q-Deformed Oscillators and D-branes on Conifold

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We study the q-deformed oscillator algebra acting on the wavefunctions of non-compact D-branes in the topological string on conifold. We find that the mirror B-model curve of conifold appears from the commutation relation of the q-deformed oscillators.

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1. Introduction

The topological string is an interesting playground to study the gauge/string duality via the geometric transition [1,2,3]. It is also interesting to study how the target space geometry is quantized in this context. Recently, it is realized that the A-model side is described by a statistical model of crystal melting [4,5], while the B-model side is reformulated as matrix models [6,7]. In both cases, a spectral curve appears either as the limit shape of molten crystal or from the loop equation of matrix model. It is expected that the spectral curve should be viewed as a "quantum Riemann surface" in the sense that the coordinates of this curve become non-commutative at finite string coupling g_s. It is argued that the natural language to deal with this phenomenon is the D-module [8,9].

In this paper, we study the non-commutative algebraic structure in the mirror B-model side of the topological string on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. As noticed in [10], there is an underlying q-deformed oscillator (or, q-oscillator for short) structure in the wavefunction of non-compact D-branes on conifold. We study the representation of q-oscillators in terms of non-commutative coordinates and show that the mirror curve of conifold appears from the commutation relation of the q-oscillators.

This paper is organized as follows. In section 2, we construct the q-oscillators $A_\pm$ acting on the D-brane wavefunctions in terms of variables obeying the commutation relation $[p, x] = g_s$. In section 3, we show that the commutation relation of q-oscillators $A_\pm$ is nothing but the mirror curve of resolved conifold. In section 4, we revisit the computation of the partition function of Chern-Simons theory using the q-oscillators. We conclude in section 5 with discussion.

1.1. Our Notations

Here we summarize our notations and elementary formulas used in the text. We denote the string coupling as g_s, and the Kähler parameter of the base $\mathbb{P}^1$ of resolved conifold as $t = g_s N$. Then we introduce the parameters q and Q by

$$q = e^{-g_s}, \quad Q = q^N = e^{-t}. \tag{1.1}$$

We also introduce the canonical pair of coordinates x and p satisfying

$$[p, x] = g_s. \tag{1.2}$$

We use the representation such that x acts as a multiplication and p acts as a derivative $p = g_s\partial_x$. In particular, when acting on a constant function "1", we get

$$x \cdot 1 = x, \quad p \cdot 1 = 0, \quad e^{ax} \cdot 1 = e^{ax}, \quad e^{bp} \cdot 1 = 1, \tag{1.3}$$

where a and b are c-number parameters. More generally, e^{bp} shifts x when acting on a function $f(x)$

$$e^{bp} f(x) = f(x + bg_s). \tag{1.4}$$

We frequently use the commutation relations such as

$$e^{ax+bp} = e^{ax}e^{bp}q^{-\frac{1}{2}ab}, \quad e^{bp}e^{ax} = e^{ax}e^{bp}q^{-ab}, \tag{1.5}$$

which follow from the the relation $e^A e^B = e^B e^A e^{[A,B]} = e^{A+B}e^{\frac{1}{2}[A,B]}$ which is valid when $[A, B]$ is a c-number.

2. Operator Representation of D-brane Wavefunctions

In this section, we consider an operator representation of the wavefunction of D-branes on conifold. The wavefunction of a D-brane on conifold in the standard framing is given by [11–17]

$$Z_N(x) = \prod_{k=0}^{\infty} \frac{1 - q^{k+\frac{1}{2}}e^{-x}}{1 - Qq^{k+\frac{1}{2}}e^{-x}} = \prod_{n=1}^{N}(1 - q^{n-\frac{1}{2}}e^{-x}) = \sum_{r=0}^{N} \begin{bmatrix} N \\ r \end{bmatrix} q^{\frac{r^2}{2}}(-1)^r e^{-rx}, \tag{2.1}$$

where the q-binomial is defined as

$$\begin{bmatrix} N \\ r \end{bmatrix} = \frac{(q)_N}{(q)_r(q)_{N-r}}, \quad (q)_r = \prod_{n=1}^{r}(1 - q^n). \tag{2.2}$$

In order to rewrite the wavefunction $Z_N(x)$ in the operator language, let us recall the q-binomial formula for the variables z and w obeying the relation $wz = qzw$

$$(z + w)^N = \sum_{r=0}^{N} \begin{bmatrix} N \\ r \end{bmatrix} z^r w^{N-r}. \tag{2.3}$$

Applying this formula for $z = -e^{-x+p}$ and $w = e^p$, we find

$$(e^p - e^{-x+p})^N = \sum_{r=0}^{N} \begin{bmatrix} N \\ r \end{bmatrix} (-1)^r e^{-rx+rp}e^{(N-r)p} = \sum_{r=0}^{N} \begin{bmatrix} N \\ r \end{bmatrix} q^{\frac{r^2}{2}}(-1)^r e^{-rx}e^{Np}. \tag{2.4}$$

In the last step, we used the commutation relation (1.5). By comparing (2.1) and (2.4), we see that the D-brane wavefunction is written as the operator $(e^p - e^{-x+p})^N$ acting on the constant function "1", according to the rule in (1.3). Namely, the D-brane wavefunction has a simple expression

$$Z_N(x) = A_+^N \cdot 1 \tag{2.5}$$

where A_+ is given by

$$A_+ = e^p - e^{-x+p}. \tag{2.6}$$

Using the commutation relation (1.5), A_+ is also written as

$$A_+ = (1 - q^{\frac{1}{2}} e^{-x}) e^p = e^p (1 - q^{-\frac{1}{2}} e^{-x}). \tag{2.7}$$

One can see that the product form of $Z_N(x)$ in (2.1) easily follows from our simple expression $Z_N(x) = A_+^N \cdot 1$. By repeatedly using the relation (1.5), we can change the ordering so that e^p comes to the rightmost position

$$\begin{aligned}
A_+^2 &= (1 - q^{\frac{1}{2}} e^{-x}) e^p (1 - q^{\frac{1}{2}} e^{-x}) e^p = (1 - q^{\frac{1}{2}} e^{-x})(1 - q^{\frac{1}{2}+1} e^{-x}) e^{2p}, \\
A_+^3 &= A_+^2 (1 - q^{\frac{1}{2}} e^{-x}) e^p = (1 - q^{\frac{1}{2}} e^{-x})(1 - q^{\frac{1}{2}+1} e^{-x})(1 - q^{\frac{1}{2}+2} e^{-x}) e^{3p}, \\
&\cdots\cdots \\
A_+^N &= (1 - q^{\frac{1}{2}} e^{-x})(1 - q^{\frac{1}{2}+1} e^{-x}) \cdots (1 - q^{\frac{1}{2}+N-1} e^{-x}) e^{Np}.
\end{aligned} \tag{2.8}$$

When acting on "1", the last expression of A_+^N gives the product form of wavefunction in (2.1).

3. q-Oscillators and D-brane Wavefunctions

In this section, we consider the q-oscillator structure of the wavefunction $Z_N(x)$. The q-oscillator structure for the Rogers-Szegö polynomials and the Stieltjes-Wigert polynomials, which are related to our wavefunction $Z_N(x)$ by a change of framing [15,16,18], was studied in [19,20].

From the definition $Z_N(x) = A_+^N \cdot 1$, it follows that A_+ acts as the raising operator

$$A_+ Z_N(x) = Z_{N+1}(x). \tag{3.1}$$

Next consider the operator lowering the index of $Z_N(x)$. From the relation

$$e^{-p} Z_N(x) = Z_N(x - g_s) = (1 - q^{-\frac{1}{2}} e^{-x}) Z_{N-1}(x), \tag{3.2}$$

and

$$Z_N(x) = (1 - q^{N-\frac{1}{2}} e^{-x}) Z_{N-1}(x), \tag{3.3}$$

one can see that the operator A_- defined by

$$A_- = \frac{q^{\frac{1}{2}} e^x (1 - e^{-p})}{1 - q}, \tag{3.4}$$

lowers the index of $Z_N(x)$ as desired:

$$A_- Z_N(x) = \frac{1 - q^N}{1 - q} Z_{N-1}(x). \tag{3.5}$$

Note that A_- annihilates the constant function "1"

$$A_- \cdot 1 = 0. \tag{3.6}$$

This implies that the constant function "1" can be identified as the vacuum of q-oscillator

$$1 \quad \leftrightarrow \quad |0\rangle. \tag{3.7}$$

Now let us see that A_+ and A_- obey the q-oscillator algebra. From (3.1) and (3.5), we find

$$\begin{aligned} A_+ A_- Z_N(x) &= \frac{1 - q^N}{1 - q} A_+ Z_{N-1}(x) = \frac{1 - q^N}{1 - q} Z_N(x), \\ A_- A_+ Z_N(x) &= A_- Z_{N+1}(x) = \frac{1 - q^{N+1}}{1 - q} Z_N(x). \end{aligned} \tag{3.8}$$

It follows that A_+ and A_- satisfy the q-oscillator algebra

$$[A_-, A_+] = q^{\widehat{N}}, \qquad A_- A_+ - q A_+ A_- = 1. \tag{3.9}$$

Here the operator $q^{\widehat{N}}$ is defined as

$$q^{\widehat{N}} A_\pm = q^{\pm 1} A_\pm q^{\widehat{N}}, \qquad q^{\widehat{N}} \cdot 1 = 1, \qquad q^{\widehat{N}} Z_N(x) = q^N Z_N(x). \tag{3.10}$$

We can directly compute the algebra of $A_\pm$ using the commutation relations (1.5) without acting them on the wavefunction as above. From the expression of $A_\pm$ in terms of variables x, p in (2.6) and (3.4), we can show that $A_\pm$ satisfy

$$\begin{aligned} A_- A_+ &= 1 + \frac{q(1 - q^{-\frac{1}{2}} e^x)(1 - e^p)}{1 - q}, \\ A_+ A_- &= \frac{(1 - q^{-\frac{1}{2}} e^x)(1 - e^p)}{1 - q}, \\ [A_-, A_+] &= 1 - (1 - q^{-\frac{1}{2}} e^x)(1 - e^p), \qquad A_- A_+ - q A_+ A_- = 1. \end{aligned} \tag{3.11}$$

Therefore, we arrive at the expression of $q^{\widehat{N}}$ in terms of x and p as

$$q^{\widehat{N}} = 1 - (1 - q^{-\frac{1}{2}} e^{x})(1 - e^{p}). \tag{3.12}$$

One can check that the RHS of (3.12) satisfies the defining properties of $q^{\widehat{N}}$ (3.10). When acting on $Z_N(x)$, the relation (3.12) leads to the following constraint on the wavefunction $Z_N(x)$:

$$1 - (1 - q^{-\frac{1}{2}} e^{x})(1 - e^{p}) = Q. \tag{3.13}$$

This agrees with the known mirror B-model curve for the conifold [21,22]. In other words, we find an interesting interpretation of the mirror curve (3.13): it represents the q-oscillator relation $[A_-, A_+] = q^{\widehat{N}}$ written in the canonical variables x, p.

3.1. Wavefunction of anti-D-Brane

In contrast to the ordinary oscillator, in the case of q-oscillator we can consider the formal inverse of $A_+ = e^{p}(1 - q^{-\frac{1}{2}} e^{-x})$

$$A_+^{-1} = (1 - q^{-\frac{1}{2}} e^{-x})^{-1} e^{-p}. \tag{3.14}$$

Repeating the similar calculation as in (2.8), we find

$$\begin{aligned} A_+^{-N} &= (1 - q^{-\frac{1}{2}} e^{-x})^{-1} e^{-p} (1 - q^{-\frac{1}{2}} e^{-x})^{-1} \cdots (1 - q^{-\frac{1}{2}} e^{-x})^{-1} e^{-p} \\ &= (1 - q^{-\frac{1}{2}} e^{-x})^{-1} (1 - q^{-\frac{1}{2}-1} e^{-x})^{-1} \cdots (1 - q^{-\frac{1}{2}-N+1} e^{-x})^{-1} e^{-Np} \\ &= \prod_{n=1}^{N} (1 - q^{n-\frac{1}{2}-N} e^{-x})^{-1} e^{-Np}. \end{aligned} \tag{3.15}$$

From this expression, we see that

$$Z_{-N}(x) = A_+^{-N} \cdot 1 = \frac{1}{Z_N(x-t)}. \tag{3.16}$$

As argued in [12], this is interpreted as the wavefunction of anti-D-brane, up to a shift of x. $Z_{-N}(x)$ has another interpretation as the wavefunction of D-brane ending on a different leg of the toric diagram of conifold [15,16,17]. The constraint equation $[A_-, A_+] = q^{\widehat{N}}$ for $Z_{-N}(x)$ reads

$$1 - (1 - q^{-\frac{1}{2}} e^{x})(1 - e^{p}) = Q^{-1}, \tag{3.17}$$

which can be rewritten as

$$1 - (1 - q^{-\frac{1}{2}} e^{x-t})(1 - e^{-p}) = Q. \tag{3.18}$$

This is the same form as the equation for $Z_N(x)$ (3.13) under the replacement $(x, p) \to (x - t, -p)$. This is consistent with the wavefunction behavior of D-brane amplitude under the change of polarization [23,15].

4. Closed String Partition Function and the q-Oscillators

In this section, we consider the partition function of closed topological string on conifold from the viewpoint of q-oscillators. Although our computation is essentially the same as [24], we emphasize that the q-oscillator structure makes the computation more transparent. There is essentially no new result in this section, but we include this for completeness.

As shown in [25,2,1], the closed string partition function of conifold is given by the $U(N)$ Chern-Simons theory on S^3. Later, it was noticed in [24] that the same partition function is written as the log-normal matrix model

$$Z_{\log} = \int_{N\times N} dM e^{-\frac{1}{2g_s}\mathrm{Tr}(\log M)^2}. \tag{4.1}$$

The orthogonal polynomial associated with this log-normal measure is known as the Stieltjes-Wigert polynomial $S_N(x)$ [26], which is given by

$$S_N(x) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} q^{k^2+\frac{1}{2}k}(-1)^k e^{-kx}. \tag{4.2}$$

In the following we will show that $S_N(x)$ is related to $Z_N(x)$ by the following change of framing

$$x \to x - p + \frac{1}{2}g_s, \quad p \to p. \tag{4.3}$$

This is realized by the conjugation by the operator $U = e^{-\frac{p^2}{2g_2}+\frac{1}{2}p}$

$$UxU^{-1} = x - p + \frac{1}{2}g_s, \quad UpU^{-1} = p. \tag{4.4}$$

Note that the operator U preserves the vacuum

$$U\cdot 1 = 1 \quad \leftrightarrow \quad U|0\rangle = |0\rangle. \tag{4.5}$$

In terms of the conjugated q-oscillators

$$\begin{aligned} B_+ &= UA_+U^{-1} = e^p - q^{\frac{1}{2}}e^{-x+2p} = e^p - q^{\frac{3}{2}}e^{-x}e^{2p} \\ B_- &= UA_-U^{-1} = \frac{q^{\frac{1}{2}}e^x(e^{-p}-e^{-2p})}{1-q} \end{aligned} \tag{4.6}$$

the Stieltjes-Wigert polynomial is written in the same form as $Z_N(x) = A_+^N \cdot 1$

$$S_N(x) = B_+^N \cdot 1. \tag{4.7}$$

Using the commutation relation (1.5), one can see that (4.7) agrees with the expression (4.2), as promised

$$S_N(x) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} (-q^{\frac{1}{2}} e^{-x+2p})^k e^{(N-k)p} \cdot 1 = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} q^{k^2+\frac{1}{2}k} (-1)^k e^{-kx}. \tag{4.8}$$

In order to calculate the partition function (4.1), we need the norm of the function $\Psi_N(x)$ defined by

$$\Psi_N(x) \equiv \Big\langle \det(e^{-x} - M) \Big\rangle = (-1)^N q^{-N^2 - \frac{N}{2}} S_N(x), \tag{4.9}$$

which is known as the FZZT wavefunction [23,27]. We see that the q-oscillator representation (4.7) simplify the computation of the norm.

The log-normal measure associated with (4.1) becomes Gaussian under the change of variable $y = e^{-x}$

$$\int_0^\infty \frac{dy}{2\pi} e^{-\frac{1}{2g_s}(\log y)^2} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\frac{x^2}{2g_s} - x}. \tag{4.10}$$

The inner product with respect to this measure is defined as

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\frac{x^2}{2g_s} - x} f(x) g(x). \tag{4.11}$$

Let us consider the adjoint of B_+ with respect to this measure

$$\langle f, B_+ g \rangle = \langle (B_+)^\dagger f, g \rangle. \tag{4.12}$$

Using the representation of B_+ in terms of x, p (4.6), the action of B_+ on a function $g(x)$ reads

$$B_+ g(x) = g(x + g_s) - q^{\frac{3}{2}} e^{-x} g(x + 2g_s). \tag{4.13}$$

Then the inner product $\langle f, B_+ g \rangle$ is written as

$$\begin{aligned} \langle f, B_+ g \rangle &= \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\frac{x^2}{2g_s} - x} f(x) \Big[g(x + g_s) - q^{\frac{3}{2}} e^{-x} g(x + 2g_s) \Big] \\ &= \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\frac{x^2}{2g_s} - x} q^{-\frac{1}{2}} e^x \Big[f(x - g_s) - f(x - 2g_s) \Big] g(x). \end{aligned} \tag{4.14}$$

From this equation, we find that the adjoint of B_+ is proportional to B_- (4.6)

$$(B_+)^\dagger = q^{-\frac{1}{2}} e^x (e^{-p} - e^{-2p}) = (q^{-1} - 1) B_-. \tag{4.15}$$

Now it is straightforward to calculate the norm of $S_N(x)$. In order to do that, it is convenient to use the bra-ket notation

$$\begin{aligned} 1 \quad &\leftrightarrow \quad |0\rangle \\ S_n(x) = B_+^n \cdot 1 \quad &\leftrightarrow \quad |n\rangle = B_+^n|0\rangle. \end{aligned} \tag{4.16}$$

Noticing that B_- satisfies the same relation as A_- (3.5) when acting on the state $|n\rangle$

$$B_-|0\rangle = 0, \quad B_-|n\rangle = \frac{1-q^n}{1-q}|n-1\rangle, \tag{4.17}$$

and using the relation between $(B_+)^\dagger$ and B_- (4.15), we find

$$(B_+)^\dagger|n\rangle = q^{-1}(1-q^n)|n-1\rangle. \tag{4.18}$$

Now let us compute the norm $\langle n|n\rangle$. First, the norm of unit function $1 = |0\rangle$ is given by

$$\langle 0|0\rangle = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\frac{x^2}{2g_s}-x} = \sqrt{\frac{g_s}{2\pi}} q^{-\frac{1}{2}}. \tag{4.19}$$

The norm of higher state $|n\rangle$ is determined by the following recursion relation

$$\langle n|n\rangle = \langle n-1|(B_+)^\dagger|n\rangle = q^{-1}(1-q^n)\langle n-1|n-1\rangle. \tag{4.20}$$

Finally, the norm of $|n\rangle$ is found to be

$$\langle n|n\rangle = \langle 0|0\rangle q^{-n} \prod_{k=1}^{n}(1-q^k) = \sqrt{\frac{g_s}{2\pi}} q^{-n-\frac{1}{2}} \prod_{k=1}^{n}(1-q^k). \tag{4.21}$$

This agrees with the known result of the norm of $S_n(x)$ with respect to the measure (4.10) [26].

The partition function of matrix model (4.1) is given by the product of the norm of $\Psi_n(x)$ in (4.9)

$$h_n = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\frac{x^2}{2g_s}-x} \Psi_n(x)^2 = q^{-2n^2-n}\langle n|n\rangle = \sqrt{\frac{g_s}{2\pi}} q^{-2(n+\frac{1}{2})^2} \prod_{k=1}^{n}(1-q^k). \tag{4.22}$$

Then the partition function of matrix model (4.1) is given by

$$Z_{\log} = \prod_{n=0}^{N-1} h_n = \eta(\widetilde{q})^N q^{-\frac{2N^3}{3}+\frac{N}{8}} \prod_{n=1}^{\infty} \left(\frac{1-Qq^n}{1-q^n}\right)^n \tag{4.23}$$

where $\eta(\widetilde{q})$ denotes the η-function

$$\eta(\widetilde{q}) = \sqrt{\frac{g_s}{2\pi}} q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = \widetilde{q}^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-\widetilde{q}^n), \tag{4.24}$$

with

$$\widetilde{q} = e^{-\frac{4\pi^2}{g_s}}. \tag{4.25}$$

On the other hand, the partition function of the $U(N)$ Chern-Simons theory on S^3 is [28]

$$Z_{CS} = \left(\frac{g_s}{2\pi}\right)^{\frac{N}{2}} \prod_{k=1}^{N-1} (q^{-\frac{k}{2}} - q^{\frac{k}{2}})^{N-k} = \eta(\widetilde{q})^N q^{-\frac{N^3}{12}+\frac{N}{24}} \prod_{n=1}^{\infty} \left(\frac{1-Qq^n}{1-q^n}\right)^n, \tag{4.26}$$

which agrees with $Z_{\log}$ up to terms in the free energy which are polynomial in t.

We note in passing that $Z_{\log}$ is written as the norm of Fermi sea state $|\Psi\rangle$

$$Z_{\log} \propto \langle\Psi|\Psi\rangle \ , \qquad |\Psi\rangle = \frac{1}{\sqrt{N!}} |0\rangle \wedge |1\rangle \wedge \cdots \wedge |N-1\rangle. \tag{4.27}$$

5. Discussion

In this paper, we find that the mirror B-model curve of resolved conifold has an interesting interpretation as the q-oscillator relation $[A_-, A_+] = q^{\widehat{N}}$ itself. It would be interesting to find the physical origin of this algebraic structure.

Recently, the partition function of the Donaldson-Thomas theory of the non-commutative version of conifold is calculated by Szendröi [29]

$$Z_{NC} = \prod_{n=1}^{\infty} \left(\frac{1-Q^{-1}q^n}{1-q^n}\right)^n \prod_{n=1}^{\infty} \left(\frac{1-Qq^n}{1-q^n}\right)^n. \tag{5.1}$$

The last factor of Z_{NC} in (5.1) is the same as the Chern-Simons partition function in (4.26), but the first factor is different

$$\eta(q)^N = q^{\frac{N}{24}} \prod_{k=1}^{\infty} (1-q^k)^N = q^{\frac{N}{24}} \prod_{n=N+1}^{\infty} (1-Q^{-1}q^n)^n \prod_{n=1}^{\infty} (1-q^n)^{-n}. \tag{5.2}$$

This difference is discussed in the context of the wall-crossing phenomena [30,31,32]. It is tempting to identify the extra factor in Z_{NC} as the effect of anti-D-branes [33]

$$\prod_{n=1}^{N} (1-Q^{-1}q^n)^n \sim \prod_{n=1}^{N} \frac{1}{\langle -n| -n\rangle}. \tag{5.3}$$

However, one should not take this relation literally, since both sides of (5.3) vanish when N is an integer. We leave this as an interesting future problem.

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