

Exceptional Askey-Wilson type polynomials through Darboux-Crum transformations

S. Odake^a and R. Sasaki^b

^a Department of Physics, Shinshu University,
Matsumoto 390-8621, Japan

^b Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan

Abstract

An alternative derivation is presented of the infinitely many exceptional Wilson and Askey-Wilson polynomials, which were introduced by the present authors in 2009. Darboux-Crum transformations intertwining the discrete quantum mechanical systems of the original and the exceptional polynomials play an important role. Infinitely many continuous Hahn polynomials are derived in the same manner. The present method provides a simple proof of the shape invariance of these systems as in the corresponding cases of the exceptional Laguerre and Jacobi polynomials.

1 Introduction

In a previous paper [1], we have derived *infinitely many exceptional orthogonal polynomials* related to the Wilson and Askey-Wilson polynomials [2, 3, 4] as the solutions of infinitely many shape invariant [5] thus exactly solvable *discrete* quantum mechanics in one dimension [6, 7, 8, 9, 10]. The corresponding Hamiltonians are the deformations of those for the Wilson and Askey-Wilson polynomials [6, 8, 9] in terms of a degree ℓ ($\ell = 1, 2, \dots$) eigenpolynomial with twisted parameters [1]. In discrete quantum mechanics, the Schrödinger equation is a second order *difference* equation, which reduces, in known cases of exactly solvable examples, to the difference equation satisfied by the polynomials belonging to the Askey scheme of hypergeometric orthogonal polynomials and their q -analogues. As stressed in our previous

publications, various concepts and formulas of these Askey scheme polynomials can be understood and formulated in a unified fashion through the framework of quantum mechanics; that is the eigenvalue problem of a hermitian (self-adjoint) linear operator (the Hamiltonian) in a certain Hilbert space. The Hamiltonian corresponding to a particular polynomial, say the Wilson or the Askey-Wilson polynomial, is specified by a choice of the potential function (2.9) as in the ordinary quantum mechanics.

In this paper we present an alternative derivation of the infinitely many exceptional Wilson and Askey-Wilson polynomials in terms of Darboux-Crum transformations [11, 12] *intertwining* the Hamiltonians of the original Wilson/Askey-Wilson polynomials with that of the corresponding exceptional polynomials. The same method is applied to the continuous Hahn polynomials [4] to construct the new infinitely many exceptional continuous Hahn polynomials indexed by a positive even integer ℓ . This is a discrete quantum mechanics version of the recent work [13] which derives the four families of infinitely many exceptional Laguerre/Jacobi polynomials [14, 15] in terms of Darboux-Crum translations *intertwining* the Hamiltonians of the exceptional polynomials with those of the well known exactly solvable Hamiltonians of the radial oscillator and the Darboux-Pöschl-Teller potentials [16, 17, 18].

The concept of exceptional (X_ℓ) polynomials was introduced by Gomez-Ullate et al [19] in 2008 as a new type of orthogonal polynomials satisfying second order differential equations. The X_ℓ polynomials start with degree ℓ ($\ell = 1, 2, \dots$) instead of the degree zero constant term, thus avoiding the constraints by Bochner's theorem [20]. They constructed the X_1 Laguerre and Jacobi polynomials [19], which are the first members of the infinitely many exceptional Laguerre and Jacobi polynomials introduced by the present authors [14, 15] in 2009. Quesne gave a quantum mechanical formulation based on shape invariance [21] and later she introduced another set of X_2 polynomials [22].

This paper is organised as follows. In section two we review the discrete quantum mechanical systems of the continuous Hahn, Wilson and Askey-Wilson polynomials in § 2.1 together with those of the corresponding exceptional polynomials in § 2.2. The properties of the deforming polynomial ξ_ℓ are discussed in some detail in § 2.3. They are important for deriving various results in the subsequent section. The main part of the paper, the Darboux-Crum transformations intertwining the original and the deformed systems are discussed in section three. The final section is for a summary and comments including the annihilation/creation operators and Rodrigues type formulas for the exceptional polynomials.

2 The original and deformed systems

Here we first recapitulate the shape invariant, therefore solvable, systems whose eigenfunctions are described by the orthogonal polynomials; the continuous Hahn, Wilson and Askey-Wilson polynomials [4], to be abbreviated as cH, W and AW, respectively. See [6] and [9] for the discrete quantum mechanical treatment of these polynomials. It is based on the factorisation method and generalisation of the Darboux-Crum transformations [11, 12]. See also the related papers [23]. The factorisation of the Askey-Wilson polynomial was also discussed in [24]. Then the deformed systems corresponding to the exceptional Askey type polynomial are summarised in § 2.2.

2.1 The original systems

Here we summarise various properties of the original Hamiltonian systems to be compared with the deformed systems which will be presented in § 2.2. Let us start with the Hamiltonians, Schrödinger equations and eigenfunctions ($x_1 < x < x_2$, $p = -i\frac{d}{dx}$):

$$\begin{aligned} \mathcal{A}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2}p}\sqrt{V^*(x; \boldsymbol{\lambda})} - e^{-\frac{\gamma}{2}p}\sqrt{V(x; \boldsymbol{\lambda})}), \\ \mathcal{A}(\boldsymbol{\lambda})^\dagger &\stackrel{\text{def}}{=} -i(\sqrt{V(x; \boldsymbol{\lambda})}e^{\frac{\gamma}{2}p} - \sqrt{V^*(x; \boldsymbol{\lambda})}e^{-\frac{\gamma}{2}p}), \end{aligned} \quad (2.1)$$

$$\mathcal{H}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\lambda})^\dagger \mathcal{A}(\boldsymbol{\lambda}), \quad (2.2)$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) \quad (n = 0, 1, 2, \dots), \quad (2.3)$$

$$\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}). \quad (2.4)$$

The set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ are

$$\text{cH} : \quad \boldsymbol{\lambda} \stackrel{\text{def}}{=} (a_1, a_2), \quad \text{Re } a_i > 0 \quad (i = 1, 2), \quad (2.5)$$

$$\begin{aligned} \text{W} : \quad \boldsymbol{\lambda} &\stackrel{\text{def}}{=} (a_1, a_2, a_3, a_4), \quad \text{Re } a_i > 0 \quad (i = 1, \dots, 4), \\ &\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \quad (\text{as a set}), \end{aligned} \quad (2.6)$$

$$\begin{aligned} \text{AW} : \quad q^\lambda &\stackrel{\text{def}}{=} (a_1, a_2, a_3, a_4), \quad |a_i| < 1, \quad (i = 1, \dots, 4), \quad 0 < q < 1, \\ &\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\} \quad (\text{as a set}), \end{aligned} \quad (2.7)$$

where $q^{(\lambda_1, \lambda_2, \dots)} \stackrel{\text{def}}{=} (q^{\lambda_1}, q^{\lambda_2}, \dots)$. The sinusoidal coordinate $\eta(x)$ is,

$$\eta(x) \stackrel{\text{def}}{=} \begin{cases} x, & x_1 = -\infty, \quad x_2 = \infty, \quad \gamma = 1 & : \text{cH} \\ x^2, & x_1 = 0, \quad x_2 = \infty, \quad \gamma = 1 & : \text{W} \\ \cos x, & x_1 = 0, \quad x_2 = \pi, \quad \gamma = \log q & : \text{AW} \end{cases}. \quad (2.8)$$

The potential function $V(x; \boldsymbol{\lambda})$ and energy eigenvalue $\mathcal{E}_n(\boldsymbol{\lambda})$ are

$$V(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} (a_1 + ix)(a_2 + ix) & : \text{cH} \\ (2ix(2ix + 1))^{-1} \prod_{j=1}^4 (a_j + ix) & : \text{W} \\ ((1 - e^{2ix})(1 - qe^{2ix}))^{-1} \prod_{j=1}^4 (1 - a_j e^{ix}) & : \text{AW} \end{cases}, \quad (2.9)$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} n(n + b_1 - 1), & b_1 \stackrel{\text{def}}{=} a_1 + a_2 + a_1^* + a_2^* & : \text{cH} \\ n(n + b_1 - 1), & b_1 \stackrel{\text{def}}{=} a_1 + a_2 + a_3 + a_4 & : \text{W} \\ (q^{-n} - 1)(1 - b_4 q^{n-1}), & b_4 \stackrel{\text{def}}{=} a_1 a_2 a_3 a_4 & : \text{AW} \end{cases}. \quad (2.10)$$

Throughout this paper we consider the potential functions, eigenfunctions, etc as analytic functions of x in the complex region containing $x_1 < \text{Re } x < x_2$. We use the $*$ -operation on an analytic function $*$: $f \mapsto f^*$ in the following sense. If $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$, then $f^*(x) \stackrel{\text{def}}{=} \sum_n a_n^* x^n$, in which a_n^* is the complex conjugation of a_n . Obviously $f^{**}(x) = f(x)$ and $f(x)^* = f^*(x^*)$. If a function satisfies $f^* = f$, we call it a ‘real’ function, for it takes real values on the real line.

The eigenfunctions are chosen real, $\phi_n^* = \phi_n$ and $\phi_0^* = \phi_0$ and $P_n^* = P_n$. The main part consists of an orthogonal polynomial $P_n(\eta; \boldsymbol{\lambda})$, a polynomial of degree n in η :

$$P_n(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} p_n(\eta; a_1, a_2, a_1^*, a_2^*) & : \text{cH} \\ W_n(\eta; a_1, a_2, a_3, a_4) & : \text{W} \\ p_n(\eta; a_1, a_2, a_3, a_4 | q) & : \text{AW} \end{cases}. \quad (2.11)$$

They are expressed in terms of the (basic) hypergeometric functions [4]:

$$\begin{aligned} \text{cH: } & p_n(\eta(x); a_1, a_2, a_1^*, a_2^*) \\ & \stackrel{\text{def}}{=} i^n \frac{(a_1 + a_1^*)_n (a_1 + a_2^*)_n}{n!} {}_3F_2 \left(\begin{matrix} -n, n + a_1 + a_2 + a_1^* + a_2^* - 1, a_1 + ix \\ a_1 + a_1^*, a_1 + a_2^* \end{matrix} \middle| 1 \right), \end{aligned} \quad (2.12)$$

$$\begin{aligned} \text{W: } & W_n(\eta(x); a_1, a_2, a_3, a_4) \\ & \stackrel{\text{def}}{=} (a_1 + a_2)_n (a_1 + a_3)_n (a_1 + a_4)_n \\ & \quad \times {}_4F_3 \left(\begin{matrix} -n, n + \sum_{j=1}^4 a_j - 1, a_1 + ix, a_1 - ix \\ a_1 + a_2, a_1 + a_3, a_1 + a_4 \end{matrix} \middle| 1 \right), \end{aligned} \quad (2.13)$$

$$\begin{aligned} \text{AW: } & p_n(\eta(x); a_1, a_2, a_3, a_4 | q) \\ & \stackrel{\text{def}}{=} a_1^{-n} (a_1 a_2, a_1 a_3, a_1 a_4; q)_n \times {}_4\phi_3 \left(\begin{matrix} q^{-n}, a_1 a_2 a_3 a_4 q^{n-1}, a_1 e^{ix}, a_1 e^{-ix} \\ a_1 a_2, a_1 a_3, a_1 a_4 \end{matrix} \middle| q; q \right), \end{aligned} \quad (2.14)$$

in which $(a)_n$ and $(a; q)_n$ are the Pochhammer symbol and its q -analogue. They are symmetric in (a_1, a_2) for cH and in (a_1, a_2, a_3, a_4) for W and AW.

These Hamiltonian systems are exactly solvable in the Schrödinger picture. Shape invariance [5] is a sufficient condition for the exact solvability. Its relations involve one more positive constant κ :

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \quad (2.15)$$

$$\boldsymbol{\delta} \stackrel{\text{def}}{=} \begin{cases} (\frac{1}{2}, \frac{1}{2}) & : \text{cH} \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & : \text{W, AW} \end{cases}, \quad \kappa \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{cH, W} \\ q^{-1} & : \text{AW} \end{cases}, \quad (2.16)$$

or equivalently,

$$V(x - i\frac{\gamma}{2}; \boldsymbol{\lambda})V^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \kappa^2 V(x; \boldsymbol{\lambda} + \boldsymbol{\delta})V^*(x - i\gamma; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.17)$$

$$V(x + i\frac{\gamma}{2}; \boldsymbol{\lambda}) + V^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \kappa(V(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + V^*(x; \boldsymbol{\lambda} + \boldsymbol{\delta})) - \mathcal{E}_1(\boldsymbol{\lambda}). \quad (2.18)$$

It is straightforward to verify these relations for the above potential functions for the cH, W and AW cases (2.9). The groundstate wavefunction $\phi_0(x; \boldsymbol{\lambda})$ is determined as a zero mode of the operator $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})\phi_0(x; \boldsymbol{\lambda}) = 0$, namely,

$$\sqrt{V^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda})} \phi_0(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \sqrt{V(x + i\frac{\gamma}{2}; \boldsymbol{\lambda})} \phi_0(x + i\frac{\gamma}{2}; \boldsymbol{\lambda}), \quad (2.19)$$

and its explicit forms are:

$$\phi_0(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \sqrt{\Gamma(a_1 + ix)\Gamma(a_2 + ix)\Gamma(a_1^* - ix)\Gamma(a_2^* - ix)} & : \text{cH} \\ \sqrt{(\Gamma(2ix)\Gamma(-2ix))^{-1} \prod_{j=1}^4 \Gamma(a_j + ix)\Gamma(a_j - ix)} & : \text{W} \\ \sqrt{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty \prod_{j=1}^4 (a_j e^{ix}; q)_\infty^{-1} (a_j e^{-ix}; q)_\infty^{-1}} & : \text{AW} \end{cases}. \quad (2.20)$$

We introduce an auxiliary function $\varphi(x)$

$$\varphi(x) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{cH} \\ 2x & : \text{W} \\ 2 \sin x & : \text{AW} \end{cases}, \quad (2.21)$$

which possesses the properties:

$$\phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \varphi(x) \sqrt{V(x + i\frac{\gamma}{2}; \boldsymbol{\lambda})} \phi_0(x + i\frac{\gamma}{2}; \boldsymbol{\lambda}), \quad (2.22)$$

$$V(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \kappa^{-1} \frac{\varphi(x - i\gamma)}{\varphi(x)} V(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}). \quad (2.23)$$

The action of the operators $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ on the eigenfunctions is

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.24)$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger \phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda}) \phi_n(x; \boldsymbol{\lambda}). \quad (2.25)$$

The factors of the energy eigenvalue, $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})$, are given by

$$f_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} n + b_1 - 1 & : \text{cH} \\ -n(n + b_1 - 1) & : \text{W} \\ q^{\frac{n}{2}}(q^{-n} - 1)(1 - b_4 q^{n-1}) & : \text{AW} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} n & : \text{cH} \\ -1 & : \text{W} \\ q^{-\frac{n}{2}} & : \text{AW} \end{cases}. \quad (2.26)$$

The forward and backward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$ are defined by

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x; \boldsymbol{\lambda}) = i\varphi(x)^{-1}(e^{\frac{\gamma}{2}p} - e^{-\frac{\gamma}{2}p}), \quad (2.27)$$

$$\mathcal{B}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^\dagger \circ \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = -i(V(x; \boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - V^*(x; \boldsymbol{\lambda})e^{-\frac{\gamma}{2}p})\varphi(x), \quad (2.28)$$

and their action on the polynomials is

$$\mathcal{F}(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.29)$$

$$\mathcal{B}(\boldsymbol{\lambda})P_{n-1}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}). \quad (2.30)$$

The second order difference operator $\tilde{\mathcal{H}}(\boldsymbol{\lambda})$ acting on the polynomial eigenfunctions is defined by

$$\begin{aligned} \tilde{\mathcal{H}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}(\boldsymbol{\lambda}) \circ \phi_0(x; \boldsymbol{\lambda}) \\ &= V(x; \boldsymbol{\lambda})(e^{\gamma p} - 1) + V^*(x; \boldsymbol{\lambda})(e^{-\gamma p} - 1), \end{aligned} \quad (2.31)$$

$$\tilde{\mathcal{H}}(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}). \quad (2.32)$$

In conventional terms, this is the difference equation determining the polynomials:

$$\begin{aligned} &V(x; \boldsymbol{\lambda})(P_n(\eta(x - i\gamma); \boldsymbol{\lambda}) - P_n(\eta(x); \boldsymbol{\lambda})) + V^*(x; \boldsymbol{\lambda})(P_n(\eta(x + i\gamma); \boldsymbol{\lambda}) - P_n(\eta(x); \boldsymbol{\lambda})) \\ &= \mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}). \end{aligned} \quad (2.33)$$

The orthogonality relation is

$$\int_{x_1}^{x_2} \phi_0(x; \boldsymbol{\lambda})^2 P_n(\eta(x); \boldsymbol{\lambda})P_m(\eta(x); \boldsymbol{\lambda})dx = h_n(\boldsymbol{\lambda})\delta_{nm}, \quad (2.34)$$

$$h_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2\pi \prod_{i,j=1}^2 \Gamma(n + a_i + a_j^*) \cdot (n!(2n + b_1 - 1)\Gamma(n + b_1 - 1))^{-1} & : \text{cH} \\ 2\pi n! (n + b_1 - 1)_n \prod_{1 \leq i < j \leq 4} \Gamma(n + a_i + a_j) \cdot \Gamma(2n + b_1)^{-1} & : \text{W} \\ 2\pi (b_4 q^{n-1}; q)_n (b_4 q^{2n}; q)_\infty (q^{n+1}; q)_\infty^{-1} \prod_{1 \leq i < j \leq 4} (a_i a_j q^n; q)_\infty^{-1} & : \text{AW} \end{cases}. \quad (2.35)$$

As shown in detail in [8] these Hamiltonian systems are exactly solvable in the Heisenberg picture, too. The positive/negative frequency parts of the exact Heisenberg operator solution of the sinusoidal coordinate $\eta(x)$ provide the *annihilation/creation* operators $a^{(\pm)}(\boldsymbol{\lambda})$ which map the eigenfunctions to the neighbouring levels

$$a^{(\pm)}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) \propto \phi_{n\pm 1}(x; \boldsymbol{\lambda}). \quad (2.36)$$

This is a disguise of the three term recurrence relations of these orthogonal polynomials. The above relations are to be contrasted with the actions of the operators $A(\boldsymbol{\lambda})$ and $A(\boldsymbol{\lambda} - \boldsymbol{\delta})^\dagger$ (2.24)–(2.25), which maps $\phi_n(x; \boldsymbol{\lambda})$ to the neighbouring levels of *shifted parameters* $\boldsymbol{\lambda} \pm \boldsymbol{\delta}$.

2.2 The deformed systems

Here we recapitulate the Hamiltonian systems of the exceptional Wilson and Askey-Wilson polynomials, which were derived by the present authors [1] in 2009. The exceptional continuous Hahn polynomials are new. For each $\ell = 1, 2, \dots$, we can construct a shape invariant system by deforming the original system ($\ell = 0$) in terms of a degree ℓ eigenpolynomial $\xi_\ell(\eta)$ of twisted parameters. We restrict the original parameter ranges (2.5)–(2.7) as follows:

$$\text{cH} : \quad a_1 > 0, \quad \ell : \text{even}, \quad (2.37)$$

$$\begin{aligned} \text{W} : \quad & a_1, a_2 \in \mathbb{R}, \quad \{a_3^*, a_4^*\} = \{a_3, a_4\} \quad (\text{as a set}), \\ & 0 < a_j < \text{Re } a_k \quad (j = 1, 2; k = 3, 4), \end{aligned} \quad (2.38)$$

$$\begin{aligned} \text{AW} : \quad & a_1, a_2 \in \mathbb{R}, \quad \{a_3^*, a_4^*\} = \{a_3, a_4\} \quad (\text{as a set}), \\ & 1 > a_j > |a_k| \quad (j = 1, 2; k = 3, 4). \end{aligned} \quad (2.39)$$

In terms of the twist operation \mathbf{t} acting on the set of parameters,

$$\mathbf{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} (-\lambda_1, \lambda_2) & : \text{cH} \\ (-\lambda_1, -\lambda_2, \lambda_3, \lambda_4) & : \text{W, AW} \end{cases}, \quad (2.40)$$

the deforming polynomial $\xi_\ell(\eta)$ is defined from the eigenpolynomial $P_\ell(\eta)$:

$$\xi_\ell(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell(\eta; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})). \quad (2.41)$$

We may need to restrict parameters further in order that $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ has no zero in the rectangular domain $x_1 \leq \text{Re } x \leq x_2$, $|\text{Im } x| \leq \frac{1}{2}|\gamma|$, which is necessary for the hermiticity of

the Hamiltonian. The potential function, the Hamiltonian and the Schrödinger equation of the deformed system are:

$$V_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} V(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x - i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (2.42)$$

$$V_\ell^*(x; \boldsymbol{\lambda}) = V^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x + i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (2.43)$$

$$\begin{aligned} \mathcal{A}_\ell(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2}p} \sqrt{V_\ell^*(x; \boldsymbol{\lambda})} - e^{-\frac{\gamma}{2}p} \sqrt{V_\ell(x; \boldsymbol{\lambda})}), \\ \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger &\stackrel{\text{def}}{=} -i(\sqrt{V_\ell(x; \boldsymbol{\lambda})} e^{\frac{\gamma}{2}p} - \sqrt{V_\ell^*(x; \boldsymbol{\lambda})} e^{-\frac{\gamma}{2}p}), \end{aligned} \quad (2.44)$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \mathcal{A}_\ell(\boldsymbol{\lambda}), \quad (2.45)$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}) = \mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}) \quad (n = 0, 1, 2, \dots), \quad \mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}). \quad (2.46)$$

The continuous Hahn polynomials are real polynomials defined on the entire real line. Therefore the odd degree members have at least one real zero, whichever coefficients we may choose. This is the reason why ℓ is restricted to even integers in the cH case (2.37).

This system is shape invariant:

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = \kappa \mathcal{A}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_{\ell,1}(\boldsymbol{\lambda}), \quad (2.47)$$

or equivalently,

$$V_\ell(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) V_\ell^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \kappa^2 V_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) V_\ell^*(x - i\gamma; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.48)$$

$$V_\ell(x + i\frac{\gamma}{2}; \boldsymbol{\lambda}) + V_\ell^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \kappa (V_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + V_\ell^*(x; \boldsymbol{\lambda} + \boldsymbol{\delta})) - \mathcal{E}_{\ell,1}(\boldsymbol{\lambda}). \quad (2.49)$$

Proof is straightforward by direct calculation. In order to derive eq. (2.49), use is made of the two properties of the deforming polynomial ξ_ℓ (2.68)–(2.69) and the factorisation of the potential (2.70). For another simple proof of shape invariance, see the discussion in the final section. The eigenfunctions are

$$\psi_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\sqrt{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})}}, \quad (2.50)$$

$$\phi_{\ell,n}(x; \boldsymbol{\lambda}) = \psi_\ell(x; \boldsymbol{\lambda}) P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) \quad (n = 0, 1, 2, \dots). \quad (2.51)$$

Here $P_{\ell,n}(\eta; \boldsymbol{\lambda})$ is a degree $\ell + n$ polynomial in η but $P_{\ell,n}(\eta(x); \boldsymbol{\lambda})$ has only n zeros in the domain $x_1 < x < x_2$. The explicit forms of $P_{\ell,n}(\eta)$ were given by eqs. (42)–(44) in [1], with eqs. (66)–(68) for W and eqs. (80)–(82) for AW. Here we present much simpler looking forms

of them, which encompasses the new cH case, too:

$$P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) = \frac{-i}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})\varphi(x)} \left(v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})P_n(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right. \\ \left. - v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})P_n(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right), \quad (2.52)$$

where $v_1(x; \boldsymbol{\lambda})$, $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$ and $\tilde{\boldsymbol{\delta}}$ will be defined in (2.71), (2.72) and (3.12). This is one of the main results of the present paper which is derived in §3. Its lowest degree member is the degree ℓ deforming polynomial itself of the shifted parameters

$$P_{\ell,0}(\eta(x); \boldsymbol{\lambda}) = \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.53)$$

which is obtained by (2.68). It is straightforward to verify that the groundstate eigenfunction

$$\phi_{\ell,0}(x; \boldsymbol{\lambda}) = \frac{\phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\sqrt{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})}} \quad (2.54)$$

is the zero mode of the operator $\mathcal{A}_\ell(\boldsymbol{\lambda})$, $\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,0}(x; \boldsymbol{\lambda}) = 0$.

The action of $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ on the eigenfunctions is

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x; \boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})\phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.55)$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{\ell,n-1}(\boldsymbol{\lambda})\phi_{\ell,n}(x; \boldsymbol{\lambda}), \quad (2.56)$$

$$f_{\ell,n}(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}), \quad b_{\ell,n-1}(\boldsymbol{\lambda}) = b_{n-1}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}). \quad (2.57)$$

Like the corresponding formulas of the original systems (2.24)–(2.25), these are simple consequences of the shape invariance and the normalisation of the eigenfunctions. In the next section, we will derive these formulas through the intertwining relations and without assuming shape invariance.

The forward shift operator $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and the backward shift operator $\mathcal{B}_\ell(\boldsymbol{\lambda})$ are defined in a similar way as before

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}_\ell(\boldsymbol{\lambda}) \circ \psi_\ell(x; \boldsymbol{\lambda}) \\ = \frac{i}{\varphi(x)\xi_\ell(\eta(x); \boldsymbol{\lambda})} \left(\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda} + \boldsymbol{\delta})e^{\frac{\gamma}{2}p} - \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda} + \boldsymbol{\delta})e^{-\frac{\gamma}{2}p} \right), \quad (2.58)$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \circ \psi_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \\ = \frac{-i}{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})} \left(V(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})e^{\frac{\gamma}{2}p} \right. \\ \left. - V^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})e^{-\frac{\gamma}{2}p} \right) \varphi(x), \quad (2.59)$$

and their action on the polynomial $P_{\ell,n}(\eta; \boldsymbol{\lambda})$ is

$$\mathcal{F}_\ell(\boldsymbol{\lambda})P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})P_{\ell,n-1}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.60)$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda})P_{\ell,n-1}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{\ell,n-1}(\boldsymbol{\lambda})P_{\ell,n}(\eta(x); \boldsymbol{\lambda}). \quad (2.61)$$

The second order difference operator $\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})$ acting on the polynomial eigenfunctions is defined by

$$\begin{aligned} \tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \mathcal{B}_\ell(\boldsymbol{\lambda})\mathcal{F}_\ell(\boldsymbol{\lambda}) = \psi_\ell(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_\ell(\boldsymbol{\lambda}) \circ \psi_\ell(x; \boldsymbol{\lambda}) \\ &= V(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})}{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})} \left(e^{\gamma p} - \frac{\xi_\ell(\eta(x - i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})} \right) \\ &\quad + V^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})}{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})} \left(e^{-\gamma p} - \frac{\xi_\ell(\eta(x + i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})} \right), \end{aligned} \quad (2.62)$$

$$\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) = \mathcal{E}_{\ell,n}(\boldsymbol{\lambda})P_{\ell,n}(\eta(x); \boldsymbol{\lambda}). \quad (2.63)$$

Again it is trivial to verify that the lowest degree polynomial $P_{\ell,0}(\eta(x); \boldsymbol{\lambda}) = \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})$ is the zero mode of $\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})$:

$$\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}) = 0. \quad (2.64)$$

The orthogonality relation is

$$\int_{x_1}^{x_2} \psi_\ell(x; \boldsymbol{\lambda})^2 P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) P_{\ell,m}(\eta(x); \boldsymbol{\lambda}) dx = h_{\ell,n}(\boldsymbol{\lambda}) \delta_{nm}, \quad (2.65)$$

$$h_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \times \begin{cases} \frac{(2a_1 + n + \ell)(a_2 + a_2^* + n + 2\ell - 1)}{(2a_1 + n)(a_2 + a_2^* + n + \ell - 1)} & : \text{cH} \\ \frac{(a_1 + a_2 + n + \ell)(a_3 + a_4 + n + 2\ell - 1)}{(a_1 + a_2 + n)(a_3 + a_4 + n + \ell - 1)} & : \text{W} \\ q^{-\ell} \frac{(1 - a_1 a_2 q^{n+\ell})(1 - a_3 a_4 q^{n+2\ell-1})}{(1 - a_1 a_2 q^n)(1 - a_3 a_4 q^{n+\ell-1})} & : \text{AW} \end{cases} \quad (2.66)$$

2.3 Properties of the deforming polynomial ξ_ℓ

Here we present three formulas of the deforming polynomial $\xi_\ell(\eta; \boldsymbol{\lambda})$ (2.67)–(2.69), which will play important roles in the derivation of various results in section three, in particular, the fundamental results of this paper (3.10) and (3.11):

$$\begin{aligned} &\left(V(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) (e^{\gamma p} - 1) + V^*(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) (e^{-\gamma p} - 1) \right) \xi_\ell(\eta(x); \boldsymbol{\lambda}) \\ &= \mathcal{E}_\ell(\mathbf{t}(\boldsymbol{\lambda})) \xi_\ell(\eta(x); \boldsymbol{\lambda}), \end{aligned} \quad (2.67)$$

$$\frac{i}{\varphi(x)}(v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})e^{\frac{\gamma}{2}p} - v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})e^{-\frac{\gamma}{2}p})\xi_\ell(\eta(x); \boldsymbol{\lambda}) = \hat{f}_{\ell,0}(\boldsymbol{\lambda})\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.68)$$

$$\begin{aligned} & \frac{-i}{\varphi(x)}(v_2(x; \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta})e^{\frac{\gamma}{2}p} - v_2^*(x; \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta})e^{-\frac{\gamma}{2}p})\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}) \\ & = \hat{b}_{\ell,0}(\boldsymbol{\lambda})\xi_\ell(\eta(x); \boldsymbol{\lambda}), \end{aligned} \quad (2.69)$$

where $v_1(x; \boldsymbol{\lambda})$, $v_2(x; \boldsymbol{\lambda})$ are the factors of the potential function $V(x; \boldsymbol{\lambda})$:

$$V(x; \boldsymbol{\lambda}) = -\sqrt{\kappa} \frac{v_1(x; \boldsymbol{\lambda})v_2(x; \boldsymbol{\lambda})}{\varphi(x)\varphi(x - i\frac{\gamma}{2})}, \quad (2.70)$$

$$v_1(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} i(a_1 + ix) & : \text{cH} \\ \prod_{j=1}^2 (a_j + ix) & : \text{W} \\ e^{-ix} \prod_{j=1}^2 (1 - a_j e^{ix}) & : \text{AW} \end{cases}, \quad v_2(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} i(a_2 + ix) & : \text{cH} \\ \prod_{j=3}^4 (a_j + ix) & : \text{W} \\ e^{-ix} \prod_{j=3}^4 (1 - a_j e^{ix}) & : \text{AW} \end{cases}. \quad (2.71)$$

The constants $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$ and $\hat{b}_{\ell,n}(\boldsymbol{\lambda})$ are given by

$$\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2a_1 + n & : \text{cH} \\ a_1 + a_2 + n & : \text{W} \\ -q^{-\frac{n-\ell}{2}}(1 - a_1 a_2 q^n) & : \text{AW} \end{cases}, \quad \hat{b}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} a_2 + a_2^* + n + 2\ell - 1 & : \text{cH} \\ a_3 + a_4 + n + 2\ell - 1 & : \text{W} \\ -q^{-\frac{n+\ell}{2}}(1 - a_3 a_4 q^{n+2\ell-1}) & : \text{AW} \end{cases}. \quad (2.72)$$

The first equation (2.67) is the difference equation for the deforming polynomial, which corresponds to (2.33). The eqs. (2.68)–(2.69) are identities relating $\xi_\ell(\eta; \boldsymbol{\lambda})$ and $\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})$. We remark that these two equations (2.68)–(2.69) imply (2.67). In similar problems in ordinary quantum mechanics, the exceptional Laguerre and Jacobi polynomials, analogous identities play important roles in proving shape invariance and other relations [27, 28, 13]. As shown in (2.12)–(2.14), the continuous Hahn, Wilson and Askey-Wilson polynomials are expressed in terms of the (basic) hypergeometric functions ${}_3F_2$, ${}_4F_3$ and ${}_4\phi_3$, respectively [4]. The identities (2.68)–(2.69) are reduced to the following identities satisfied by the (basic) hypergeometric functions: $(\alpha, \alpha', \alpha_1, \dots, \alpha_4 : \text{generic parameters})$

$$\begin{aligned} & (\alpha_1 + ix) {}_3F_2\left(\begin{matrix} -\alpha, \alpha', \alpha_1 + 1 + ix \\ \alpha_3, \alpha_1 + \alpha_2 + 1 \end{matrix} \middle| 1\right) + (\alpha_2 - ix) {}_3F_2\left(\begin{matrix} -\alpha, \alpha', \alpha_1 + ix \\ \alpha_3, \alpha_1 + \alpha_2 + 1 \end{matrix} \middle| 1\right) \\ & = (\alpha_1 + \alpha_2) {}_3F_2\left(\begin{matrix} -\alpha, \alpha', \alpha_1 + ix \\ \alpha_3, \alpha_1 + \alpha_2 \end{matrix} \middle| 1\right), \end{aligned} \quad (2.73)$$

$$\begin{aligned} & \frac{-i}{2x} \left((\alpha_1 + ix)(\alpha_2 + ix) {}_4F_3\left(\begin{matrix} -\alpha, \alpha', \alpha_1 + 1 + ix, \alpha_1 - ix \\ \alpha_1 + \alpha_2 + 1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_4 \end{matrix} \middle| 1\right) \right. \\ & \quad \left. - (\alpha_1 - ix)(\alpha_2 - ix) {}_4F_3\left(\begin{matrix} -\alpha, \alpha', \alpha_1 + ix, \alpha_1 + 1 - ix \\ \alpha_1 + \alpha_2 + 1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_4 \end{matrix} \middle| 1\right) \right) \\ & = (\alpha_1 + \alpha_2) {}_4F_3\left(\begin{matrix} -\alpha, \alpha', \alpha_1 + ix, \alpha_1 - ix \\ \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_4 \end{matrix} \middle| 1\right), \end{aligned} \quad (2.74)$$

$$\frac{-i}{2 \sin x} \left(e^{-ix}(1 - \alpha_1 e^{ix})(1 - \alpha_2 e^{ix}) {}_4\phi_3\left(\begin{matrix} \alpha^{-1}, \alpha', \alpha_1 q e^{ix}, \alpha_1 e^{-ix} \\ \alpha_1 \alpha_2 q, \alpha_1 \alpha_3, \alpha_1 \alpha_4 \end{matrix} \middle| q; q \right) \right.$$

$$\begin{aligned}
& -e^{ix}(1-\alpha_1e^{-ix})(1-\alpha_2e^{-ix}) {}_4\phi_3\left(\begin{matrix} \alpha^{-1}, \alpha', \alpha_1e^{ix}, \alpha_1qe^{-ix} \\ \alpha_1\alpha_2q, \alpha_1\alpha_3, \alpha_1\alpha_4 \end{matrix} \middle| q; q\right) \\
& = -(1-\alpha_1\alpha_2) {}_4\phi_3\left(\begin{matrix} \alpha^{-1}, \alpha', \alpha_1e^{ix}, \alpha_1e^{-ix} \\ \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4 \end{matrix} \middle| q; q\right).
\end{aligned} \tag{2.75}$$

These identities can be easily verified by the series definition of the (basic) hypergeometric functions.

3 Intertwining relations

Here we demonstrate that the Hamiltonian system of the original polynomials reviewed in §2.1 and the deformation summarised in §2.2 are intertwined by a discrete version of the Darboux-Crum transformation. This provides simple expressions of the eigenfunctions of the deformed systems (2.52) in terms of those of the original system, which is exactly solvable. It also delivers a simple proof of the shape invariance of the deformed system.

3.1 General setting

For well-defined operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$, let us define a pair of Hamiltonians $\hat{\mathcal{H}}_\ell^{(\pm)}(\boldsymbol{\lambda})$

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}), \quad \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger, \tag{3.1}$$

and consider their the Schrödinger equations, that is, the eigenvalue problems:

$$\hat{\mathcal{H}}_\ell^{(\pm)}(\boldsymbol{\lambda}) \hat{\phi}_{\ell,n}^{(\pm)}(x; \boldsymbol{\lambda}) = \hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda}) \hat{\phi}_{\ell,n}^{(\pm)}(x; \boldsymbol{\lambda}) \quad (n = 0, 1, 2, \dots). \tag{3.2}$$

By definition, all the eigenfunctions must be square integrable. Obviously the pair of Hamiltonians are intertwined:

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}), \tag{3.3}$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) = \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}). \tag{3.4}$$

If $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) \neq 0$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) \neq 0$, then the two systems are exactly isospectral and there is one-to-one correspondence between the eigenfunctions:

$$\hat{\mathcal{E}}_{\ell,n}^{(+)}(\boldsymbol{\lambda}) = \hat{\mathcal{E}}_{\ell,n}^{(-)}(\boldsymbol{\lambda}), \tag{3.5}$$

$$\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) \propto \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}), \quad \hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) \propto \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}). \tag{3.6}$$

It should be stressed in the ordinary setting of Crum's theorem, the zero mode of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ is the groundstate of $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$. In that case, $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$ and $\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda})$ are iso-spectral except for the groundstate (essentially iso-spectral) of $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$, since it is annihilated by $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ [25].

In the following we will present the explicit forms of the operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$, which intertwine the original systems in § 2.1 and the deformed systems in § 2.2.

3.2 Intertwining the original and the deformed systems of the polynomials

The potential function \hat{V}_ℓ is the original potential function V with *twisted parameters* and multiplicatively deformed by the deforming polynomial ξ_ℓ :

$$\hat{V}_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} V(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) \frac{\xi_\ell(\eta(x - i\gamma); \boldsymbol{\lambda})}{\xi_\ell(\eta(x); \boldsymbol{\lambda})}, \quad (3.7)$$

$$\hat{V}_\ell^*(x; \boldsymbol{\lambda}) = V^*(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) \frac{\xi_\ell(\eta(x + i\gamma); \boldsymbol{\lambda})}{\xi_\ell(\eta(x); \boldsymbol{\lambda})}, \quad (3.8)$$

$$\begin{aligned} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2p}} \sqrt{\hat{V}_\ell^*(x; \boldsymbol{\lambda})} - e^{-\frac{\gamma}{2p}} \sqrt{\hat{V}_\ell(x; \boldsymbol{\lambda})}), \\ \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger &\stackrel{\text{def}}{=} -i(\sqrt{\hat{V}_\ell(x; \boldsymbol{\lambda})} e^{\frac{\gamma}{2p}} - \sqrt{\hat{V}_\ell^*(x; \boldsymbol{\lambda})} e^{-\frac{\gamma}{2p}}). \end{aligned} \quad (3.9)$$

It is illuminating to compare these potential functions (3.7)–(3.8) with those of the original (2.9) and deformed (2.42)–(2.43) systems. Again it is obvious that the overall normalisation of the deforming polynomial ξ_ℓ is immaterial for the deformation. For this choice of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$, one of the pair of Hamiltonians $\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})$ (3.1) becomes proportional to the original Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ (2.2) with $\boldsymbol{\lambda} \rightarrow \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}$ and the partner Hamiltonian $\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda})$ is proportional to the deformed Hamiltonian $\mathcal{H}_\ell(\boldsymbol{\lambda})$ (2.45):

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{H}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})), \quad (3.10)$$

$$\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{H}_\ell(\boldsymbol{\lambda}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})), \quad (3.11)$$

where $\hat{\kappa}_\ell(\boldsymbol{\lambda})$ and $\tilde{\boldsymbol{\delta}}$ are given by

$$\hat{\kappa}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{cH, W} \\ (a_1 a_2 q^\ell)^{-1} & : \text{AW} \end{cases}, \quad \tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \begin{cases} (\frac{1}{2}, -\frac{1}{2}) & : \text{cH} \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) & : \text{W, AW} \end{cases}. \quad (3.12)$$

The multiplicative and additive constants are common to the pair of Hamiltonians (3.10)–(3.11). These are the main results of this paper. Like the corresponding formulas in ordinary quantum mechanics [13], these fundamental results can be obtained by explicit calculation.

The three formulas in § 2.3 are essential. That is, the difference equation satisfied by the deforming polynomial $\xi_\ell(\eta(x); \boldsymbol{\lambda})$, (2.67) and the two identities relating the deforming polynomial $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ to its shifted one $\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})$, (2.68) and (2.69).

It is instructive to verify that the zero modes of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$ do not belong to the Hilbert space of the eigenfunctions. In fact, the zero mode $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ is

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\chi = 0, \quad \chi = \xi_\ell(\eta(x); \boldsymbol{\lambda})\phi_0(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})). \quad (3.13)$$

It has at least one pole in the rectangular domain $x_1 \leq \operatorname{Re} x \leq x_2$, $|\operatorname{Im} x| \leq \frac{1}{2}|\gamma|$, therefore it cannot belong to the Hilbert space. The zero mode of $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$ is

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \rho = 0, \quad \rho = \frac{\phi_0(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}))^{-1}}{\sqrt{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})V^*(x - i\frac{\gamma}{2}; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}))}}, \quad (3.14)$$

which is obviously non-square integrable. This situation is the discrete analogue of the ‘broken susy’ case in ordinary quantum mechanics in the terminology of supersymmetric quantum mechanics [17, 25].

Based on the results (3.10)–(3.11), we have

$$\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) = \phi_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}), \quad \hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) = \phi_{\ell,n}(x; \boldsymbol{\lambda}), \quad (3.15)$$

$$\hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})). \quad (3.16)$$

Then it is trivial to verify $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) \neq 0$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) \neq 0$. For, if one of the eigenfunction is annihilated by $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ ($\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$), the left hand side of (3.10)((3.11)) vanishes, whereas the right hand side is $\hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}))$ times the eigenfunction, which is obviously non-vanishing. Note that $\mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})$.

The correspondence of the pair of eigenfunctions $\hat{\phi}_{\ell,n}^{(\pm)}(x)$ is expressed as

$$\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) = \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda})}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})\hat{f}_{\ell,n}(\boldsymbol{\lambda})}}, \quad \hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) = \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda})}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda})}}. \quad (3.17)$$

We introduce operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ defined by

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda})^{-1} \circ \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})}} \circ \phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}), \quad (3.18)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})^{-1} \circ \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})}} \circ \psi_\ell(x; \boldsymbol{\lambda}). \quad (3.19)$$

The operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ are expressed explicitly by using the concrete forms of $V(x; \boldsymbol{\lambda})$, $\psi_\ell(x; \boldsymbol{\lambda})$ and $\phi_0(x; \boldsymbol{\lambda})$:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) = \frac{-i}{\varphi(x)} \left(v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) e^{\frac{\gamma}{2}p} - v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) e^{-\frac{\gamma}{2}p} \right), \quad (3.20)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) = \frac{1}{\xi_\ell(\eta(x); \boldsymbol{\lambda})} \frac{-i}{\varphi(x)} \left(v_2(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{\frac{\gamma}{2}p} - v_2^*(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{-\frac{\gamma}{2}p} \right). \quad (3.21)$$

Here is a technical remark on (3.20). In deriving the explicit form of the operator $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ in (3.20), one extracts the factorised potential function $v_1(x)$ and $v_1^*(x)$ from the corresponding expression of the square root of the *twisted potential* $\sqrt{V(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}))}$ and $\sqrt{V^*(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}))}$. Thus the choice of the argument is a subtle problem, in particular, for the cH case, in which only one factor undergoes the sign change. In other (W and AW) cases, sign change occur in two factors and thus the effect cancels out. Let us consider the twisted potential of cH case

$$V(x; \mathbf{t}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})) = (-a_1 - \frac{\ell}{2} + ix)(a_2 + \frac{\ell}{2} + ix).$$

For positive x close to the origin ($|x| \ll 1$), we choose $-a_1 - \frac{\ell}{2} + ix$ to have an argument close to $+\pi$. Then its $*$ -operation $-a_1 - \frac{\ell}{2} - ix = -(a_1 + \frac{\ell}{2} + ix)$ has an argument close to $-\pi$. This would mean $\sqrt{-(a_1 + \frac{\ell}{2} + ix)^2} = -i(a_1 + \frac{\ell}{2} + ix) = -v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})$, instead of the naively obtained $i(a_1 + \frac{\ell}{2} + ix) = v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})$.

The operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ act as the forward and backward shift operators connecting the original polynomials $P_n(\eta)$ and the exceptional polynomials $P_{\ell,n}(\eta)$:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \hat{f}_{\ell,n}(\boldsymbol{\lambda}) P_{\ell,n}(\eta(x); \boldsymbol{\lambda}), \quad (3.22)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) = \hat{b}_{\ell,n}(\boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}). \quad (3.23)$$

The former relation (3.22) with the explicit form of $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (3.20) provides the new explicit expression (2.52) of the exceptional orthogonal polynomials, which is one of the main results of this paper.

Other simple consequences of these relations are

$$\hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda}) \hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{b}_{\ell,n}(\boldsymbol{\lambda}), \quad \mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}) = \hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{b}_{\ell,n}(\boldsymbol{\lambda}) - \hat{f}_{\ell,0}(\boldsymbol{\lambda}) \hat{b}_{\ell,0}(\boldsymbol{\lambda}). \quad (3.24)$$

The normalisation constant $h_{\ell,n}(\boldsymbol{\lambda})$ (2.66) of the exceptional polynomials is related to that of the original polynomial $h_n(\boldsymbol{\lambda})$ (2.35):

$$h_{\ell,n}(\boldsymbol{\lambda}) = \frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda}) \hat{f}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{b}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}). \quad (3.25)$$

In the second equality we have used the explicit forms of $h_n(\boldsymbol{\lambda})$ (2.35). Eq. (3.25) is shown in the following way:

$$\begin{aligned}
& \hat{\kappa}_\ell(\boldsymbol{\lambda}) \hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{f}_{\ell,m}(\boldsymbol{\lambda}) \int_{x_1}^{x_2} dx \phi_{\ell,n}(x; \boldsymbol{\lambda}) \phi_{\ell,m}(x; \boldsymbol{\lambda}) \\
& \stackrel{(i)}{=} \int_{x_1}^{x_2} dx \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \phi_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \cdot \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \phi_m(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \\
& \stackrel{(ii)}{=} \int_{x_1}^{x_2} dx \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \phi_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \cdot \phi_m(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \\
& \stackrel{(iii)}{=} \hat{\mathcal{E}}_{\ell,n}^{(+)}(\boldsymbol{\lambda}) \int_{x_1}^{x_2} dx \phi_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \phi_m(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \\
& \stackrel{(iv)}{=} \hat{\kappa}_\ell(\boldsymbol{\lambda}) \hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{b}_{\ell,n}(\boldsymbol{\lambda}) h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \delta_{nm}. \tag{3.26}
\end{aligned}$$

Here we have used (3.17) and (3.15) in (i), (3.2) and (3.15) in (iii), (3.24) and (2.34) in (iv). In order to show (ii), we need to shift the integration contour to the imaginary direction. It is allowed if $\hat{V}_\ell(x; \boldsymbol{\lambda}) \phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})^2$ has no pole in the rectangular domain $x_1 \leq \text{Re } x \leq x_2$, $0 \leq \frac{\text{Im } x}{\gamma} \leq \frac{1}{2}$. This condition is fulfilled if the deforming polynomial $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ has no zero in the rectangular domain $x_1 \leq \text{Re } x \leq x_2$, $|\text{Im } x| \leq \frac{1}{2}|\gamma|$, which is indeed the case.

3.3 Other intertwining relations

It is interesting to note that the operator $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ intertwines those of the original and deformed systems $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})$:

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}) \mathcal{A}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \mathcal{A}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}), \tag{3.27}$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})^\dagger = \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}). \tag{3.28}$$

In terms of the definitions of the forward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ (2.27), $\mathcal{F}_\ell(\boldsymbol{\lambda})$ (2.58), $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (3.18), and $\mathcal{B}(\boldsymbol{\lambda})$ (2.28), $\mathcal{B}_\ell(\boldsymbol{\lambda})$ (2.59), the above relations are rewritten as:

$$\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})} \hat{\mathcal{F}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}) \mathcal{F}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})} \mathcal{F}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}), \tag{3.29}$$

$$\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})} \hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) \mathcal{B}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})} \mathcal{B}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{F}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}). \tag{3.30}$$

These relations can be proven by explicit calculation with the help of the three formulas of the deforming polynomial $\xi_\ell(\eta; \boldsymbol{\lambda})$ (2.67)–(2.69) in § 2.3.

By applying $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})$ to (2.24) and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ to (2.25) together with the use of (3.27), (3.28) and (3.17), we obtain

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}) = \sqrt{\frac{\hat{\kappa}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})}{\hat{\kappa}_\ell(\boldsymbol{\lambda})}} f_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \frac{\hat{f}_{\ell,n-1}(\boldsymbol{\lambda} + \boldsymbol{\delta})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} \phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})$$

$$= f_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})\phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (3.31)$$

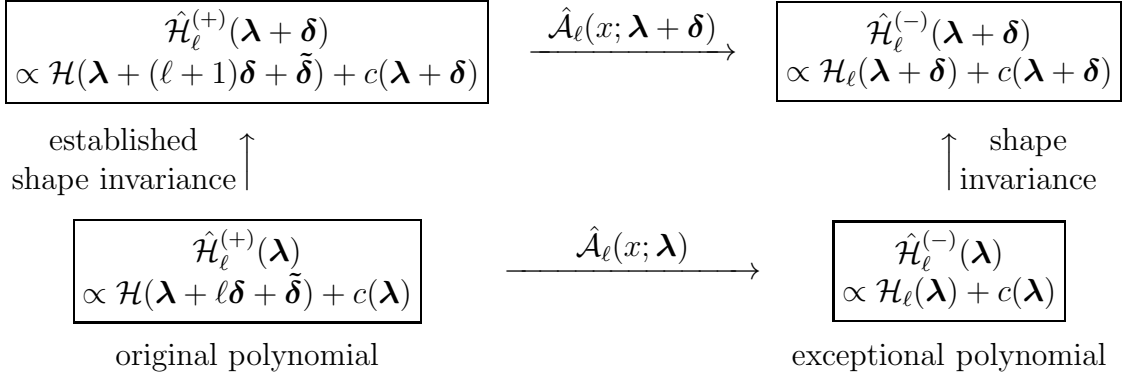
$$\begin{aligned} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) &= \sqrt{\frac{\hat{\kappa}_\ell(\boldsymbol{\lambda})}{\hat{\kappa}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})}} b_{n-1}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \frac{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n-1}(\boldsymbol{\lambda} + \boldsymbol{\delta})} \phi_{\ell,n}(x; \boldsymbol{\lambda}) \\ &= b_{n-1}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})\phi_{\ell,n}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}). \end{aligned} \quad (3.32)$$

In the calculation use is made of the explicit forms of $\hat{\kappa}_\ell(\boldsymbol{\lambda})$, $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$, $f_n(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$ in the second equalities. This provides a proof of (2.55)–(2.57) without recourse to the shape invariance. Likewise the above intertwining relations of the forward-backward shift operators (3.29)–(3.30) give the simple proof of (2.60)–(2.61), respectively, again without recourse to the shape invariance.

4 Summary and Comments

The Darboux-Crum transformations intertwining the Hamiltonians of the continuous Hahn, Wilson and Askey-Wilson polynomials with those of the corresponding exceptional polynomials are constructed in a unified fashion. This gives a much simpler expressions (2.52) of the exceptional Wilson and Askey-Wilson polynomials than those given in a previous paper [1]. The exceptional continuous Hahn polynomials are new. See the recent work of Gomez-Ullate et al [26]. It provides the iso-spectral Darboux-Crum transformations intertwining the Hamiltonians of the X_ℓ Laguerre polynomials to that of the radial oscillator and shows the shape invariance.

The present paper is a discrete version of the recent work [13] which provides the Darboux-Crum transformations intertwining the Hamiltonians of the radial oscillator/Darboux-Pöschl-Teller potential (the Laguerre and Jacobi polynomials) with those of the corresponding exceptional polynomials [14, 15, 27]. This offers a simple proof of the shape invariance of the ℓ -th exceptional polynomials through the established shape invariance of the original polynomials as depicted in the following commutative diagram:



Two ways of proving shape invariance of the exceptional polynomials system.

The Darboux-Crum transformation also supplies the annihilation/creation operators for the exceptional polynomial systems through those $a^{(\pm)}(\boldsymbol{\lambda})$ for the original system (2.36)

$$\hat{\mathcal{A}}_\ell(x; \boldsymbol{\lambda}) a^{(\pm)}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \hat{\mathcal{A}}_\ell(x; \boldsymbol{\lambda})^\dagger, \quad (4.1)$$

which map $\phi_{\ell,n}(x; \boldsymbol{\lambda})$ to $\phi_{\ell,n\pm 1}(x; \boldsymbol{\lambda})$. The analogous formulas work for the exceptional Laguerre and Jacobi polynomials [13]. As shown in [8, 9], the annihilation/creation operators together with the Hamiltonian constitute dynamical symmetry algebra of an exactly solvable system, for example, the q -oscillator algebra [29]. It would be an interesting challenge to clarify the structure of the dynamical symmetry algebras associated with the exceptional Askey type polynomials. Likewise the three term recurrence relations for the original polynomials are mapped to those of the exceptional polynomials [28]. However, their significance and utility are as yet unclear.

In [27], the shape invariance of the infinitely many potentials for the exceptional Laguerre and Jacobi polynomials are attributed to as many cubic identities among the original polynomials. The corresponding identities for the cH, W and AW polynomials are given by (2.49). These identities are quartic in ξ_ℓ and can be proven by using (2.68) and (2.69).

Rodrigues type formulas for the exceptional polynomials $P_{\ell,n}$ are obtained by multiple applications of the backward shift operators for the exceptional polynomials $\mathcal{B}_\ell(\boldsymbol{\lambda})$ (2.61) or multiple applications of the backward shift operators for the original polynomials $\mathcal{B}(\boldsymbol{\lambda})$ (2.30) followed by the intertwining forward shift operator $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (3.22)

$$P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) = \prod_{k=0}^{n-1} \frac{\mathcal{B}_\ell(\boldsymbol{\lambda} + k\boldsymbol{\delta})}{b_{n-1-k}(\boldsymbol{\lambda} + (\ell + k)\boldsymbol{\delta})} \cdot \xi_\ell(\eta(x); \boldsymbol{\lambda} + (n + 1)\boldsymbol{\delta}) \quad (4.2)$$

$$= \frac{\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} \prod_{k=0}^{n-1} \frac{\mathcal{B}(\boldsymbol{\lambda} + (\ell + k)\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})}{b_{n-1-k}(\boldsymbol{\lambda} + (\ell + k)\boldsymbol{\delta})} \cdot 1, \quad (4.3)$$

where $\prod_{k=0}^{n-1} a_k = a_0 a_1 \cdots a_{n-1}$ is the ordered product notation of operators. In this connection, it is interesting to compare the forward-backward shift operators for the original polynomials (2.27)–(2.28), the exceptional polynomials (2.58)–(2.59) and the intertwining ones (3.20)–(3.21). For the original polynomials, the forward shift operator $\mathcal{F}(\boldsymbol{\lambda})$ (2.27) is trivial, whereas the potential dependence is contained in the backward shift operator $\mathcal{B}(\boldsymbol{\lambda})$ (2.28). For the exceptional polynomials, the forward shift operator $\mathcal{F}_\ell(\boldsymbol{\lambda})$ (2.58) is solely determined by the deforming polynomial ξ_ℓ and the potential function enters in the backward shift operator $\mathcal{B}_\ell(\boldsymbol{\lambda})$ (2.59). The intertwining ones are really twisted. The twisted part $v_1(x)$ of the potential function enters in $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (3.20), whereas the untwisted part $v_2(x)$ remains in $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ (3.21).

Certain generating functions of the exceptional polynomials are easily constructed from those of the original polynomials through the main result (2.52). Suppose a generating function of the original orthogonal polynomials P_n is given by

$$G(t, x; \boldsymbol{\lambda}) = \sum_{n=0}^{\infty} \alpha_n(\boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda}) t^n, \quad (4.4)$$

in which $\alpha_n(\boldsymbol{\lambda})$ is a constant. For explicit forms, see [4], eqs. (1.4.11)–(1.4.13) for cH, (1.1.12)–(1.1.15) for W and (3.1.13)–(3.1.15) for AW. Then eq. (2.52) gives the corresponding generating function of the exceptional orthogonal polynomials $P_{\ell,n}$,

$$\begin{aligned} & \sum_{n=0}^{\infty} \alpha_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \hat{f}_{\ell,n}(\boldsymbol{\lambda}) P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) t^n \\ &= \frac{-i}{\varphi(x)} \left(v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) G(t, x - i\frac{\gamma}{2}; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right. \\ & \quad \left. - v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) G(t, x + i\frac{\gamma}{2}; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right). \end{aligned} \quad (4.5)$$

It is well known that the Laguerre polynomials are obtained from the Jacobi polynomials in a certain limit. Likewise the Wilson polynomials are produced by a certain limit from the Askey-Wilson polynomials. The corresponding limiting relations for the exceptional Laguerre and Wilson polynomials are discussed in [15] and [28], respectively. Here we comment on the limiting relations among the exceptional continuous Hahn polynomials and the exceptional Wilson polynomials. The continuous Hahn polynomials are obtained from the Wilson

polynomials in the following limit:

$$\lim_{L \rightarrow \infty} (-2L)^{-n} W_n((x+L)^2; \alpha_1 - iL, \alpha_3 + iL, \alpha_2 - iL, \alpha_4 + iL) = n! p_n(x; \alpha_1, \alpha_2, \alpha_3, \alpha_4). \quad (4.6)$$

By taking $x^W = x^{\text{cH}} + L$ and $\lambda^W = (a_1^{\text{cH}} - iL, a_1^{\text{cH}} + iL, a_2^{\text{cH}} - iL, a_2^{\text{cH}*} + iL)$, and after appropriate overall rescaling, various quantities of the exceptional Wilson systems reduce to those of the exceptional continuous Hahn system in this $L \rightarrow \infty$ limit.

The idea of the infinitely many exceptional Laguerre and Jacobi polynomials [14] was obtained while studying various possibility of generating exactly solvable quantum mechanical systems from the known shape invariant ones, *e.g.* the radial oscillator and DPT potentials. Adler's modification [30, 31] of Crum's theorem is the most comprehensive way to generate infinite variety of exactly solvable systems from a known ones [32, 25]. After the formulation of the discrete quantum mechanics version of Crum's theorem [33, 34, 35], its modification à la Adler is now published [36]. Its Appendix has many formulas reminiscent of those given in this paper.

There are two types of discrete quantum mechanics. In one of them, as discussed in this paper, difference operators cause shifts in the pure imaginary direction [9]. The formulation of the other type of discrete quantum mechanics, in which difference operators cause real shifts, is provided in [10]. The corresponding eigenfunctions of the known solvable systems consist of the orthogonal polynomials of a discrete variable, for example, the (q -) Racah polynomials, [4, 37]. It is a good challenge to construct the exceptional polynomials corresponding to these orthogonal polynomials of a discrete variable.

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