

RATIONAL VISIBILITY OF A LIE GROUP IN THE MONOID OF SELF-HOMOTOPY EQUIVALENCES OF A HOMOGENEOUS SPACE

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ABSTRACT. Let M be a homogeneous space admitting a left translation by a connected Lie group G . The adjoint to the action gives rise to a map from G to the monoid of self-homotopy equivalences of M . The purpose of this paper is to investigate the injectivity of the homomorphism which is induced by the adjoint map on the rational homotopy. In particular, *the visible degrees* are determined explicitly for all the cases of simple Lie groups and their associated homogeneous spaces of rank one which are classified by Oniscik.

1. INTRODUCTION

The study of rational visibility problems we here consider is motivated by work due to Kedra and McDuff [18] in which symplectic topological methods are effectively used. In this paper, we deal with such problems relying upon algebraic models for spaces and maps, which are viewed as complements of those developed and used in recent work on rational homotopy of functions spaces [4, 6, 7, 8, 21, 22, 23].

Let $f : X \rightarrow Y$ be a map between connected spaces whose fundamental groups are abelian. We say that X is *rationally visible* in Y with respect to the map f if the induced map $f_* \otimes 1 : \pi_i(X) \otimes \mathbb{Q} \rightarrow \pi_i(Y) \otimes \mathbb{Q}$ is injective for any $i \geq 1$. Let $\text{aut}_1(M)$ denote the identity component of the monoid of self-homotopy equivalences of a space M . Let G be a connected Lie group and M an appropriate homogeneous space M admitting a left translation by G . We then define a map of monoids

$$\lambda_{G,M} : G \rightarrow \text{aut}_1(M)$$

by $\lambda_{G,M}(g)(x) = gx$ for $g \in G$ and $x \in M$. In this paper, we investigate the rational visibility of G in $\text{aut}_1(M)$ with respect to the map $\lambda_{G,M}$.

The monoid map $\lambda_{G,M}$ factors through the identity component $\text{Homeo}_1(M)$ of the monoid of homeomorphisms of M as well as the identity component $\text{Diff}_1(M)$ of the space of diffeomorphisms of M . Therefore the rational visibility of G in $\text{aut}_1(M)$ implies that of G in $\text{Homeo}_1(M)$ and $\text{Diff}_1(M)$. We also expect that non-trivial characteristic classes of the classifying spaces $B\text{aut}_1(M)$, $B\text{Homeo}_1(M)$ and $B\text{Diff}_1(M)$ can be obtained through the study of rational visibility. Very little is known about the (rational) homotopy of the groups $\text{Homeo}_1(M)$ and $\text{Diff}_1(M)$ for a general manifold M ; see [10] for the calculation of $\pi_i(\text{Diff}_1(S^n)) \otimes \mathbb{Q}$ for i in some

2000 Mathematics Subject Classification: 55P62, 57R19, 57R20, 57T35.

Key words and phrases. Self-homotopy equivalence, homogeneous space, Sullivan model.

This research was partially supported by a Grant-in-Aid for Scientific Research (C)20540070 from Japan Society for the Promotion of Science.

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range. Then such implication and expectation inspire us to consider the visibility problems of Lie groups. We refer the reader to papers [12] and [38] for the study of rational homotopy types of $\text{aut}_1(M)$ itself and related function spaces.

The key device for the study of rational visibility is the function space model due to Brown and Szczarba [5] and Haefliger [15]. Especially, an explicit rational model for the map $\lambda_{G,M}$ is constructed by using that for the evaluation map described in [7] and [19]; see Theorem 3.3. By analyzing such elaborate models, we obtain a recognition principle for rational visibility in Theorem 3.1 below. We here emphasize that not only does our machinery developed in this paper allow us to give other proofs to results in [18], [32], [34] and [39] concerning rational visibility but also it leads us to an unifying way of looking at the visibility problem explicitly as is seen in Tables 1 and 2 below.

In the rest of this section, we state our results.

Theorem 1.1. *Let G be a simply-connected Lie group and T a torus in G which is not necessarily maximal. Then G is rationally visible in $\text{aut}_1(G/T)$ with respect to the map $\lambda_{G,G/T}$ defined by the left translation of G/T by G .*

Theorem 1.1 is a generalization of the result [32, Proposition 2.4], in which T is assumed to be the maximal torus of G . We mention that the result due to Notbohm and Smith plays an important role in the proof of the uniqueness of fake Lie groups with a maximal torus; see [31, Section 1]. Theorem 1.1 is deduced from Theorem 1.2 below, which gives a tractable criterion for the rational visibility.

In order to describe Theorem 1.2, we fix notations. Let G be a connected Lie group and U a closed connected subgroup of G . Let $B\iota : BU \rightarrow BG$ be the map induced by the inclusion $\iota : U \rightarrow G$. We assume that the rational cohomology of BG is a polynomial algebra, say $H^*(BG; \mathbb{Q}) \cong \mathbb{Q}[c_1, \dots, c_k]$. In what follows, we write $H^*(X)$ for the cohomology of a space X with coefficients in the rational field.

Consider the Lannes division functor $(H^*(BU) : H^*(G/U))$ in the category of differential graded algebras (DGA's). Then the functor is regarded as a quotient of the free algebra $\wedge(H^*(BU) \otimes H_*(G/U))$, which in turn is isomorphic to $\wedge(QH^*(BU) \otimes H_*(G/U))$ as an algebra, where $QH^*(BU)$ denotes the vector space of indecomposable elements; see Section 2 for more details. Under the isomorphism, we can define an algebra map $u : (H^*(BU) : H^*(G/U)) \rightarrow \mathbb{Q}$ by $u(h \otimes b_*) = \langle j^*(h), b_* \rangle$, where $j : G/U \rightarrow BU$ is the fibre inclusion of the fibration $G/U \xrightarrow{j} BU \xrightarrow{B\iota} BG$. Moreover define M_u to be the ideal of $(H^*(BU) : H^*(G/U))$ generated by the set

$$\{\eta \mid \deg \eta < 0\} \cup \{\eta - u(\eta) \mid \deg \eta = 0\}.$$

Let $\pi : H^*(BU) \otimes H_*(G/U) \rightarrow (H^*(BU) : H^*(G/U))$ denote the composite of the inclusion $H^*(BU) \otimes H_*(G/U) \rightarrow \wedge(H^*(BU) \otimes H_*(G/U))$ and the projection.

A recognition principle for rational visibility, Theorem 3.1 below, enables one to deduce the following result.

Theorem 1.2. *With the above notation, assume that for $c_{i_1}, \dots, c_{i_s} \in \{c_1, \dots, c_k\}$, there are elements $c_{j_1}, \dots, c_{j_s} \in H^*(BG)$ and $u_{1*}, \dots, u_{s*} \in H^{\geq 1}(G/U)$ such that*

$$\pi((B\iota)^*(c_{i_t}) \otimes 1_*) \equiv \pi((B\iota)^*(c_{j_t}) \otimes u_{t*})$$

for $t = 1, \dots, s$ modulo decomposable elements in $(H^*(BG) : H^*(G/U))/M_u$. Then there exists a map $\rho : \times_{j=1}^s S^{\deg c_{j_j} - 1} \rightarrow G$ such that $\times_{j=1}^s S^{\deg c_{j_j} - 1}$ is rationally visible in $\text{aut}_1(G/U)$ with respect to the map $(\lambda_{G,G/U}) \circ \rho$. In particular,

if $(B\iota)^*(c_{i_1}), \dots, (B\iota)^*(c_{i_s})$ are decomposable elements, then $\pi((B\iota)^*(c_{i_t}) \otimes 1_*) \equiv 0$ in $(H^*(BG) : H^*(G/U))/M_u$ for $t = 1, \dots, s$ and hence one obtains the same conclusion.

For a Lie group G and a homogeneous space M which admits a left translation by G , put $n(G) := \{i \in \mathbb{N} \mid \pi_i(G) \otimes \mathbb{Q} \neq 0\}$ and define the set $vd(G, M)$ of *visible degrees* by

$$vd(G, M) = \{i \in n(G) \mid (\lambda_{G, M})_* : \pi_i(G) \otimes \mathbb{Q} \rightarrow \pi_i(\text{aut}_1(M)) \otimes \mathbb{Q} \text{ is injective}\}.$$

As for monoids of homeomorphisms and of diffeomorphisms, we benefit by the study of rational visibility. In fact, we have an immediate but very important corollary.

Corollary 1.3. *If $l \in vd(G, M)$, then there exists an element with infinite order in $\pi_l(\text{Diff}_1(M))$ and $\pi_l(\text{Homeo}_1(M))$.*

Example 1.4. Since $SO(d+1)/SO(d)$ is homeomorphic to the sphere S^d , we can define the maps $\lambda_{SO(d+1), S^d} : SO(d+1) \rightarrow \text{aut}_1(S^d)$ by left translations. The Haefliger, Brown and Szczarba model for the function space $\text{aut}_1(S^d)$ allows us to deduce that $\text{aut}_1(S^{2m+1}) \simeq_{\mathbb{Q}} S^{2m+1}$ and $\text{aut}_1(S^{2m}) \simeq_{\mathbb{Q}} S^{4m-1}$; see Example 2.4 below. Therefore $\lambda_{SO(d+1), S^d}$ is not injective on the rational homotopy in general. However it follows that the induced maps

$$(\lambda_{SO(2m+2), S^{2m+1}})_* : \pi_{2m+1}(SO(2m+2)) \otimes \mathbb{Q} \rightarrow \pi_{2m+1}(\text{aut}_1(S^{2m+1})) \otimes \mathbb{Q},$$

$$(\lambda_{SO(2m+1), S^{2m}})_* : \pi_{4m-1}(SO(2m+1)) \otimes \mathbb{Q} \rightarrow \pi_{4m-1}(\text{aut}_1(S^{2m})) \otimes \mathbb{Q}$$

are injective. In fact it is well known that, as algebras, $H^*(BSO(2m+1)) \cong \mathbb{Q}[p_1, \dots, p_m]$ and $H^*(BSO(2m+2)) \cong \mathbb{Q}[p_1, \dots, p_m, \chi]$, where $\deg p_j = 4j$ and $\deg \chi = 2m+2$. Moreover for inclusions $\iota_1 : SO(2m+1) \rightarrow SO(2m+2)$ and $\iota_2 : SO(2m) \rightarrow SO(2m+1)$, we see that $(B\iota_1)^*(\chi) = 0$ and $(B\iota_2)^*(p_m) = \chi^2$; see [30]. Thus Theorem 1.2 yields that $vd(SO(2m+2), S^{2m+1}) = \{2m+1\}$ and that $vd(SO(2m+1), S^{2m}) = \{4m-1\}$.

The result [1, 1.1.5 Lemma] allows one to conclude that the map $SO(d+1) \rightarrow \text{Diff}_1(S^d)$ induced by the left translations is injective on the homotopy group. This implies that the inclusion $\text{Diff}_1(S^d) \rightarrow \text{aut}_1(S^d)$ is surjective on the rational homotopy group.

Theorem 3.1 yields another proof of results due to Kedra and McDuff [18] and Sasao [34].

Theorem 1.5. [18, Proposition 4.8][34] *Let M be the flag manifold of the form $U(m)/U(m_1) \times \dots \times U(m_l)$. Then $SU(m)$ is rationally visible in $\text{aut}_1(M)$ with respect to the map $\lambda_{SU(m), M}$ given by the left translations; that is, $vd(SU(m), M) = n(SU(m)) = \{3, 5, \dots, 2m-1\}$. In particular, the localized map*

$$(\lambda_{SU(m), U(m)/U(m-1) \times U(1)})_{\mathbb{Q}} : SU(m)_{\mathbb{Q}} \rightarrow \text{aut}_1(\mathbb{C}P^{m-1})_{\mathbb{Q}}$$

is a homotopy equivalence.

Furthermore, the same argument as in the proof of Theorem 1.5 allows one to establish the following result.

Theorem 1.6. *Let M be the flag manifold $Sp(m)/Sp(m_1) \times \dots \times Sp(m_l)$. Then $vd(Sp(m), M) = \{7, 11, \dots, 4m-1\}$. In particular, the 3-connected cover $Sp(m)\langle 3 \rangle$*

is rationally visible in $\text{aut}_1(M)$ with respect to $\lambda_{Sp(m),M} \circ \pi$, where $\pi : Sp(m)\langle 3 \rangle \rightarrow Sp(m)$ is the projection.

Let G be a compact connected simple Lie group and U a closed connected subgroup for which G/U is a simply-connected homogeneous space of rank one; that is, its rational cohomology is generated by a single element. In order to illustrate usefulness of Theorems 1.2 and 3.1, by applying the results, we determine visible degrees of G in $\text{aut}_1(G/U)$ for each couple (G, U) classified by Oniscik in [33, Theorems 2 and 4].

In the following table, we first list such homogeneous spaces of the form G/U with a simple Lie group G and its subgroup U , which is not diffeomorphic to spheres or projective spaces, together with the sets $vd(G, G/U)$.

(G, U, index)	$(G/U)_{\mathbb{Q}}$	$vd(G, G/U)$	$n(G)$
(1) $(SO(2n+1), SO(2n-1) \times SO(2), 1)$	$\mathbb{C}P^{2n-1}$	$\{3, \dots, 4n-1\}$	$\{3, \dots, 4n-1\}$
(2) $(SO(2n+1), SO(2n-1), 1)$	S^{4n-1}	$\{4n-1\}$	$\{3, \dots, 4n-1\}$
(3) $(SU(3), SO(3), 4)$	S^5	$\{5\}$	$\{3, 5\}$
(4) $(Sp(2), SU(2), 10)$	S^7	$\{7\}$	$\{3, 7\}$
(5) $(G_2, SO(4), (1, 3))$	$\mathbb{H}P^2$	$\{11\}$	$\{3, 11\}$
(6) $(G_2, U(2), 3)$	$\mathbb{C}P^5$	$\{3, 11\}$	$\{3, 11\}$
(7) $(G_2, SU(2), 3)$	S^{11}	$\{11\}$	$\{3, 11\}$
(6)' $(G_2, U(2), 1)$	$\mathbb{C}P^5$	$\{3, 11\}$	$\{3, 11\}$
(7)' $(G_2, SU(2), 1)$	S^{11}	$\{11\}$	$\{3, 11\}$
(8) $(G_2, SO(3), 4)$	S^{11}	$\{11\}$	$\{3, 11\}$
(9) $(G_2, SO(3), 28)$	S^{11}	$\{11\}$	$\{3, 11\}$

Table 1

Here the value of the index of the inclusion $j : U \rightarrow G$ is regarded as the integer i by which the induced map $j_* : H_3(U; \mathbb{Z}) \rightarrow H_3(G; \mathbb{Z}) = \mathbb{Z}$ is multiplication; see the proof of [33, Lemma 4]. The second column denotes the rational homotopy type of G/U corresponding a triple (G, U, i) . The homogeneous spaces G/U for the cases (6)' and (7)' are diffeomorphic to those for the cases (1) and (2) with $n = 3$, respectively. Moreover, the homogeneous spaces are not diffeomorphic each other except for the cases (6)' and (7)'.

The following table describes visible degrees of a simple Lie group G in $\text{aut}_1(G/U)$ for which G/U is of rank one and diffeomorphic to the sphere or the projective space, where the second column denotes the diffeomorphism type of the homogeneous space G/U for the triple (G, U, i) .

(G, U, index)	G/U	$vd(G, G/U)$	$n(G)$
(10) $(SU(n+1), SU(n), 1)$	S^{2n+1}	$\{2n+1\}$	$\{3, \dots, 2n+1\}$
(11) $(SU(n+1), S(U(n) \times U(1)), 1)$	$\mathbb{C}P^n$	$\{3, \dots, 2n+1\}$	$\{3, \dots, 2n+1\}$
(12) $(SO(2n+1), SO(2n), 1)$	S^{2n}	$\{4n-1\}$	$\{3, \dots, 4n-1\}$
(13) $(SO(9), SO(7), 1)$	S^{15}	$\{15\}$	$\{3, 7, 11, 15\}$
(14) $(Spin(7), G_2, 1)$	S^7	$\{7\}$	$\{3, 7, 11\}$
(15) $(Sp(n), Sp(n-1), 1)$	S^{4n-1}	$\{4n-1\}$	$\{3, \dots, 4n-1\}$
(16) $(Sp(n), Sp(n-1) \times S^1, 1)$	$\mathbb{C}P^{2n-1}$	$\{3, \dots, 4n-1\}$	$\{3, \dots, 4n-1\}$
(17) $(Sp(n), Sp(n-1) \times Sp(1), 1)$	$\mathbb{H}P^{n-1}$	$\{7, \dots, 4n-1\}$	$\{3, \dots, 4n-1\}$
(18) $(SO(2n), SO(2n-1), 1)$	S^{2n-1}	$\{2n-1\}$	$\{3, \dots, 4n-5, 2n-1\}$
(19) $(F_4, Spin(9), 1)$	$\mathcal{L}P^2$	$\{23\}$	$\{3, 11, 15, 23\}$
(20) $(G_2, SU(3), 1)$	S^6	$\{11\}$	$\{3, 11\}$

Table 2

Here $\mathcal{L}P^2$ stands for the Cayley plane.

The former half of Theorem 1.2, namely the Lannes functor argument, does work well enough when determining the set $vd(G_2, G_2/U(2))$ of visible degrees in case (6) in Table 1; see Section 8. Observe that for the cases (12) and (18) the results follow from those in Example 1.4. We are aware that in the above tables G is rationally visible in $\text{aut}_1(G/U)$ if and only if G/U has the rational homotopy type of the complex projective space. It should be mentioned that for the map $\lambda_* : \pi_*(F_4) \otimes \mathbb{Q} \rightarrow \pi_*(\text{aut}_1(\mathcal{L}P^2)) \otimes \mathbb{Q}$, the restriction $(\lambda_*)_{15}$ is not injective though the vector space $\pi_{15}(\text{aut}_1(\mathcal{L}P^2)) \otimes \mathbb{Q}$ and $\pi_{15}(F_4) \otimes \mathbb{Q}$ are non-trivial; see Section 8. Moreover, Corollary 1.3 enables us to obtain non-trivial elements with infinite order in $\pi_l(\text{Diff}_1(M))$ and $\pi_l(\text{Homeo}_1(M))$ for each homogeneous space M described in Tables 1 and 2.

Let X be a space and $\mathcal{H}_{H,X}$ the monoid of all homotopy equivalences that act trivially on the rational homology of X . The result [18, Proposition 4.8] asserts that if X is generalized flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$, then the map $B\psi_{SU(m)} : BSU(m) \rightarrow B\mathcal{H}_{H,X}$ arising from the left translations is injective on the rational homotopy. Let $\iota : \text{aut}_1(X) \rightarrow \mathcal{H}_{H,X}$ be the inclusion. Since $B\psi_{SU(m)} = B\iota \circ B\lambda_{SU(m),X}$, the result [18, Proposition 4.8] yields Theorem 1.5. Theorem 1.7 below guarantees that the converse also holds; that is, the result due to Kedra and McDuff is deduced from Theorem 1.5; see Section 7.

Before describing Theorem 1.7, we recall an F_0 -space, which is a simply-connected finite complex with finite-dimensional rational homotopy and trivial rational cohomology in odd degree. For example, a homogeneous space G/T for which G is a connected Lie group and T is a maximal torus of G is an F_0 -space.

Theorem 1.7. *Let X be an F_0 -space or a space having the rational homotopy type of the product of odd dimensional spheres and G a connected topological group which acts on X . Then $(B\lambda_{G,X})_* : H_*(BG) \rightarrow H_*(B\text{aut}_1(X))$ is injective if and only if so is $(B\psi)_* : H_*(BG) \rightarrow H_*(B\mathcal{H}_{H,X})$. Here $\psi : G \rightarrow \mathcal{H}_{H,X}$ denotes the morphism of monoids induced by the action of G on X .*

We now provide an overview of the rest of the paper. In Section 2, we recall a model for the evaluation map of a function space from [19], [7] and [17]. In Section 3, a rational model for the map $\lambda_{G,M}$ mentioned above is constructed. Section 4 is devoted to the study of a model for the left translation of a Lie group on a homogeneous space. In Section 5, we prove Theorem 1.2. Theorem 1.5 is proved in Section 6. In Section 7, we prove Theorem 1.7. The results on visible degrees in Tables 1 and 2 are verified in Section 8. In Appendix, Section 9, the group cohomology of $\text{Diff}_1(M)$ for an appropriate homogeneous space M is discussed. By using Theorem 1.2, we find a non-trivial class in the group cohomology.

2. PRELIMINARIES

The tool for the study of the rational visibility problem is a rational model for the evaluation map $ev : \text{aut}_1(M) \times M \rightarrow M$, which is described in terms of the rational model due to Brown and Szczarba [5] and Haefliger [15]. For the convenience of the reader and to make notation more precise, we recall from [7] and [19] the model for the evaluation map. We shall use the same terminology as in [3] and [11].

Throughout the paper, for an augmented algebra A , we write QA for the space $\overline{A}/\overline{A} \cdot \overline{A}$ of indecomposable elements, where \overline{A} denotes the augmentation ideal. For a DGA (A, d) , let d_0 denote the linear part of the differential.

In what follows, we assume that a space is nilpotent and has the homotopy type of a connected CW complex with rational homology of finite type unless otherwise explicitly stated. We denote by $X_{\mathbb{Q}}$ the localization of a nilpotent space X .

Let A_{PL} be the simplicial commutative cochain algebra of polynomial differential forms with coefficients in \mathbb{Q} ; see [3] and [11, Section 10]. Let \mathcal{A} and $\Delta\mathcal{S}$ be the category of DGA's and that of simplicial sets, respectively. Let $\text{DGA}(A, B)$ and $\text{Simpl}(K, L)$ denote the hom-sets of the categories \mathcal{A} and $\Delta\mathcal{S}$, respectively. Following Bousfield and Gugenheim [3], we define functors $\Delta : \mathcal{A} \rightarrow \Delta\mathcal{S}$ and $\Omega : \Delta\mathcal{S} \rightarrow \mathcal{A}$ by $\Delta(A) = \text{DGA}(A, A_{PL})$ and by $\Omega(K) = \text{Simpl}(K, A_{PL})$.

Let (B, d_B) be a connected, locally finite DGA and B_* denote the differential graded coalgebra defined by $B_q = \text{Hom}(B^{-q}, \mathbb{Q})$ for $q \leq 0$ together with the coproduct D and the differential d_{B^*} which are dual to the multiplication of B and to the differential d_B , respectively. We denote by I the ideal of the free algebra $\wedge(\wedge V \otimes B_*)$ generated by $1 \otimes 1_* - 1$ and all elements of the form

$$a_1 a_2 \otimes \beta - \sum_i (-1)^{|a_2||\beta'_i|} (a_1 \otimes \beta'_i)(a_2 \otimes \beta''_i),$$

where $a_1, a_2 \in \wedge V$, $\beta \in B_*$ and $D(\beta) = \sum_i \beta'_i \otimes \beta''_i$. Observe that $\wedge(\wedge V \otimes B_*)$ is a DGA with the differential $d := d_A \otimes 1 \pm 1 \otimes d_{B^*}$. The result [5, Theorem 3.5] implies that the composite $\rho : \wedge(V \otimes B_*) \hookrightarrow \wedge(\wedge V \otimes B_*) \rightarrow \wedge(\wedge V \otimes B_*)/I$ is an isomorphism of graded algebras. Moreover, it follows that [5, Theorem 3.3] that $dI \subset I$. Thus $(\wedge(V \otimes B_*), \delta = \rho^{-1}d\rho)$ is a DGA. Observe that, for an element $v \in V$ and a cycle $e \in B_*$, if $d(v) = v_1 \cdots v_m$ with $v_i \in V$ and $D^{(m-1)}(e_j) = \sum_j e_{j_1} \otimes \cdots \otimes e_{j_m}$, then

$$(2.1) \quad \delta(v \otimes e) = \sum_j \pm (v_1 \otimes e_{j_1}) \cdots (v_m \otimes e_{j_m}).$$

Here the sign is determined by the Koszul rule; that is, $ab = (-1)^{\deg a \deg b} ba$ in a graded algebra. Let F be the ideal of $E := \wedge(V \otimes B_*)$ generated by $\bigoplus_{i < 0} E^i$ and $\delta(E^{-1})$. Then E/F is a free algebra and $(E/F, \delta)$ is a Sullivan algebra (not necessarily connected), see the proofs of [5, Theorem 6.1] and of [7, Proposition 19].

Remark 2.1. The result [5, Corollary 3.4] implies that there exists a natural isomorphism $\text{DGA}(\wedge(\wedge V \otimes B_*)/I, C) \cong \text{DGA}(\wedge V, B \otimes C)$ for any DGA C . Then $\wedge(\wedge V \otimes B_*)/I$ is regarded as the Lannes division functor $(\wedge V : B)$ by definition.

The singular simplicial set of a topological space U is denoted by ΔU and let $|X|$ be the geometrical realization of a simplicial set X . By definition, $A_{PL}(U)$ the DGA of polynomial differential forms on U is given by $A_{PL}(U) = \Omega \Delta U$. Given spaces X and Y , we denote by $\mathcal{F}(X, Y)$ the space of continuous maps from X to Y . The connected component of $\mathcal{F}(X, Y)$ containing a map $f : X \rightarrow Y$ is denoted by $\mathcal{F}(X, Y; f)$.

Let $\alpha : A = (\wedge V, d) \xrightarrow{\cong} A_{PL}(Y) = \Omega \Delta Y$ be a Sullivan model (not necessarily minimal) for Y and $\beta : (B, d) \xrightarrow{\cong} A_{PL}(X)$ a Sullivan model for X for which B is connected and locally finite. For the function space $\mathcal{F}(X, Y)$ which is considered below, we assume that

$$(2.2) \quad \dim \bigoplus_{q \geq 0} H^q(X; \mathbb{Q}) < \infty \quad \text{or} \quad \dim \bigoplus_{i \geq 2} \pi_i(Y) \otimes \mathbb{Q} < \infty.$$

Then the proof of [19, Proposition 4.3] enables us to deduce the following lemma; see also [7].

Lemma 2.2. (i) Let $\{b_j\}$ and $\{b_{j*}\}$ be a basis of B and its dual basis of B_* , respectively and $\tilde{\pi} : \wedge(A \otimes B_*) \rightarrow (\wedge(A \otimes B_*)/I)/F$ denote the projection. Define a map $m(ev) : A \rightarrow (\wedge(A \otimes B_*)/I)/F \otimes B$ by

$$m(ev)(x) = \sum_j (-1)^{\tau(|b_j|)} \tilde{\pi}(x \otimes b_{j*}) \otimes b_j,$$

for $x \in A$, where $\tau(n) = [(n+1)/2]$, the greatest integer in $(n+1)/2$. Then $m(ev)$ is a well-defined DGA map.

(ii) There exists a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times X_{\mathbb{Q}} & \xrightarrow{ev} & Y_{\mathbb{Q}} \\ \Theta \times 1 \uparrow & & \parallel \\ |\Delta(E/F)| \times |\Delta(B)| & \xrightarrow{|\Delta m(ev)|} & |\Delta(A)| \end{array}$$

in which Θ is the homotopy equivalence described in [5, Sections 2 and 3]; see also [19, (3.1)].

We next recall a Sullivan model for a connected component of a function space. Choose a basis $\{a'_k, b'_k, c'_j\}_{k,j}$ for B_* so that $d_{B_*}(a'_k) = b'_k$, $d_{B_*}(c'_j) = 0$ and $c'_0 = 1$. Moreover we take a basis $\{v_i\}_{i \geq 1}$ for V such that $\deg v_i \leq \deg v_{i+1}$ and $d(v_{i+1}) \in \wedge V_i$, where V_i is the subvector space spanned by the elements v_1, \dots, v_i . The result [5, Lemma 5.1] ensures that there exist free algebra generators w_{ij} , u_{ik} and v_{ik} such that

$$(2.3) \quad w_{i0} = v_i \otimes 1 \text{ and } w_{ij} = v_i \otimes c'_j + x_{ij}, \text{ where } x_{ij} \in \wedge(V_{i-1} \otimes B_*),$$

$$(2.4) \quad \delta w_{ij} \text{ is in } \wedge(\{w_{sl}; s < i\}),$$

$$(2.5) \quad u_{ik} = v_i \otimes a'_k \text{ and } \delta u_{ik} = v_{ik}.$$

We then have a inclusion

$$(2.6) \quad \gamma : E := (\wedge(w_{ij}), \delta) \hookrightarrow (\wedge(V \otimes B_*), \delta),$$

which is a homotopy equivalence with a retract

$$(2.7) \quad r : (\wedge(V \otimes B_*), \delta) \rightarrow E;$$

see [5, Lemma 5.2] for more details. Let q be a Sullivan representative for a map $f : X \rightarrow Y$; that is, q fits into the homotopy commutative diagram

$$\begin{array}{ccc} \wedge W & \xrightarrow{\simeq} & A_{PL}(X) \\ q \uparrow & & \uparrow_{A_{PL}(f)} \\ \wedge V & \xrightarrow{\simeq} & A_{PL}(Y). \end{array}$$

Moreover we define a 0-simplex $\tilde{u} \in \Delta(\wedge(\wedge V \otimes B_*)/I)_0$ by

$$(2.8) \quad \tilde{u}(a \otimes b) = (-1)^{\tau(|a|)} b(q(a)),$$

where $a \in \wedge V$ and $b \in B_*$. Put $u = \Delta(\gamma)\tilde{u}$. Let M_u be the ideal of E generated by the set $\{\eta \mid \deg \eta < 0\} \cup \{\delta \eta \mid \deg \eta = 0\} \cup \{\eta - u(\eta) \mid \deg \eta = 0\}$. Then we see that $(E/M_u, \delta)$ is an explicit model for the connected component $F(X, Y; f)$; see [5, Theorem 6.1] and [17, Section 3]. The proof of [19, Proposition 4.3] and [17, Remark 3.4] allow us to deduce the following proposition; see also [7].

Proposition 2.3. *With the same notation as in Lemma 2.2, we define a map $m(ev) : A = (\wedge V, d) \rightarrow (E/M_u, \delta) \otimes B$ by*

$$m(ev)(x) = \sum_j (-1)^{\tau(|b_j|)} \pi \circ r(x \otimes b_{j*}) \otimes b_j,$$

for $x \in A$, where $\pi : E \rightarrow E/M_u$ denotes the natural projection. Then $m(ev)$ is a model for the evaluation map $ev : \mathcal{F}(X, Y; f) \times X \rightarrow Y$; that is, there exists a homotopy commutative diagram

$$\begin{array}{ccc} A_{PL}(Y) & \xrightarrow{A_{PL}(ev)} & A_{PL}(\mathcal{F}(X, Y; f) \times X) \\ \alpha \uparrow \simeq & & \uparrow \simeq \\ A & \xrightarrow{m(ev)} & (E/M_u, \delta) \otimes B \\ & & \simeq \uparrow \xi \otimes \beta \\ & & A_{PL}(\mathcal{F}(X, Y; f)) \otimes A_{PL}(X) \end{array}$$

in which $\xi : (E/M_u, \delta) \xrightarrow{\simeq} A_{PL}(\mathcal{F}(X, Y; f))$ is the Sullivan model for $\mathcal{F}(X, Y; f)$ due to Brown and Szczarba [5].

We call the DGA $(E/M_u, \delta)$ the Haefliger-Brown-Szczarba model (HBS-model for short) for the function space $\mathcal{F}(X, Y; f)$.

Example 2.4. Let M be a space whose rational cohomology is isomorphic to the truncated algebra $\mathbb{Q}[x]/(x^m)$, where $\deg x = l$. Recall the model $(E/M_u, \delta)$ for $\text{aut}_1(M)$ mentioned in [17, Example 3.6]. Since the minimal model for M has the form $(\wedge(x, y), d)$ with $dy = x^m$, it follows that

$$E/M_u = \wedge(x \otimes 1_*, y \otimes (x^s)_*; 0 \leq s \leq m-1)$$

with $\delta(x \otimes 1_*) = 0$ and $\delta(y \otimes (x^s)_*) = (-1)^s \binom{m}{s} (x \otimes 1_*)^{m-s}$, where $\deg x \otimes 1_* = l$ and $\deg(y \otimes (x^s)_*) = lm - ls - 1$. Then the rational model $m(ev)$ for the evaluation map $ev : \text{aut}_1(M) \times M \rightarrow M$ is given by $m(ev)(x) = (x \otimes 1_*) \otimes 1 + 1 \otimes x$ and

$$m(ev)(y) = \sum_{s=0}^{m-1} (-1)^s (y \otimes (x^s)_*) \otimes x^s + 1 \otimes y.$$

Remark 2.5. We here describe variants of the HBS-model for a function space.

(i) Let $\wedge \tilde{V} \xrightarrow{\simeq} A_{PL}(Y)$ be a Sullivan model (not necessarily minimal) and $B \xrightarrow{\simeq} A_{PL}(X)$ a Sullivan model of finite type. We recall the homotopy equivalence $\gamma : E \rightarrow \tilde{E} = \wedge(\wedge V \otimes B_*)/I$ mentioned in (2.6). Let $\tilde{u} \in \Delta(\tilde{E})_0$ be a 0-simplex and u a 0-simplex of E defined by composing \tilde{u} with the quasi-isomorphism γ . Then the induced map $\tilde{\gamma} : E/M_u \rightarrow \tilde{E}/M_{\tilde{u}}$ is a quasi-isomorphism. In fact the results [5, Theorem 6.1] and [7, Proposition 19] imply that the projections onto the quotient DGA's E/M_u and $\tilde{E}/M_{\tilde{u}}$ induce homotopy equivalences $\Delta(p) : \Delta(E/M_u) \rightarrow \Delta(E)_u$ and $\Delta(\tilde{p}) : \Delta(\tilde{E}/M_{\tilde{u}}) \rightarrow \Delta(\tilde{E})_{\tilde{u}}$, respectively. Here K_v denotes the connected component containing the vertex v for a simplicial set K , namely, the set of simplices all of whose faces are at v .

Then we have a commutative diagram

$$\begin{array}{ccc} \pi_*(|\Delta(E/M_u)|) & \xrightarrow[\cong]{|\Delta(p)|} & \pi_*(|\Delta(E)|, |u|) \\ |\Delta(\bar{\gamma})|_* \uparrow & & \uparrow |\Delta(\gamma)|_* \\ \pi_*(|\Delta(\tilde{E}/M_{\tilde{u}})|) & \xrightarrow[\cong]{|\Delta(\bar{p})|} & \pi_*(|\Delta(\tilde{E})|, |\tilde{u}|). \end{array}$$

Since γ is a homotopy equivalence, it follows that $|\Delta(\gamma)|_*$ is an isomorphism and hence so is $|\Delta(\bar{\gamma})|_*$. This yields that $|\Delta(\bar{\gamma})|$ is a homotopy equivalence. By virtue of the Sullivan-de Rham equivalence Theorem [3, 9.4], we see that $\bar{\gamma}$ is a quasi-isomorphism.

As in Lemma 2.2, we define the DGA map $\widetilde{m}(ev) : (\wedge V, d) \rightarrow \widetilde{E}/\widetilde{F} \otimes B$ and let $m(ev) : (\wedge V, d) \rightarrow \widetilde{E}/M_{\tilde{u}} \otimes B$ be the DGA defined by $m(ev) = \pi \otimes 1 \circ \widetilde{m}(ev)$. We then have a homotopy commutative diagram

$$\begin{array}{ccc} & & E/M_u \otimes B \\ & \nearrow^{m(ev)} & \\ \wedge V & & \simeq \downarrow \bar{\gamma} \otimes 1 \\ & \searrow_{m(ev)} & \widetilde{E}/M_{\tilde{u}} \otimes B. \end{array}$$

In fact the homotopy between $id_{\widetilde{E}}$ and $\gamma \circ r$ defined in [5, Lemma 5.2] induces a homotopy between $id_{\widetilde{E}/\widetilde{F}}$ and $\gamma \circ r : \widetilde{E}/\widetilde{F} \rightarrow E/F \rightarrow \widetilde{E}/\widetilde{F}$. It is immediate that $r \circ \gamma = id_{E/F}$. Let $m(ev)' : \wedge V \rightarrow E/F \otimes B$ be the DGA defined as in Proposition 2.3. Then it follows that

$$\begin{aligned} \bar{\gamma} \otimes 1 \circ m(ev) &= \bar{\gamma} \otimes 1 \circ \pi \otimes 1 \circ m(ev)' \\ &= \pi \otimes 1 \circ \gamma \otimes 1 \circ r \otimes 1 \circ \widetilde{m}(ev) \\ &\simeq \pi \otimes 1 \circ \widetilde{m}(ev) = m(ev). \end{aligned}$$

(ii) In the case where X is formal, we have a more tractable model for $\mathcal{F}(X, Y; f)$. Suppose that X is a formal space with a minimal model $(B, d_B) = (\wedge W', d)$. Then there exists a quasi-isomorphism $k : (\wedge W', d) \rightarrow H^*(B)$ which is surjective; see [9, Theorem 4.1]. With the notation mentioned above, let $\{e_j\}_j$ be a basis for the homology $H(B_*)$ of the differential graded coalgebra $B_* = (\wedge W')_*$ and $\{v_i\}_i$ a basis for V . Then it follows from the proof of [5, Theorem 1.9] that the subalgebra $\mathbb{Q}\{v_i \otimes e_j\}$ is closed for the differential δ and that the inclusion $\mathbb{Q}\{v_i \otimes e_j\} \rightarrow \wedge(W \otimes B_*) = \widetilde{E}$ gives rise to a homotopy equivalence

$$\gamma : E' := (\wedge(v_i \otimes e_j), \delta) \rightarrow (\wedge(W \otimes B_*), \delta) = \widetilde{E}.$$

In fact, the elements w_{ij} in (2.3) can be chosen so that $w_{i0} = v_i \otimes 1_*$ and $w_{ij} = v_i \otimes e_j$ for $j \geq 1$. Moreover we see that there exists a retraction $r : \wedge(W \otimes B_*) \rightarrow E'$ which is the homotopy inverse of γ . Thus Proposition 2.3 remains true after replacing E by E' . Here the 0-simplex $\tilde{u} \in \Delta(\wedge(W \otimes B_*))_0$ needed in the construction of the model for $\mathcal{F}(X, Y; f)$ has the same form as in (2.8).

We conclude this section with some comments on models for a connected component of a function space and related maps.

In the original construction in [15] and [7] of a model for a function space $\mathcal{F}(X, Y)$, it is assumed that the source space X admits a finite dimensional model.

Indeed the construction of a model for the evaluation map in [7, Theorem 1] requires existence of such a model for the space X . As described in Lemma 2.2 and Proposition 2.3, our construction only needs the assumption (2.2). Thus our model for a function space endowed with a model for evaluation map is viewed as a generalization of that in [7].

The arguments in [5, Section 7] and [7] on a model for a connected component of $\mathcal{F}(X, Y)$ begin with a 0-simplex; that is, the considered component is that containing a map f which corresponds to the given 0-simplex via a sequence of weak equivalences between the singular simplicial set of $\mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$ and the simplicial set $\Delta(E/F)$; see [17, (2.3)]. On the other hand, for any given map $f : X \rightarrow Y$, an explicit form of a 0-simplex corresponding to f is clarified in [17, Remark 3.4] with (2.8). Thus our constructions in this section complement the basic constructions in rational homotopy theory of function spaces due to Buijs and Murillo [7].

Observe that $\text{aut}_1(X)$ is nothing but the function space $\mathcal{F}(X, X; id_M)$. Moreover, for a manifold M , the function space $\text{aut}_1(M)$ satisfies the assumption (2.2). Thus we have explicit models for $\text{aut}_1(X)$ and for the evaluation map according to the procedure in this section. Adding such the models, we moreover provide an elaborate model for the map $\lambda_{G, M}$ mentioned in Introduction in the next section; see Theorem 3.1 below.

3. A RATIONAL MODEL FOR THE MAP λ INDUCED BY LEFT TRANSLATION

Let M be a space admitting an action of Lie group G on the left. We define the map $\lambda : G \rightarrow \text{aut}_1(M)$ by $\lambda(g)(x) = gx$. The subjective in this section is to construct an algebraic model for the map

$$in \circ \lambda : G \rightarrow \text{aut}_1(M) \rightarrow \mathcal{F}(M, M),$$

where $in : \text{aut}_1(M) \rightarrow \mathcal{F}(M, M)$ denotes the inclusion. To this end we use a model for the evaluation map

$$ev : \mathcal{F}(X, Y) \times X \rightarrow Y$$

defined by $ev(f)(x) = f(x)$ for $f \in \mathcal{F}(X, Y)$ and $x \in X$, which is considered in [19] and [7].

Let G be a connected Lie group, U a closed subgroup of G and K a closed subgroup which contains U . Let $(\wedge V_G, d)$ and $(\wedge W, d)$ denote a minimal model for G and a Sullivan model for the homogeneous space G/U , respectively. Let $\lambda : G \rightarrow \mathcal{F}(G/U, G/K)$ be the adjoint of the composite of the left translation $G \times G/U \rightarrow G/U$ and projection $p : G/U \rightarrow G/K$. Observe that the map λ coincides with the composite

$$p_* \circ in \circ \lambda_{G, G/U} : G \rightarrow \text{aut}_1(G/U) \rightarrow \mathcal{F}(G/U, G/U) \rightarrow \mathcal{F}(G/U, G/K).$$

We shall construct a model for λ by using the HBS-model for $\mathcal{F}(G/U, G/K; p)$ mentioned in Remark 2.5 (i), a Sullivan representative

$$\zeta' : \wedge W \rightarrow \wedge V_G \otimes \wedge W'$$

for the composite $G \times G/U \rightarrow G/K$ of the left translation $G \times G/U \rightarrow G/U$ and the projection $p : G/U \rightarrow G/K$.

Let A , B and C be connected DGA's. Recall from [5, Section 3] the bijection $\Psi : (A \otimes B_*, C)_{DG} \xrightarrow{\cong} (A, C \otimes B)_{DG}$ defined by

$$\Psi(w)(a) = \sum_j (-1)^{\tau(|b_j|)} w(a \otimes b_{j*}) \otimes b_j.$$

Consider the case where $A = (\wedge W, d)$, $B = (\wedge W', d)$ and $C = (\wedge V_G, d)$. Moreover define a map $\tilde{\mu} : \wedge(A \otimes B_*) \rightarrow \wedge V_G$ by

$$(3.1) \quad \tilde{\mu}(y \otimes b_{j*}) = (-1)^{\tau(|b_j|)} \langle \zeta'(y), b_{j*} \rangle,$$

where $\langle \cdot, b_{j*} \rangle : \wedge V_G \otimes \wedge W' \rightarrow \wedge V_G$ is a map defined by $\langle x \otimes a, b_{j*} \rangle = x \cdot \langle a, b_{j*} \rangle$. Then we see that $\Psi(\tilde{\mu}) = \zeta'$. Hence it follows from [5, Theorem 3.3] that

$$\tilde{\mu} : \tilde{E} := \wedge(A \otimes B_*)/I \rightarrow \wedge V_G$$

is a well-defined DGA map. We define an augmentation $\tilde{u} : \tilde{E} \rightarrow \mathbb{Q}$ by $\tilde{u} = \varepsilon \circ \tilde{\mu}$, where $\varepsilon : \wedge V_G \rightarrow \mathbb{Q}$ is the augmentation. It is readily seen that $\tilde{\mu}(M_{\tilde{u}}) = 0$. Thus we see that $\tilde{\mu}$ induces a DGA map $\tilde{\mu} : \tilde{E}/M_{\tilde{u}} \rightarrow \wedge V_G$.

The result [16, Theorem 3.11] asserts that the map

$$e_{\sharp} : \mathcal{F}(G/U, (G/K); p) \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$$

is a localization. Thus we have the localization $\lambda_{\mathbb{Q}} : G_{\mathbb{Q}} \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$. Observe that $\lambda_{\mathbb{Q}}$ fits into the homotopy commutative diagram

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\lambda_{\mathbb{Q}}} & \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p) \\ e \uparrow & & \uparrow e_{\sharp} \\ G & \xrightarrow{\lambda} & \mathcal{F}(G/U, (G/K); p), \end{array}$$

where e denotes the localization map.

We then have a recognition principle for rational visibility.

Theorem 3.1. *Let $\{x_i\}_i$ be a basis for the image of the induced map*

$$H^*(Q(\tilde{\mu})) : H^*(Q(\tilde{E}/M_{\tilde{u}}), \delta_0) \rightarrow H^*(Q(\wedge V_G), d_0) = V_G.$$

Then there exists a map $\rho : \times_{j=1}^s S^{\deg x_i} \rightarrow G$ such that the map

$$(\lambda_{\mathbb{Q}} \circ \rho_{\mathbb{Q}})_* : \pi_*((\times_{j=1}^s S^{\deg x_i})_{\mathbb{Q}}) \rightarrow \pi_*(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}), e \circ p)$$

is injective. Moreover $(\lambda_{\mathbb{Q}})_ : \pi_i(G_{\mathbb{Q}}) \rightarrow \pi_i(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}), e \circ p)$ is injective if and only if $H^i(Q(\tilde{\mu}))$ is surjective.*

In order to prove Theorem 3.1, it suffices to show that $\tilde{\mu} : \tilde{E}/M_{\tilde{u}} \rightarrow \wedge V_G$ is a Sullivan model for the map $G_{\mathbb{Q}} \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$.

We first observe that the diagram

$$(3.2) \quad \begin{array}{ccc} \wedge V_G \otimes \wedge W' & \xleftarrow{\tilde{\mu} \otimes 1} & (\wedge(A \otimes B_*)/I)/F \otimes \wedge W' \\ & \searrow \zeta' & \nearrow m(ev) \\ & \wedge W & \end{array}$$

is commutative. Thus Lemma 2.2 enables us to obtain a commutative diagram (3.3)

$$(3.3) \quad \begin{array}{ccc} |\Delta \wedge V_G| \times |\Delta \wedge W'| & \xrightarrow{(\Theta \circ |\Delta \tilde{\mu}|) \times 1} & \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}) \times (G/U)_{\mathbb{Q}} \\ & \searrow \text{action}_{\mathbb{Q}} & \swarrow ev \\ & |\Delta \wedge W| = (G/K)_{\mathbb{Q}} & \end{array}$$

Observe that the assumption (2.2) is satisfied in the case where we here consider.

Since the restriction $|\Delta \zeta'|_{|* \times |\Delta \wedge W|}$ is homotopic to $p_{\mathbb{Q}}$, it follows from the commutativity of the diagram (3.3) that $p_{\mathbb{Q}} \simeq \Theta \circ |\Delta \tilde{\mu}|(*)$. This implies that $\Theta \circ |\Delta \tilde{\mu}|$ maps $G_{\mathbb{Q}}$ into the function space $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; p_{\mathbb{Q}})$.

Lemma 3.2. *Let $\lambda_{\mathbb{Q}} : G_{\mathbb{Q}} \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ be the localized map of λ mentioned above and $e^{\sharp} : \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; p_{\mathbb{Q}}) \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ the map induced by the localization $e : (G/U) \rightarrow (G/U)_{\mathbb{Q}}$. Then*

$$e^{\sharp} \circ \Theta \circ |\Delta \tilde{\mu}| \simeq \lambda_{\mathbb{Q}} : G_{\mathbb{Q}} \rightarrow \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; e \circ p).$$

Proof. Consider the commutative diagram

$$(3.4) \quad \begin{array}{ccc} [G \times G/U, G/K] & \xrightarrow{\theta} & [G, \mathcal{F}(G/U, G/K)] \\ e_* \downarrow & & \downarrow (e_{\sharp})_* \\ [G \times G/U, (G/K)_{\mathbb{Q}}] & \xrightarrow{\theta} & [G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})] \\ (e \times e)^* \uparrow \approx & & \uparrow e^* \\ [G_{\mathbb{Q}} \times (G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}] & & [G_{\mathbb{Q}}, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})] \\ & \searrow \theta & \approx \uparrow (e^{\sharp})_* \\ & & [G_{\mathbb{Q}}, \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}})] \end{array}$$

in which θ is the adjoint map and e stands for the localization map. It follows from the diagram (3.3) that $\theta(\text{action}_{\mathbb{Q}}) = \Theta \circ |\Delta \tilde{\mu}|$. Moreover we have $\theta(\text{action}) = e_{\sharp} \circ \lambda = \lambda_K \circ e$. Thus the commutativity of the diagram (3.4) implies that $e^*([e^{\sharp} \circ \Theta \circ |\Delta \tilde{\mu}|]) = e^*([\lambda_{\mathbb{Q}}])$ in $[G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})]$. Since G is connected, it follows that $(e^{\sharp}) \circ \Theta \circ |\Delta \tilde{\mu}| \circ e \simeq \lambda_{\mathbb{Q}} \circ e : G \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$. The fact that $e_{\sharp} : \mathcal{F}(G/U, (G/K); p) \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ is the localization yields that the induced map $e^* : [G_{\mathbb{Q}}, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)] \rightarrow [G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)]$ is bijective. This completes the proof. \square

Before proving Theorem 3.1, we recall some maps. For a simplicial set K , there exists a natural homotopy equivalence $\xi_K : K \rightarrow \Delta|K|$, which is defined by $\xi_K(\sigma) = t_{\sigma} : \Delta^n \rightarrow \{\sigma\} \times \Delta \rightarrow |K|$. This gives rise to a quasi-isomorphism $\xi_A : \Omega \Delta A \xrightarrow{\cong} \Omega \Delta|\Delta A|$. Moreover, we can define a bijection $\eta : \text{DGA}(A, \Omega K) \xrightarrow{\cong} \text{Simp}(K, \Delta A)$ by $\eta : \phi \mapsto f; f(\sigma)(a) = \phi(a)(\sigma)$, where $a \in A$ and $\sigma \in K$. We observe that $\eta^{-1}(id) : A \rightarrow \Omega \Delta A$ is a quasi-isomorphism if A is a connected Sullivan algebra; see [3, 10.1. Theorem].

Proof of Theorem 3.1. Let $\pi : \tilde{E} \rightarrow \tilde{E}/M_{\tilde{u}}$ be the projection. With the same notation as above, we then have a commutative diagram

$$(3.5) \quad \begin{array}{ccc} |\Delta(\wedge(W \otimes B_*)/F)| & \xrightarrow[\simeq]{\Theta} & \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}) \\ \nearrow^{|\Delta(\tilde{\mu})|} & \uparrow^{|\Delta\pi|} & \uparrow \\ |\Delta(\wedge V_G)| & \xrightarrow[|\Delta(\tilde{\mu})|]{} & |\Delta(\tilde{E}/M_{\tilde{u}})| \xrightarrow[|\Delta\pi|]{\simeq} |(\Delta\tilde{E})_{\tilde{u}}| \xrightarrow[\Theta]{\simeq} \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1, \tilde{u})])), \end{array}$$

where $[(1, \tilde{u})] \in |\Delta\tilde{E}|$ is the element whose representative is $(1, \tilde{u}) \in \Delta^0 \times (\Delta\tilde{E})_0$. Lemma 3.2 yields that

$$(3.6) \quad e^{\sharp} \circ \Theta \circ |\Delta\pi| \circ |\Delta\tilde{\mu}| \simeq e^{\sharp} \circ \Theta \circ |\Delta\tilde{\mu}| \simeq \lambda_{\mathbb{Q}}.$$

Thus we see that e^{\sharp} maps $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1, \tilde{u})]))$ to $\mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e^{\sharp} \circ \Theta([(1, \tilde{u})]))$, which is the connected component containing $\text{Im}(\lambda_{\mathbb{Q}})$. This implies that $\mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e^{\sharp} \circ \Theta([(1, \tilde{u})])) = \mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e \circ p)$. Therefore, by the naturality of maps η and ξ_A , we have a diagram

$$(3.7) \quad \begin{array}{ccc} A_{PL}(G_{\mathbb{Q}}) & \xleftarrow{A_{PL}(\lambda_{\mathbb{Q}})} & A_{PL}(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)) \\ \parallel & & \downarrow ((e^{\sharp})^*) \\ & & A_{PL}(\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1, \tilde{u})]))) \\ & & \downarrow \Theta^* \\ A_{PL}(|\Delta \wedge V_G|) & \xleftarrow{|\Delta\tilde{\mu}|^*} & A_{PL}(|\Delta(\tilde{E}/M_{\tilde{u}})|) = \Omega\Delta(\tilde{E}/M_{\tilde{u}}) \\ \uparrow \simeq^{t' := (\xi_{\wedge V_G})\eta^{-1}(id)} & & \simeq \uparrow \xi_{\tilde{E}/M_{\tilde{u}}}\eta^{-1}(id) =: t \\ \wedge V_G & \xleftarrow{\tilde{\mu}} & \tilde{E}/M_{\tilde{u}} \end{array}$$

in which the upper square is homotopy commutative and the lower square is strictly commutative. Lifting Lemma allows us to obtain a DGA map $\varphi : \tilde{E}/M_{\tilde{u}} \rightarrow A_{PL}(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}))$ such that $\Theta^* \circ ((e^{\sharp})^*) \circ \varphi \simeq t$ and hence $A_{PL}(\lambda_{\mathbb{Q}}) \circ \varphi \simeq t' \circ \tilde{\mu}$.

Given a space X , let $u : A \rightarrow A_{PL}(X)$ be a DGA map from a Sullivan algebra A . Let $[f]$ be an element of $\pi_n(X)$ and $\iota : (\wedge Z, d) \xrightarrow{\simeq} A_{PL}(S^n)$ the minimal model. By taking a Sullivan representative $\tilde{f} : A \rightarrow \wedge Z$ with respect to u , namely a DGA map satisfying the condition that $\iota \circ \tilde{f} \simeq A_{PL}(f) \circ u$, we define a map $\nu_u : \pi_n(X) \rightarrow \text{Hom}(H^n Q(A), \mathbb{Q})$ by $\nu_u([f]) = H^n Q(\tilde{f}) : H^n Q(A) \rightarrow H^n Q(\wedge Z) = \mathbb{Q}$. By virtue of [3, 6.4 Proposition], in particular, we have a commutative diagram

$$\begin{array}{ccc} \pi_n(G_{\mathbb{Q}}) & \xrightarrow{\lambda_{\mathbb{Q}}} & \pi_n(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)) \\ \nu_{t'} \downarrow \cong & & \cong \downarrow \nu_{\varphi} \\ \text{Hom}((V_G)^n, \mathbb{Q}) & \xrightarrow[HQ(\tilde{\mu})^*]{} & \text{Hom}(H^n Q(\tilde{E}/M_{\tilde{u}}), \mathbb{Q}). \end{array}$$

in which $\nu_{t'}$ and ν_{φ} are an isomorphism; see [3, 8.13 Proposition]. There exists an element $[f_i] \otimes q$ in $\pi_*(G) \otimes \mathbb{Q}$ which corresponds to the dual element x_i^* via the isomorphism $\pi_*(G) \otimes \mathbb{Q} \cong \pi_*(G_{\mathbb{Q}}) \xrightarrow{\nu_{t'}} \text{Hom}((V_G)^n, \mathbb{Q})$ for any $i = 1, \dots, s$. The required map $\rho : \times_{j=1}^s S^{\deg x_i} \rightarrow G$ is defined by the composite of the map $\times_{j=1}^s f_j$ and the product $\times_{j=1}^s G \rightarrow G$. \square

The existence of the commutative diagram (3.7) up to homotopy yields the following result.

Theorem 3.3. *The DGA map $\tilde{\mu} : \tilde{E}/M_{\tilde{u}} \rightarrow \wedge V_G$ is a model for the map $\lambda : G \rightarrow \mathcal{F}(G/U, G/K; p)$, namely a Sullivan representative in the sense of [11, Definition, page 154].*

4. A MODEL FOR THE LEFT TRANSLATION

In order to prove Theorems 1.2 and 1.5, a more explicit model for the map $\lambda_{G,M} : G \rightarrow \text{aut}_1(M)$ is required. To this end, we refine the model of the left translation described in the proof of Theorem 3.1.

We first observe that the cohomology $H^*(BU; \mathbb{Q})$ is isomorphic to a polynomial algebra with finite generators, say $H^*(BU; \mathbb{Q}) \cong \mathbb{Q}[h_1, \dots, h_l]$. We consider a commutative diagram of fibrations

$$\begin{array}{ccccc} & & G & \xlongequal{\quad} & G \\ & & \downarrow i & & \downarrow \\ G/U & \xleftarrow[\simeq]{h} & G \times_U E_U & \longrightarrow & E_G \\ & & \downarrow \bar{\pi} & & \downarrow \pi \\ & & BU & \xrightarrow[B\iota]{} & BG \end{array}$$

in which $h : G \times_U E_U \rightarrow G/U$ is a homotopy equivalence defined by $h([g, e]) = [g]$. This diagram yields a Sullivan model $(\wedge W, d)$ for G/U which has the form $(\wedge(h_1, \dots, h_l, x_1, \dots, x_k), d)$ with $dx_j = (B\iota)^*c_j$; see [11, Proposition 15.16] for the details. Moreover we have a model $(\wedge V_G, d)$ for G of the form $(\wedge(x_1, \dots, x_k), 0)$. Since $h \circ i$ is nothing but the projection $\pi : G \rightarrow G/U$, it follows that the natural projection $\rho : (\wedge(h_1, \dots, h_l, x_1, \dots, x_k), d) \rightarrow (\wedge(x_1, \dots, x_k), 0)$ is a Sullivan model for the map π .

Let $\beta : G \times (G \times_U E_U) \rightarrow G \times_U E_G$ be the action of G on $G \times_U E_U$. Then the left translation $tr : G \times G/U \rightarrow G/U$ coincides with β up to the homotopy equivalence $h : (G \times_U E_U) \rightarrow G/U$ mentioned above. Thus in order to obtain a model for the linear action, it suffices to construct a model for β . Recall the fibration $G \rightarrow G \times_U E_U \xrightarrow{\bar{\pi}} BU$ and the universal fibration $G \rightarrow E_G \xrightarrow{\pi} BG$. We here consider a commutative diagram

$$(4.1) \quad \begin{array}{ccccc} & & G \times (G \times_U E_U) & \xrightarrow{1 \times f} & G \times E_G \\ & \swarrow \beta & \downarrow \bar{\pi}' & \searrow \alpha & \downarrow \pi' \\ G \times_U E_U & \xrightarrow{f} & E_G & & BG \\ \downarrow \bar{\pi} & \swarrow = & \downarrow \pi & \searrow = & \downarrow \pi \\ BU & \xrightarrow[B\iota]{} & BU & \xrightarrow[B\iota]{} & BG \end{array}$$

in which π' and $\bar{\pi}'$ are fibrations with the same fibre $G \times G$ and the restrictions $\alpha|_{\text{fibre}} : G \times G \rightarrow G$ and $\beta|_{\text{fibre}} : G \times (G \times_U E_U) \rightarrow (G \times_U E_U)$ are the multiplication on G and the action of G , respectively. Let $i : (\wedge V_{BG}, 0) \rightarrow \wedge(\widetilde{V}_{BU}, d)$ be a Sullivan model for $B\iota$. In particular, we can choose such a model so that

$$\wedge \widetilde{V}_{BU} = \wedge(c_1, \dots, c_m) \otimes \wedge(h_1, \dots, h_l) \otimes \wedge(\tau_1, \dots, \tau_m)$$

and $d(\tau_i) = B\iota(c_i) - c_i$. By the construction of a model for pullback fibration mentioned in [11, page 205], we obtain a diagram

$$(4.2) \quad \begin{array}{ccccc} & & \wedge Z & \xleftarrow{v'} & \wedge W' \\ & \nearrow \tilde{\beta} & \uparrow u' & \xleftarrow{\tilde{\alpha}} & \uparrow \\ \wedge V & \xleftarrow{v} & \wedge V' & \xrightarrow{\tilde{\alpha}} & \wedge W' \\ \uparrow u & & \downarrow & & \uparrow \\ \wedge \widetilde{V}_{BU} & \xleftarrow{i} & \wedge \widetilde{V}_{BG} & \xrightarrow{i} & \wedge V_{BG} \\ \uparrow & \xrightarrow{=} & \uparrow & \xrightarrow{=} & \uparrow \\ \wedge \widetilde{V}_{BU} & \xleftarrow{i} & \wedge \widetilde{V}_{BG} & \xrightarrow{i} & \wedge V_{BG} \end{array}$$

in which vertical arrows are Sullivan models for the fibrations in the diagram (4.1). Observe that squares are commutative except for the top square. Let $\Psi : \wedge Z \rightarrow A_{PL}(G \times (G \times_U E_U))$ be the Sullivan model with which Sullivan representatives in (4.2) are constructed. The argument in [11, page 205] allows us to choose homotopies, which makes maps v , $\tilde{\beta}$, v' and $\tilde{\alpha}$ Sullivan representatives for the corresponding maps, so that all of them are relative with respect to $\wedge V_{BG}$. This implies that $\Psi \circ \beta \circ v \simeq \Psi \circ v' \circ \tilde{\alpha} \text{ rel } \wedge V_{BG}$. By virtue of Lifting lemma [11, Proposition 14.6], we have a homotopy $H : \tilde{\beta} \circ v \simeq v' \circ \tilde{\alpha} \text{ rel } \wedge V_{BG}$. Thus we have a homotopy commutative diagram

$$\begin{array}{ccc} \wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V}_{BU} & \xrightarrow{u \cdot v} & \wedge V \\ \tilde{\alpha} \otimes 1 \downarrow & & \downarrow \tilde{\beta} \\ \wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V}_{BU} & \xrightarrow{u' \cdot v'} & \wedge Z \end{array}$$

in which horizontal arrows are quasi-isomorphisms; see [11, (15.9) page 204]. In fact the homotopy $K : \wedge \widetilde{V}_{BU} \otimes_{\wedge V_{BG}} \wedge V' \rightarrow \wedge W \otimes \wedge(t, dt)$ is given by $K = (\tilde{\beta} \circ u) \cdot H$. Observe that $\tilde{\beta} \circ u = u'$. Thus we have a model $\tilde{\alpha} \otimes 1$ for $\tilde{\beta}$ and hence for the left translation.

The model $\tilde{\alpha} \otimes 1$ can be replaced by more tractable one. In fact, recalling the model (\widetilde{V}_{BU}, d) for BU mentioned above, it is readily seen that the map $s : \wedge \widetilde{V}_{BU} \rightarrow \wedge V_{BU} = \wedge(h_1, \dots, h_l)$, which is defined by $s(c_i) = (B\iota)^*(c_i)$, $s(h_i) = h_i$ and $s(\tau_j) = 0$, is a quasi-isomorphism and is compatible with $\wedge V_{BG}$ -action. Observe that the Sullivan representative for $B\iota : BU \rightarrow BG$ is also denoted by $(B\iota)^*$. Thus we have a commutative diagram

$$\begin{array}{ccc} \wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V}_{BU} & \xleftarrow{1 \otimes s} & \wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V}_{BU} \\ \zeta := \tilde{\alpha} \otimes 1 \downarrow & & \downarrow \tilde{\alpha} \otimes 1 \\ \wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V}_{BU} & \xleftarrow{1 \otimes s} & \wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V}_{BU} \end{array}$$

in which the DGA maps $1 \otimes s$ are quasi-isomorphisms. As usual, the Lifting lemma enables us to deduce the following lemma.

Lemma 4.1. *The DGA map $\zeta := \tilde{\alpha} \otimes 1$ is a Sullivan representative for the left translation $tr : G \times G/U \rightarrow G/U$.*

In order to construct a model for tr more explicitly, we proceed to construct that for α .

Lemma 4.2. *There exists a Sullivan representative ψ for α such that a diagram*

$$\begin{array}{ccc} & \wedge(x_1, \dots, x_l) \otimes \wedge V_{BG} = \wedge V' & \\ & \uparrow i_1 & \downarrow \psi \\ \wedge V_{BG} & & \wedge(x_1, \dots, x_l) \otimes \wedge(x_1, \dots, x_l) \otimes \wedge V_{BG} = \wedge W' \\ & \downarrow i_2 & \end{array}$$

is commutative and $\psi(x_i) = x_i \otimes 1 \otimes 1 + 1 \otimes x_i \otimes 1 + \sum_n X_n \otimes X'_n C_n$ for some monomials $X_n \in \wedge(x_1, \dots, x_l)$, $X'_n \in \wedge^+(x_1, \dots, x_l)$ and monomials $C_n \in \wedge^+ V_{BG}$. Here i_1 and i_2 denote Sullivan models for p and p' , respectively.

Proof. We first observe that $d(x_i \otimes 1) = 0$ and $d(1 \otimes x_i) = c_i \in \wedge(c_1, \dots, c_l) = \wedge V_{BG}$ in $\wedge W'$. It follows from [11, 15.9] that there exists a DGA map ψ which makes the diagram commutative. We write

$$\psi(x_i) = x_i \otimes 1 \otimes 1 + 1 \otimes x_i \otimes 1 + \sum_n X_n \otimes X'_n C_n + \sum_n \tilde{X}_n \otimes \tilde{X}'_n + \sum_n X''_n \otimes C''_n$$

with monomial bases, where $X_n, X''_n \in \wedge(x_1, \dots, x_l) \otimes 1 \otimes 1$, $X'_n \in 1 \otimes \wedge^+(x_1, \dots, x_l) \otimes 1$, $\tilde{X}_n \otimes \tilde{X}'_n \in \wedge(x_1, \dots, x_l) \otimes \wedge(x_1, \dots, x_l) \otimes 1$ and $C_n, C''_n \in \wedge^+ V_{BG}$. The map $\wedge(x_1, \dots, x_l) \rightarrow \wedge(x_1, \dots, x_l) \otimes \wedge(x_1, \dots, x_l)$ induced by ψ is a Sullivan representative for the product of G . This allows us to conclude that \tilde{X}_n and \tilde{X}'_n are in $\wedge^+(x_1, \dots, x_l)$. Since ψ is a DGA map, it follows that

$$dx_i = \psi(dx_i) = dx_i + \sum_n X_n \otimes d(X'_n)C_n + \sum_n \tilde{X}_n \otimes d(\tilde{X}'_n).$$

This implies that $\sum_n X_n \otimes d(X'_n)C_n = 0$ and $\sum_n \tilde{X}_n \otimes d(\tilde{X}'_n) = 0$. Since the map $d : \wedge^+(x_1, \dots, x_l) \rightarrow \wedge(x_1, \dots, x_l) \otimes \wedge V_{BG}$ is a monomorphism, it follows that $\sum_n \tilde{X}_n \otimes \tilde{X}'_n = 0$. We write $C''_n = c_{i_n}^{k_n} \tilde{C}_n$, where $k_n \geq 1$. Define a homotopy

$$H : \wedge(x_1, \dots, x_l) \otimes \wedge V_{BG} \rightarrow \wedge(x_1, \dots, x_l) \otimes \wedge(x_1, \dots, x_l) \otimes \wedge V_{BG} \otimes \wedge(t, dt)$$

by $H(c_i) = c_i \otimes 1$ and

$$\begin{aligned} H(x_i) &= x_i \otimes 1 \otimes 1 + 1 \otimes x_i \otimes 1 + \sum_n X_n \otimes X'_n C_n \\ &\quad - \sum_n X''_n \otimes x_{i_n} \otimes c_{i_n}^{k_n-1} \tilde{C}_n \otimes dt + \sum_n X''_n \otimes 1 \otimes c_{i_n}^{k_n} \tilde{C}_n \otimes t. \end{aligned}$$

Put $\tilde{\psi} = (\varepsilon_0 \otimes 1) \circ \psi$. We see that $\tilde{\psi} \simeq \psi \text{ rel } \wedge V_{BG}$. This completes the proof. \square

5. PROOF OF THEOREM 1.2

We prove Theorem 1.2 by means of the model for the left translation described in the previous section.

Proof of Theorem 1.2. We adapt Theorem 3.1. We recall the Sullivan model $(\wedge W, d)$ for G/U mentioned in Section 4. Observe that $(\wedge W, d)$ has the form

$$(\wedge W, d) = (\wedge(h_1, \dots, h_l, x_1, \dots, x_k), d)$$

with $dx_j = (B\iota)^* c_j$.

Let $l : (H^*(BU), 0) \rightarrow (\wedge W, d)$ be the inclusion and

$$k : (\wedge W, d) \twoheadrightarrow (\wedge(h_1, \dots, h_l)/(dx_1, \dots, dx_l), 0) \twoheadrightarrow (H^*(G/U), 0)$$

the DGA map defined by $k(h_i) = (-1)^{\tau(|h_i|)}h_i$ and $k(x_i) = 0$. Recall the DGA $\tilde{E} = \wedge(\wedge W \otimes (\wedge W)_*)/I$ and the DGA map $\tilde{\mu} : \tilde{E} \rightarrow \wedge V_G$ mentioned in Section 3, where we use the model $\zeta : \wedge W \rightarrow \wedge V_G \otimes \wedge W$ for the action $G \times G/U \rightarrow G/U$ constructed in Lemmas 4.1 and 4.2 in order to define $\tilde{\mu}$; see (3.1). Consider the composite

$$\begin{aligned} \theta : (H^*(BU) : H^*(G/U)) &= \wedge(H^*(BU) \otimes H_*(G/U))/I \\ &\xrightarrow{l \otimes 1} \wedge(\wedge W \otimes H_*(G/U))/I \xrightarrow{1 \otimes k^\sharp} \wedge(\wedge W \otimes (\wedge W)_*)/I = \tilde{E}. \end{aligned}$$

Let $\tilde{u} : \tilde{E} \rightarrow \mathbb{Q}$ be an augmentation defined by $\tilde{u} = \varepsilon \circ \tilde{\mu}$, where $\varepsilon : \wedge V_G \rightarrow \mathbb{Q}$ is the augmentation. Then we have $\theta(M_u) \subset M_{\tilde{u}}$. In fact, since $i^*(h_i) = (-1)^{\tau(|h_i|)}k \circ l(h_i)$ and $\langle h_i, k^\sharp b_* \rangle = \langle \zeta h_i, b_* \rangle$ for $h_i \in H^*(BU)$, it follows that

$$\begin{aligned} \theta(h_i \otimes b_* - u(h_i \otimes b_*)) &= h_i \otimes k^\sharp b_* - \langle i^* h_i, b_* \rangle \\ &= h_i \otimes k^\sharp b_* - (-1)^{\tau(|h_i|)} \langle k h_i, b_* \rangle \\ &= h_i \otimes k^\sharp b_* - (-1)^{\tau(|h_i|)} \langle \zeta h_i, b_* \rangle \\ &= h_i \otimes k^\sharp b_* - \tilde{u}(h_i \otimes k^\sharp b_*). \end{aligned}$$

Consider an element $z := x_{i_t} \otimes 1_* - (-1)^{\tau(|u_{t*}|)}x_{j_t} \otimes k^\sharp(u_{t*}) \in Q(\tilde{E}/M_{\tilde{u}})$. For any $\alpha \in \wedge W$, $\langle \alpha, d^\sharp k^\sharp u_{t*} \rangle = \langle k d \alpha, u_{t*} \rangle = 0$. Therefore we see that, in $Q(\tilde{E}/M_{\tilde{u}})$,

$$\delta_0(z) = dx_{i_t} \otimes 1_* - (-1)^{\tau(|u_{t*}|)}dx_{j_t} \otimes k^\sharp(u_{t*}) = \theta((Bl)^*(c_{i_t}) \otimes 1_* - (Bl)^*(c_{j_t}) \otimes u_{t*}) = 0.$$

The last equality follows from the assumption that $(Bl)^*(c_{i_t}) \otimes 1_* \equiv (Bl)^*(c_{j_t}) \otimes u_{t*}$ modulo decomposable elements in $(H^*(BU) : H^*(G/U))/M_u$. By using the notation in Lemma 4.2, we see that

$$\begin{aligned} H^*Q(\tilde{\mu})(z) &= \langle \zeta x_{i_t}, 1_* \rangle - \langle \zeta x_{j_t}, k^\sharp u_{t*} \rangle \\ &= \langle x_{i_t} \otimes 1, 1_* \rangle - \langle \sum X_n \otimes X'_n C_n, k^\sharp u_{t*} \rangle \\ &= x_{i_t} - \sum X_n \langle k(X'_n)C_n, u_{t*} \rangle = x_{i_t}. \end{aligned}$$

Observe that $k(X'_n) = 0$. By virtue of Theorem 3.1, we have the result. \square

Remark 5.1. As for the latter half of Theorem 1.2, namely, in the case where $(Bl)^*(c_{i_1}), \dots, (Bl)^*(c_{i_s})$ are decomposable, we have a very simple proof of the assertion. In fact, the composite of the evaluation map $ev_0 : \text{aut}_1(G/U) \rightarrow G/U$ and the map $\lambda : G \rightarrow \text{aut}_1(G/U)$ is nothing but the projection $\pi : G \rightarrow G/U$. We consider the model $\eta : (\wedge W, d) \rightarrow (\wedge V_G, 0)$ for π mentioned in the proof of Theorem 1.2. Then we see that $HQ(\rho)(x_{i_t}) = x_{i_t}$ for the map $HQ(\rho) : HQ(\wedge W) \rightarrow HQ(\wedge V_G) = V_G$. Observe that $x_{i_t} \in HQ(\wedge W)$ since $(Bl)^*(c_{i_t})$ is decomposable. The same argument as the proof of Theorem 3.1 enables us to conclude that there is a map $\rho : \times_{t=1}^s S^{\deg c_{i_t} - 1} \rightarrow G$ such that $\pi_* \circ \rho_* : \pi_*(\times_{t=1}^s S_{\mathbb{Q}}^{\deg c_{i_t} - 1}) \rightarrow \pi_*(G_{\mathbb{Q}})$ is injective. Thus $\lambda_* \circ \rho_*$ is injective in the rational homotopy.

Remark 5.2. In the proof of Theorem 1.2, we construct a model for G of the form $(\wedge(x_1, \dots, x_k), 0)$. By virtue of [11, Proposition 15.13], we can choose the elements x_j so that $\sigma^*(c_j) = x_j$, where $\sigma^* : H^*(BG) \xrightarrow{\pi^*} H^*(E_G, G) \xleftarrow{\cong} H^*(G)$ denotes the cohomology suspension.

In the rest of this section, we describe a suitable model for $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ for proving Theorems 1.5 and 1.6.

Let G be a connected Lie group, U a connected maximal rank subgroup and K another connected maximal rank subgroup which contains U . We recall from Section 2 a Sullivan model for the connected component $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ containing the composite $e \circ p$ of the function space $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}})$, where $e : G/K \rightarrow (G/K)_{\mathbb{Q}}$ is the localization map.

Let $\iota_1 : K \rightarrow G$ and $\iota_2 : U \rightarrow K$ be the inclusions and put $\iota = \iota_1 \circ \iota_2$. Let $\varphi_U : (\wedge W', d) \xrightarrow{\cong} \Omega\Delta(G/U)$ and $\varphi_K : (\wedge \widetilde{W}, d) \xrightarrow{\cong} \Omega\Delta(G/K)$ be the Sullivan models for G/U and G/K , respectively, mentioned in the proof of Theorem 1.2; that is, $(\wedge W', d) = (\wedge(h_1, \dots, h_l, x_1, \dots, x_k), d)$ with $d(x_i) = (B\iota)^*(c_i)$ and $(\wedge \widetilde{W}, d) = (\wedge(e_1, \dots, e_s, x_1, \dots, x_k), d)$ with $d(x_i) = (B\iota_1)^*(c_i)$. By applying Lifting Lemma to the commutative diagram

$$\begin{array}{ccccc} \wedge V_{BK} & \xrightarrow{(B\iota_2)^*} & \wedge V_{BU} & \xrightarrow{\quad} & \wedge W' \\ \downarrow & & & & \downarrow \varphi_U \\ \wedge \widetilde{W} & \xrightarrow{\varphi_K} & \Omega\Delta(G/K) & \xrightarrow{\Omega\Delta(p)} & \Omega\Delta(G/U), \end{array}$$

we have a diagram

$$(5.1) \quad \begin{array}{ccccc} H^*(G/U) & \xleftarrow[\simeq]{k} & \wedge W' & \xrightarrow[\simeq]{} & \Omega\Delta(G/U) \\ p^* \uparrow & & \varphi \uparrow & & \uparrow \Omega\Delta(p) \\ H^*(G/K) & \xleftarrow[\simeq]{l} & \wedge \widetilde{W} & \xrightarrow[\simeq]{} & \Omega\Delta(G/K) \end{array}$$

in which the right square is homotopy commutative and the left that is strictly commutative. In particular, $k(x_i) = 0$, $l(x_i) = 0$ and $\varphi(e_i) = (B\iota_2)^*e_i$.

Let $w : \wedge W \rightarrow \wedge \widetilde{W}$ be a minimal model for $(\wedge \widetilde{W}, d)$ and $k^\sharp : (H^*(G/U))^\sharp \rightarrow (\wedge W')^\sharp$ the dual to the map k .

As in Remark 2.5(ii), we construct the DGA E' by using $(\wedge W', d) = (B, d_B)$ and $(\wedge W, d)$. Then we have a sequence of quasi-isomorphisms

$$E' \xrightarrow[\simeq]{\gamma := 1 \otimes k^\sharp} \wedge(\wedge W \otimes (\wedge W')_*)/I \xrightarrow[\simeq]{w \otimes 1} \wedge(\wedge \widetilde{W} \otimes (\wedge W')_*)/I = \widetilde{E}.$$

Moreover, we choose a model ζ' for the action $G \times G/U \xrightarrow{tr} G/U \xrightarrow{p} G/K$ defined by the composite $\zeta' : \wedge \widetilde{W} \xrightarrow{\zeta} \wedge V_G \otimes \wedge \widetilde{W} \xrightarrow{1 \otimes \varphi} \wedge V_G \otimes \wedge W'$, where ζ is the Sullivan representative for the left translation tr mentioned in Lemmas 4.1 and 4.2. Then the map ζ' deduces a model

$$(5.2) \quad \widetilde{\mu} : E'/M_u \rightarrow \wedge V_G$$

for $\lambda : G \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ as in Theorem 3.1. Observe that

$$(5.3) \quad \widetilde{\mu}(v_i \otimes e_j) = (-1)^{\tau(e_j)} \langle (1 \otimes \varphi)\zeta w(v_i), k^\sharp e_j \rangle \quad \text{and} \quad u = \varepsilon \circ \widetilde{\mu},$$

where $\varepsilon : \wedge V_G \rightarrow \mathbb{Q}$ denotes the augmentation. In the next section, we shall prove Theorem 1.5 by using the model $\widetilde{\mu} : E'/M_u \rightarrow \wedge V_G$.

6. PROOF OF THEOREM 1.5

Let G and U be the Lie group $U(m+k)$ and a maximal rank subgroup of the form $U(m_1) \times \cdots \times U(m_s) \times U(k)$, respectively. Without loss of generality, we can assume that $m_1 \geq \cdots \geq m_s \geq k$. Let K be the subgroup $U(m) \times U(k)$ of U , where $m = m_1 + \cdots + m_s$. Then the Leray-Serre spectral sequence with coefficients in the rational field for the fibration $p : G/U \rightarrow G/K$ with fibre K/U collapses at the E_2 -term because the cohomologies of G/K and of K/U are algebras generated by elements with even degree. Therefore it follows that the induced map $p^* : H^*(G/K) \rightarrow H^*(G/U)$ is a monomorphism. In order to prove Theorem 1.5, we apply Theorem 3.1 to the function space $\mathcal{F}(G/U, G/K, p)$.

Let $P = \{S_1, \dots, S_n\}$ be a family consisting of subsets of the finite ordered set $\{1, \dots, s\}$ which satisfies the condition that $x < y$ whenever $x \in S_i$ and $y \in S_{i+1}$. Define $\sharp^l P$ to be the number of elements of the set $\{S_j \in P \mid |S_j| = l\}$. Let k be a fixed integer. We call such the family P a (i_1, \dots, i_k) -type block partition of $\{1, \dots, s\}$ if $\sharp^l P = i_l$ for $1 \leq l \leq k$. Let $Q_{i_1, \dots, i_k}^{(s)}$ denote the number of (i_1, \dots, i_k) -type block partitions of $\{1, \dots, s\}$.

We construct a minimal model explicitly for the homogeneous space $U(m+k)/U(m) \times U(k)$. Assume that $m \geq k$. As in the proof of Theorem 1.2, we have a Sullivan model for $U(m+k)/U(m) \times U(k)$ of the form

$$(\wedge \widetilde{W}, d) = (\wedge(\tau_1, \dots, \tau_{m+k}, c_1, \dots, c_k, c'_1, \dots, c'_m), d)$$

with $d\tau_l = \sum_{i+j=l} c'_i c_j$.

Lemma 6.1. *There exists a sequence of quasi-isomorphisms*

$$\wedge \widetilde{W} \xleftarrow{\cong} \wedge W_{(1)} \xleftarrow{\cong} \cdots \xleftarrow{\cong} \wedge W_{(s)} \xleftarrow{\cong} \cdots \xleftarrow{\cong} \wedge W_{(m)}$$

in which, for any s , $(\wedge W_{(s)}, d_{(s)})$ is a DGA of the form

$$\wedge W_{(s)} = \wedge(\tau_{s+1}, \dots, \tau_{m+k}, c_1, \dots, c_k, c'_{s+1}, \dots, c'_m) \quad \text{with}$$

$$\begin{aligned} d_{(s)}\tau_l &= c'_l + c'_{l-1}c_1 + \cdots + c'_{s+1}c_{l-(s+1)} \\ &+ \sum_{i_1+2i_2+\cdots+ki_k=s} (-1)^{i_1+\cdots+i_k} Q_{i_1, \dots, i_k}^{(s)} c_1^{i_1} \cdots c_k^{i_k} c_{l-s} \\ &+ \sum_{i_1+2i_2+\cdots+ki_k=s-1} (-1)^{i_1+\cdots+i_k} Q_{i_1, \dots, i_k}^{(s-1)} c_1^{i_1} \cdots c_k^{i_k} c_{l-(s-1)} \\ &+ \cdots + (-c_1)c_{l-1} + c_l \end{aligned}$$

for $s+1 \leq l \leq m+k$, where $c_i = 0$ for $i < 0$ or $i > k$.

Proof. We shall prove this lemma by induction on the integer s . We first observe that $d\tau_2 = c'_2 - c_1c_1 + c_2$ in $\wedge W_{(1)}$ because $Q_{i_1}^{(1)} = 1$. Define a map $\varphi : \wedge W_{(1)} \rightarrow \wedge \widetilde{W}$ by $\varphi(c_i) = c_i$, $\varphi(c'_j) = c'_j$ and $\varphi(\tau_2) = \tau_2 - \tau_1c_1$. Since $d\tau_1 = c'_1 + c_1$ in $\wedge W$, it follows that φ is a well-defined quasi-isomorphism. Suppose that $(\wedge W_{(s)}, d_{(s)})$ in the lemma can be constructed for some $s \leq m-1$. In particular, we have

$$d_{(s)}\tau_{s+1} = c'_{s+1} + \sum_{0 \leq j \leq s} \sum_{i_1+2i_2+\cdots+ki_k=j} (-1)^{i_1+\cdots+i_k} Q_{i_1, \dots, i_k}^{(j)} c_1^{i_1} \cdots c_k^{i_k} c_{s+1-j}.$$

Claim 1.

$$Q_{i_1, \dots, i_k}^{(s+1)} = Q_{i_1-1, i_2, \dots, i_k}^{(s)} + Q_{i_1, i_2-1, \dots, i_k}^{(s-1)} + \cdots + Q_{i_1, \dots, i_k-1}^{(s+1-k)}.$$

Claim 1 implies that

$$d_{(s)}\tau_{s+1} = c'_{s+1} - \sum_{i_1+2i_2+\dots+ki_k=s+1} (-1)^{i_1+\dots+i_k} Q_{i_1,\dots,i_k}^{(s+1)} c_1^{i_1} \dots c_k^{i_k}.$$

We define $d_{(s+1)}\tau_{l+1}$ in $\wedge W_{(s+1)}$ by replacing the factor c'_{s+1} which appears in $d_{(s)}\tau_{l+1}$ with $c'_{s+1} - d_{(s)}\tau_{s+1}$, namely,

$$\begin{aligned} d_{(s+1)}\tau_{l+1} &= c'_{l+1} + c'_l c_1 + \dots + c'_{s+2} c_{(l+1)-(s+2)} \\ &\quad + \sum_{i_1+2i_2+\dots+ki_k=s+1} (-1)^{i_1+\dots+i_k} Q_{i_1,\dots,i_k}^{(s+1)} c_1^{i_1} \dots c_k^{i_k} c_{l-s} \\ &\quad + \dots + (-c_1)c_l + c_{l+1}. \end{aligned}$$

Moreover define a map $\varphi : \wedge W_{(s+1)} \rightarrow \wedge W_{(s)}$ by $\varphi(c_i) = c_i$, $\varphi(c'_j) = c'_j$ and $\varphi(\tau_{l+1}) = \tau_{l+1} - \tau_{s+1}c_{l+1-(s+1)}$. It is readily seen that φ is a well-defined DGA map. The usual spectral sequence argument enables us to deduce that φ is a quasi-isomorphism. This finishes the proof. \square

Proof of Claim 1. Let $\{P_l\}$ denote the family of all (i_1, \dots, i_k) -type block partitions of $\{1, \dots, s+1\}$. We write $P_l = \{S_1^{(l)}, \dots, S_n^{(l)}\}$. Then $\{P_l\}$ is represented as the disjoint union of the families of (i_1, \dots, i_k) -type block partitions whose last sets $S_n^{(l)}$ consist of j elements, namely, $\{P_l\} = \coprod_{1 \leq j \leq k} \{P_l \mid |S_n^{(l)}| = j\}$. It follows that

$$\left| \{P_l \mid |S_n^{(l)}| = j\} \right| = Q_{i_1, \dots, i_{j-1}, i_j-1, i_{j+1}, \dots, i_k}^{(s+1-j)}.$$

We have the result. \square

Recall the minimal model $(\wedge W_{(m)}, d)$ for G/K in Lemma 6.1. We see that $\deg d\tau_{m+1} = \deg c_1^m c_1 = 2(m+1)$ and that $d\alpha = 0$ for any element α with $\deg \alpha \leq 2m+1$. This yields that $c_1^m \neq 0$ in $H^*(G/K; \mathbb{Q})$. As mentioned before Lemma 6.1, the induced map $p^* : H^*(G/K) \rightarrow H^*(G/U)$ is injective. Therefore we have $(p^*c_1)^s \neq 0$ for $s \leq m$.

Let $\tilde{\mu} : \tilde{E}/M_u \rightarrow \wedge V_G$ be the model for the map $\lambda : G \rightarrow \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ mentioned in the previous section; see (5.2) and (5.3). The following four lemmas are keys to proving Theorem 1.5. The proofs are deferred to the end of this section.

Lemma 6.2. $\delta_0(\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_*) = (-1)^m c_{m-s+1}$ if $m \neq s$.

Lemma 6.3. $\tilde{\mu}(\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_*) = 0$ if $m \neq s$.

Lemma 6.4. $\delta_0(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = (-1)^s s c_{m-s+1}$.

Lemma 6.5. $\tilde{\mu}(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = \tau_{m-s+1}$.

Proof of Theorem 1.5. By virtue of Lemmas 6.2, 6.3, 6.4 and 6.5, we have

$$\begin{aligned} &\delta_0((-1)^m \tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_* - \frac{(-1)^s}{s} \tau_{m+1} \otimes ((p^*c_1)^s)_*) \\ &= (-1)^m (-1)^m c_{m-s+1} - \frac{(-1)^s}{s} (-1)^s s c_{m-s+1} = 0 \quad \text{and} \\ &\tilde{\mu}((-1)^m \tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_* - \frac{(-1)^s}{s} \tau_{m+1} \otimes ((p^*c_1)^s)_*) \\ &= -\frac{(-1)^s}{s} \tau_{m-s+1}, \end{aligned}$$

where $s \leq m - 1$. Theorem 3.1 implies that

$$(\lambda_{\mathbb{Q}})_i : \pi_i(G_{\mathbb{Q}}) \rightarrow \pi_i(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}, e \circ p))$$

is injective for $i = \deg \tau_1, \dots, \deg \tau_m$. Since $d\tau_l = \sum_{i+j} c'_i c_j$ in $(\wedge W)$, it follows that $d\tau_l$ is decomposable for $l \geq M + 1$. Therefore Theorem 1.2 yields that $(\lambda_{\mathbb{Q}})_i$ is also injective for $i = \deg \tau_{m+1}, \dots, \deg \tau_{m+k}$.

The latter half of Theorem 1.5 is obtained by comparing the dimension of rational homotopy groups. In fact, it follows from the rational model for $\text{aut}_1(\mathbb{C}P^{m-1})$ mentioned in Example 2.4 that

$$\begin{aligned} \pi_*(\text{aut}_1(\mathbb{C}P^{m-1}) \otimes \mathbb{Q})^{\sharp} &\cong H_*(Q(\tilde{E}/M_u), \delta_0) \\ &\cong \mathbb{Q}\{y \otimes 1_*, y \otimes (x^1)_*, \dots, y \otimes (x^{m-2})_*\}. \end{aligned}$$

This implies that $\dim \pi_i(\text{aut}_1(\mathbb{C}P^{m-1}) \otimes \mathbb{Q}) = 1 = \dim \pi_i(SU(m)) \otimes \mathbb{Q}$ for $i = 3, \dots, 2m - 1$. The result follows from the first assertion. This completes the proof. \square

We conclude this section with proofs of Lemmas 6.2, 6.3, 6.4 and 6.5.

Proof of Lemma 6.2. We regard the free algebra $\wedge(c_1, \dots, c_l)$ as a primitively generated Hopf algebra. Observe that $(c_i^s)_* = \frac{1}{s!}((c_i)_*)^s$. Recall the 0-simplex u in $\Delta E'$ mentioned in (5.3). We have $u(c_j \otimes (p^* c_1)_*) = 0$ if $j \neq 1$ and

$$\begin{aligned} u(c_1 \otimes (p^* c_1)_*) &= (-1)^{\tau(|p^* c_1|)} k^{\sharp}(p^*(c_1)_*)(\varphi \circ w(c_1)) \\ &= (-1)((p^*(c_1)_*)k \circ \varphi \circ w(c_1)) = (-1)((p^*(c_1)_*)p^* c_1) = -1. \end{aligned}$$

For the map k and q , see the diagram (5.1) and the ensuing paragraph. Thus it follows that

$$\begin{aligned} &\delta_0(\tau_{m+(m-s+1)} \otimes ((p^* c_1)^m)_*) \\ &= c_1^m c_{m-s+1} \cdot D^{(m)}(p^* c_1^m)_* = c_1^m c_{m-s+1} \cdot \frac{1}{m!} D^{(m)}(p^* c_1)_*^m \\ &= \frac{1}{m!} c_1^m c_{m-s+1} \cdot \left((p^* c_1)_* \otimes 1 \otimes \dots \otimes 1 + 1 \otimes (p^* c_1)_* \otimes 1 \otimes \dots \otimes 1 \right. \\ &\quad \left. + \dots + 1 \otimes \dots \otimes 1 \otimes (p^* c_1)_* \right)^m \\ &= \frac{1}{m!} c_1^m c_{m-s+1} \cdot (\dots + m!(p^* c_1)_* \otimes \dots \otimes (p^* c_1)_* \otimes 1 + \dots) \\ &= u(c_1 \otimes (p^* c_1)_*) \cdot \dots \cdot u(c_1 \otimes (p^* c_1)_*) c_{m-s+1} = (-1)^m c_{m-s+1}. \end{aligned}$$

\square

Proof of Lemma 6.3. Recall the quasi-isomorphism $\varphi_{s+1} : \wedge W_{(s+1)} \rightarrow \wedge W_{(s)}$ in the proof of Lemma 6.1 which is defined by $\varphi(\tau_{l+1}) = \tau_{s+1} - \tau_{l+1} c_{l+1-(s+1)}$. Let w denote the composite $\varphi_1 \circ \dots \circ \varphi_m : \wedge W = \wedge W_{(m)} \rightarrow \wedge \tilde{W}$. It is readily seen that $w(\tau_{m+(m-s+1)})$ does not have the element c_1^m as a factor if $s \neq m$. Hence using the DGA map ζ' in Lemma 4.1, we have

$$\tilde{\mu}(\tau_{m+(m-s+1)} \otimes ((p^* c_1)^m)_*) = (-1)^{\tau(|p^* c_1^m|)} \langle (1 \otimes \varphi) \zeta w(\tau_{m+(m-s+1)}), k^{\sharp}(p^* c_1^m)_* \rangle = 0.$$

See (5.1) for the notations. Observe that $H^*(G/K) \cong H^*(\wedge W) \cong \mathbb{Q}[c_1, \dots, c_k]$ for $* \leq 2m$. This completes the proof. \square

Proof of Lemma 6.4. From Lemma 6.1, we see that in $\wedge W_{(m)}$,

$$\begin{aligned} d\tau_{m+1} &= \sum_{i_1+2i_2+\dots+ki_k=m} (-1)^{i_1+\dots+i_k} Q_{i_1,\dots,i_k}^{(m)} c_1^{i_1} \dots c_k^{i_k} c_1 \\ &+ \sum_{i_1+2i_2+\dots+ki_k=m-1} (-1)^{i_1+\dots+i_k} Q_{i_1,\dots,i_k}^{(m-1)} c_1^{i_1} \dots c_k^{i_k} c_2 \\ &+ \dots + \sum_{i_1+2i_2+\dots+ki_k=l} (-1)^{i_1+\dots+i_k} Q_{i_1,\dots,i_k}^{(l)} c_1^{i_1} \dots c_k^{i_k} c_{m-l+1} \\ &+ \dots \end{aligned}$$

Suppose that $c_1^{i_1} \dots c_k^{i_k} c_{m-l+1} \otimes ((p^*c_1)^s)_* \neq 0$ in $Q(\tilde{E}/M_u)$, where $i_1 + 2i_2 + \dots + ki_k = l$. Then we have

- (1) $l = m$ and $c_1^{i_1} \dots c_k^{i_k} = c_1^{s-1} c_{m-s+1}$ or
- (2) $l \neq m$, $l = s$ and $c_1^{i_1} \dots c_k^{i_k} = c_1^s$.

It follows that $(-1)^{i_1+\dots+i_k} Q_{i_1,\dots,i_k}^{(m)} c_1^{s-1} c_{m-s+1} c_1 = (-1)^{s-1+1} (s-1) c_1^s c_{m-s+1}$ if $(i_1, \dots, i_k) = (s-1, 0, \dots, 0, 1, 0, \dots, 0)$ with $i_{m-s+1} = 1$ and that $Q_{i_1,\dots,i_k}^{(s)} c_1^s c_{m-s+1} = (-1)^s \cdot 1 \cdot c_1^s c_{m-s+1}$ if $(i_1, \dots, i_k) = (s, 0, \dots, 0)$. This fact allows us to conclude that $\delta_0(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = (-1)^s (s-1) c_{m-s+1} + (-1)^s c_{m-s+1} = (-1)^s s c_{m-s+1}$. We have the result. \square

Proof of Lemma 6.5. In order to compute $\tilde{\mu}$, we determine $\langle (1 \otimes \varphi) \zeta w(\tau_{m+1}), k^\sharp(p^*c_1^s)_* \rangle$. With the same notation as in the proof of Lemma 6.3, we have $w(\tau_{m+1}) = \dots + (-1)^s \tau_{m-s+1} c_1^s + \dots$. Lemmas 4.1 and 4.2 imply that

$$\begin{aligned} \zeta(\tau_{m-s+1} c_1^s) &= \psi \otimes 1(\tau_{m-s+1} \otimes c_1^s) \\ &= (\tau_{m-s+1} \otimes 1 \otimes 1 + 1 \otimes \tau_{m-s+1} \otimes 1 + \sum_n X_n \otimes X'_n C_n) c_1^s. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \tilde{\mu}(\tau_{m+1} \otimes ((p^*c_1)^s)_*) &= (-1)^{\tau(|p^*c_1^s|)} \langle (1 \otimes \varphi) \zeta w(\tau_{m+1}), k^\sharp(p^*c_1^s)_* \rangle \\ &= (-1)^{s+s} \langle (1 \otimes \varphi) \zeta(\tau_{m-s+1} c_1^s), k^\sharp(p^*c_1^s)_* \rangle \\ &= \tau_{m-s+1} \langle \varphi(c_1^s), k^\sharp(p^*c_1^s)_* \rangle + \langle \varphi(\tau_{m-s+1} c_1^s), k^\sharp(p^*c_1^s)_* \rangle \\ &\quad + \sum_n X_n \langle \varphi(X'_n C_n c_1^s), k^\sharp(p^*c_1^s)_* \rangle \\ &= \tau_{m-s+1} \langle k\varphi(c_1^s), (p^*c_1^s)_* \rangle + \langle k\varphi(\tau_{m-s+1} c_1^s), (p^*c_1^s)_* \rangle \\ &\quad + \sum_n X_n \langle k\varphi(X'_n C_n c_1^s), (p^*c_1^s)_* \rangle \\ &= \tau_{m-s+1}. \end{aligned}$$

The last equality is extracted from the commutativity of the diagram (5.1). This completes the proof. \square

7. PROOF OF THEOREM 1.7.

This section is devoted to proving Theorem 1.7. The inclusion $\iota : \text{aut}_1(X) \rightarrow \mathcal{H}_{H,X}$ induces the map $B\iota : \text{Baut}_1(X) \rightarrow B\mathcal{H}_{H,X}$ with $B\iota \circ B\lambda_{G,X} = B\psi$. Therefore if $B\psi$ is injective on homology, then so is $B\lambda_{G,X}$.

We shall prove the ‘‘only if’’ part by using the general categorical construction of a classifying space due to May [24, Section 12] and by applying a part of the argument in the proof of [25, Theorem 3.2] to our case.

We here recall briefly the notion of a \mathcal{O} -graph; see [25, page 68] for more detail. Let \mathcal{O} be a discrete topological space. Define a \mathcal{O} -graph to be a space \mathcal{A} together with maps $S : \mathcal{A} \rightarrow \mathcal{O}$ and $T : \mathcal{A} \rightarrow \mathcal{O}$. The space \mathcal{O} itself is regarded as \mathcal{O} -graph with arrows S and T the identity map. Let $\mathcal{O}Gr$ be the category of \mathcal{O} -graphs whose morphisms are maps $h : \mathcal{A} \rightarrow \mathcal{A}'$ compatible with maps S and T . Observe that the pullback construction with respect to S and T makes $\mathcal{O}Gr$ a monoidal category. In fact, for \mathcal{O} -graphs \mathcal{A} and \mathcal{A}' , $\mathcal{A} \square \mathcal{A}'$ is defined by $\{(a, a') \in \mathcal{A} \times \mathcal{A}' \mid Sa = Ta'\}$. Let \mathcal{X} and \mathcal{Y} be a left \mathcal{O} -graph and a right \mathcal{O} -graph, respectively; that is, \mathcal{X} is a space with a map $T : \mathcal{X} \rightarrow \mathcal{O}$ and the space \mathcal{Y} admits only a map $S : \mathcal{Y} \rightarrow \mathcal{O}$.

Let \mathcal{M} be a monoid in $\mathcal{O}Gr$ the category of \mathcal{O} -graphs and $B(\mathcal{Y}, \mathcal{M}, \mathcal{X})$ denote the two-sided bar construction in the sense of May [24, Section 12], which is the geometric realization of the simplicial space B_* with $B_j = \mathcal{Y} \square \mathcal{M}^{\square j} \square \mathcal{X}$. We regard a topological monoid G as that in $\mathcal{O}Gr$ with $\mathcal{O} = \{x\}$ the space of a point. Then the classifying space BG we consider here is regarded as the bar construction $B(x, G, x)$.

Proof of the “only if” part of Theorem 1.7. Let $\iota' : \mathcal{H}_{H,X} \rightarrow \mathcal{F}(X, X)$ be the inclusion and $e_* : \mathcal{F}(X, X) \rightarrow \mathcal{F}(X, X_{\mathbb{Q}})$ the map induced by the localization $e : X \rightarrow X_{\mathbb{Q}}$. Since X is an F_0 -space or a space having the rational homotopy type of the product of odd dimensional spheres by assumption, it follows from [2, 3.6 Corollary] and [11, Proposition 32.16] that the natural map $[X, X_{\mathbb{Q}}] \rightarrow \text{Hom}(H^*(X_{\mathbb{Q}}; \mathbb{Q}), H^*(X; \mathbb{Q}))$ is bijective. We see that $e \circ \varphi \simeq e$ for any $\varphi \in \mathcal{H}_{H,X}$. Therefore the composite $e_* \circ \iota'$ factors through the connected component $\mathcal{F}(X, X_{\mathbb{Q}}; e)$ of $\mathcal{F}(X, X_{\mathbb{Q}})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{H,X} & \xrightarrow{e_* \circ \iota'} & \mathcal{F}(X, X_{\mathbb{Q}}; e) \xrightarrow{\simeq} \text{aut}_1(X_{\mathbb{Q}}) \\ \uparrow \iota & \nearrow e_* & \\ \text{aut}_1(X) & & \end{array}$$

in which the induced map e^* is a homotopy equivalence.

Define \mathcal{O} to be the discrete space with two points x and y . Let \mathcal{M} be the monoid in $\mathcal{O}Gr$ defined by $\mathcal{M}(x, x) = \text{aut}_1(X)$, $\mathcal{M}(y, y) = \text{aut}_1(X_{\mathbb{Q}})$ and $\mathcal{M}(x, y) = \mathcal{F}(X, X_{\mathbb{Q}}; e)$ with $\mathcal{M}(y, x)$ empty. Arrows $S, T : \mathcal{M}(a, b) \rightarrow \mathcal{O}$ are defined by $S(z) = a$ and $T(z) = b$ for $z \in \mathcal{M}(a, b)$. Moreover we define another monoid \mathcal{M}' in $\mathcal{O}Gr$ by $\mathcal{M}'(x, x) = \mathcal{H}_{H,X}$, $\mathcal{M}'(y, y) = \text{aut}_1(X_{\mathbb{Q}})$, $\mathcal{M}'(x, y) = \mathcal{F}(X, X_{\mathbb{Q}}; e)$ and $\mathcal{M}'(y, x) = \phi$ with arrows defined immediately as mentioned above.

The inclusions $i : \text{aut}_1(X) \rightarrow \mathcal{M}$, $j : \text{aut}_1(X_{\mathbb{Q}}) \rightarrow \mathcal{M}$, $i' : \mathcal{H}_{H,X} \rightarrow \mathcal{M}'$ and $j' : \text{aut}_1(X_{\mathbb{Q}}) \rightarrow \mathcal{M}'$ induce the maps between classifying spaces which fit into the commutative diagram

$$(7.1) \quad \begin{array}{ccccc} & & B\mathcal{H}_{H,X} & \xrightarrow{Bi'} & B(\mathcal{O}, \mathcal{M}', \mathcal{O}) & \xleftarrow{Bj'} & & \\ & \nearrow B\psi & \uparrow B\iota & & \uparrow B\tilde{\iota} & \simeq & & \\ BG & & B\text{aut}_1(X) & \xrightarrow{Bi} & B(\mathcal{O}, \mathcal{M}, \mathcal{O}) & \xleftarrow{Bj} & B\text{aut}_1(X_{\mathbb{Q}}), & \\ & \searrow B\lambda_{G,X} & & & & & & \end{array}$$

where $\tilde{\iota} : \mathcal{M} \rightarrow \mathcal{M}'$ is the morphism of monoids in $\mathcal{O}Gr$ induced by the inclusion $\iota : \text{aut}_1(X) \rightarrow \mathcal{H}_{H,X}$. The proof of [25, Theorem 3.2] enables us to conclude that maps Bj and Bj' are homotopy equivalences. The map $\Omega((Bj)^{-1} \circ (Bi))$ coincides with the composite $(e^*)^{-1} \circ e_* : \text{aut}_1(X) \rightarrow \mathcal{F}(X, X_{\mathbb{Q}}; e) \rightarrow \text{aut}_1(X_{\mathbb{Q}})$ up to weak equivalence; see [25, Theorem 3.2(i)]. Moreover the map $e_* : \text{aut}_1(X) \rightarrow$

$\mathcal{F}(X, X_{\mathbb{Q}}; e)$ is a localization; see [16]. These facts yield that $\pi_*(\Omega Bi) \otimes \mathbb{Q}$ is an isomorphism and hence so is $\pi_*(Bi) \otimes \mathbb{Q}$. Thus the localized map $(Bi)_{\mathbb{Q}}$ is a weak equivalence. This implies that $(Bi)_* : H_*(Baut_1(X); \mathbb{Q}) \rightarrow H_*(B(\mathcal{O}, \mathcal{M}, \mathcal{O}); \mathbb{Q})$ is an isomorphism. The commutative diagram (7.1) enables us to conclude that $H_*(B\psi; \mathbb{Q})$ is injective if so is $H_*(B\lambda_{G,X}; \mathbb{Q})$. This completes the proof. \square

As we pointed out in the introduction, [18, Proposition 4.8] follows from Theorems 1.5 and 1.7. In fact, suppose that M is the flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$ and $G = SU(m)$. Then as is seen in Remark 7.1 below $(\lambda_{G,M})_* : \pi_*(BG) \otimes \mathbb{Q} \rightarrow \pi_*(Baut_1(M)) \otimes \mathbb{Q}$ is injective if and only if $(B\lambda_{G,M})^* : H^*(BG) \rightarrow H^*(Baut_1(M))$ is surjective.

Remark 7.1. Suppose that M is a homogeneous space of the form G/H for which $\text{rank } G = \text{rank } H$. The main theorem in [35] due to Shiga and Tezuka implies that $\pi_{2i}(\text{aut}_1(M)) \otimes \mathbb{Q} = 0$ for any i . Thus $H^*(Baut_1(M); \mathbb{Q})$ is a polynomial algebra generated by the graded vector space $(sV)^{\sharp}$, where $(sV)_l = \pi_{l-1}(\text{aut}_1(M))$. Therefore the dual map to the Hurewicz homomorphism $\Xi^{\sharp} : H^*(Baut_1(M); \mathbb{Q}) \rightarrow \text{Hom}(\pi_*(Baut_1(M)), \mathbb{Q})$ induces an isomorphism on the vector space of indecomposable elements; see [11, page 173] for example. Thus the commutative diagram

$$\begin{array}{ccc} H^*(BG; \mathbb{Q}) & \xleftarrow{(B\lambda_{G,M})^*} & H^*(Baut_1(M); \mathbb{Q}) \\ \Xi^{\sharp} \downarrow & & \downarrow \Xi^{\sharp} \\ \text{Hom}(\pi_*(BG), \mathbb{Q}) & \xleftarrow{(B\lambda_{G,M})_*^{\sharp}} & \text{Hom}(\pi_*(Baut_1(M)), \mathbb{Q}) \end{array}$$

yields that the map $(B\lambda_{G,M})^*$ is surjective if G is rationally visible in $\text{aut}_1(M)$. We also see that the induced map $(B\psi)_* : H_j(BG) \rightarrow H_j(B\mathcal{H}_{H,G/U})$ is injective for each triple (G, U, i) in Tables 1 and 2 if $j \in \text{vd}(G, G/U)$.

8. THE SETS $\text{vd}(G, G/U)$ OF VISIBLE DEGREES IN TABLES 1 AND 2

In this section, we deal with the visible degrees described in Tables 1 and 2 in Introduction.

For the case where the homogeneous space G/U has the rational homotopy type of the sphere, the assertions on the visible degrees follow from the latter half of Theorem 1.2. In fact, the argument in Example 1.4 does work well to obtain such results. The details are left to the reader. The results for (11) and for (17) follow from Theorems 1.5 and 1.6, respectively. We are left to verify the visible degrees for the cases (1), (5), (6), (6)', (16) and (19).

(1). It is well-known that $(Bi)^*(p_i) = (-1)^i(\chi^2 p'_{i-1} + p'_i)$ for the induced map $(Bi)^* : H^*(BSO(2m+1)) \rightarrow H^*(B(SO(2) \times SO(2m-1)))$, where p'_i is the i th Pontrjagin class in $H^*(B(SO(2m-1))) \cong \mathbb{Q}[p'_1, \dots, p'_{m-1}]$; see [30].

We can construct a Sullivan model $(\wedge W, d)$ for the Grassmann manifold $M := SO(2m+1)/SO(2) \times SO(2m-1)$ for which $\wedge W = \wedge(\chi, p'_1, \dots, p'_{m-1}, \tau_2, \tau_4, \dots, \tau_{2m})$ and $d(\tau_{2i}) = (-1)^i(\chi^2 p'_{i-1} + p'_i)$ for $1 \leq i \leq m$. We see that there exists a quasi-isomorphism $w : (\wedge(\chi, \tau_{2m}), d\tau_{2m} = -\chi^{2m}) \rightarrow (\wedge W, d)$ such that $w(\chi) = \chi$ and

$$w(\tau_{2m}) = \chi^{2(m-1)}\tau_2 + \cdots + \chi^2\tau_{2(m-1)} + \tau_{2m}.$$

In view of the rational model $\tilde{\mu} : E'/M_u \rightarrow \wedge V_G$ for $\lambda_{G,M} : SO(2m+1) \rightarrow \text{aut}_1(M)$ mentioned in (5.2) and Theorem 3.3, it follows from Lemma 4.2 that

$$\begin{aligned} \tilde{\mu}(\tau_{2m} \otimes (\chi^{2l})_*) &= (-1)^{\tau(|\chi^{2l}|)} \langle \zeta \circ w(\tau_{2m}), (\chi^{2l})_* \rangle \\ &= \langle \chi^{2(m-l)} \tau_2 + \cdots + \chi^2 \tau_{2(m-1)} + \tau_{2m}, (\chi^{2l})_* \rangle \\ &= \tau_{2(m-l)}, \end{aligned}$$

where ζ is the Sullivan representative for the action $SO(2m-1) \times M \rightarrow M$ described in Lemma 4.1. We have the result.

The same argument does work well to prove the result for the case (16).

(19). Let $\iota : Spin(9) \rightarrow F_4$ be the inclusion map. Without loss of generality, we can assume that the induce map

$$(B\iota)^* : H^*(BF_4; \mathbb{Q}) = \mathbb{Q}[y_4, y_{12}, y_{16}, y_{24}] \rightarrow H^*(BSpin(9); \mathbb{Q}) = \mathbb{Q}[y_4, y_8, y_{12}, y_{16}]$$

satisfies the condition that $(B\iota)^*(y_i) = y_i$ for $i = 4, 12, 16$ and $(B\iota)^*(y_{24}) = y_8^3$, where $\deg y_i = i$. This fact follows from a usual argument with the Eilenberg-Moore spectral sequence for the fibration $\mathcal{L}P^2 \rightarrow BSpin(9) \xrightarrow{B\iota} BF_4$. By virtue of Lemmas 4.1 and 4.2, we see that there exists a model for the linear action $F_4 \times \mathcal{L}P^2 \rightarrow \mathcal{L}P^2$ of the form

$$\zeta : (\wedge(x'_{23}) \otimes \wedge(y_8), d) \rightarrow (\wedge(x_3, x_{11}, x_{15}, x_{23}) \otimes \wedge(x'_{23} \otimes \wedge(y_8), d')$$

with $\zeta(x'_{23}) = x_{23} \otimes 1 \otimes 1 + 1 \otimes x'_{23} \otimes 1$, where $d(x'_{23}) = y_8^3$, $d'(x_j) = 0$ for $j = 3, 11, 15, 23$. In fact, for dimensional reasons, we write $\zeta(x'_{23}) = 1 \otimes x'_{23} \otimes 1 + x_{23} \otimes 1 \otimes 1 + cx_{15} \otimes 1 \otimes y_8$ with a rational number c . By definition, we see that $\zeta = \psi \otimes 1$, where ψ denotes the DGA map in Lemma 4.2. Since the image of each element with degree less than 24 by $(B\iota)^*$ does not have the element y_8 as a factor, it follows that $c = 0$. Observe that $\wedge V_{BF_4}$ -action on $\wedge V_{BSpin(9)}$ is induced by the map $(B\iota)^*$. The dual to the map $(\lambda_*)_i : \pi_i(F_4) \otimes \mathbb{Q} \rightarrow \pi_i(\text{aut}_1(F_4/Spin(9))) \otimes \mathbb{Q}$ is regarded as the induced map

$$H(Q(\tilde{\mu})) : H^*(Q(\tilde{E}/M_u), \delta_0) \rightarrow V_G = \mathbb{Q}\{x_3, x_{11}, x_{15}, x_{23}\}$$

in Theorem 3.1. We see that $Q(\tilde{E}/M_u) = \mathbb{Q}\{y_8 \otimes 1_*, x'_{23} \otimes 1_*, x'_{23} \otimes (y_8)_*, x'_{23} \otimes (y_8^2)_*\}$, $\delta_0(x'_{23} \otimes (y_8^2)_*) = 3y_8 \otimes 1_*$, $\delta_0(x'_{23} \otimes 1_*) = \delta_0(x'_{23} \otimes (y_8^1)_*) = 0$; see Example 2.4. Moreover the direct computation with (3.1) shows that $Q(\tilde{\mu})(x'_{23} \otimes 1_*) = \pm x_{23}$ and $Q(\tilde{\mu})(x'_{23} \otimes (y_8)_*) = 0$. This implies that $vd(F_4, \mathcal{L}P^2) = \{23\}$.

(5). The inclusion $\iota : SO(4) \rightarrow G_2$ induces the ring homomorphism

$$(B\iota)^* : H^*(BG_2) \cong \mathbb{Q}[y_4, y_{12}] \rightarrow H^*(BSO(4)) \cong \mathbb{Q}[p_1, \chi],$$

where $\deg p_1 = 4$ and $\deg \chi = 4$. It is immediate that $(B\iota)^*(y_{12})$ is decomposable for dimensional reasons. From Example 2.4, we see that $\pi_*(\text{aut}_1(\mathbb{H}P^2)) \cong \mathbb{Q}\{y \otimes 1_*, y \otimes (x^1)_*\}$, where $\deg y \otimes 1_* = 11$ and $\deg y \otimes (x^1)_* = 7$. It follows from Theorem 1.2 that $vd(G_2, G_2/SO(4)) = \{11\}$.

(6). Let T^2 be the standard maximal torus of $U(2)$. We assume that $G_2 \supset U(2) \supset T^2$ without loss of generality. Then the inclusion $W(G_2) \supset W(U(2))$ of Weyl groups gives the inclusions

$$\begin{array}{ccccc} \mathbb{Q}[t_1, t_2]^{W(G_2)} & \twoheadrightarrow & \mathbb{Q}[t_1, t_2]^{W(U(2))} & \twoheadrightarrow & \mathbb{Q}[t_1, t_2] \\ \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\ H^*(BG_2) & & H^*(BU(2)) & & H^*(BT^2). \end{array}$$

The result [36, page 212, Example 3] implies that there exist generators y_4, y_{12} of $H(BG_2)$ such that $H(BG_2) \cong \mathbb{Q}[y_4, y_{12}]$ and $y_4 = t_1^2 - t_1 t_2 + t_2^2$, $y_{12} = (t_1 t_2^2 - t_1^2 t_2)^2$ in $\mathbb{Q}[t_1, t_2]^{W(G_2)}$. Since the Chern classes $c_1, c_2 \in H^*(BU(2))$ are regarded as $t_1 + t_2$ and $t_1 t_2$, respectively in $\mathbb{Q}[t_1, t_2]^{W(U(2))}$, it follows that

$$(B\iota)^*(y_4) = c_1^2 - 3c_2 \quad \text{and} \quad (B\iota)^*(y_{12}) = c_1^2 c_2^2 - 4c_2^3,$$

where $\iota : U(2) \rightarrow G_2$ is the inclusion. Put $\tilde{c}_2 = -\frac{1}{3}c_1^2 + c_2$. Then we see that $(B\iota)^*(-\frac{1}{3}y_4) = \tilde{c}_2$ and

$$(B\iota)^*(y_{12}) = -\frac{1}{27}c_1^6 - \frac{2}{3}c_1^4 \tilde{c}_2 - 3c_1^2 \tilde{c}_2^2 - 4\tilde{c}_2^3.$$

By the direct computation implies that

$$\begin{aligned} & (B\iota)^*(-\frac{1}{3}y_4) \otimes 1_* - (B\iota)^*(y_{12}) \otimes (-\frac{3}{2})(c_1^4)_* \\ &= \tilde{c}_2 \otimes 1_* + \frac{3}{2} \left(-\frac{1}{27}c_1^6 - \frac{2}{3}c_1^4 \tilde{c}_2 - 3c_1^2 \tilde{c}_2^2 - 4\tilde{c}_2^3 \right) \otimes (c_1^4)_* \\ &\equiv \tilde{c}_2 \otimes 1_* - \tilde{c}_2 \otimes 1_* \equiv 0 \end{aligned}$$

modulo decomposable elements in $(H^*(BU(2)) : H^*(G_2/U(2)))/M_u$. It is immediate that $(B\iota)^*(y_{12})$ is decomposable. By virtue of Theorem 1.2, we have $vd(G_2, G_2/U(2)) = \{3, 11\}$. The same argument works well to obtain the result for the case (6)'.

Acknowledgments. I am particularly grateful to Kojin Abe, Kohhei Yamaguchi and Masaki Kameko for valuable comments on this work and to Hiroo Shiga for drawing my attention to this subject. I thank the anonymous referee of a previous version of this paper for showing me a very simple proof of the latter half of Theorem 1.2, which is described in Remark 5.1.

9. APPENDIX. EXTENSIONS OF CHARACTERISTIC CLASSES

For a space X , let X^δ denote the space with the discrete topology whose underlying set is the same as that of X . Let M be a homogeneous space admitting an action of a connected Lie group G . In this section, we consider cohomology classes of $B(\text{Diff}_1(M))^\delta$ as well as those of $B(\text{aut}_1(M))^\delta$, which detect familiar characteristic classes via the induced map

$$(B\lambda)^* : H^*(B(\text{Diff}_1(M))^\delta; \mathbb{Q}) \rightarrow H^*(BG^\delta; \mathbb{Q}).$$

Let G be a real semi-simple connected Lie group with finitely many components and $h : G \rightarrow G_{\mathbb{C}}$ the complexification of G . One has a commutative diagram

$$\begin{array}{ccccc} H^*(BG_{\mathbb{C}}) & \xrightarrow{h^*} & H^*(BG) & \xrightarrow{j^*} & H^*(B(G^\delta)) \\ & \nearrow B\lambda^* & \uparrow & & \uparrow \\ H^*(B\text{aut}_1(G/U)) & \longrightarrow & H^*(B\text{Diff}_1(G/U)) & \longrightarrow & H^*(B(\text{Diff}_1(G/U))^\delta), \end{array}$$

where $j : G^\delta \rightarrow G$ stands for the natural map. The result [27, THEOREM 2] asserts that the kernel of j^* is equal to the ideal generated by the positive dimensional elements in $\text{Im}h^*$.

As an example, we consider the case where $G = SL(2m; \mathbb{R})$ and U is a maximal rank subgroup of $SO(2m)$ with $(QH^*(BU; \mathbb{Q}))^{2m} = 0$, for example U is a maximal torus of $SO(2m)$. Observe that $G \cong SO(2m)$ and $G_{\mathbb{C}} \cong SU(2m)$.

Then Milnor's result mentioned above allows us to conclude that the Euler class χ of $H^*(BSL(2m; \mathbb{R}))$ survives in $H^*(B(G^\delta))$; see [29]. Moreover Theorem 1.2 yields that $(B\lambda)^* : H^i(Baut_1(G/U)) \rightarrow H^i(BG)$ is surjective for $i = 2m$; see also Remark 7.1. Thus the class $\chi \in H^*(B(G)^\delta)$ is extendable to an element $\tilde{\chi}$ of $H^*(B(\text{Diff}_1(G/U))^\delta)$. Let p_m be the m th Pontrjagin class. Then we see that $h^*(c_{2m}) = p_m$ and $p_m = \chi^2$; see [30]. This yields that $\chi^2 = 0$ in $H^*(B(G^\delta))$. It remains to show whether $\tilde{\chi}^2$ is zero in $H^*(B(\text{Diff}_1(G/U))^\delta)$.

Remark 9.1. The result [27, Corollary] yields that the induced homomorphism $(Bj)^* : H^*(BG; \mathbb{Z}) \rightarrow H^*(BG^\delta; \mathbb{Z})$ is injective. Thus the same argument as above does work well to find a nontrivial element in the cohomology $H^*(B(\text{Diff}_1(G/U))^\delta)$ for an appropriate subgroup U of G if $H^*(BG; \mathbb{Z})$ is torsion free.

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