Backward Shifts on Function Algebras

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Abstract

J. R. Holub introduced the concept of backward shift on Banach spaces. We show that an infinite-dimensional function algebra does not admit a backward shift. Moreover, we define a backward quasi-shift as a weak type of a backward shift, and show that a function algebra A does not admit it, under the assumption that the Choquet boundary of A has at most finitely many isolated points.

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1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and T a bounded linear operator on $\mathcal{H}$. We call T a *(forward) shift* on $\mathcal{H}$, if there is a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \ldots$. Also, we call T a *backward shift* on $\mathcal{H}$, if there is a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ such that $Te_1 = 0$ and $Te_n = e_{n-1}$ for $n = 2, 3, \ldots$. In [5], R. M. Crownover introduced a shift on a Banach space, as a generalization of a forward shift on $\mathcal{H}$. The isometric shifts on various function spaces have been studied in [1], [6], [8], [14] and so on. In [10], J. R. Holub gave a similar generalization for a backward shift, as follows:

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Definition. Let $\mathcal{B}$ be a Banach space and T a bounded linear operator on $\mathcal{B}$. We write $\ker T$ to denote the kernel $\{f \in \mathcal{B} : Tf = 0\}$. We call T a *backward shift* on $\mathcal{B}$ if T satisfies the following conditions:

(i) The dimension of $\ker T$ is 1.

(ii) The induced operator $\hat{T} : f + \ker T \mapsto Tf$ from the quotient space $\mathcal{B}/\ker T$ into $\mathcal{B}$ is an isometry.

(iii) $\bigcup_{n=1}^{\infty} \ker T^n$ is dense in $\mathcal{B}$.

In this paper, we are concerned with this backward shift. Also, we say that T is a *backward quasi-shift* on $\mathcal{B}$, if T satisfies (i) and (ii) only.

Holub discussed the problem of the existence of backward shifts on various function spaces. One of the spaces consists of continuous functions. Let X be a compact Hausdorff space. By $C(X)$, we denote the Banach space of all continuous functions on X, equipped with the uniform norm. M. Rajagopalan and K. Sundaresan proved that $C(X)$ does not admit a backward shift if X is infinite (The case that $C(X)$ consists of real-valued functions was proved in [12] and the complex-value case was in [13]). A further generalization was given by M. Rajagopalan, T. M. Rassias and K. Sundaresan ([11]).

In this paper, we consider $C(X)$ as the *Banach algebra* of all continuous *complex-valued* functions on X, and deal with a function algebra as a generalization of $C(X)$. Recall that a function algebra A on X is a uniformly closed subalgebra of $C(X)$ which contains the constants and *separates* the points of X, that is, for each pair of distinct points $x_1, x_2 \in X$, there exists $f \in A$ such that $f(x_1) \neq f(x_2)$. The book [3] is a good reference on function algebras. In [2] and [7], J. Araujo and J. J. Font studied the finite-codimensional isometries on function algebras.

The main result in this paper is the following:

Theorem 1.1. *An infinite-dimensional function algebra does not admit a backward shift.*

This is a generalization of the Rajagopalan-Sundaresan theorem mentioned above. Here the adjective "infinite-dimensional" is crucially necessary, because a finite-dimensional space always admits a backward shift. Note that backward shifts on finite-dimensional spaces are not surjective. On the other hand, backward shifts on infinite-dimensional spaces are always surjective (see [12, Proposition 1.2]).

We also prove the following theorem:

Theorem 1.2. *Let A be a function algebra. Suppose that the Choquet boundary of A has at most finitely many isolated points. Then A does not admit a surjective backward quasi-shift.*

2. Lemmas

This section is devoted to the preparation for the proof of Theorems 1.1 and 1.2. Throughout this section, X is a compact Hausdorff space and A is a function algebra on X. Also, we use the following notations: Let $\mathbb{C}$ be a set of all complex numbers, and put $\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$. For a normed linear space $\mathcal{S}$, we use the symbol ball $\mathcal{S}$ to denote the closed unit ball of $\mathcal{S}$, and write $\mathcal{S}^*$ for the dual space of $\mathcal{S}$.

[Step 1] We first define a measure on X which is an extreme point of a certain measure space.

Let $M(X)$ denote the Banach space of all complex regular Borel measures on X, with the total variation norm. A simple example of a measure in $M(X)$ is a *point mass* δ_p concentrated at $p \in X$. We know that $\|\delta_p\| = 1$.

Now, we use δ_p to construct another measure. Take $u \in C(X)$ and put $S(u) = \{x \in X : u(x) \neq 0\}$. Choose distinct points $p, q \in S(u)$. We put

$$k_{upq} = \frac{u(q)}{|u(p)| + |u(q)|},$$

and define a measure λ_{upq} on X by

$$\lambda_{upq} = k_{upq}\delta_p - k_{uqp}\delta_q.$$

Since $|k_{upq}| + |k_{uqp}| = 1$, it follows that

$$\|\lambda_{upq}\| \leq |k_{upq}|\,\|\delta_p\| + |k_{uqp}|\,\|\delta_q\| = 1.$$

We characterize the measure λ_{upq}, as follows:

Lemma 2.1. *Let $\mu \in M(X)$ and $u \in C(X)$. Suppose that p and q are distinct points in $S(u)$. Then $\mu = \lambda_{upq}$ if and only if μ satisfies the following conditions:*

$$\mu(\{p\}) = k_{upq}, \quad \mu(\{q\}) = -k_{uqp} \quad \text{and} \quad \|\mu\| \leq 1. \tag{2.1}$$

Moreover, $\|\lambda_{upq}\| = 1$ and $|\lambda_{upq}|(X \setminus \{p, q\}) = 0$.

Proof. It is clear that $\mu = \lambda_{upq}$ satisfies (2.1). For the "if" part, suppose that μ satisfies (2.1). Then we have

$$\begin{aligned} 0 \leq |\mu|(X \setminus \{p, q\}) &= |\mu|(X) - |\mu|(\{p\}) - |\mu|(\{q\}) \\ &= \|\mu\| - |\mu(\{p\})| - |\mu(\{q\})| \\ &= \|\mu\| - |k_{upq}| - |k_{uqp}| = \|\mu\| - 1 \leq 0. \end{aligned}$$

Thus we obtain

$$\|\mu\| = 1 \quad \text{and} \quad |\mu|(X \setminus \{p, q\}) = 0.$$

Now let us show $\mu = \lambda_{upq}$. Take a Borel set E in X arbitrarily. If $p, q \notin E$, then $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X \setminus \{p, q\}) = 0$, and hence $\mu(E) = 0 = \lambda_{upq}(E)$. If $p \in E$ and $q \notin E$, then $\mu(E \setminus \{p\}) = 0$, and so

$$\mu(E) = \mu(E \setminus \{p\}) + \mu(\{p\}) = k_{upq} = \lambda_{upq}(E).$$

If $p \notin E$ and $q \in E$, we can see $\mu(E) = \lambda_{upq}(E)$ similarly. Finally, if $p, q \in E$, then $\mu(E \setminus \{p,q\}) = 0$, and so

$$\mu(E) = \mu(E \setminus \{p,q\}) + \mu(\{p\}) + \mu(\{q\}) = k_{upq} - k_{uqp} = \lambda_{upq}(E).$$

In any case, we obtain $\mu(E) = \lambda_{upq}(E)$. All is proven. $\square$

For $u \in C(X)$, we define a subspace $M([u]^{\perp})$ of $M(X)$ by

$$M([u]^{\perp}) = \left\{ \mu \in M(X) : \int_X u d\mu = 0 \right\}.$$

Lemma 2.2. *If $u \in C(X)$, and if p and q are distinct points in $S(u)$, then λ_{upq} is an extreme point of* $\operatorname{ball} M([u]^{\perp})$.

Proof. By Lemma 2.1, $|\lambda_{upq}|(X \setminus \{p,q\}) = 0$, and so

$$\begin{aligned} \int_X u d\lambda_{upq} &= \int_{\{p,q\}} u d\lambda_{upq} = u(p)\lambda_{upq}(\{p\}) + u(q)\lambda_{upq}(\{q\}) \\ &= u(p)k_{upq} - u(q)k_{uqp} = \frac{u(p)u(q)}{|u(p)| + |u(q)|} - \frac{u(q)u(p)}{|u(q)| + |u(p)|} = 0. \end{aligned}$$

Hence $\lambda_{upq} \in M([u]^{\perp})$. Since $\|\lambda_{upq}\| \leq 1$, we get $\lambda_{upq} \in \operatorname{ball} M([u]^{\perp})$.

Let us show that λ_{upq} is an extreme point of $\operatorname{ball} M([u]^{\perp})$. Assume that

$$\lambda_{upq} = t\mu + (1-t)\nu, \tag{2.2}$$

where $\mu, \nu \in \operatorname{ball} M([u]^{\perp})$ and $0 < t < 1$. We first observe the equations:

$$|\mu(\{p\})| + |\mu(\{q\})| = |\nu(\{p\})| + |\nu(\{q\})| = 1, \tag{2.3}$$

$$\arg \mu(\{p\}) = \arg \nu(\{p\}) \quad \text{and} \quad \arg \mu(\{q\}) = \arg \nu(\{q\}). \tag{2.4}$$

Indeed, we have

$$\begin{aligned} 1 &= |k_{upq}| + |k_{uqp}| \\ &= |\lambda_{upq}(\{p\})| + |\lambda_{upq}(\{q\})| \\ &= \big|t\mu(\{p\}) + (1-t)\nu(\{p\})\big| + \big|t\mu(\{q\}) + (1-t)\nu(\{q\})\big| \\ &\leq t|\mu(\{p\})| + (1-t)|\nu(\{p\})| + t|\mu(\{q\})| + (1-t)|\nu(\{q\})| \\ &= t\big(|\mu(\{p\})| + |\mu(\{q\})|\big) + (1-t)\big(|\nu(\{p\})| + |\nu(\{q\})|\big) \\ &\leq t\|\mu\| + (1-t)\|\nu\| \\ &\leq t + (1-t) = 1. \end{aligned}$$

Thus all above inequalities become equalities. Note that the inequality in the fourth line follows from the triangle inequality; $|\alpha+\beta| \leq |\alpha|+|\beta|$, where equality holds if and only if $\arg\alpha = \arg\beta$ or $\alpha\beta = 0$. Hence we obtain (2.4). Moreover the instance of equality in the last three lines implies (2.3).

Next, we show that

$$u(p)\mu(\{p\}) + u(q)\mu(\{q\}) = u(p)\nu(\{p\}) + u(q)\nu(\{q\}) = 0. \tag{2.5}$$

By (2.3), we have $|\mu|(X \setminus \{p, q\}) = |\mu|(X) - |\mu|(\{p\}) - |\mu|(\{q\}) = \|\mu\| - 1 \leq 0$, and so

$$0 = \int_X u d\mu = \int_{\{p,q\}} u d\mu = u(p)\mu(\{p\}) + u(q)\mu(\{q\}).$$

Similarly, we get $u(p)\nu(\{p\}) + u(q)\nu(\{q\}) = 0$.

By (2.5), $\mu(\{q\}) = -(u(p)/u(q))\, \mu(\{p\})$. Inserting this into (2.3) gives

$$|\mu(\{p\})| = \frac{|u(q)|}{|u(p)| + |u(q)|} = |k_{upq}|.$$

In the same way, we get $|\nu(\{p\})| = |k_{upq}|$. Hence $|\mu(\{p\})| = |\nu(\{p\})|$. Combining with the first equation in (2.4), we obtain $\mu(\{p\}) = \nu(\{p\})$. Hence (2.2) leads to $\mu(\{p\}) = \nu(\{p\}) = \lambda_{upq}(\{p\}) = k_{upq}$. By a similar argument, we can see that $\mu(\{q\}) = \nu(\{q\}) = \lambda_{upq}(\{q\}) = -k_{uqp}$. Here we recall that $\|\mu\| \leq 1$ and $\|\nu\| \leq 1$. By Lemma 2.1, we obtain $\mu = \nu = \lambda_{upq}$. Thus (2.2) implies $\lambda_{upq} = \mu = \nu$, and hence λ_{upq} is an extreme point. □

[Step 2] We here summarize our tools about the Choquet boundary of a function algebra.

Let $\varphi \in A^*$. The Hahn-Banach theorem and the Riesz representation theorem guarantee the existence of a measure $\mu \in M(X)$ such that

$$\varphi(f) = \int_X f d\mu \quad \text{for all } f \in A \quad \text{and} \quad \|\varphi\| = \|\mu\|.$$

Such a μ is called a *representing measure* for φ. We should note that a representing measure for φ is not always determined uniquely.

For each $p \in X$, an *evaluation functional* τ_p on A is defined by $\tau_p(f) = f(p)$ for all $f \in A$. We know that $\tau_p \in A^*$ and $\|\tau_p\| = \tau_p(1) = 1$. Also, we easily see that the point mass δ_p is one of the representing measures for τ_p. We recall that the *Choquet boundary* of A, which is denoted by $\mathrm{Ch}(A)$, is the set of all $p \in X$ such that δ_p is the only representing measure for τ_p.

The next lemma seems to be known:

Lemma 2.3. *Let $\varphi \in \operatorname{ball} A^*$. Then φ is an extreme point of $\operatorname{ball} A^*$ if and only if there exist $p \in \mathrm{Ch}(A)$ and $\alpha \in \mathbb{T}$ such that $\varphi = \alpha\tau_p$.*

Sketch of proof. To prove the "if" part, it suffices to show that for $p \in \mathrm{Ch}(A)$, τ_p is an extreme point of $\operatorname{ball} A^*$. Assume that $\tau_p = t\varphi + (1 - t)\psi$, where $\varphi, \psi \in \operatorname{ball} A^*$ and $0 < t < 1$. Let μ and ν be representing measures for φ and ψ, respectively. Then the measure $t\mu + (1 - t)\nu$ is a representing measure for τ_p, and so $t\mu + (1 - t)\nu = \delta_p$. By [4, Theorem V.8.4], we see that $\mu = \nu = \delta_p$, and hence $\varphi = \psi = \tau_p$.

For the "only if" part, let φ be an extreme point of $\operatorname{ball} A^*$. Using the method in [9, Page 145], we can find $p \in X$ and $\alpha \in \mathbb{T}$ such that $\varphi = \alpha\tau_p$. Here, we easily see that τ_p is an extreme point of the set $\{\varphi \in A^* : \|\varphi\| = \varphi(1) = 1\}$. Hence it follows from [3, Theorem 2.2.8] that $p \in \operatorname{Ch}(A)$. $\square$

There is another characterization of $\operatorname{Ch}(A)$; the Bishop-deLueew theorem, which states: A point $p \in X$ belongs to $\operatorname{Ch}(A)$ if and only if for each neighborhood U of p and for each $\varepsilon > 0$, there exists $g \in \operatorname{ball} A$ such that $g(p) > 1 - \varepsilon$ and $|g(x)| < \varepsilon$ for all $x \in X \setminus U$ (see [3, Theorem 2.3.4]).

Lemma 2.4. *Let p be an isolated point of $\operatorname{Ch}(A)$. Then there exists $f \in A$ such that $f(p) = 1$ and $f(x) = 1$ for all $x \in \operatorname{Ch}(A) \setminus \{p\}$.*

Proof. Since p is isolated in $\operatorname{Ch}(A)$, we find a neighborhood U of p in X so that $U \cap \operatorname{Ch}(A) = \{p\}$. Then the Bishop-deLueew theorem gives a sequence of functions $\{f_n\} \subset \operatorname{ball} A$ such that $f_n(p) > 1 - 1/2^n$ and $|f_n(x)| < 1/2^n$ for all $x \in X \setminus U$. This sequence satisfies $\sup\{|f_m(x) - f_n(x)| : x \in \operatorname{Ch}(A)\} \leq 1/2^{n-1}$ whenever $m > n$. Since $\|f\| = \sup\{|f(x)| : x \in \operatorname{Ch}(A)\}$ for all $f \in A$, it follows that $\{f_n\}$ is a Cauchy sequence in A. By the completeness of A, there exists $f \in A$ such that $\|f_n - f\| \to 0$. This function f must have the desired properties. $\square$

Lemma 2.5. *Let p and q be distinct points in $\operatorname{Ch}(A)$, and let $\alpha, \beta \in \mathbb{T}$. Then for each neighborhood W of $\{p, q\}$ and each $\varepsilon > 0$, there exists $f \in \operatorname{ball} A$ such that $|f(p) - \alpha| < \varepsilon$, $|f(q) - \beta| < \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus W$.*

Proof. Choose disjoint open sets U and V so that $p \in U \subset W$, $q \in V \subset W$. By the Bishop-deLeeuw theorem, there exist $g, h \in \operatorname{ball} A$ such that

$$\begin{array}{llll} g(p) > 1 - \varepsilon & \text{and} & |g(x)| < \varepsilon & \text{for } x \in X \setminus U, \\ h(q) > 1 - \varepsilon & \text{and} & |h(x)| < \varepsilon & \text{for } x \in X \setminus V. \end{array}$$

Then we have

$$|\alpha g(x) + \beta h(x)| \leq \begin{cases} \|g\| + |h(x)| \leq 1 + \varepsilon & \text{if } x \in U, \\ |g(x)| + \|h\| \leq \varepsilon + 1 & \text{if } x \in X \setminus U. \end{cases}$$

Now, we define a function $f \in \operatorname{ball} A$ by $f = (\alpha g + \beta h)/(1 + \varepsilon)$. Then we have $|f(p) - \alpha| < 3\varepsilon/(1 + \varepsilon)$, because

$$\begin{aligned} |f(p) - \alpha| &= \left| \frac{(\alpha g(p) + \beta h(p)) - \alpha(1 + \varepsilon)}{1 + \varepsilon} \right| \\ &\leq \frac{|\alpha|\,|g(p) - 1| + |\beta|\,|h(p)| + |\alpha|\varepsilon}{1 + \varepsilon} < \frac{3\varepsilon}{1 + \varepsilon}. \end{aligned}$$

Similarly, we obtain $|f(q) - \beta| < 3\varepsilon/(1 + \varepsilon)$. Furthermore, if $x \in X \setminus W$, then $|g(x)| < \varepsilon$ and $|h(x)| < \varepsilon$, so that $|f(x)| < 2\varepsilon/(1 + \varepsilon)$. Finally, we only have to arrange a positive number ε to find the desired function f. $\square$

[Step 3] Let us consider the functional on A that is represented by the measure λ_{upq}. For each $u \in A$ and for each pair of distinct points $p, q \in S(u)$, we define the bounded linear functional θ_{upq} on A by

$$\theta_{upq} = k_{upq}\tau_p - k_{uqp}\tau_q,$$

where the constants k_{upq}, k_{uqp} are defined in Step 1, and τ_p, τ_q are the evaluation functional defined in Step 2.

Lemma 2.6. *Let $u \in A$, and let p and q be distinct points in $S(u) \cap \mathrm{Ch}(A)$. Then*

(i) *For each neighborhood W of $\{p, q\}$ and each $\varepsilon > 0$, there exists $f \in \mathrm{ball}\, A$ such that $|\theta_{upq}(f)| > 1 - \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus W$.*

(ii) $\|\theta_{upq}\| = 1$.

Proof. To see (i), take $\alpha = |u(q)|/u(q)$ and $\beta = -|u(p)|/u(p)$ in Lemma 2.5. Then the resulting function f in $\mathrm{ball}\, A$ satisfies $|f(x)| < \varepsilon$ for all $x \in X \setminus W$. It also satisfies $|f(p) - \alpha| < \varepsilon$ and $|f(q) - \beta| < \varepsilon$, so that

$$\begin{aligned}
1 - |\theta_{upq}(f)| &\leq |\theta_{upq}(f) - 1| = |k_{upq}f(p) - k_{uqp}f(q) - (|k_{upq}| + |k_{uqp}|)| \\
&= |k_{upq}f(p) - k_{uqp}f(q) - k_{upq}\alpha + k_{uqp}\beta| \\
&\leq |k_{upq}|\,|f(p) - \alpha| + |k_{uqp}|\,|f(q) - \beta| \\
&< |k_{upq}|\varepsilon + |k_{uqp}|\varepsilon = \varepsilon.
\end{aligned}$$

Thus (i) is proved.

For (ii), note that $\|\theta_{upq}\| \leq |k_{upq}|\,\|\tau_p\| + |k_{uqp}|\,\|\tau_q\| = |k_{upq}| + |k_{uqp}| = 1$. Also, the function f in (i) satisfies $\|\theta_{upq}\| \geq |\theta_{upq}(f)| > 1 - \varepsilon$. Since ε is arbitrary, we get $\|\theta_{upq}\| \geq 1$. □

Lemma 2.7. *Let $u \in A$, and let p and q be distinct points in $S(u) \cap \mathrm{Ch}(A)$. Then λ_{upq} is the only representing measure for θ_{upq}.*

Proof. For any $f \in A$, we have

$$\theta_{upq}(f) = k_{upq}\tau_p(f) - k_{uqp}\tau_q(f) = k_{upq}\int_X f\,d\delta_p - k_{uqp}\int_X f\,d\delta_q = \int_X f\,d\lambda_{upq}.$$

Also, Lemma 2.6 (ii) and Lemma 2.1 yield $\|\theta_{upq}\| = 1 = \|\lambda_{upq}\|$. Therefore, λ_{upq} is a representing measure for θ_{upq}.

Let us show the uniqueness of λ_{upq}. Let μ be another representing measure for θ_{upq}. For each neighborhood W of $\{p, q\}$ and each $\varepsilon > 0$, Lemma 2.6 (i) gives a function $f \in \mathrm{ball}\, A$ such that $|\theta_{upq}(f)| > 1 - \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus W$. Then we have

$$\begin{aligned}
1 - \varepsilon < |\theta_{upq}(f)| &= \left|\int_X f\,d\mu\right| \leq \left|\int_W f\,d\mu\right| + \left|\int_{X\setminus W} f\,d\mu\right| \\
&\leq \|f\|\,|\mu|(W) + \varepsilon|\mu|(X \setminus W) \leq |\mu|(W) + \varepsilon(1 - |\mu|(W)) \\
&= (1 - \varepsilon)|\mu|(W) + \varepsilon,
\end{aligned}$$

so that

$$|\mu|(W) \geq \frac{1-2\varepsilon}{1-\varepsilon}.$$

Letting $\varepsilon \to 0$, we get $|\mu|(W) \geq 1$, and the regularity of μ forces $|\mu|(\{p,q\}) = 1$. Since $|\mu|(X) = \|\mu\| = \|\theta_{upq}\| = 1$, it follows that $|\mu|(X \setminus \{p,q\}) = 0$. Hence, for each $f \in A$, we have

$$\begin{aligned} k_{upq} f(p) - k_{uqp} f(q) = \theta_{upq}(f) &= \int_X f d\mu = \int_{\{p,q\}} f d\mu \\ &= f(p)\mu(\{p\}) + f(q)\mu(\{q\}). \end{aligned}$$

Taking $f \in A$ so that $f(p) = 1$ and $f(q) = 0$, we obtain $k_{upq} = \mu(\{p\})$. While, taking f so that $f(p) = 0$ and $f(q) = 1$ yields $-k_{uqp} = \mu(\{q\})$. Moreover, we know $\|\mu\| = 1$. Finally, we appeal to Lemma 2.1 to get $\mu = \lambda_{upq}$. □

[Step 4] In this step, we show the functional version of Lemma 2.2. For $u \in A$, we put

$$[u] = \{\alpha u : \alpha \in \mathbb{C}\}$$

and

$$[u]^{\perp} = \{\varphi \in A^* : \varphi(u) = 0\}.$$

Lemma 2.8. *If $u \in A$, and if p and q are distinct points in $S(u) \cap \mathrm{Ch}(A)$, then θ_{upq} is an extreme point of* $\mathrm{ball}[u]^{\perp}$.

Proof. Since

$$\theta_{upq}(u) = k_{upq}\tau_p(u) - k_{uqp}\tau_q(u) = \frac{u(q)u(p)}{|u(p)| + |u(q)|} - \frac{u(p)u(q)}{|u(q)| + |u(p)|} = 0,$$

it follows $\theta_{upq} \in [u]^{\perp}$. Combining with Lemma 2.6 (ii), we get $\theta_{upq} \in \mathrm{ball}[u]^{\perp}$.

Next, we show that θ_{upq} is an extreme point of $\mathrm{ball}[u]^{\perp}$. Assume that

$$\theta_{upq} = t\varphi + (1-t)\psi,$$

where $\varphi, \psi \in \mathrm{ball}[u]^{\perp}$ and $0 < t < 1$. Take representing measures μ and ν for φ and ψ, respectively. Put $\lambda = t\mu + (1-t)\nu$. Then for any $f \in A$, we have

$$\int_X f d\lambda = t\int_X f d\mu + (1-t)\int_X f d\nu = t\varphi(f) + (1-t)\psi(f) = \theta_{upq}(f).$$

This implies

$$|\theta_{upq}(f)| = \left|\int_X f d\lambda\right| \leq \int_X |f| d|\lambda| \leq \|f\|\,\|\lambda\|,$$

and so $\|\theta_{upq}\| \leq \|\lambda\|$. Also, $\|\mu\| = \|\varphi\| \leq 1$ and $\|\nu\| = \|\psi\| \leq 1$, and hence

$$\|\lambda\| \leq t\|\mu\| + (1-t)\|\nu\| \leq 1 = \|\theta_{upq}\|.$$

Therefore, $\|\theta_{upq}\| = \|\lambda\|$. As a consequence, λ is a representing measure for θ_{upq}, and Lemma 2.7 shows that $\lambda = \lambda_{upq}$. Thus we obtain

$$\lambda_{upq} = t\mu + (1-t)\nu. \tag{2.6}$$

Since φ and ψ belong to $[u]^{\perp}$, it follows that

$$\int_X u d\mu = \varphi(u) = 0 \quad \text{and} \quad \int_X u d\nu = \psi(u) = 0.$$

Hence $\mu, \nu \in \operatorname{ball} M([u]^{\perp})$. Recall from Lemma 2.2 that λ_{upq} is an extreme point of $\operatorname{ball} M([u]^{\perp})$. Then (2.6) leads to $\lambda_{upq} = \mu = \nu$. Thus we have

$$\theta_{upq}(f) = \int_X f d\lambda_{upq} = \int_X f d\mu = \varphi(f)$$

for all $f \in A$, that is, $\theta_{upq} = \varphi$. Similarly, we get $\theta_{upq} = \psi$. We reach the desired equation $\theta_{upq} = \varphi = \psi$. $\square$

[Step 5] In this step, we investigate the distance $\|\varphi - \psi\|$ for $\varphi, \psi \in \operatorname{ball} A^*$.

Lemma 2.9. *If p and q are distinct points in $\operatorname{Ch}(A)$ and if $\alpha, \beta \in \mathbb{T}$, then*

$$\|\alpha\tau_p - \beta\tau_q\| = 2.$$

Proof. It is clear that $\|\alpha\tau_p - \beta\tau_q\| \leq 2$. For the reverse inequality, take $\varepsilon > 0$. Lemma 2.5 gives a function $f \in \operatorname{ball} A$ such that $|f(p) - \overline{\alpha}| < \varepsilon$ and $|f(q) + \overline{\beta}| < \varepsilon$. Then we have

$$\begin{aligned} 2 - |\alpha\tau_p(f) - \beta\tau_q(f)| &\leq |\alpha\tau_p(f) - \beta\tau_q(f) - 2| \\ &= |\alpha(f(p) - \overline{\alpha}) - \beta(f(q) + \overline{\beta})| \\ &\leq |\alpha|\,|f(p) - \overline{\alpha}| + |\beta|\,|f(q) + \overline{\beta}| < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Therefore, $\|\alpha\tau_p - \beta\tau_q\| \geq |\alpha\tau_p(f) - \beta\tau_q(f)| > 2 - 2\varepsilon$. Since ε is arbitrary, we get $\|\alpha\tau_p - \beta\tau_q\| \geq 2$. $\square$

Lemma 2.10. *Let $u \in A$. If the set $S(u) \cap \operatorname{Ch}(A)$ contains at least three distinct points, then there exist extreme points φ and ψ of $\operatorname{ball}[u]^{\perp}$ such that*

(i) $\|\varphi - \psi\| < 2$, *and*

(ii) *φ and ψ are linearly independent.*

Proof. By hypothesis, we find three distinct points p, q and r in $S(u) \cap \operatorname{Ch}(A)$. Then we may assume that

$$\arg u(p) \neq \arg(-u(q)). \tag{2.7}$$

For, if there exist no such points p and q, then three equations

$$\arg u(p) = \arg(-u(q)),\ \arg u(q) = \arg(-u(r)) \text{ and } \arg u(r) = \arg(-u(p))$$

hold simultaneously, which is impossible. Now, put $\varphi = \theta_{upr}$ and $\psi = \theta_{uqr}$. By Lemma 2.8, φ and ψ are extreme points of $\mathrm{ball}[u]^\perp$.

Let us show (i). By (2.7),

$$\arg k_{urp} \neq \arg(-k_{urq}).$$

Therefore, the triangle inequality $|k_{urp} - k_{urq}| < |k_{urp}| + |k_{urq}|$ holds strictly. Hence we have

$$\begin{aligned}\|\varphi - \psi\| = \|\theta_{upr} - \theta_{uqr}\| &= \|(k_{upr}\tau_p - k_{urp}\tau_r) - (k_{uqr}\tau_q - k_{urq}\tau_r)\| \\ &= \|k_{upr}\tau_p - (k_{urp} - k_{urq})\tau_r - k_{uqr}\tau_q\| \\ &\leq |k_{upr}| + |k_{urp} - k_{urq}| + |k_{uqr}| \\ &< |k_{upr}| + |k_{urp}| + |k_{urq}| + |k_{uqr}| = 2.\end{aligned}$$

To verify (ii), assume $\alpha\varphi + \beta\psi = 0$ and $\alpha, \beta \in \mathbb{C}$. Then, for any $f \in A$, we have

$$\begin{aligned}0 = \alpha\varphi(f) + \beta\psi(f) &= \alpha\big(k_{upr}\tau_p(f) - k_{urp}\tau_r(f)\big) + \beta\big(k_{uqr}\tau_q(f) - k_{urq}\tau_r(f)\big) \\ &= \alpha k_{upr} f(p) - (\alpha k_{urp} + \beta k_{urq}) f(r) + \beta k_{uqr} f(q).\end{aligned}$$

Taking $f \in A$ so that $f(p) = 1$ and $f(q) = f(r) = 0$, we have $0 = \alpha k_{upr}$. Noting $k_{upr} \neq 0$, we get $\alpha = 0$. On the other hand, if we take $f \in A$ so that $f(q) = 1$ and $f(p) = f(r) = 0$, then we get $\beta = 0$. Thus φ and ψ are linearly independent. □

[Step 6] The preceding two lemmas yield the following lemma:

Lemma 2.11. *Let $u \in A$. If the set $S(u) \cap \mathrm{Ch}(A)$ contains at least three distinct points, then $[u]^\perp$ is not linearly isometric to A^*.*

Proof. Assume that $[u]^\perp$ is linearly isometric to A^*. Then there is a linear isometry T of $[u]^\perp$ onto A^*. Consider extreme points φ and ψ of $\mathrm{ball}[u]^\perp$ described in Lemma 2.10. Then $T\varphi$ and $T\psi$ become extreme points of $\mathrm{ball}\, A^*$. Hence Lemma 2.3 shows $T\varphi = \alpha\tau_p$ and $T\psi = \beta\tau_q$, where $p, q \in \mathrm{Ch}(A)$ and $\alpha, \beta \in \mathbb{T}$.

If $p \neq q$, Lemma 2.9 implies that $\|T\varphi - T\psi\| = \|\alpha\tau_p - \beta\tau_q\| = 2$. Since T is an isometry, $\|\varphi - \psi\| = 2$, which contradicts the condition (i) in Lemma 2.10.

On the other hand, if $p = q$, then we have

$$T(\beta\varphi - \alpha\psi) = \beta T\varphi - \alpha T\psi = \beta\alpha\tau_p - \alpha\beta\tau_q = \alpha\beta(\tau_p - \tau_p) = 0.$$

Since T is injective, it follows that $\beta\varphi - \alpha\psi = 0$. Note that $\alpha, \beta \neq 0$. This contradicts the linear independence of φ and ψ from Lemma 2.10 (ii). Consequently, $[u]^\perp$ is not linearly isometric to A^*. □

[Step 7] Let us consider a backward quasi-shift on A.

Lemma 2.12. *Suppose that there exists a surjective backward quasi-shift T on A. If $f \in \bigcup_{n=1}^{\infty} \ker T^n$, then $S(f) \cap \mathrm{Ch}(A)$ is a finite set. In particular, if $\ker T = [u]$, then $S(u) \cap \mathrm{Ch}(A)$ is finite.*

Proof. Since $\ker T$ is one-dimensional, we can write $\ker T = [u]$, where $u \in A$ and $u \neq 0$. Since the induced operator $\hat{T} : f + [u] \mapsto Tf$ is a linear isometry from $A/[u]$ onto A, the adjoint operator $\hat{T}^*$ is a linear isometry from A^* onto $(A/[u])^*$. Note that $(A/[u])^*$ is linearly isometric to $[u]^\perp$, via the linear isometry $\sigma : (A/[u])^* \to [u]^\perp$ defined by $(\sigma(\Phi))(f) = \Phi(f + [u])$ for all $f \in A$ and $\Phi \in (A/[u])^*$. Thus we have

$$\begin{aligned}((\sigma \circ \hat{T}^*)\varphi)(f) &= (\sigma(\hat{T}^*\varphi))(f) = (\hat{T}^*\varphi)(f + [u]) \\ &= \varphi\big(\hat{T}(f + [u])\big) = \varphi(Tf) = (T^*\varphi)(f)\end{aligned}$$

for all $f \in A$ and $\varphi \in A^*$. Hence $\sigma \circ \hat{T}^* = T^*$, and so T^* is a linear isometry from A^* onto $[u]^\perp$.

Once we have seen that $[u]^\perp$ is linearly isometric to A^*, Lemma 2.11 says that the number of elements of $S(u) \cap \mathrm{Ch}(A)$ is less than 2. Of course, $S(u) \cap \mathrm{Ch}(A)$ is finite.

To prove the lemma, we show the following assertion for all $n = 1, 2, \ldots$:

$$\text{If } f \in \ker T^n, \text{ then } S(f) \cap \mathrm{Ch}(A) \text{ is a finite set.} \tag{2.8}$$

We adopt an induction on n.

First, consider the case $n = 1$. If $f \in \ker T = [u]$, then $f = \alpha u$ for some $\alpha \in \mathbb{C}$. Hence

$$S(f) \cap \mathrm{Ch}(A) = S(\alpha u) \cap \mathrm{Ch}(A) \subset S(u) \cap \mathrm{Ch}(A).$$

Since $S(u) \cap \mathrm{Ch}(A)$ is finite, so is $S(f) \cap \mathrm{Ch}(A)$. Thus (2.8) is true when $n = 1$.

For the inductive step, assume that (2.8) is valid for some n. We must show that if $f \in \ker T^{n+1}$, then $S(f) \cap \mathrm{Ch}(A)$ is finite. Put $g = Tf$. Then $g \in \ker T^n$, and the assumption (2.8) implies that $S(g) \cap \mathrm{Ch}(A)$ is finite.

Consider the set P of all $p \in \mathrm{Ch}(A)$ such that there exist $q \in S(g) \cap \mathrm{Ch}(A)$ and $\alpha \in \mathbb{T}$ satisfying $T^*(\alpha\tau_q) = \tau_p$. We know that for each $p \in P$, the pair (q, α) as above is uniquely determined, because T^* is injective. Thus we can define the map $\pi : P \to S(g) \cap \mathrm{Ch}(A)$ by $\pi(p) = q$, where $p \in P$, $q \in S(g) \cap \mathrm{Ch}(A)$, $\alpha \in \mathbb{T}$ and $T^*(\alpha\tau_q) = \tau_p$. Let us show that π is injective. If not, there exist $p, p' \in P$ such that $\pi(p) = \pi(p')$ $(= q)$. Then $T^*(\alpha\tau_q) = \tau_p$ and $T^*(\alpha'\tau_q) = \tau_{p'}$ for some $\alpha, \alpha' \in \mathbb{T}$. Take a function f so that $f(p) = 1$ and $f(p') = 0$. Then we have

$$\begin{aligned}1 = f(p) = \tau_p(f) &= (T^*(\alpha\tau_q))(f) \\ &= \frac{\alpha}{\alpha'}(T^*(\alpha'\tau_{q'}))(f) = \frac{\alpha}{\alpha'}\tau_{p'}(f) = \frac{\alpha}{\alpha'}f(p') = 0,\end{aligned}$$

which is a contradiction. Hence $\pi : P \to S(g) \cap \mathrm{Ch}(A)$ is injective, and so the number of the elements of P is less than that of the elements of $S(g) \cap \mathrm{Ch}(A)$. Since $S(g) \cap \mathrm{Ch}(A)$ is finite, so is P.

Next, we show the inclusion:

$$S(f) \cap \mathrm{Ch}(A) \subset (S(u) \cap \mathrm{Ch}(A)) \cup P. \tag{2.9}$$

For this, it suffices to show that if $p \in S(f) \cap \mathrm{Ch}(A)$ and if $p \notin S(u)$, then $p \in P$. Since $p \notin S(u)$, $\tau_p(u) = u(p) = 0$, and so $\tau_p \in [u]^\perp$. Using Lemma 2.3, we easily see that τ_p is an extreme point of ball$[u]^\perp$. Since T^* is a linear isometry from A^* onto $[u]^\perp$, we find an extreme point φ of ball A^* such that $T^*\varphi = \tau_p$, and Lemma 2.3 gives the form $\varphi = \alpha\tau_q$, where $q \in \mathrm{Ch}(A)$ and $\alpha \in \mathbb{T}$. Thus $T^*(\alpha\tau_q) = \tau_p$. Also, $p \in S(f)$ implies

$$\alpha g(q) = \alpha\tau_q(g) = (\alpha\tau_q)(Tf) = (T^*(\alpha\tau_q))(f) = \tau_p(f) = f(p) \neq 0,$$

and so $q \in S(g)$. Thus we arrive at $p \in P$, and the inclusion (2.9) is established.

We now know that both $S(u) \cap \mathrm{Ch}(A)$ and P are finite. Therefore, (2.9) implies that $S(f) \cap \mathrm{Ch}(A)$ is finite. This accomplishes the inductive step and completes the proof. □

3. Proofs of Theorems

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let A be an infinite-dimensional function algebra on a compact Hausdorff space X. The linear space $\{f|_{\mathrm{Ch}(A)} : f \in A\}$ is isomorphic to A, and it is also infinite-dimensional. Hence $\mathrm{Ch}(A)$ must have infinitely many points. Thus the compact set X contains an accumulation point p of $\mathrm{Ch}(A)$. In other words, there exists a net $\{p_i\}$ consisting of infinitely many points of $\mathrm{Ch}(A)$ such that $\{p_i\}$ converges to p.

Now, assume that there exists a backward shift T on A. From the comment in Introduction, we know that T is a surjective backward quasi-shift on A. Take $f \in \bigcup_{n=1}^\infty \ker T^n$ arbitrarily. By Lemma 2.12, the set $S(f) \cap \mathrm{Ch}(A)$ is finite. So, we may assume that $\{p_i\} \subset \mathrm{Ch}(A) \setminus S(f)$. Then, for each i, we have $f(p_i) = 0$, and the continuity of f shows that $f(p) = 0$. Thus we have

$$\|1 - f\| \geq |1 - f(p)| = 1.$$

Since this holds for all $f \in \bigcup_{n=1}^\infty \ker T^n$, the constant function 1 cannot lie in the closure of $\bigcup_{n=1}^\infty \ker T^n$. Hence, $\bigcup_{n=1}^\infty \ker T^n$ is not dense in A. This contradicts the fact that T is a backward shift, and the theorem is proved. □

Proof of Theorem 1.2. Assume that there exists a surjective backward quasi-shift T on A. Since $\ker T$ is one-dimensional, we can write $\ker T = [u]$, where $u \in A$ and $u \neq 0$. Note that $S(u)$ is open in X and that $S(u) \cap \mathrm{Ch}(A)$ is finite by Lemma 2.12. We see that all points in $S(u) \cap \mathrm{Ch}(A)$ are isolated points of $\mathrm{Ch}(A)$. While, $u \neq 0$ implies that $S(u) \cap \mathrm{Ch}(A)$ is non-empty. As a consequence, there exists at least one isolated point of $\mathrm{Ch}(A)$.

Now, let m be the number of isolated points of $\mathrm{Ch}(A)$. We show that the dimension of $\ker T^{m+1}$ is less than m. Write down all isolated points of $\mathrm{Ch}(A)$ as $p_1, \ldots, p_m$. For each $j = 1, \ldots, m$, Lemma 2.4 gives us a function $f_j \in A$ such that $f_j(p_j) = 1$ and $f_j(x) = 0$ for all $x \in \mathrm{Ch}(A) \setminus \{p_j\}$. Pick $f \in \ker T^{m+1}$ arbitrarily. By Lemma 2.12, $S(f) \cap \mathrm{Ch}(A)$ is finite, and so we again see that all points in $S(f) \cap \mathrm{Ch}(A)$ are isolated points of $\mathrm{Ch}(A)$, that is, $S(f) \cap \mathrm{Ch}(A) \subset \{p_1, \ldots, p_m\}$. Hence, if we put $\alpha_j = f(p_j)$ for each $j = 1, \ldots, m$, then

$$\begin{aligned} f|_{\mathrm{Ch}(A)} &= \alpha_1 f_1|_{\mathrm{Ch}(A)} + \cdots + \alpha_m f_m|_{\mathrm{Ch}(A)} \\ &= (\alpha_1 f_1 + \cdots + \alpha_m f_m)|_{\mathrm{Ch}(A)}, \end{aligned}$$

which implies $f = \alpha_1 f_1 + \cdots + \alpha_m f_m$. Thus every $f \in \ker T^{m+1}$ is written as a linear combination of $f_1, \ldots, f_m$, and we conclude that the dimension of $\ker T^{m+1}$ is less than m.

Now note that

$$[u] = \ker T \subset \ker T^2 \subset \cdots \subset \ker T^m \subset \ker T^{m+1}.$$

As a consequence of the preceding paragraph, we must have $\ker T^N = \ker T^{N+1}$ for some $N \in \{0, 1, \ldots, m\}$. Since T^N, like T, is surjective, we find $h \in A$ with $T^N h = u$. Then $T^{N+1} h = T(T^N h) = Tu = 0$ and so $h \in \ker T^{N+1} = \ker T^N$. Hence $u = T^N h = 0$, a contradiction. □

4. Examples

In this section, we exhibit three examples related with Theorems 1.1 and 1.2. The first is an example of a surjective backward quasi-shift which is not a backward shift.

Example 4.1. Let c denote the Banach algebra of all convergent sequences with the supremum norm. Define an operator T on c by $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. It is easily seen that T is a surjective backward quasi-shift on c. However, T is not a backward shift, because it does not satisfy (iii). Next, we identify c with $C(X)$, where X is the one-point compactification of the natural numbers. Thus we know that $C(X)$ can admit a surjective backward quasi-shift, for some X.

The next example deals with the L^∞-spaces.

Example 4.2. Let $L^\infty(\Omega, \mu)$ be the Banach algebra of essentially bounded measurable functions on a finite measure space (Ω, μ), with the essential supremum norm. It is well known that $L^\infty(\Omega, \mu)$ is isometrically isomorphic to $C(X)$, where X is the maximal ideal space of $L^\infty(\Omega, \mu)$. If the measure μ has at most finitely many atoms, then X has at most finitely many isolated points, and so Theorem 1.2 shows that $L^\infty(\Omega, \mu)$ does not admit a surjective backward quasi-shift.

In the last example, we discuss the question whether the disc algebra admits an isometric shift or a backward shift.

Example 4.3. Let $A(\mathbb{D})$ be the disc algebra, that is, the function algebra of all continuous functions on the closed unit disc which are analytic in the open unit disc. The isometric shifts on $A(\mathbb{D})$ are characterized by T. Takayama and J. Wada [14]. A typical example of it is the multiplication operator T:

$$(Tf)(z) = zf(z) \quad \text{for all } z \text{ and } f \in A(\mathbb{D}).$$

This example suggests to us that the following operator T may be a backward shift:

$$(Tf)(z) = \begin{cases} \frac{f(z)-f(0)}{z} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0, \end{cases} \quad \text{for all } f \in A(\mathbb{D}).$$

It is easy to see that T is surjective and satisfies the conditions (i) and (iii) in the definition of backward shift. But T is not a backward shift. Indeed, T does not satisfy (ii), because $\ker T$ is the subspace of constant functions, and the function $f(z) = z^2 + z$ satisfies that

$$\inf\{\|f + g\| : g \in \ker T\} \leq \left\| f - \frac{1}{2} \right\| = \sqrt{\frac{27}{8}} < 2 = \|Tf\|.$$

Moreover, Theorem 1.2 implies that $A(\mathbb{D})$ does not admit a surjective backward quasi-shift, because $\mathrm{Ch}(A(\mathbb{D}))$ is the unit circle $\mathbb{T}$ which has no isolated points.

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