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Our main object of study is a certain degree-one cohomology class of the space $\mathcal{K}_3$ of long knots in $\mathbb{R}^3$. We describe this class in terms of graphs and configuration space integrals, showing the vanishing of some anomalous obstructions. To show that this class is not zero, we integrate it over a cycle studied by Gramain. As a corollary, we establish a relation between this class and ($\mathbb{R}$-valued) Casson's knot invariant. These are $\mathbb{R}$-versions of the results which were previously proved by Teiblyum, Turchin and Vassiliev over $\mathbb{Z}/2$ in a different way from ours.

1. Introduction

A *long knot* in $\mathbb{R}^n$ is an embedding $f : \mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ that agrees with the standard inclusion $\iota(t) = (t, 0, \dots, 0)$ outside $[-1, 1]$. We denote by $\mathcal{K}_n$ the space of long knots in $\mathbb{R}^n$ equipped with C^∞-topology.

In [Cattaneo et al. 2002] a cochain map $I : \mathcal{D}^* \to \Omega^*_{DR}(\mathcal{K}_n)$ from a certain *graph complex* $\mathcal{D}^*$ was constructed for $n > 3$. The cocycles of $\mathcal{K}_n$ corresponding to *trivalent graph cocycles* via I generalize an integral expression of finite type invariants for (long) knots in $\mathbb{R}^3$ [Altschuler and Freidel 1997; Bott and Taubes 1994; Kohno 1994; Volić 2007]. In [Sakai 2008] the author found a *nontrivalent* graph cocycle $\Gamma \in \mathcal{D}^*$ and proved that, when $n > 3$ is odd, it gives a nonzero cohomology class $[I(\Gamma)] \in H^{3n-8}_{DR}(\mathcal{K}_n)$. On the other hand, when $n = 3$, some obstructions to I being a cochain map (called *anomalous obstructions*; see for example [Volić 2007, Section 4.6]) may survive, so even the closedness of $I(\Gamma)$ was not clear. However, the obstructions for trivalent graph cocycles X (of "even orders") in fact vanish [Altschuler and Freidel 1997], hence the map I still yields closed zero-forms $I(X)$ of $\mathcal{K}_3$ (they are finite type invariants). This raises our hope

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that all obstructions for any graphs may vanish and hence the map I could be a cochain map even when $n = 3$.

In this paper we will show (in Theorem 2.4) that the obstructions for the non-trivalent graph cocycle Γ mentioned above also vanish, hence the map I yields the first example of a closed one-form $I(\Gamma)$ of $\mathcal{K}_3$. To show that $[I(\Gamma)] \in H^1_{DR}(\mathcal{K}_3)$ is not zero, we will study in part how $I(\Gamma)$ fits into a description of the homotopy type of $\mathcal{K}_3$ given in [Budney 2010; 2007; Budney and Cohen 2009]. It is known that on each component $\mathcal{K}_3(f)$ that contains $f \in \mathcal{K}_3$, there exists a one-cycle G_f called the *Gramain cycle* [Gramain 1977; Budney 2010; Turchin 2006; Vassiliev 2001]. The Kronecker pairing gives an isotopy invariant $V : f \mapsto \langle I(\Gamma),\ G_f \rangle$. We show in Theorem 3.1 that V coincides with *Casson's knot invariant* v_2, which is characterized as the coefficient of z^2 in the Alexander–Conway polynomial. This result will be generalized in Theorem 3.6 for one-cycles obtained by using an action of *little two-cubes operad* on the space $\tilde{\mathcal{K}}_3$ of *framed* long knots [Budney 2007].

Closely related results have appeared in [Turchin 2006; Vassiliev 2001], where the $\mathbb{Z}/2$-reduction of a cocycle v^1_3 of $\mathcal{K}_n$ ($n \geq 3$), appearing in the E_1-term of Vassiliev's spectral sequence [Vassiliev 1992], was studied. A natural quasi-isomorphism $\mathcal{D}^* \to E_0 \otimes \mathbb{R}$ maps our cocycle Γ to v^1_3. In this sense, our results can be seen as "lifts" of those in [Turchin 2006; Vassiliev 2001] to $\mathbb{R}$.

The invariant v_2 can also be interpreted as the linking number of colinearity manifolds [Budney et al. 2005]. Notice that in each formulation (including the one in this paper) the value of v_2 is computed by counting some colinearity pairs on the knot.

2. Construction of a close differential form

Configuration space integral. We review briefly how we can construct (closed) forms of $\mathcal{K}_n$ from graphs. For full details see [Cattaneo et al. 2002; Volić 2007].

Let X be a *graph* in the sense of those references (see Figure 1 for examples). Let v_{i} and v_{f} be the numbers of the *interval vertices* (or *i-vertices* for short; those on the specified oriented line) and the *free* vertices (or *f-vertices*; those which are not interval vertices) of X, respectively. With X we associate a configuration space

$$C_X := \left\{ \begin{array}{l} (f; x_1, \dots, x_{v_{\mathrm{i}}}; x_{v_{\mathrm{i}}+1}, \dots, x_{v_{\mathrm{i}}+v_{\mathrm{f}}}) \\ \in \mathcal{K}_n \times \mathrm{Conf}\,(\mathbb{R}^1, v_{\mathrm{i}}) \times \mathrm{Conf}\,(\mathbb{R}^n, v_{\mathrm{f}}) \end{array} \middle| \begin{array}{l} f(x_i) \neq x_j \text{ for any} \\ 1 \leq i \leq v_{\mathrm{i}} < j \leq v_{\mathrm{i}} + v_{\mathrm{f}} \end{array} \right\},$$

where $\mathrm{Conf}\,(M, k) := M^{\times k} \setminus \bigcup_{1 \leq i < j \leq k} \{x_i = x_j\}$ for a space M.

Let e be the number of the edges of X. Define $\omega_X \in \Omega^{(n-1)e}_{DR}(C_X)$ as the wedge of closed $(n-1)$-forms $\varphi^*_\alpha \mathrm{vol}_{S^{n-1}}$, where $\varphi_\alpha : C_X \to S^{n-1}$ is the *Gauss map*, which assigns a unit vector determined by two points in $\mathbb{R}^n$ corresponding to the vertices adjacent to an edge α of X (for an i-vertex corresponding to $x_i \in \mathbb{R}^1$, we consider the point $f(x_i) \in \mathbb{R}^n$). Here we assume that $\mathrm{vol}_{S^{n-1}}$ is "(anti)symmetric", namely $i^*\mathrm{vol}_{S^{n-1}} = (-1)^n \mathrm{vol}_{S^{n-1}}$ for the antipodal map $i : S^{n-1} \to S^{n-1}$. Then $I(X) \in \Omega_{DR}^{(n-1)e-v_\mathrm{i}-nv_\mathrm{f}}(\mathcal{K}_n)$ is defined by

$$I(X) := (\pi_X)_* \omega_X,$$

the integration along the fiber of the natural fibration $\pi_X : C_X \to \mathcal{K}_n$. This fiber is a subspace of $\mathrm{Conf}(\mathbb{R}^1, v_\mathrm{i}) \times \mathrm{Conf}(\mathbb{R}^n, v_\mathrm{f})$. Such integrals converge, since the fiber can be compactified in such a way that the forms $\varphi_\alpha^* \mathrm{vol}_{S^{n-1}}$ are still well-defined on the compactification [Bott and Taubes 1994, Proposition 1.1]. We extend I linearly onto $\mathcal{D}^*$, a cochain complex spanned by graphs. The differential δ of $\mathcal{D}^*$ is defined as a signed sum of graphs obtained by "contracting" the edges one at a time.

One of the results of [Cattaneo et al. 2002] states that $I : \mathcal{D}^* \to \Omega_{DR}^*(\mathcal{K}_n)$ is a cochain map if $n > 3$. The proof is outlined as follows. By the generalized Stokes theorem, $dI(X) = \pm(\pi_X^\partial)_* \omega_X$, where π_X^∂ is the restriction of π_X to the codimension one strata of the boundary of the (compactified) fiber of π_X. Each codimension one stratum corresponds to a collision of subconfigurations in C_X, or equivalently to $A \subset V(X) \cup \{\infty\}$ (here $V(X)$ is the set of vertices of X) with a consecutiveness property: if two i-vertices p, q are in A, then all the other i-vertices between p and q are in A. Here "$\infty \in A$" means that the points x_l ($l \in A$) escape to infinity. When $\infty \notin A$, the interior $\mathrm{Int}\,\Sigma_A$ of the corresponding stratum Σ_A to A is described by the pullback square

(2-1)
$$\begin{array}{ccccc} & & \mathrm{Int}\,\Sigma_A & \longrightarrow & \hat{B}_A \\ & \swarrow^{\pi_X^{\partial_A}} & \downarrow & & \downarrow \rho_A \\ \mathcal{K}_n & \underset{\pi_{X/X_A}}{\longleftarrow} & C_{X/X_A} & \underset{D_A}{\longrightarrow} & B_A \end{array}$$

Here

- X_A is the maximal subgraph of X with $V(X_A) = A$, and X/X_A is a graph obtained by collapsing the subgraph X_A to a single vertex v_A;
- $B_A = S^{n-1}$ if A contains at least one i-vertex, and $B_A = \{*\}$ otherwise;
- if A consists of i-vertices $i_1, \dots, i_s$ ($s > 0$) and f-vertices $i_{s+1}, \dots, i_{s+t}$, then

$$\hat{B}_A := \left\{ \begin{array}{l} (v; (x_{i_1}, \dots, x_{i_s}; x_{i_{s+1}}, \dots, x_{i_{s+t}})) \\ \in S^{n-1} \times \mathrm{Conf}(\mathbb{R}^1, s) \times \mathrm{Conf}(\mathbb{R}^n, t) \end{array} \middle| \begin{array}{l} x_{i_p} v \neq x_{i_q} \text{ for any} \\ 1 \le p \le s < q \le s+t \end{array} \right\} \Big/ \sim,$$

where $\sim$ is defined by

$$\big(v; (x_{i_1}, \dots, x_{i_s}; x_{i_{s+1}}, \dots, x_{i_{s+t}})\big) \sim \\ \big(v; (a(x_{i_1} + r), \dots, a(x_{i_s} + r); a(x_{i_{s+1}} + rv), \dots, a(x_{i_{s+t}} + rv))\big),$$

for any $a \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}$ (if A consists only of t f-vertices, then

$$\hat{B}_A := \mathrm{Conf}\,(\mathbb{R}^n, t)/(\mathbb{R}^1_{>0} \rtimes \mathbb{R}^n),$$

where $\mathbb{R}^1_{>0} \rtimes \mathbb{R}^n$ acts on $\mathrm{Conf}\,(\mathbb{R}^n, t)$ by scaling and translation);

- ρ_A is the natural projection;
- when A contains at least one i-vertex, $D_A : C_{X/X_A} \to S^{n-1}$ maps $(f; (x_i))$ to $f'(x_{v_A})/|f'(x_{v_A})|$.

We omit the case $\infty \in A$; see [Cattaneo et al. 2002, Appendix].

By properties of fiber integrations and pullbacks, the integration of ω_X along $\mathrm{Int}\,\Sigma_A$ can be written as $(\pi_{X/X_A})_*(\omega_{X/X_A} \wedge D_A^*(\rho_A)_*\hat{\omega}_{X_A})$, where $\hat{\omega}_{X_A} \in \Omega^*_{DR}(\hat{B}_A)$ is defined similarly to $\omega_X \in \Omega^*_{DR}(C_X)$.

The stratum Σ_A is called *principal* if $|A| = 2$, *hidden* if $|A| \geq 3$, and *infinity* if $\infty \in A$. Since two-point collisions correspond to contractions of edges, we have $dI(X) = I(\delta X)$ modulo the integrations along hidden and infinity faces. When $n > 3$, the hidden/infinity contributions turn out to be zero; in fact $(\rho_A)_*\hat{\omega}_{X_A} = 0$ if $n > 3$ and if A is not principal; see [Cattaneo et al. 2002, Appendix] or the next example. This proves that the map I is a cochain map if $n > 3$.

Example 2.1. Here we show one example of vanishing of an integration along a hidden face Σ_A. Let X be the seventh graph in Figure 1 and $A := \{1, 4, 5\}$. Then in (2-1), $B_A = S^{n-1}$ since A contains an i-vertex 1, and

$$\hat{B}_A = \{(v; x_1; x_4, x_5) \in S^{n-1} \times \mathbb{R}^1 \times \mathrm{Conf}\,(\mathbb{R}^n, 2) \mid x_1 v \neq x_4, x_5\}/\sim,$$

where $(v; x_1; x_4, x_5) \sim (v; a(x_1 + r); a(x_4 + rv), a(x_5 + rv))$ for any $a > 0$ and $r \in \mathbb{R}^1$. The subgraph X_A consists of three vertices $1, 4, 5$ and three edges 14, 15 and 45. The open face $\mathrm{Int}\,\Sigma_A$, where three points $f(x_1)$, x_4 and x_5 collide with each other, is a hidden face and is described by the square (2-1). Then the integration of ω_X along $\mathrm{Int}\,\Sigma_A$ is $(\pi_{X/X_A})_*(\omega_{X/X_A} \wedge D_A^*(\rho_A)_*\hat{\omega}_{X_A})$, where

$$\hat{\omega}_{X_A} = \varphi_{14}^*\mathrm{vol}_{S^{n-1}} \wedge \varphi_{15}^*\mathrm{vol}_{S^{n-1}} \wedge \varphi_{45}^*\mathrm{vol}_{S^{n-1}} \in \Omega^{3(n-1)}_{DR}(\hat{B}_A),$$

$$\varphi_{1j} := \frac{x_j - x_1 v}{|x_j - x_1 v|} \quad (j = 4, 5), \qquad \varphi_{45} := \frac{x_5 - x_4}{|x_5 - x_4|}.$$

In this case we can prove that $(\rho_A)_*\hat{\omega}_{X_A} = 0$, hence the integration of ω_X along $\mathrm{Int}\,\Sigma_A$ vanishes. Indeed a fiberwise involution $\chi : \hat{B}_A \to \hat{B}_A$ defined by

$$\chi(v; x_1; x_4, x_5) := (v; x_1; 2x_1 v - x_4, 2x_1 v - x_5)$$

preserves the orientation of the fiber but $\chi^*\hat{\omega}_{X_A} = -\hat{\omega}_{X_A}$ (here we use that $\mathrm{vol}_{S^{n-1}}$ is antisymmetric), hence we have $(\rho_A)_*\hat{\omega}_{X_A} = -(\rho_A)_*\hat{\omega}_{X_A}$.

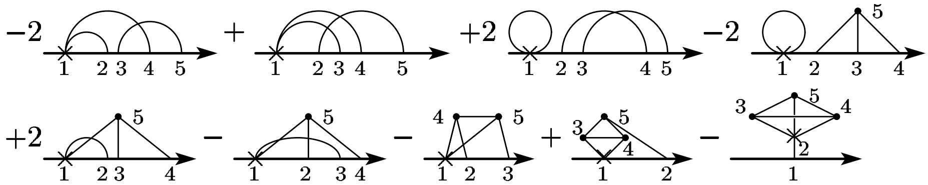


Figure 1. A graph cocycle Γ.

Nontrivalent cocycle. It is shown in [Cattaneo et al. 2002] that, when $n > 3$, the induced map I on cohomology restricted to the space of trivalent graph cocycles is injective. In [Sakai 2008], the author gave the first example of a nontrivalent graph cocycle Γ (Figure 1) which also gives a nonzero class $[I(\Gamma)] \in H^{3n-8}_{DR}(\mathcal{K}_n)$ when $n > 3$ is odd.

In Figure 1, nontrivalent vertices and trivalent f-vertices are marked by $\times$ and $\bullet$, respectively, and other crossings are not vertices. Here we say an i-vertex v is trivalent if there is exactly one edge emanating from v other than the specified oriented line. Each edge ij $(i < j)$ is oriented so that i is the initial vertex.

Remark 2.2. An analogous nontrivalent graph cocycle for the space of embeddings $S^1 \hookrightarrow \mathbb{R}^n$ for even $n \geq 4$ can be found in [Longoni 2004].

If $n = 3$, integrations along some hidden faces (called *anomalous contributions*) might survive, so the map I might fail to be a cochain map. However, nonzero anomalous contributions arise from limited hidden faces.

Theorem 2.3. *Let X be a graph and $A \subset V(X) \cup \{\infty\}$ be such that Σ_A is not principal. When $n = 3$, the integration of ω_X along Σ_A can be nonzero only if the subgraph X_A is trivalent.*

Our main theorem is proved by using Theorem 2.3.

Theorem 2.4. *$I(\Gamma) \in \Omega^1_{DR}(\mathcal{K}_3)$ is a closed form.*

Proof. We call the nine graphs in Figure 1 $\Gamma_1, \dots, \Gamma_9$, respectively. The graphs Γ_i, $i \neq 3, 4, 9$, do not contain trivalent subgraphs X_A satisfying the *consecutive property*; see the paragraph just before (2-1). So $dI(\Gamma_i) = I(d\Gamma_i)$ for $i \neq 3, 4, 9$ by Theorem 2.3.

Possibly the integration of ω_{Γ_i} $(i = 3, 4, 9)$ along Σ_A $(A := \{2, \dots, 5\})$ might survive, since the corresponding subgraph X_A is trivalent. However, we can prove $(\rho_A)_* \hat{\omega}_{X_A} = 0$ (and hence $dI(\Gamma_i) = I(d\Gamma_i)$) as follows: $(\rho_A)_* \hat{\omega}_{X_A} = 0$ for Γ_3, because there is a fiberwise free action of $\mathbb{R}_{>0}$ on $\hat{B}_A$ given by translations of x_2 and x_4 [Volić 2007, Proposition 4.1] which preserves $\hat{\omega}_{X_A}$. Thus $(\rho_A)_* \hat{\omega}_{X_A} = 0$ by dimensional reason. The proof for Γ_4 has appeared in [Bott and Taubes 1994, page 5271]; $\hat{\omega}_{X_A} = 0$ on $\hat{B}_A$ since the image of the Gauss map $\varphi : B_A \to (S^2)^3$ corresponding to three edges of X_A is of positive codimension. As for Γ_9, $(\rho_A)_* \hat{\omega}_{X_A} = 0$ follows from $\deg(\rho_A)_* \hat{\omega}_{X_A} = 4$ which exceeds $\dim B_A$ (in fact $B_A = \{*\}$ in this case). □

Proof of Theorem 2.3. Let A be a subset of $V(X)$ with $|A| \geq 3$ or $\infty \in A$, and X_A is nontrivalent. We must show the vanishing of the integrations along the nonprincipal face Σ_A of the fiber of $C_X \to \mathcal{K}_3$. To do this it is enough to show $(\rho_A)_* \hat{\omega}_{X_A} = 0$. By dimensional arguments [Cattaneo et al. 2002, (A.2)] the contributions of infinite faces vanish. So below we consider the hidden faces Σ_A with $|A| \geq 3$.

If X_A has a vertex of valence ≤ 2, then $(\rho_A)_* \hat{\omega}_{X_A} = 0$ is proved by dimensional arguments or existence of a fiberwise symmetry of B_A which reverses the orientation of the fiber of $\rho_A : \hat{B}_A \to B_A$ but preserves the integrand $\hat{\omega}_{X_A}$ (like χ from Example 2.1, see also [Cattaneo et al. 2002, Lemmas A.7–A.9]).

Next, consider the case that there is a vertex of X_A of valence ≥ 4. Let e, s and t be the numbers of the edges, the i-vertices and the f-vertices of X_A, respectively. Then $\deg \hat{\omega}_{X_A} = 2e$ and the dimension of the fiber of ρ_A is $s + 3t - k$, where $k = 2$ or 4 according to whether $s > 0$ or $s = 0$ [Cattaneo et al. 2002, (A.1)]. Thus $(\rho_A)_* \hat{\omega}_{X_A} \in \Omega^*_{DR}(B_A)$ is of degree $2e - s - 3t + k$. It is not difficult to see $2e - s - 3t > 0$ because at least one vertex of X_A is of valence ≥ 4. Hence $\deg(\rho_A)_* \hat{\omega}_{X_A}$ exceeds $\dim B_A$ ($= 0$ or 2) and hence $(\rho_A)_* \hat{\omega}_{X_A} = 0$.

Thus only the integrations along Σ_A with X_A trivalent can survive. □

Remark 2.5. Every finite type invariant v for long knots in $\mathbb{R}^3$ can be written as a sum of $I(\Gamma_v)$ (Γ_v is a trivalent graph cocycle) and some "correction terms" which kill the contributions of hidden faces corresponding to trivalent subgraphs [Altschuler and Freidel 1997; Bott and Taubes 1994; Kohno 1994; Volić 2007]. So by Theorem 2.3 the problem whether $I : \mathcal{D}^* \to \Omega^*_{DR}(\mathcal{K}_3)$ is a cochain map or not is equivalent to the problem whether one can eliminate all the correction terms from integral expressions of finite type invariants.

3. Evaluation on some cycles

Here we will show that $[I(\Gamma)] \in H^1_{DR}(\mathcal{K}_3)$ restricted to some components of $\mathcal{K}_3$ is not zero.

We introduce two assumptions to simplify computations.

Assumption 1. The support of (antisymmetric) vol_{S^2} is contained in a sufficiently small neighborhood of the poles $(0, 0, \pm 1)$ as in [Sakai 2008]. So only the configurations with the images of the Gauss maps lying in a neighborhood of $(0, 0, \pm 1)$ can nontrivially contribute to various integrals below. Presumably $[I(\Gamma)] \in H^1_{DR}(\mathcal{K}_3)$ may be independent of choices of vol_{S^2} [Cattaneo et al. 2002, Proposition 4.5].

Assumption 2. Every long knot in $\mathbb{R}^3$ is contained in xy-plane except for over-arc of each crossing, and each over-arc is in $\{0 \le z \le h\}$ for a sufficiently small $h > 0$ so that the projection onto xy-plane is a regular diagram of the long knot.

The Gramain cycle. For any $f \in \mathcal{K}_3$, we denote by $\mathcal{K}_3(f)$ the component of $\mathcal{K}_3$ which contains f. Regarding $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and fixing f, we define the map $G_f : S^1 \to \mathcal{K}_3(f)$, called the *Gramain cycle*, by $G_f(s)(t) := R(s)f(t)$, where $R(s) \in SO(3)$ is the rotation by the angle s fixing the "long axis" (the x-axis). G_f generates an infinite cyclic subgroup of $\pi_1(\mathcal{K}_3(f))$ if f is nontrivial [Gramain 1977]. The homology class $[G_f] \in H_1(\mathcal{K}_3(f))$ is independent of the choice of f in the connected component; if $f_t \in \mathcal{K}_3$ $(0 \le t \le 1)$ is an isotopy connecting f_0 and f_1, then $G_{f_t} : [0, 1] \times S^1 \to \mathcal{K}_3$ gives a homotopy between G_{f_0} and G_{f_1}. Therefore the Kronecker pairing gives an isotopy invariant $V(f) := \langle I(\Gamma), G_f\rangle$ for long knots.

Theorem 3.1. *The invariant V is equal to Casson's knot invariant v_2.*

Corollary 3.2. $[I(\Gamma)|_{\mathcal{K}_3(f)}] \in H^1_{DR}(\mathcal{K}_3(f))$ *is not zero if $v_2(f) \neq 0$.* □

We will prove two statements that characterize Casson's knot invariant: V is of finite type of order two and $V(3_1) = 1$, where 3_1 is the long trefoil knot. To do this, we will represent G_f using a *Browder operation*, as in [Sakai 2008].

Little cubes action. Let $\tilde{\mathcal{K}}_n$ be the space of *framed* long knots in $\mathbb{R}^n$ (embeddings $\tilde{f} : \mathbb{R}^1 \times D^{n-1} \hookrightarrow \mathbb{R}^n$ that are standard outside $[-1, 1] \times D^{n-1}$). There is a homotopy equivalence $\Phi : \tilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$ [Budney 2007] that maps $\tilde{f}$ to the pair $(\tilde{f}|_{\mathbb{R}^1\times\{(0,0)\}}, \operatorname{fr}\tilde{f})$, where the *framing number* $\operatorname{fr}\tilde{f}$ is defined as the linking number of $\tilde{f}|_{\mathbb{R}^1\times\{(0,0)\}}$ with $\tilde{f}|_{\mathbb{R}^1\times\{(1,0)\}}$. Since $\operatorname{fr}\tilde{f}$ is additive under the connected sum, Φ is a homotopy equivalence of H-spaces. In general, $\tilde{\mathcal{K}}_n \simeq \mathcal{K}_n \times \Omega\, SO(n-1)$ as H-spaces, where Ω stands for the based loop space functor.

In [Budney 2007] an action of the *little two-cubes operad* on the space $\tilde{\mathcal{K}}_n$ was defined. Its second stage gives a map $S^1 \times (\tilde{\mathcal{K}}_n)^2 \to \tilde{\mathcal{K}}_n$ up to homotopy, which is given as "shrinking one knot f and sliding it along another knot g by using the framing, and repeating the same procedure with f and g exchanged" [Budney 2007, Figure 2]. Fixing a generator of $H_1(S^1)$, we obtain the *Browder operation* $\lambda : H_p(\tilde{\mathcal{K}}_n) \otimes H_q(\tilde{\mathcal{K}}_n) \to H_{p+q+1}(\tilde{\mathcal{K}}_n)$, which is a graded Lie bracket satisfying the Leibniz rule with respect to the product induced by the connected sum. The author proved in [Sakai 2008] that $\langle I(\Gamma), r_*\lambda(e, v)\rangle = 1$ when $n > 3$ is odd, where $r : \tilde{\mathcal{K}}_n \to \mathcal{K}_n$ is the forgetting map, $e \in H_{n-3}(\tilde{\mathcal{K}}_n)$ comes from the space of framings, and $v \in H_{2(n-3)}(\tilde{\mathcal{K}}_n)$ is the first nonzero class of $\mathcal{K}_n$ represented by a map $(S^{n-3})^{\times 2} \to \mathcal{K}_n$ (see below).

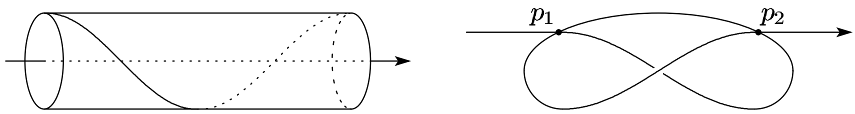


Figure 2. The cycles e and $v = v(T)$.

The case $n = 3$. In [Sakai 2008] the assumption $n > 3$ was used only to deduce the closedness of $I(\Gamma)$ from the results of Cattaneo et al. [2002]. The cycles e and v are defined even when $n = 3$:

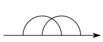
- Under the homotopy equivalence $\tilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$, the zero-cycle e is given by $(\iota, 1)$ where ι is the trivial long knot ($\iota(t) = (t, 0, 0)$ for any $t \in \mathbb{R}^1$).
- The zero-cycle $v = v(T)$ is given by $\sum_{\varepsilon_i=\pm1} \varepsilon_1\varepsilon_2 T_{\varepsilon_1,\varepsilon_2}$, where $T = 3_1$ and $T_{\varepsilon_1,\varepsilon_2}$ is T with its crossing p_i, for $i = 1, 2$ changed to be positive if $\varepsilon_i = +1$ and negative if $\varepsilon_i = -1$ (see Figure 2).

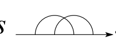
Notice that, for any $f \in \mathcal{K}_3$ and any pair (p_1, p_2) of its crossings, an analogous zero-cycle $v = v(f; p_1, p_2)$ can be defined.

Regard $f \in \mathcal{K}_3$ as a zero-cycle of $\tilde{\mathcal{K}}_3$ (with $\mathrm{fr}\, f = 0$) and consider $r_*\lambda(e, f)$. During a knot f "going through" e, f rotates once around the x-axis. Thus the one-cycle $r_*\lambda(e, f)$ is homologous to the Gramain cycle G_f. This leads us to the fact that, for $v = v(f; p_1, p_2)$, the one-cycle $r_*\lambda(e, v)$ is homologous to the sum $\sum_{\varepsilon_i=\pm1} \varepsilon_1\varepsilon_2 G_{f_{\varepsilon_1,\varepsilon_2}}$. This is why we can apply the method in [Sakai 2008] to compute

$$D^2V(f) := \sum_{\varepsilon_j=\pm1} \varepsilon_1\varepsilon_2 V(f_{\varepsilon_1,\varepsilon_2}) = \sum_{\varepsilon_j=\pm1} \varepsilon_1\varepsilon_2 \langle I(\Gamma), G_{f_{\varepsilon_1,\varepsilon_2}}\rangle = \langle I(\Gamma), r_*\lambda(e, v(f))\rangle.$$

Recall that our graph cocycle Γ is a sum of nine graphs $\Gamma_1, \dots, \Gamma_9$ (see Figure 1). By Assumption 1, the integration $\langle I(\Gamma_i), G_f\rangle$ can be computed by "counting" the configurations with all the images of the Gauss maps corresponding to edges of Γ_i being around the poles of S^2. Lemma 3.4 below was proved in such a way in [Sakai 2008] when $n > 3$. Since $[v(f)] \in H_0(\mathcal{K}_3(f))$ is independent of small $h > 0$ (see Assumption 2), we may compute $D^2V(f)$ in the limit $h \to 0$.

Definition 3.3. We say that a pair (p_1, p_2) of crossings of f *respects the diagram*  if there exist $t_1 < t_2 < t_3 < t_4$ where $f(t_1)$ and $f(t_3)$ correspond to p_1, while $f(t_2)$ and $f(t_4)$ correspond to p_2. The notion of (p_1, p_2) respecting  or  is defined analogously.

Lemma 3.4 [Sakai 2008]. *Suppose that* (p_1, p_2) *respects* . *Then, in the limit* $h \to 0$, $P_i(f) := \sum_{\varepsilon_j=\pm1} \varepsilon_1\varepsilon_2 \langle I(\Gamma_i), G_{f_{\varepsilon_1,\varepsilon_2}}\rangle$ *converges to zero for* $i \neq 2$, *and* $P_2(f)$ *converges to* 1. *Thus* $D^2V(f) = 1$.

Outline of proof. Let $\hat{C}_{\Gamma_i} \to S^1$ be the pullback of $C_{\Gamma_i} \to \mathscr{K}_3$ via G_f, and let $\hat{G}_f : \hat{C}_{\Gamma_i} \to C_{\Gamma_i}$ be the lift of G_f. By the properties of pullbacks and fiber integrations,

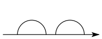
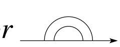
$$P_i(f) = \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \varepsilon_2 \int_{\hat{C}_{\Gamma_i}} \hat{G}^*_{f_{\varepsilon_1, \varepsilon_2}} \omega_{\Gamma_i}. \tag{3-1}$$

Let $t_1 < \cdots < t_4$ be such that $f(t_1)$ and $f(t_3)$ correspond to p_1, while $f(t_2)$ and $f(t_4)$ correspond to p_2. Define the subspace $C'_{\Gamma_i} \subset \hat{C}_{\Gamma_i}$ as consisting of $(G_f(s); (x_j))$ $(s \in S^1)$ such that, for each $j = 1, 2$, there is a pair (l, m) of i-vertices of Γ_i such that x_l is on the over-arc of p_j, x_m is on the under-arc of p_j, and there is a sequence of edges in Γ_i from l to m.

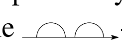
First observation: The integration over $\hat{C}_{\Gamma_i} \setminus C'_{\Gamma_i}$ does not essentially contribute to $P_i(f)$ in the limit $h \to 0$. This is because, over $\hat{C}_{\Gamma_i} \setminus C'_{\Gamma_i}$, the integrals in (3-1) are well defined and continuous even when $h = 0$ (p_j becomes a double point), so two terms in $P_i(f)$ corresponding to $\varepsilon_j = \pm 1$ cancel each other. This implies $\lim_{h \to 0} P_i(f) = 0$ for $i = 7, 8, 9$, since $C'_{\Gamma_i} = \varnothing$ if $\sharp\{\text{i-vertices}\} \le 3$.

Second observation: Consider the configurations $(x_i) \in C'_{\Gamma_i}$ such that, for any pair (l, m) of i-vertices of Γ_i with x_l on the over-arc of p_j and x_m on the under-arc of p_j, all the points x_k (k is in a sequence in Γ_i from l to m) are not near p_j. Such configurations also do not essentially contribute to $P_i(f)$ in the limit $h \to 0$, by the same reason as above. This implies $\lim_{h \to 0} P_i(f) = 0$ for $i = 4, 5, 6$; the configurations $(x_l) \in C'_{\Gamma_i}$ $(4 \le i \le 6)$ must be such that the point $x_l \in \mathbb{R}^1$ $(1 \le l \le 4)$ is near t_l. By the second observation, the "free point" x_5 must be near p_1 or p_2. But then $\omega_{\Gamma_i} = 0$, since at least one Gauss map φ_{l5} has its image outside the support of vol_{S^2} (see Assumption 1). Thus $\lim_{h \to 0} P_i(f) = 0$.

Finally consider the $P_i(f)$, for $i = 1, 2, 3$. For $i = 1$ we have $\omega_{\Gamma_1} = 0$ over C'_{Γ_1}, since the Gauss map corresponding to the edge 12 has its image outside of the support of vol_{S^2}. The same reasoning, using the loop edge 11, shows that $\omega_{\Gamma_3} = 0$ over C'_{Γ_3}. Only $P_2(f)$ survives, since the configurations with x_1 near t_1, x_2 near t_2, x_3 and x_4 near t_3, and x_5 near t_4, contribute nontrivially to the integral [Sakai 2008, Lemma 4.6]. □

Lemma 3.5. *If (p_1, p_2) respects*  *or* *, then $D^2 V(f) = 0$.*

Proof. For $i = 4, \dots, 9$, we see in the same way as in Lemma 3.4 that $P_i(f)$ approaches 0 as $h \to 0$. That $\lim_{h \to 0} P_i(f)$ for $i = 2, 3$ and the -case for $i = 1$ is proved by the first observation in the proof of Lemma 3.4.

In the -case for $P_1(f)$ over C'_{Γ_1} only the configurations with x_j near t_j, with $j = 1, 2, 3$, and x_5 near t_4 may essentially contribute to $P_1(f)$; in this case the edges 12 and 35 join the over/under arcs of p_1 and p_2 respectively. However, the Gauss map φ_{14} cannot have its image in the support of vol_{S^2}, so ω_{Γ_1} vanishes. □

Proof of Theorem 3.1. For three crossings (p_1, p_2, p_3) of $f \in \mathcal{K}_3$, consider the third difference

$$D^3V(f) := \sum_{\varepsilon_j=\pm 1} \varepsilon_1\varepsilon_2\varepsilon_3 V(f_{\varepsilon_1,\varepsilon_2,\varepsilon_3}) = D^2V(g_{+1}) - D^2V(g_{-1}),$$

where $g_{\pm 1} := f_{+1,+1,\pm 1}$ and $D^2V(g_{\pm 1})$ are taken with respect to (p_1, p_2). Since the pair (p_1, p_2) of g_{+1} respects the same diagram as (p_1, p_2) of g_{-1}, we have $D^2V(g_{+1}) = D^2V(g_{-1})$ by the above Lemmas 3.4, 3.5. Thus $D^3V = 0$ and hence V is finite type of order two. Moreover $V(\iota) = 0$ for the trivial long knot ι since $\mathcal{K}_3(\iota)$ is contractible [Hatcher 1983]; therefore $G_\iota \sim 0$, and $V(3_1) = 1$ by Lemma 3.4 and $V(\iota) = 0$. These properties uniquely characterize Casson's knot invariant v_2. □

The Browder operations. We denote a framed long knot corresponding to (f, k) under the equivalence $\tilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$ by $f^k \in \tilde{\mathcal{K}}_3$ (unique up to homotopy). As mentioned above, the Gramain cycle can be written as $[G_f] = [r_*\lambda(f^k, \iota^1)]$ (k may be arbitrary). Below we will evaluate $I(\Gamma)$ on more general cycles $r_*\lambda(f^k, g^l)$ of $\mathcal{K}_3$ for any nontrivial $f, g \in \mathcal{K}_3$ and $k, l \in \mathbb{Z}$. This generalizes Theorem 3.1.

Theorem 3.6. *We have* $\langle I(\Gamma),\ r_*\lambda(f^k, g^l)\rangle = l v_2(f) + k v_2(g)$ *for any* $f, g \in \mathcal{K}_3$ *and* $k, l \in \mathbb{Z}$.

Corollary 3.7. *If at least one of* $v_2(f)$ *and* $v_2(g)$ *is not zero, then*

$$[I(\Gamma)|_{\mathcal{K}_3(f\sharp g)}] \in H^1_{DR}(\mathcal{K}_3(f\sharp g)) \neq 0,$$

where $\sharp$ *stands for the connected sum.*

Proof. This is because $r_*\lambda(f^k, g^l)$ is a one-cycle of $\mathcal{K}_3(f\sharp g)$ for any $k, l \in \mathbb{Z}$. Since $v_2(f)$ or $v_2(g)$ is not zero, there exist some k, l such that $l v_2(f) + k v_2(g) \neq 0$, so $\langle I(\Gamma),\ r_*\lambda(f^k, g^l)\rangle \neq 0$ by Theorem 3.6. □

Remark 3.8. If $v_2(f) = -v_2(g)$, then $v_2(f\sharp g) = 0$ since it is known that v_2 is additive under $\sharp$. Hence we cannot deduce $[I(\Gamma)|_{\mathcal{K}_3(f\sharp g)}] \neq 0$ from Corollary 3.2. Moreover if $v_2(f) = -v_2(g) \neq 0$, then Corollary 3.7 implies $[I(\Gamma)|_{\mathcal{K}_3(f\sharp g)}] \neq 0$.

To prove Theorem 3.6, first we remark that $f^m \sim f^0\sharp\iota^m$. Since λ satisfies the Leibniz rule, $\lambda(f^k, g^l)$ is homologous to

$$\lambda(f^0, g^0)\sharp\iota^{k+l} + \lambda(f^0, \iota^l)\sharp g^k + \lambda(\iota^k, g^0)\sharp f^l + \lambda(\iota^k, \iota^l)\sharp f^0\sharp g^0.$$

Since by definition $r_*\lambda(f^k, \iota^m) \sim mG_f$ ($k, m \in \mathbb{Z}$) and $G_\iota \sim 0$,

$$r_*\lambda(f^k, g^l) \sim r_*\lambda(f^0, g^0) + lG_f\sharp g + kf\sharp G_g. \tag{3-2}$$

Notice that $\sharp$ makes $\mathcal{K}_3$ an H-space and induces a coproduct Δ on $H^*_{DR}(\mathcal{K}_3)$.

Lemma 3.9. $\Delta([I(\Gamma)]) = 1 \otimes [I(\Gamma)] + [I(\Gamma)] \otimes 1 \in H^*_{DR}(\mathcal{K}_3)^{\otimes 2}.$

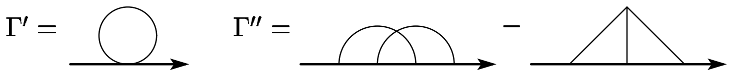


Figure 3. Graph cocycles Γ' and Γ''.

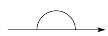
Proof. $\mathscr{D}$ also admits Δ defined as a "separation" of the graphs by removing a point from the specified oriented line [Cattaneo et al. 2005, Section 3.2]. Theorem 6.3 of [Cattaneo et al. 2005] shows, without using $n > 3$, that $(I \otimes I)\Delta(X) = \Delta I(X)$ if X satisfies $dI(X) = I(\delta X)$.

As for our graphs in Figure 1, $\Delta\Gamma_i = 1 \otimes \Gamma_i + \Gamma_i \otimes 1$ ($i \neq 3, 4$) and

$$\Delta(\Gamma_3 - \Gamma_4) = 1 \otimes (\Gamma_3 - \Gamma_4) + (\Gamma_3 - \Gamma_4) \otimes 1 + \Gamma' \otimes \Gamma'' + \Gamma'' \otimes \Gamma',$$

where Γ' and Γ'' are as shown in Figure 3. Thus

$$\Delta I(\Gamma) = 1 \otimes I(\Gamma) + I(\Gamma) \otimes 1 + I(\Gamma') \otimes I(\Gamma'') + I(\Gamma'') \otimes I(\Gamma').$$

But in fact $\Gamma' = \delta\Gamma_0$ where $\Gamma_0 =$ , and $I(\Gamma') = dI(\Gamma_0)$ since there is no hidden face in the boundary of the fiber of π_{Γ_0}. □

By (3-2), Lemma 3.9 and Theorem 3.1,

$$\langle I(\Gamma),\, r_*\lambda(f^k, g^l)\rangle = \langle I(\Gamma),\, r_*\lambda(f^0, g^0)\rangle + l v_2(f) + k v_2(g).$$

Thus it suffices to prove Theorem 3.6 in the case $k = l = 0$.

Proof of Theorem 3.6. Fix g and regard $\langle I(\Gamma),\, r_*\lambda(f^0, g^0)\rangle$ as an invariant $V_g(f)$ of f. We choose two crossings p_1 and p_2 from the diagram of f in xy-plane, and compute $D^2 V_g(f) := \sum_{\varepsilon_1,\varepsilon_2} \varepsilon_1\varepsilon_2 \langle I(\Gamma), r_*\lambda(f^0_{\varepsilon_1,\varepsilon_2}, g^0)\rangle$ in the limit $h \to 0$ as on page 414. If this is zero for any (p_1, p_2), then the arguments similar to that in the proof of Theorem 3.1 show that V_g is of order two and takes the value zero for the trefoil knot, thus identically $V_g = 0$ for any g. This will complete the proof.

We will compute each $P'_i := \sum_{\varepsilon=\pm 1} \langle I(\Gamma_i), r_*\lambda(f^0_{\varepsilon_1,\varepsilon_2}, g^0)\rangle$ $(1 \le i \le 9)$ in the limit $h \to 0$. The two observations appearing in the proof of Lemma 3.4 allow us to conclude $P'_i \to 0$ for $4 \le i \le 9$ in the same way as before, so we compute P'_i for $i = 1, 2, 3$ below. We may concentrate on the integration over C'_{Γ_i} by the first observation. Recall $C'_{\Gamma_i} \subset S^1 \times \mathrm{Conf}\,(\mathbb{R}^1, s) \times \mathrm{Conf}\,(\mathbb{R}^3, t)$ by definition. We take the S^1-parameter $\alpha \in S^1 = \mathbb{R}^1/2\pi\mathbb{Z}$ so that g goes through f during $0 \le \alpha \le \pi$, and f goes through g during $\pi \le \alpha \le 2\pi$.

First consider the integration over $0 \le \alpha \le \pi$. We may shrink g sufficiently small. Then the sliding of g through f does not affect the integration, so almost all the integrations converge to zero for the same reasons as in Lemmas 3.4 and 3.5. Only the configurations $(x_i) \in C'_{\Gamma_1}$ with x_1 and x_2 near p_1 may essentially contribute to P'_1 when g comes around p_1; the form $\varphi^*_{12}\mathrm{vol}_{S^2}$ may detect the knotting of g. However, the two terms for $\varepsilon_1 = \pm 1$ cancel each other.

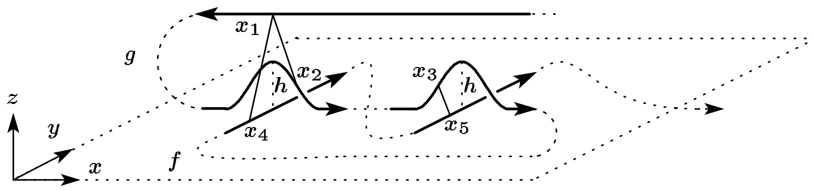
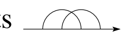


Figure 4. When f comes near an under-arc of g.

Next consider the integration over $\pi \leq \alpha \leq 2\pi$. There may be two types of contributions to P_i'. One type comes from the configurations in which all the points on the knot concentrate in a neighborhood of f. Such a contribution depends only on the framing number $\operatorname{fr} g$ of g, not on the global knotting of g. Since $\operatorname{fr} g^0 = 0$ here, such configurations do not essentially contribute to P_i'.

The other possible contributions arise when f comes near the crossings of g. For example, consider the case that (p_1, p_2) respects . When f comes near a crossing of g, a configuration $(x_1, \ldots, x_5) \in C_{\Gamma_1}$ as in Figure 4 is certainly in C'_{Γ_1}, so it may contribute to P_1'.

However, such contributions converge to zero in the limit $h \to 0$, because x_1 cannot be near p_1 (see the second observation in the proof of Lemma 3.4). For Γ_3, we should take the configuration $(x_1, \ldots, x_5)$ with x_j $(2 \leq j \leq 5)$ near t_{j-1} into account; but in this case the Gauss map φ_{11} cannot have the image in the support of vol_{S^2}. In such ways we can check that all such contributions of Γ_i $(i = 1, 2, 3)$ can be arbitrarily small. $\square$

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KEIICHI SAKAI
DEPARTMENT OF MATHEMATICAL SCIENCES
SHINSHU UNIVERSITY
3-1-1 ASAHI, MATSUMOTO 390-8621
JAPAN

ksakai@math.shinshu-u.ac.jp
http://math.shinshu-u.ac.jp/~ksakai/index.html

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