Progress of Theoretical Physics, Vol. 127, No. 2, February 2012

A Note on the Partition Function of ABJM Theory on S^3

Kazumi Okuyama

Department of Physics, Shinshu University, Matsumoto 390-8621, Japan

(Received November 10, 2011; Revised December 15, 2011)

We study the partition function Z of $U(N)_k \times U(N)_{-k}$ Chern-Simons matter theory (ABJM theory) on S^3 which is recently obtained by the localization method. We evaluate the eigenvalue integral in Z exactly for the $N = 2$ case. We find that Z has a different dependence on k for even k and odd k. We comment on the possible implication of this result in the context of AdS/CFT correspondence.

Subject Index: 121, 125, 183

§1. Introduction

In the seminal paper,[1)] the theory on the N coincident M2-branes on the orbifold $\mathbb{R}^8/\mathbb{Z}_k$ was identified as the $d = 3$ $\mathcal{N} = 6$ $U(N)_k \times U(N)_{-k}$ Chern-Simons matter theory (ABJM theory). Recently, the partition function $Z_{N,k}$ of ABJM theory on S^3 was obtained by the localization method,[2)] and $Z_{N,k}$ was given in the form of a matrix integral. The behavior of $Z_{N,k}$ has been analyzed previously[3)–6)] in the 't Hooft limit

$$k, N \to \infty, \quad t = \frac{N}{k} = \text{fixed} , \tag{1·1}$$

and it was shown that the free energy $F = -\log Z_{N,k}$ exhibits the correct $N^{\frac{3}{2}}$ scaling as predicted by the holographic dual gravity theory. The ABJM theory in the 't Hooft limit is holographically dual to the type IIA theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, which appears from the S^1 reduction of the M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ when $k \gg N^{\frac{1}{5}}$.

However, if we are interested in the dynamics of M2-branes in the truly M-theory regime, or in the strong coupling regime of type IIA theory, we need to know the behavior of $Z_{N,k}$ at finite k, since the IIA string coupling is inversely proportional to k. Of particular interest is the ABJM theory at $k = 1$, which is conjectured to describe the M2-branes on the flat eleven dimensional Minkowski space. Therefore we might want to develop a technique to analyze the partition function $Z_{N,k}$ in the M-theory regime where

$$N \to \infty, \quad k = \text{finite} . \tag{1·2}$$

This regime was studied in 7) by the saddle point method for the eigenvalue integral.

In this paper we find the exact partition function $Z_{N,k}$ for $N = 2$ with finite k by performing the eigenvalue integral explicitly for the $N = 2$ case. We find that the result depends on the parity of k:

$$Z_{2,\,\mathrm{odd}\,k} = \frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2} - \frac{s}{k}\right)\tan^2\frac{\pi s}{k} + \frac{(-1)^{\frac{k-1}{2}}}{\pi} , \tag{1·3}$$

$$Z_{2,\,\mathrm{even}\,k}=\frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2}-\frac{s}{k}\right)^2\tan^2\frac{\pi s}{k}\ . \tag{1·4}$$

For both even k and odd k cases, the summation over s has a natural interpretation as the effect of $\mathbb{Z}_k$ orbifolding of $\mathbb{R}^8/\mathbb{Z}_k$.

This paper is organized as follows. In §2, we first rewrite the partition function of ABJM theory on S^3 in terms of the integrals associated with the cyclic permutations. Then we consider the grand partition function of ABJM theory, following the similar analysis of the matrix integrals which arise from the dimensional reduction of super Yang-Mills theories to 0-dimension.[8),9)] We also comment on the mirror description of the partition function of ABJM theory. In §3, we compute the partition function of $U(2)_k\times U(2)_{-k}$ ABJM theory and find that the result depends on the parity of k. In §4, we speculate the possible implication of this result in the context of AdS/CFT correspondence. In Appendices A and B, we present the details of the calculation of integrals used in §3.

§2. Structure of the partition function of ABJM theory on S^3

2.1. *Grand partition function of ABJM theory*

Recently, by applying the localization method of 10), the partition function of general $\mathcal{N}=2$ Chern-Simons matter theories on S^3 with the gauge group G and the matter chiral multiplet in a representation $R\oplus R^*$ was obtained in a form of matrix integral[2),*)]

$$Z=\frac{1}{|W|}\int da\, e^{-i\pi k a^2}\frac{\det_{Ad}(\sinh\pi a)}{\det_R(\cosh\pi a)}\ . \tag{2·1}$$

Above, the integral of a is over the Cartan subalgebra of G, $|W|$ is the order of the Weyl group of G, and k is the Chern-Simons coupling which is quantized to be an integer. Note that a originates from the constant mode of the real scalar field in the vector multiplet.

Since the ABJM theory is the $d=3$ $U(N)_k\times U(N)_{-k}$ Chern-Simons theory with bi-fundamental matter multiplets, its partition function on S^3 is given by

$$Z_{N,k}=\frac{1}{(N!)^2}\int d^N\sigma d^N\widetilde{\sigma}\,\Delta(\sigma,\widetilde{\sigma})^2\,e^{i\pi k(\sigma^2-\widetilde{\sigma}^2)}\ , \tag{2·2}$$

where σ^2 is the shorthand for $\sum_{i=1}^N\sigma_i^2$, and similarly $\widetilde{\sigma}^2=\sum_{i=1}^N\widetilde{\sigma}_i^2$, and $\Delta(\sigma,\widetilde{\sigma})$ is given by

$$\Delta(\sigma,\widetilde{\sigma})=\frac{\prod_{i<j}\sinh\pi(\sigma_i-\sigma_j)\sinh\pi(\widetilde{\sigma}_i-\widetilde{\sigma}_j)}{\prod_{i,j}\cosh\pi(\sigma_i-\widetilde{\sigma}_j)}\ . \tag{2·3}$$

*) The partition function of the theory with matter multiplet in the non-self-conjugate representation was obtained in 12) and 13). Note that (2·1) is valid only when the R-charge carried by the matter multiplet is 1/2. The partition function for the case of non-canonical R-charge q was also calculated in 12) and 13). See also 14) for a recent review on the localization technique in $d=3$ theories.

Using the Cauchy identity[11)]

$$\Delta(\sigma,\widetilde{\sigma}) = \sum_{\rho\in S_N} (-1)^\rho \prod_{i=1}^{N} \frac{1}{\cosh\pi(\sigma_i - \widetilde{\sigma}_{\rho(i)})} \,, \tag{2·4}$$

the partition function is rewritten as

$$Z_{N,k} = \frac{1}{N!}\int d^N\sigma d^N\widetilde{\sigma}\, e^{i\pi k(\sigma^2-\widetilde{\sigma}^2)} \sum_{\rho\in S_N} (-1)^\rho \prod_{i=1}^{N} \frac{1}{\cosh\pi(\sigma_i-\widetilde{\sigma}_i)\cosh\pi(\sigma_i - \widetilde{\sigma}_{\rho(i)})} \,. \tag{2·5}$$

The sum over permutations can be simplified by noting that the integral depends only on the conjugacy class of permutation. The conjugacy class of permutation ρ is labeled by the cycle of length ℓ and the number d_ℓ of such cycles contained in ρ

$$[\rho] = [1^{d_1}2^{d_2}\cdots N^{d_N}] \equiv \Big[\prod_\ell \ell^{d_\ell}\Big], \quad N = \sum_\ell \ell d_\ell \,. \tag{2·6}$$

The number of elements in the conjugacy class $[\rho]$ and the signature are given by

$$\#[\rho] = \frac{N!}{\prod_\ell \ell^{d_\ell} d_\ell!}, \quad (-1)^\rho = (-1)^{\sum_\ell d_\ell(\ell-1)} \,. \tag{2·7}$$

One can show that the integral in (2·5) is decomposed into the integral associated with the cyclic permutation

$$Z_{N,k} = \sum_{d_\ell\ge 0, \sum \ell d_\ell = N} \prod_{\ell=1}^{N} \frac{1}{d_\ell!}\left[\frac{(-1)^{\ell-1}A_{\ell,k}}{\ell}\right]^{d_\ell} \,, \tag{2·8}$$

where $A_{\ell,k}$ denotes the integral coming from the cycle of length ℓ

$$A_{\ell,k} = \int d^\ell\sigma d^\ell\widetilde{\sigma}\, e^{i\pi k(\sigma^2-\widetilde{\sigma}^2)} \prod_{i=1}^{\ell} \frac{1}{\cosh\pi(\sigma_i-\widetilde{\sigma}_i)\cosh\pi(\sigma_i-\widetilde{\sigma}_{i+1})} \,. \tag{2·9}$$

Here the mod-ℓ identification $\widetilde{\sigma}_{\ell+1}\equiv\widetilde{\sigma}_1$ should be understood.

By introducing the chemical potential μ for N, the grand partition function is defined by

$$\mathcal{Z}_k(\mu) = \sum_{N=0}^{\infty} e^{\mu N} Z_{N,k} \,. \tag{2·10}$$

From (2·8) one can easily see that $\mathcal{Z}_k(\mu)$ is exponentiated after summing over d_ℓ's

$$\mathcal{Z}_k(\mu) = \exp\left[\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} e^{\mu\ell} A_{\ell,k}\right] \,. \tag{2·11}$$

Once we know the grand partition function, we can recover the fixed N partition function from the integral of $\mathcal{Z}_k(\mu)$ by analytically continuing the chemical potential to a pure imaginary value $\mu = i\theta$

$$Z_{N,k} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta}\mathcal{Z}_k(i\theta) \,. \tag{2·12}$$

It would be interesting to see whether the grand partition function of ABJM theory has a hidden integrable structure as in 9).

2.2. *Mirror description of ABJM theory*

By the mirror symmetry, the ABJM theory is dual to a theory without Chern-Simons term. More concretely, the mirror of ABJM theory is a $U(N)$ super Yang-Mills theory with matter hypermultiplets in certain representations of $U(N)$. As discussed in 11), the partition function on S^3 is a useful tool to check this type of mirror symmetry. The key relation to prove the equality of partition functions of the original theory and its mirror is the following identity,

$$\int dx \frac{e^{2\pi i x\sigma}}{\cosh \pi x} = \frac{1}{\cosh \pi\sigma} . \tag{2·13}$$

Using this relation, the partition function of ABJM theory $Z_{N,k}$ is rewritten as

$$Z_{N,k} = \frac{k^{2N}}{N!}\int d^N\sigma d^N\widetilde{\sigma} d^N x d^N y \sum_{\rho\in S_N}(-1)^\rho \frac{e^{i\pi k\sum_{i=1}^N[\sigma_i^2-\widetilde{\sigma}_i^2+2x_i(\sigma_i-\widetilde{\sigma}_i)+2y_i(\sigma_i-\widetilde{\sigma}_{\rho(i)})]}}{\prod_{i=1}^N \cosh \pi k x_i \cosh \pi k y_i} . \tag{2·14}$$

After doing the Gaussian integral for $\sigma, \widetilde{\sigma}$ and using the identity (2·13) again for the y-integral, (2·14) becomes

$$Z_{N,k} = \frac{1}{N!}\int \prod_{i=1}^N dx_i \sum_{\rho\in S_N}(-1)^\rho \prod_{i=1}^N \frac{1}{\cosh \pi k x_i \cosh \pi(x_i - x_{\rho(i)})} . \tag{2·15}$$

Applying the Cauchy identity for the sum over permutations, we arrive at the mirror expression of the partition function of ABJM theory

$$Z_{N,k} = \frac{1}{N!}\int \prod_{i=1}^N dx_i \frac{\prod_{i<j}\sinh^2 \pi(x_i - x_j)}{\prod_i \cosh \pi k x_i \prod_{i,j}\cosh \pi(x_i - x_j)} . \tag{2·16}$$

From this, we can read off the matter content of the mirror of ABJM theory. When $k = 1$, the mirror theory is the $U(N)$ super Yang-Mills theory with one adjoint and one fundamental hypermultiplets, where the factors $1/\prod_{i,j}\cosh \pi(x_i - x_j)$ and $1/\prod_i \cosh \pi x_i$ in (2·16) are the 1-loop determinant of those hypermultiplets, respectively.[11)] When $k \geq 2$ it is not clear whether the factor $1/\prod_i \cosh k\pi x_i$ can be interpreted as the 1-loop determinant of hypermultiplet in some representation R. In particular it is different from the 1-loop determinant of hypermultiplet in the k^{th} symmetric product of fundamental representations.

The grand partition function of the mirror theory of ABJM theory has the same form as (2·11), and the contribution from the cycle of length ℓ in the mirror description is given by

$$A_{\ell,k} = \int d^\ell x \prod_{i=1}^\ell \frac{1}{\cosh \pi k x_i \cosh \pi(x_i - x_{i+1})} . \tag{2·17}$$

§3. Partition function of $U(2)_k \times U(2)_{-k}$ ABJM theory

In this section, we study the partition function $Z_{2,k}$ of $U(2)_k \times U(2)_{-k}$ ABJM theory. Since this model is conjecture to describe the dynamics of two M2-branes on $\mathbb{R}^8/\mathbb{Z}_k$, we expect that some information of the two-body interaction of M2-branes is contained in the partition function $Z_{2,k}$. Therefore, the study of the partition function of $U(2)_k \times U(2)_{-k}$ theory would be a modest first step toward the understanding of the still mysterious multiple M2-brane dynamics.*)

Here we evaluate the eigenvalue integral of $Z_{N,k}$ in (2·5) explicitly for the $N = 2$ case. To do that, we first rewrite $Z_{2,k}$ as a combination of the integral $A_{\ell,k}$ coming from the cyclic permutation of length ℓ as shown in (2·8)

$$Z_{2,k} = \frac{1}{2}\Big[(A_{1,k})^2 - A_{2,k}\Big] . \tag{3·1}$$

Although $A_{2,k}$ is originally written as an integral over four variables (2·9), after some computation this four-variable integral can be reduced to a single variable integral. We find that $A_{1,k}$ and $A_{2,k}$ are given by (see Appendix A for details)

$$A_{1,k} = \frac{1}{k} , \tag{3·2}$$

$$A_{2,k} = \int_{-\infty}^{\infty} d\lambda \frac{2\lambda}{\sinh \pi k\lambda \cosh^2 \pi\lambda} = \frac{1}{k^2} - \int_{-\infty}^{\infty} d\lambda \frac{2\lambda}{\sinh \pi k\lambda} \frac{\sinh^2 \pi\lambda}{\cosh^2 \pi\lambda} . \tag{3·3}$$

Plugging this into (3·1), we obtain

$$Z_{2,k} = \int_{-\infty}^{\infty} d\lambda \frac{\lambda}{\sinh \pi k\lambda} \frac{\sinh^2 \pi\lambda}{\cosh^2 \pi\lambda} . \tag{3·4}$$

Note that λ is related to the original variables (up to permutation) as

$$\lambda = \sigma_1 - \widetilde{\sigma}_1 . \tag{3·5}$$

As explained in Appendix B, the remaining λ-integral can be evaluated by picking up the residues of the poles of $\frac{1}{\sinh \pi k\lambda}$ and $\frac{1}{\cosh^2 \pi\lambda}$. It turns out that the result depends on the parity of k

$$Z_{2,\,\mathrm{odd}\,k} = \frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2} - \frac{s}{k}\right)\tan^2 \frac{\pi s}{k} + \frac{(-1)^{\frac{k-1}{2}}}{\pi} , \tag{3·6}$$

$$Z_{2,\,\mathrm{even}\,k} = \frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2} - \frac{s}{k}\right)^2 \tan^2 \frac{\pi s}{k} . \tag{3·7}$$

In the above expression of $Z_{2,\,\mathrm{even}\,k}$, the $s = \frac{k}{2}$ term should be understood as the limit

$$\lim_{s\to\frac{k}{2}} \frac{1}{k}(-1)^{s-1}\left(\frac{1}{2} - \frac{s}{k}\right)^2 \tan^2 \frac{\pi s}{k} = \frac{(-1)^{\frac{k}{2}-1}}{k\pi^2} . \tag{3·8}$$

*) In a slightly different context, the exact evaluation of the partition function of $U(2)$ IIB matrix model was reported in 15) and 16).

Let us consider the physical interpretation of this result (3·7). For both even k and odd k cases, the sum over s comes from the poles at $\sinh \pi k\lambda = 0$. It is natural to interpret this sum as the effect of the $\mathbb{Z}_k$ orbifolding of $\mathbb{R}^8/\mathbb{Z}_k$. On the other hand, the second term $\frac{(-1)^{\frac{k-1}{2}}}{\pi}$ in $Z_{2,\,\mathrm{odd}\,k}$ comes from the pole at $\cosh \pi\lambda = 0$. This pole corresponds to the zero of the 1-loop determinant of the bi-fundamental hypermultiplet, so it represents a singularity on the space of vector multiplet scalar fields where one of the bi-fundamental hypermultiplet becomes massless. However, the location of the singularity is at the imaginary value of the scalar field

$$\sigma_1 - \widetilde{\sigma}_1 = \frac{i}{2}\ , \tag{3·9}$$

and hence this singularity is not realized in the physical theory. We should also mention that the poles coming from the $\frac{1}{\sinh \pi k\lambda}$ factor do not correspond to the zeros of the 1-loop determinant of the hypermultiplets in the original ABJM theory. Those poles effectively show up only after integrating out some of the variables σ_i and $\widetilde{\sigma}_i$, which are coupled via the Chern-Simons term $e^{\pi i k(\sigma^2-\widetilde{\sigma}^2)}$.

From (3·3), we see that $A_{2,k}$ is positive. Therefore, we find the inequality*)

$$Z_{2,k} < \frac{1}{2}(Z_{1,k})^2\ , \tag{3·13}$$

where $Z_{1,k} = A_{1,k} = \frac{1}{k}$ is the partition function of $U(1)_k \times U(1)_{-k}$ theory. From this inequality (3·13), it is tempting to draw a conclusion that the binding energy of two M2-branes is negative and M2-branes tend to dissociate into a configuration of two separated M2-branes. However, we think this is not the correct interpretation. When the ABJM theory is put on S^3, the bi-fundamental matter multiplets acquire a mass term from the coupling to the curvature of S^3, and hence the moduli space corresponding to the freely moving M2-branes on $\mathbb{R}^8/\mathbb{Z}_k$ is lifted. Therefore, the free energy of ABJM theory on S^3 is not a suitable measure of the binding energy of M2-branes on flat $\mathbb{R}^{1,2} \times \mathbb{R}^8/\mathbb{Z}_k$. Rather, the partition function on S^3 is a natural quantity to consider in the context of the Euclidean version of AdS/CFT duality,

*) The normalization of the partition function in 3) is different from ours by the factor of 2 in the 1-loop determinant. Namely, the partition function in 3) is related to ours by the replacement $\sinh \to 2\sinh, \cosh \to 2\cosh$

$$Z^{(\mathrm{DMP})}_{N,k} = \frac{1}{(N!)^2}\int d^N\sigma d^N\widetilde{\sigma}\, e^{i\pi k(\sigma^2-\widetilde{\sigma}^2)} \left[\frac{\prod_{i<j} 2\sinh \pi(\sigma_i-\sigma_j)\cdot 2\sinh \pi(\widetilde{\sigma}_i-\widetilde{\sigma}_j)}{\prod_{i,j} 2\cosh \pi(\sigma_i-\widetilde{\sigma}_j)}\right]^2 . \tag{3·10}$$

One can easily see that the difference between $Z^{(\mathrm{DMP})}_{N,k}$ and ours is just the overall factor 2^{-2N}

$$Z^{(\mathrm{DMP})}_{N,k} = 2^{-2N} Z^{(\mathrm{ours})}_{N,k}\ . \tag{3·11}$$

However, this factor drops out when taking the ratio of $(Z_{1,k})^2$ and $Z_{2,k}$

$$\frac{(Z^{(\mathrm{DMP})}_{1,k})^2}{Z^{(\mathrm{DMP})}_{2,k}} = \frac{(Z^{(\mathrm{ours})}_{1,k})^2}{Z^{(\mathrm{ours})}_{2,k}}\ . \tag{3·12}$$

Therefore, the statement $Z_{2,k} < \frac{1}{2}(Z_{1,k})^2$ has a physical meaning regardless of the normalization we choose.

where S^3 appears as the boundary of Euclidean AdS_4. In the next concluding section we discuss a possible implication of our result in the context of $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality.

§4. Discussion

As discussed in 1), the ABJM theory is dual to the M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with the metric

$$ds^2 = \frac{R^2}{4} ds^2_{\mathrm{AdS}_4} + R^2 ds^2_{S^7/\mathbb{Z}_k} \ , \tag{4·1}$$

where the radius of curvature R is given by

$$\left(\frac{R}{l_p}\right)^6 = 32\pi^2 kN \ . \tag{4·2}$$

The classical $d = 11$ supergravity description is valid when the radius of $S^7/\mathbb{Z}_k$ is much larger than the eleven-dimensional Planck length l_p

$$l_p \ll \frac{R}{k} \quad \to \quad k^5 \ll N \ . \tag{4·3}$$

In particular, the large N limit of ABJM theory with k fixed to a finite integer is in the regime of (4·3).

On the ABJM theory side, it seems that the even/odd k difference of the behavior of the partition function $Z_{N,k}$ persists for $N > 2$. This is because, in the integral of $A_{\ell,k}$ in (A.6), the pole of the form $\frac{1}{\sinh \pi k\lambda}$ related to the $\mathbb{Z}_k$ orbifolding appears also for general $\ell > 2$ in the same way as $A_{2,k}$ by integrating out some of the variables in σ_i and $\widetilde{\sigma}_i$ coupled through the Chern-Simons term, and the remaining integral over λ depends on the parity of k. Since the partition function $Z_{N,k}$ is written as a combination of $A_{\ell,k}$ (2·8), $Z_{N,k}$ also depends on the parity of k, unless some miraculous cancellation happens. But we think that is unlikely and the dependence on the parity of k is not an artifact of $Z_{2,k}$ but the general property of $Z_{N,k}$ for all $N \geq 2$.

If we believe in the duality between the ABJM theory and M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, this difference of even/odd k must be encoded in the M-theory dual, perhaps in a very subtle way. However, so far there is no known indication of this difference in the supergravity approximation of M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. Even if we take account of the wrapped brane configuration in this background, the bulk theory seems to be insensitive to the parity of k. In fact, the BPS configuration of M5-branes wrapped on the 3-cycle in $S^7/\mathbb{Z}_k$ is characterized by the homology class

$$H_3(S^7/\mathbb{Z}_k) = \mathbb{Z}_k \ , \tag{4·4}$$

which is interpreted as the fractional M2-brane charge.[17)] Clearly, this charge does not distinguish the parity of k. It might be the case that the even/odd k difference appears in the bulk theory as some sort of quantum effects in M-theory, which cannot be seen in the supergravity approximation. If this is true, it would be nice to understand this effect better.

In the regime where

$$N^{\frac{1}{5}} \ll k \ll N \ , \tag{4·5}$$

the bulk theory is described by the type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. On the CFT side, this regime is related to the 't Hooft limit of ABJM theory (1·1), and the classical type IIA supergravity description becomes good when the 't Hooft coupling $t = \frac{N}{k}$ is large. When comparing the free energy $F = -\log Z_{N,k}$ of ABJM theory and the classical action of the bulk supergravity theory, we need to perform an analytic continuation of $Z_{N,k}$ as a function of k and N. In particular, when determining the eigenvalue distribution for the matrix integral (2·2) in the 't Hooft limit, the analytic continuation in k is implicitly assumed.

Our result suggests that the analyticity in k is not obvious a priori, even in the large N regime. In some cases of Chern-Simons-matter theories, the analytic continuation in k requires the deformation of integration contour. However, the integral representation of $Z_{2,k}$ in (3·4) is well-defined for $k \in \mathbb{R}$ without changing the integration contour of λ. From this integral representation (3·4), one can see that $Z_{2,k}$ decreases monotonically as a function of k,*) and the expression (3·4) for $k \in \mathbb{R}$ serves as an interpolating function of our result (3·7) for integer k. It would be nice to see if similar analytic continuation is possible for $N > 2$ without deforming the integration contour.

Acknowledgements

I would like to thank Kazuo Hosomichi for a nice lecture on the localization method in supersymmetric gauge theories at Chubu Summer School 2011. This work is supported in part by a JSPS Grant-in-Aid for Young Scientists ((B) 23740178).

Appendix A
—— *Computation of $A_{\ell,k}$* ——

In this appendix, by performing the integration of two variables, we rewrite the 2ℓ-variable integral $A_{\ell,k}$ given in (2·9) into the integral of $2(\ell-1)$ variables. Using this expression, we find $A_{1,k} = \frac{1}{k}$. We also find the expression of $A_{2,k}$ as a single variable integral.

A.1. *Writing $A_{\ell,k}$ as the integral of $2(\ell-1)$ variables*

For readers convenience, we repeat the integral $A_{\ell,k}$ in (2·9)

$$A_{\ell,k} = \int d^\ell\sigma d^\ell\widetilde{\sigma}\, e^{i\pi k(\sigma^2 - \widetilde{\sigma}^2)} \prod_{i=1}^{\ell} \frac{1}{\cosh\pi(\sigma_i - \widetilde{\sigma}_i)\cosh\pi(\sigma_i - \widetilde{\sigma}_{i+1})} \ . \tag{A·1}$$

This integral can be simplified by the following change of variables

$$(\sigma_1, \cdots, \sigma_\ell, \widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_\ell) \to (\lambda_1, \cdots, \lambda_\ell, \widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_{\ell-1}, \widetilde{\sigma}_\ell) \tag{A·2}$$

*) We would like to thank the referee of Prog. Theor. Phys. for pointing this out.

where

$$\lambda_i = \sigma_i - \widetilde{\sigma}_i \quad (i = 1, \cdots, \ell) \ , \qquad \widetilde{\lambda}_i = \sigma_i - \widetilde{\sigma}_{i+1} \quad (i = 1, \cdots, \ell - 1) \ . \tag{A·3}$$

In terms of these new variables, the integral becomes

$$A_{\ell,k} = \int d^\ell \lambda d^{\ell-1} \widetilde{\lambda} d\widetilde{\sigma}_\ell \prod_{i=1}^{\ell} \frac{1}{\cosh \pi \lambda_i} \prod_{i=1}^{\ell-1} \frac{1}{\cosh \pi \widetilde{\lambda}_i} \cdot \frac{1}{\cosh \pi (\sum_{i=1}^{\ell} \lambda_i - \sum_{i=1}^{\ell-1} \widetilde{\lambda}_i)}$$
$$\times \exp\left(2\pi k i \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} \lambda_j \widetilde{\lambda}_i + 2\pi k i \sum_{i=1}^{\ell} \lambda_i \widetilde{\sigma}_\ell \right) \ . \tag{A·4}$$

Since the variable $\widetilde{\sigma}_\ell$ appears only in the exponent, the $\widetilde{\sigma}_\ell$ integral is just a δ-function

$$\int d\widetilde{\sigma}_\ell \exp\left(2\pi k i \sum_{i=1}^{\ell} \lambda_i \widetilde{\sigma}_\ell \right) = \frac{1}{k} \delta \left(\sum_{i=1}^{\ell} \lambda_i \right) \ . \tag{A·5}$$

After integrating out λ_ℓ by setting $\lambda_\ell = -\sum_{i=1}^{\ell-1} \lambda_i$ by the above δ-function, we obtain

$$A_{\ell,k} = \frac{1}{k} \int d^{\ell-1} \lambda d^{\ell-1} \widetilde{\lambda} \prod_{i=1}^{\ell-1} \frac{1}{\cosh \pi \lambda_i \cosh \pi \widetilde{\lambda}_i} \cdot \frac{1}{\cosh \pi (\sum_{i=1}^{\ell-1} \lambda_i) \cosh \pi (\sum_{i=1}^{\ell-1} \widetilde{\lambda}_i)}$$
$$\times \exp\left(2\pi k i \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} \lambda_j \widetilde{\lambda}_i \right) \ . \tag{A·6}$$

A.2. *$A_{1,k}$ and $A_{2,k}$*

Let us look closely at the expression (A·6) for $\ell = 1, 2$. For $\ell = 1$, there is no integral and the result is simply

$$A_{1,k} = \frac{1}{k} \ . \tag{A·7}$$

For $\ell = 2$, the original four-variable integral is reduced to a two-variable integral

$$A_{2,k} = \frac{1}{k} \int d\lambda d\widetilde{\lambda} \frac{e^{2\pi k i \lambda \widetilde{\lambda}}}{\cosh^2 \pi \lambda \cosh^2 \pi \widetilde{\lambda}} \ . \tag{A·8}$$

The $\widetilde{\lambda}$-integral can be done by closing the contour in the upper half plane when $\lambda > 0$, or the lower half plane when $\lambda < 0$, and the result turns out to be independent of the sign of λ

$$A_{2,k} = \int d\lambda \frac{1}{\cosh^2 \pi \lambda} \frac{2\lambda}{\sinh \pi k \lambda} \ . \tag{A·9}$$

Using the relation $\frac{1}{\cosh^2 \pi \lambda} = 1 - \tanh^2 \pi \lambda$, (A·9) can be further rewritten as

$$A_{2,k} = \int d\lambda \frac{2\lambda}{\sinh \pi k \lambda} - \int d\lambda \frac{2\lambda}{\sinh \pi k \lambda} \tanh^2 \pi \lambda$$
$$= \frac{1}{k^2} - \int d\lambda \frac{2\lambda}{\sinh \pi k \lambda} \tanh^2 \pi \lambda \ . \tag{A·10}$$

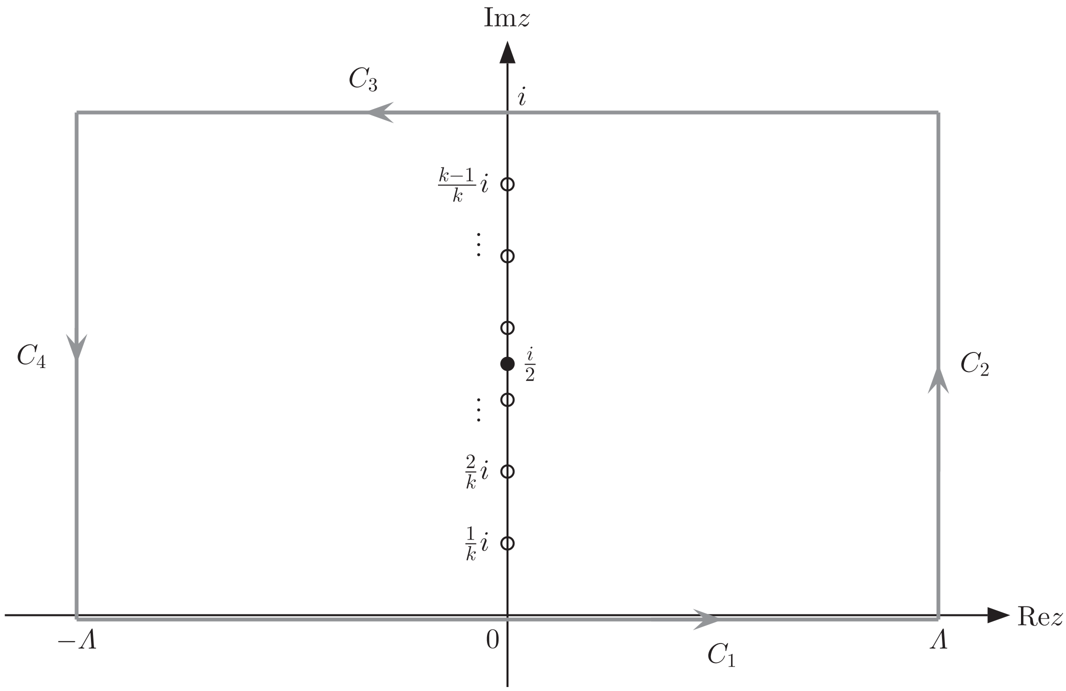

Fig. 1. This is the contour $C = C_1 + C_2 + C_3 + C_4$ used in Appendix B. C_1 and C_3 are the horizontal lines at Im$z = 0$ and Im$z = 1$, respectively. C_2 and C_4 are the vertical segments at $|\text{Re}z| = \Lambda$, and we will take the limit $\Lambda \to \infty$ at the end of computation.

The partition function $Z_{N,k}$ is given by a combination of $A_{\ell,k}$ (2·8). For the $N = 2$ case, we find

$$Z_{2,k} = \frac{1}{2}\left[(A_{1,k})^2 - A_{2,k}\right] = \int_{-\infty}^{\infty} d\lambda \frac{\lambda}{\sinh \pi k\lambda} \tanh^2 \pi\lambda \ . \qquad \text{(A·11)}$$

Note that from (A·3) the variable λ corresponds to $\sigma_1 - \widetilde{\sigma}_1$.

Appendix B
—— *Evaluation of $Z_{2,k}$* ——

In this appendix, we evaluate the partition function of the $U(2)_k \times U(2)_{-k}$ ABJM theory found in Appendix A (A·11)

$$Z_{2,k} = \int_{-\infty}^{\infty} d\lambda \frac{\lambda}{\sinh \pi k\lambda} \frac{\sinh^2 \pi\lambda}{\cosh^2 \pi\lambda} \ . \qquad \text{(B·1)}$$

In order to evaluate this integral, we consider a contour integral of some holomorphic functions along the contour $C = C_1 + C_2 + C_3 + C_4$ depicted in Fig. 1. We will specify the relevant holomorphic functions shortly, which are closely related to the integrand of (B·1). It turns out that there are two types of poles inside C: the first type is the poles of $\frac{1}{\sinh \pi k z}$, and the second type is the pole of $\frac{1}{\cosh^2 \pi z}$. We call those poles the "sh-type" poles and the "ch-type" pole, respectively. Namely,

$$\begin{aligned} &\text{sh-type poles} : \sinh \pi k z = 0 \quad \to z = \frac{s}{k} i \ , \quad (s = 1, \cdots, k-1) \\ &\text{ch-type pole} : \cosh \pi z = 0 \quad \to z = \frac{i}{2} \ . \end{aligned} \tag{B·2}$$

When k is odd, the pole $z = \frac{i}{2}$ does not appear in the set of sh-type poles. On the other hand, when k is even, $z = \frac{i}{2}$ is also a pole of $\frac{1}{\sinh \pi k z}$. Therefore, we have to analyze the even k case and the odd k case separately.

B.1. *Odd k case*

Let us consider the odd k case first. In order to evaluate the integral (B·1), we introduce the holomorphic function

$$f(z) = \frac{1}{2} \frac{z - \frac{i}{2}}{\sinh \pi k z} \frac{\sinh^2 \pi z}{\cosh^2 \pi z} \ . \tag{B·3}$$

Since $f(z)$ is regular at $z = 0$ and $z = i$, there is no pole on the contour C. Note also that all poles in (B·2) are simple poles of $f(z)$. In particular, $z = \frac{i}{2}$ is a simple pole of $f(z)$ due to the factor of $z - \frac{i}{2}$ in (B·3).

In the contour integral of $f(z)$ along C, the contributions of C_2 and C_4 become zero in the limit of $\Lambda \to \infty$

$$\lim_{\Lambda \to \infty} \int_{C_2} dz f(z) = \lim_{\Lambda \to \infty} \int_{C_4} dz f(z) = 0 \ . \tag{B·4}$$

As for the integral along C_1 and C_3, one can easily see that the limit $\Lambda \to \infty$ exists and leads to a finite result. Hence, in what follows we will not indicate the limit $\Lambda \to \infty$ explicitly and we will only write the result of $\Lambda = \infty$. The integral along C_1 is

$$\begin{aligned} \int_{C_1} dz f(z) &= \int_{-\infty}^{\infty} d\lambda f(\lambda) = \int_{-\infty}^{\infty} d\lambda \frac{1}{2} \frac{\lambda - \frac{i}{2}}{\sinh \pi k \lambda} \frac{\sinh^2 \pi \lambda}{\cosh^2 \pi \lambda} \\ &= \int_{-\infty}^{\infty} d\lambda \frac{1}{2} \frac{\lambda}{\sinh \pi k \lambda} \frac{\sinh^2 \pi \lambda}{\cosh^2 \pi \lambda} \ . \end{aligned} \tag{B·5}$$

Here the term proportional to $-\frac{i}{2}$ vanishes, because the integrand of that term is an odd function of λ. Therefore, we find

$$\int_{C_1} dz f(z) = \frac{1}{2} Z_{2,k} \ . \tag{B·6}$$

For the integral along C_3, we parametrize z as

$$z = i - \lambda \ . \quad (\lambda \in \mathbb{R}, -\infty < \lambda < \infty) \tag{B·7}$$

Using the following property of the function $f(z)$

$$f(i - \lambda) = -f(\lambda) \ , \tag{B·8}$$

we find

$$\int_{C_3} dz f(z) = -\int_{-\infty}^{\infty} d\lambda f(i-\lambda) = \int_{-\infty}^{\infty} d\lambda f(\lambda) = \frac{1}{2} Z_{2,k} \ . \tag{B·9}$$

Combining (B·4), (B·6) and (B·9), we find that the partition function $Z_{2,k}$ is equal to the integral $\oint_C dz f(z)$

$$\oint_C dz f(z) = \sum_{a=1}^{4} \int_{C_a} dz f(z) = Z_{2,k} \ . \tag{B·10}$$

On the other hand, by the Cauchy residue theorem this integral $\oint_C dz f(z)$ can be written as a sum of residues of the poles inside C

$$\oint_C dz f(z) = 2\pi i \mathrm{Res}_{z=\frac{i}{2}} f(z) + 2\pi i \sum_{s=1}^{k-1} \mathrm{Res}_{z=\frac{s}{k}i} f(z) \ . \tag{B·11}$$

Putting everything together, we arrive at our final result

$$Z_{2,k} = Z_{2,k}^{(\mathrm{sh})} + Z_{2,k}^{(\mathrm{ch})} \ , \tag{B·12}$$

where

$$\begin{aligned} Z_{2,k}^{(\mathrm{sh})} &= 2\pi i \sum_{s=1}^{k-1} \mathrm{Res}_{z=\frac{s}{k}i} f(z) = \frac{1}{k} \sum_{s=1}^{k-1} (-1)^{s-1} \left(\frac{1}{2} - \frac{s}{k} \right) \tan^2 \frac{\pi s}{k} \ , \\ Z_{2,k}^{(\mathrm{ch})} &= 2\pi i \mathrm{Res}_{z=\frac{i}{2}} f(z) = \frac{(-1)^{\frac{k-1}{2}}}{\pi} \ . \end{aligned} \tag{B·13}$$

In the above expression of $Z_{2,k}^{(\mathrm{sh})}$, using the symmetry under $s \to k-s$, one can show that the sum over the latter half of $s \in [\frac{k+1}{2}, k-1]$ is the same as the sum over the first half of $s \in [1, \frac{k-1}{2}]$. Therefore, the sh-type part can be written as the twice of the sum over $s \in [1, \frac{k-1}{2}]$

$$Z_{2,k}^{(\mathrm{sh})} = \frac{2}{k} \sum_{s=1}^{\frac{k-1}{2}} (-1)^{s-1} \left(\frac{1}{2} - \frac{s}{k} \right) \tan^2 \frac{\pi s}{k} \ . \tag{B·14}$$

B.2. *Even k case*

Next we consider the even k case. When k is even, $z = \frac{i}{2}$ is a triple zero of the function $\sinh \pi k z \cosh^2 \pi z$ which appears in the denominator of the integral (B·1). Therefore, in order to make $z = \frac{i}{2}$ a simple pole, we consider a function $h(z)$ with a factor $(z - \frac{i}{2})^2$

$$h(z) = \frac{i}{2} \frac{\left(z - \frac{i}{2}\right)^2}{\sinh \pi k z} \frac{\sinh^2 \pi z}{\cosh^2 \pi z} \ . \tag{B·15}$$

Let us consider the integral of $h(z)$ along the contour C in Fig. 1. As in the previous subsection, we can see that the contributions from the vertical segments C_2 and C_4 vanish

$$\lim_{\Lambda \to \infty} \int_{C_2} dz\, h(z) = \lim_{\Lambda \to \infty} \int_{C_4} dz\, h(z) = 0 \ . \tag{B·16}$$

For the integral along C_1, from the parity of the integrand under $\lambda \to -\lambda$, only the term linear in λ survives

$$\int_{C_1} dz\, h(z) = \int_{-\infty}^{\infty} d\lambda\, h(\lambda) = \int_{-\infty}^{\infty} d\lambda \frac{1}{2} \frac{i\lambda^2 + \lambda - \frac{i}{4}}{\sinh \pi k\lambda} \cdot \frac{\sinh^2 \pi\lambda}{\cosh^2 \pi\lambda} = \frac{1}{2} Z_{2,k} \ . \tag{B·17}$$

For the integral along C_3, using the property

$$h(i - \lambda) = -h(\lambda) \tag{B·18}$$

we find

$$\int_{C_3} dz\, h(z) = -\int_{-\infty}^{\infty} d\lambda\, h(i-\lambda) = \int_{-\infty}^{\infty} d\lambda\, h(\lambda) = \frac{1}{2} Z_{2,k} \ . \tag{B·19}$$

Therefore, the integral $\oint_C dz\, h(z)$ is equal to the partition function $Z_{2,k}$

$$\oint_C dz\, h(z) = \sum_{a=1}^{4} \int_{C_a} dz\, h(z) = Z_{2,k} \ . \tag{B·20}$$

By the Cauchy residue theorem, $Z_{2,k}$ is written as a sum of residues of the poles inside C

$$Z_{2,k} = \oint_C dz\, h(z) = 2\pi i \sum_{s=1\ (s\neq \frac{k}{2})}^{k-1} \mathrm{Res}_{z=\frac{s}{k}i} h(z) + 2\pi i \mathrm{Res}_{z=\frac{i}{2}} h(z) \ , \tag{B·21}$$

where

$$\begin{aligned} 2\pi i \sum_{s=1\ (s\neq \frac{k}{2})}^{k-1} \mathrm{Res}_{z=\frac{s}{k}i} h(z) &= \frac{1}{k} \sum_{s=1\ (s\neq \frac{k}{2})}^{k-1} (-1)^{s-1} \left(\frac{1}{2} - \frac{s}{k}\right)^2 \tan^2 \frac{\pi s}{k} \ , \\ 2\pi i \mathrm{Res}_{z=\frac{i}{2}} h(z) &= \frac{(-1)^{\frac{k}{2}-1}}{k\pi^2} \ . \end{aligned} \tag{B·22}$$

The residue of the pole $z = \frac{i}{2}$ can be included in the sum of s as the $s = \frac{k}{2}$ term, with the understanding of taking the limit

$$\lim_{s\to\frac{k}{2}} \frac{1}{k} (-1)^{s-1} \left(\frac{1}{2} - \frac{s}{k}\right)^2 \tan^2 \frac{\pi s}{k} = \frac{(-1)^{\frac{k}{2}-1}}{k\pi^2} \ . \tag{B·23}$$

Since this term scales as k^{-1}, it seems natural to identify this term as a part of sh-type contribution. Therefore, one can think that the partition function for even k consists solely of the sh-type part

$$Z_{2,k} = \frac{1}{k} \sum_{s=1}^{k-1} (-1)^{s-1} \left(\frac{1}{2} - \frac{s}{k}\right)^2 \tan^2 \frac{\pi s}{k} \ . \tag{B·24}$$

As in the case of odd k, the sum over s can be reduced to the half range by using the symmetry under $s \to k - s$

$$Z_{2,k} = \frac{2}{k} \sum_{s=1}^{\frac{k}{2}-1} (-1)^{s-1} \left(\frac{1}{2} - \frac{s}{k}\right)^2 \tan^2 \frac{\pi s}{k} + \frac{(-1)^{\frac{k}{2}-1}}{k\pi^2} \ . \tag{B·25}$$

B.3. *Some examples of $Z_{2,k}$ for low k's*

To see the behavior of the partition function of $U(2)_k \times U(2)_{-k}$ ABJM theory, here we list the values of $Z_{2,k}$ from $k=1$ to $k=8$

$$
\begin{aligned}
Z_{2,1} &= \frac{1}{\pi}\,, & Z_{2,2} &= \frac{1}{2\pi^2}\,,\\
Z_{2,3} &= \frac{1}{3} - \frac{1}{\pi}\,, & Z_{2,4} &= \frac{1}{32} - \frac{1}{4\pi^2}\,,\\
Z_{2,5} &= \frac{10-8\sqrt{5}}{25} + \frac{1}{\pi}\,, & Z_{2,6} &= -\frac{5}{324} + \frac{1}{6\pi^2}\,,\\
Z_{2,7} &= \frac{5\tan^2\frac{\pi}{7} - 3\tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}}{49} - \frac{1}{\pi}\,, & Z_{2,8} &= \frac{13-8\sqrt{2}}{128} - \frac{1}{8\pi^2}\,.
\end{aligned}
$$

It is curious to observe that the orbifold part of $Z_{2,5}$ and $Z_{2,8}$ are not rational numbers, and $Z_{2,7}$ cannot be written in a simple form as a combination of the elementary functions of $k(=7)$, such as k^α with some power α. It would be nice to find a closed form expression of $Z_{2,k}$ as a function of k.

References

1) O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, J. High Energy Phys. **10** (2008), 091, arXiv:0806.1218.
2) A. Kapustin, B. Willett and I. Yaakov, J. High Energy Phys. **03** (2010), 089, arXiv:0909.4559.
3) N. Drukker, M. Marino and P. Putrov, Commun. Math. Phys. **306** (2011), 511, arXiv:1007.3837.
4) N. Drukker, M. Marino and P. Putrov, arXiv:1103.4844.
5) H. Fuji, S. Hirano and S. Moriyama, J. High Energy Phys. **08** (2011), 001, arXiv:1106.4631.
6) R. C. Santamaria, M. Marino and P. Putrov, arXiv:1011.6281.
7) C. P. Herzog, I. R. Klebanov, S. S. Pufu and T. Tesileanu, Phys. Rev. D **83** (2011), 046001, arXiv:1011.5487.
8) G. W. Moore, N. Nekrasov and S. Shatashvili, Commun. Math. Phys. **209** (2000), 77, hep-th/9803265.
9) V. A. Kazakov, I. K. Kostov and N. A. Nekrasov, Nucl. Phys. B **557** (1999), 413, hep-th/9810035.
10) V. Pestun, arXiv:0712.2824.
11) A. Kapustin, B. Willett and I. Yaakov, J. High Energy Phys. **10** (2010), 013, arXiv:1003.5694.
12) D. L. Jafferis, arXiv:1012.3210.
13) N. Hama, K. Hosomichi and S. Lee, J. High Energy Phys. **03** (2011), 127, arXiv:1012.3512.
14) M. Marino, arXiv:1104.0783.
15) T. Suyama and A. Tsuchiya, Prog. Theor. Phys. **99** (1998), 321, hep-th/9711073.
16) M. B. Green and M. Gutperle, Phys. Lett. B **398** (1997), 69, hep-th/9612127.
17) O. Aharony, O. Bergman and D. L. Jafferis, J. High Energy Phys. **11** (2008), 043, arXiv:0807.4924.