] N. M. Borštnik and H. B. Nielsen, J. Math. Phys. (N.Y.) **44**, 4817 (2003).
[54] A. B. Bračič and N. S. M. Borštnik, Phys. Rev. D **74**, 073013 (2006).
[55] M. Y. Khlopov, *Cosmoparticle Physics* (World Scientific, Singapore, 1999).