PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 9, September 2011, Pages 3191–3193
S 0002-9939(2011)11026-8
Article electronically published on March 24, 2011

FAITHFUL REPRESENTATIONS OF ASSOCIATION SCHEMES

AKIHIDE HANAKI

(Communicated by Jim Haglund)

ABSTRACT. Every character of an association scheme can be considered as a faithful character of some quotient scheme. Also we will show that a faithful character of an association scheme determines a thin closed subset which is cyclic as a finite group.

1. INTRODUCTION

Let G be a finite group and let χ be a character of G. We can consider that χ is a faithful character of $G/\mathrm{Ker}(\chi)$. If G has a faithful irreducible character, then the center of G is cyclic. These are well known facts in group representation theory. We will generalize them to characters of association schemes.

Every character of an association scheme can be considered as a faithful character of some quotient scheme (Theorem 2.1). Also we will show that a faithful character of an association scheme determines a thin closed subset which is cyclic as a finite group (Theorem 3.1).

Let (X, S) be an association scheme in the sense of [7] or [3]. We will denote the valency of $s \in S$ by n_s. Let T be a closed subset of S. Put $e_T = {n_T}^{-1} \sum_{t \in T} \sigma_t$. Then e_T is an idempotent of $\mathbb{C}S$. It is known that $\mathbb{C}(S/\!\!/T) \cong e_T \mathbb{C}S e_T$ as algebras by $\sigma_{s^T} \mapsto (n_{s^T}/n_s) e_T \sigma_s e_T$ (see [4]).

We denote the identity matrix by E.

2. FAITHFUL REPRESENTATIONS

Let (X, S) be an association scheme, and let $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ be a representation of (X, S) affording a character φ. Define

$$K(\Phi) = \{s \in S \mid \Phi(\sigma_s) = n_s E\}$$

and

$$K(\varphi) = \{s \in S \mid \varphi(\sigma_s) = n_s \varphi(1)\}.$$

It is known that $K(\Phi) = K(\varphi)$ (see [1, section 3]). Note that $K(\Phi)$ is closed but not necessarily normal. We say that Φ or φ is *faithful* if $K(\Phi) = \{1\}$.

If $K(\Phi)$ is a normal closed subset of S, then there is a natural algebra epimorphism $\pi : \mathbb{C}S \to \mathbb{C}(S/\!\!/K(\Phi))$ and Φ can be considered as a representation of $S/\!\!/K(\Phi)$. But $K(\Phi)$ is not necessarily normal, and the natural map $\pi : \mathbb{C}S \to \mathbb{C}(S/\!\!/K(\Phi))$ is not an algebra homomorphism in general (see [6]).

Received by the editors August 19, 2010.
2010 *Mathematics Subject Classification.* Primary 05E30.
Key words and phrases. Association scheme, representation, character, faithful.

The next theorem is the main result in this section.

Theorem 2.1. *Let (X,S) be an association scheme, and let $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ be a representation of (X,S). Suppose that T is a closed subset contained in $K(\Phi)$. Then we can define a representation $\Phi' : \mathbb{C}(S/\!\!/T) \to M_n(\mathbb{C})$ by $\Phi'(\sigma_{s^T}) = (n_{s^T}/n_s)\Phi(\sigma_s)$. Moreover, Φ' is faithful if $T = K(\Phi)$.*

Proof. If Ψ is an irreducible component of Φ, then $K(\Phi) \subseteq K(\Psi)$. Thus, without loss of generality, we may suppose that Φ is irreducible.

Let φ be the character afforded by Φ. Let e_φ be the primitive central idempotent of $\mathbb{C}S$ corresponding to φ. By the assumption on T, we have $e_\varphi e_T = e_\varphi = e_T e_\varphi$.

We will show that Φ' is well-defined. Suppose $s^T = u^T$. We have

$$\begin{aligned}\frac{1}{n_s}\Phi(\sigma_s) &= \Phi(e_\varphi \frac{1}{n_s}\sigma_s e_\varphi) = \Phi(e_\varphi e_T \frac{1}{n_s}\sigma_s e_T e_\varphi)\\ &= \Phi(e_\varphi e_T \frac{1}{n_u}\sigma_u e_T e_\varphi) = \Phi(e_\varphi \frac{1}{n_u}\sigma_u e_\varphi) = \frac{1}{n_u}\Phi(\sigma_u).\end{aligned}$$

This means that Φ' is well-defined.

We show that Φ' is an algebra homomorphism. We use the isomorphism $\mathbb{C}(S/\!\!/T) \cong e_T\mathbb{C}Se_T$ and identify them. Then $\Phi'(e_T\sigma_s e_T) = \Phi(\sigma_s)$. We have

$$\begin{aligned}\Phi'((e_T\sigma_s e_T)(e_T\sigma_u e_T)) &= \Phi(\sigma_s e_T \sigma_u) = \Phi(\sigma_s e_T)\Phi(\sigma_u)\\ &= \Phi(\sigma_s e_T e_\varphi)\Phi(\sigma_u) = \Phi(\sigma_s e_\varphi)\Phi(\sigma_u)\\ &= \Phi(\sigma_s)\Phi(\sigma_u) = \Phi'(e_T\sigma_s e_T)\Phi'(e_T\sigma_u e_T).\end{aligned}$$

Finally, we will show that Φ' is faithful if $T = K(\Phi)$. Suppose $s^T \in K(\Phi')$. Then $E = {n_{s^T}}^{-1}\Phi'(\sigma_{s^T}) = {n_s}^{-1}\Phi(\sigma_s)$. So $s \in K(\Phi)$ and $s^T = 1^T$. Now Φ' is faithful. □

Corollary 2.2. *Let (X,S) be an association scheme, and let φ be a character of (X,S). Suppose that T is a closed subset contained in $K(\varphi)$. Then ${n_u}^{-1}\varphi(\sigma_u) = {n_s}^{-1}\varphi(\sigma_s)$ for any $u \in TsT$.*

Proof. This is obtained by the fact that Φ' in Theorem 2.1 is well-defined. □

3. Faithful representations and closed subsets

Let (X,S) be an association scheme, and let $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ be a representation of (X,S) affording a character φ. Define

$$Z(\varphi) = \{s \in S \mid |\varphi(\sigma_s)| = n_s\varphi(1)\}.$$

Then $Z(\varphi)$ is a closed subset of S containing $K(\varphi)$ (see [2, Proposition 3.2 and 3.3]). For $s \in S$, $s \in Z(\varphi)$ if and only if $\Phi(\sigma_s) = \varepsilon_s n_s E$ for some root of unity ε_s.

The following theorem is a generalization of [5, Theorem 2.32 (a)].

Theorem 3.1. *Let φ be a faithful character of (X,S). Then $Z(\varphi)$ is thin and cyclic as a finite group.*

In the rest of this section, (X,S) is an association scheme and $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ is a faithful representation of (X,S) affording a character φ. For $u \in Z(\varphi)$ we define a root of unity ε_u by $\Phi(\sigma_u) = \varepsilon_u n_u E$ or equivalently by $\varphi(\sigma_u) = \varepsilon_u n_u \varphi(1)$. We need a lemma.

Lemma 3.2. *If $u, v \in Z(\varphi)$ and $u \neq v$, then $\varepsilon_u \neq \varepsilon_v$. Moreover, $Z(\varphi)$ is thin.*

Proof. For $u, v \in Z(\varphi)$, suppose $\varepsilon_u = \varepsilon_v$. Then $\Phi(\sigma_u)\Phi(\sigma_{v^*}) = n_u n_{v^*} E$. Now $w \in K(\varphi) = \{1\}$ for any $w \in uv^*$. So $uv^* = \{1\}$. This means that $u = v$ and u is thin. □

Proof of Theorem 3.1. If ξ is an irreducible constituent of φ, then $Z(\varphi) \subseteq Z(\xi)$. So we may suppose that φ is irreducible. We consider $Z(\varphi)$ as a finite group. Then, by Lemma 3.2, $\sigma_u \mapsto \varepsilon_u$ is an irreducible faithful character of an abelian group $Z(\varphi)$. So $Z(\varphi)$ is cyclic. □

We remark that if φ is a faithful irreducible character of a finite group G, then $Z(\varphi)$ is just the center of G. But for a character of an association scheme, it is not true in general.

Acknowledgement

The author would like to thank the referee for many helpful comments.

References

1. A. Hanaki, *Characters of association schemes and normal closed subsets*, Graphs Combin. **19** (2003), no. 3, 363–369. MR2007898 (2004h:05132)
2. ______, *Nilpotent schemes and group-like schemes*, J. Combin. Theory Ser. A **115** (2008), no. 2, 226–236. MR2382513 (2008m:05314)
3. ______, *Representations of finite association schemes*, European J. Combin. **30** (2009), no. 6, 1477–1496. MR2535397 (2010g:05395)
4. A. Hanaki and M. Hirasaka, *Theory of Hecke algebras to association schemes*, SUT J. Math. **38** (2002), no. 1, 61–66. MR1919947 (2003g:05134)
5. I. M. Isaacs, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. MR0460423 (57:417)
6. B. Xu, *On morphisms of association schemes*, J. Algebra **322** (2009), no. 4, 1013–1028. MR2537669 (2010i:05358)
7. P.-H. Zieschang, *An algebraic approach to association schemes*, Lecture Notes in Mathematics, vol. 1628, Springer-Verlag, Berlin, 1996. MR1439253 (98h:05185)

Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
E-mail address: hanaki@shinshu-u.ac.jp