

Terwilliger algebras of direct and wreath products of association schemes

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Abstract

We will consider structures of Terwilliger algebras of direct and wreath products of association schemes. In general, it is difficult to determine the structure of the Terwilliger algebras though they are known to be semisimple \mathbb{C} -algebras. But, we get the structure of Terwilliger algebras of these cases under some assumptions.

Keywords: Terwilliger algebra, representation, association scheme

1. Introduction

The Terwilliger algebra is a new algebraic tool for the study of association schemes, introduced by Terwilliger in 1992 [6], [7] and [8]. In a sense, Terwilliger algebras can contain combinatorial information more than adjacency algebras. However, we don't know general theory for the structure of Terwilliger algebras. In general, it is difficult to determine irreducible representations of Terwilliger algebras although the algebras are known to be semisimple over the complex number field.

In [2], Bhattacharyya et al. determined the structure of Terwilliger algebras of repeated wreath products of class-one schemes. They computed all irreducible representations concretely. In this article, we will determine all irreducible representations of wreath products of arbitrary schemes by class-

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one schemes or thin schemes using information of smaller schemes. The main result in [2] is obtained by repeating our result.

2. Association schemes and Terwilliger algebras

Let X be a finite set, S a collection of non-empty subsets of $X \times X$. We say that (X, S) is an *association scheme* if the following conditions hold:

- (1) $\bigcup_{s \in S} s = X \times X$ and $s \cap t = \emptyset$ if $s \neq t$.
- (2) Put $1 = \{(x, x) \mid x \in X\}$. Then, $1 \in S$.
- (3) For $s \in S$, put $s^* = \{(y, x) \mid (x, y) \in s\} \in S$. Then $s^* \in S$.
- (4) For all $s, t, u \in S$ and all $x, y \in X$,

$$p_{st}^u = \#\{z \in X \mid (x, z) \in s, (z, y) \in t\}$$

is constant whenever $(x, y) \in u$.

We call $p_{ss^*}^1$ the *valency* of S and write it by n_s .

Let (X, S) denote an association scheme. Let $M_X(\mathbb{C})$ denote a \mathbb{C} -algebra of matrices with complex entries, where the rows and columns are indexed by elements in X . For $s \in S$, let σ_s denote the matrix in $M_X(\mathbb{C})$ that has entries

$$(\sigma_s)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in s, \\ 0 & \text{otherwise.} \end{cases}$$

We call σ_s the *adjacency matrix* of $s \in S$.

Then $\bigoplus_{s \in S} \mathbb{C}\sigma_s$ becomes a subalgebra of $M_X(\mathbb{C})$ by the condition (4). We call $\bigoplus_{s \in S} \mathbb{C}\sigma_s$ the *adjacency algebra* of S , and write it by $\mathcal{A}(S)$.

Definition 2.1 (thin scheme). Let G is a finite group. For each $g \in G$, put $s_g = \{(\alpha, \beta) \in G \times G \mid \alpha^{-1}\beta = g\}$. Then $(G, \{s_g \mid g \in G\})$ is an association scheme. We call this a *thin scheme*.

Definition 2.2 (class-one scheme). Let X is finite set. We define relations $1 = \{(x, x) \mid x \in X\}$ and $t = \{(x, y) \mid x \neq y\}$. Then $(X, \{1, t\})$ is an association scheme. We call this a *class-one scheme*.

Let (X, S) be an association scheme. For $U \subset X$, we denote by ε_U the diagonal matrix in $M_X(\mathbb{C})$ with entries $(\varepsilon_U)_{xx} = 1$ if $x \in U$ and $(\varepsilon_U)_{xx} = 0$ otherwise.

The *Terwilliger algebra* of (X, S) with respect to $x_0 \in X$ is defined as a subalgebra of $M_X(\mathbb{C})$ generated by $\{\sigma_s \mid s \in S\} \cup \{\varepsilon_{x_0s} \mid s \in S\}$ (see [6],[7], and [8]). The Terwilliger algebra will be denoted by $\mathcal{T}(X, S, x_0)$ or $\mathcal{T}(S)$ briefly. Since $\mathcal{A}(S)$ and $\mathcal{T}(S)$ are closed under transposed conjugate, they are semisimple \mathbb{C} -algebras. The set of irreducible characters of $\mathcal{T}(S)$ and $\mathcal{A}(S)$ will be denoted by $\text{Irr}(\mathcal{T}(S))$ and $\text{Irr}(\mathcal{A}(S))$, respectively. The *trivial character* $1_{\mathcal{A}(S)}$ of $\mathcal{A}(S)$ is a map $\sigma_s \mapsto n_s$ and the corresponding central primitive idempotent is $|X|^{-1}J_X$, where J_X is the all-one matrix. The *trivial character* $1_{\mathcal{T}(S)}$ of $\mathcal{T}(S)$ corresponds to the central primitive idempotent $\sum_{s \in S} n_s^{-1} \varepsilon_{x_0s} J_X \varepsilon_{x_0s}$ of $\mathcal{T}(S)$. For $\chi \in \text{Irr}(\mathcal{A}(S))$ or $\text{Irr}(\mathcal{T}(S))$, e_χ will be the corresponding central primitive idempotent of $\mathcal{A}(S)$ or $\mathcal{T}(S)$.

For $Y \subset X$ and $s \in S$, set

$$s_Y = s \cap (Y \times Y)$$

and set

$$S_Y = \{s_Y \mid s \in S, s_Y \neq \emptyset\}.$$

In general, (Y, S_Y) is not necessary an association scheme. When (Y, S_Y) is an association scheme, we say that Y *induces an association scheme* (Y, S_Y) .

3. Terwilliger algebras of direct products

Let (X, S) and (Y, T) be association schemes. We will consider the direct product $(X \times Y, S \times T)$ of (X, S) and (Y, T) (for example, see [5]). The adjacency matrix of $(s, t) \in S \times T$ is given by the Kronecker product $\sigma_s \otimes \sigma_t$.

We fix $x_0 \in X$ and $y_0 \in Y$ and consider the Terwilliger algebras of (X, S) and (Y, T) with respect to x_0 and y_0 , respectively. We will determine the structure of the Terwilliger algebra of $(X \times Y, S \times T)$ with respect to (x_0, y_0) .

Theorem 3.1. *We have*

$$\mathcal{T}(X \times Y, S \times T, (x_0, y_0)) \cong \mathcal{T}(X, S, x_0) \otimes_{\mathbb{C}} \mathcal{T}(Y, T, y_0).$$

Proof. First we confirm notation. We will consider $\mathcal{T}(S \times T) \subset M_{X \times Y}(\mathbb{C})$ and $\mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T) \subset M_X(\mathbb{C}) \otimes_{\mathbb{C}} M_Y(\mathbb{C}) \cong M_{X \times Y}(\mathbb{C})$. We will identify

$M_{X \times Y}(\mathbb{C})$ with $M_X(\mathbb{C}) \otimes_{\mathbb{C}} M_Y(\mathbb{C})$ by the natural way and prove that $\mathcal{T}(S \times T) = \mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T)$.

For $(s, t) \in S \times T$, $\sigma_{(s,t)} = \sigma_s \otimes \sigma_t \in \mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T)$. For $(s, t) \in S \times T$, $\varepsilon_{(x_0, y_0)(s,t)} = \varepsilon_{x_0 s \times y_0 t} = \varepsilon_{x_0 s} \otimes \varepsilon_{y_0 t} \in \mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T)$. Since $\mathcal{T}(S \times T)$ and $\mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T)$ are algebras and $\mathcal{T}(S \times T)$ is generated by $\sigma_{(s,t)}$'s and $\varepsilon_{(x_0, y_0)(s,t)}$'s, we can say that $\mathcal{T}(S \times T) \subset \mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T)$.

For $s \in S$,

$$\sigma_s \otimes I_Y = \sigma_{(s,1)} \in \mathcal{T}(S \times T)$$

and

$$\varepsilon_{x_0 s} \otimes I_Y = \varepsilon_{x_0 s \times Y} = \sum_{t \in T} \varepsilon_{x_0 s \times y_0 t} = \sum_{t \in T} \varepsilon_{x_0 s} \otimes \varepsilon_{y_0 t} \in \mathcal{T}(S \times T).$$

Now, for an arbitrary $\alpha \in \mathcal{T}(S)$, $\alpha \otimes I_Y \in \mathcal{T}(S \times T)$. Similarly, for an arbitrary $\beta \in \mathcal{T}(T)$, $I_X \otimes \beta \in \mathcal{T}(S \times T)$. So $\alpha \otimes \beta = (\alpha \otimes \beta) = (\alpha \otimes I_Y)(I_X \otimes \beta) \in \mathcal{T}(S \times T)$. We can say that $\mathcal{T}(S) \otimes_{\mathbb{C}} \mathcal{T}(T) \subset \mathcal{T}(S \times T)$. \square

4. Wreath products

Let (X, S) and (Y, T) be association schemes. For $s \in S$, set $\tilde{s} = \{((x, y), (x', y)) \mid (x, x') \in s, y \in Y\}$. For $t \in T$, set $\bar{t} = \{((x, y), (x', y')) \mid x, x' \in X, (y, y') \in t\}$. Also set $S \wr T = \{\tilde{s} \mid s \in S\} \cup \{\bar{t} \mid t \in T \setminus \{1\}\}$. Then $(X \times Y, S \wr T)$ is an association scheme and called the *wreath product* of (X, S) by (Y, T) (see [5]). For the adjacency matrices, we have

$$\sigma_{\tilde{s}} = \sigma_s \otimes I_Y, \quad \sigma_{\bar{t}} = J_X \otimes \sigma_t$$

where I_Y is the identity matrix and J_X is the all-one matrix. We fix $x_0 \in X$ and $y_0 \in Y$. Note that

$$\begin{aligned} (x_0, y_0)\tilde{s} &= (x_0 s, y_0) = \{(x, y_0) \mid x \in x_0 s\}, \\ (x_0, y_0)\bar{t} &= (X, y_0 t) = \{(x, y) \mid x \in X, y \in y_0 t\} \end{aligned}$$

and

$$\begin{aligned} \varepsilon_{(x_0, y_0)\tilde{s}} &= \varepsilon_{x_0 s} \otimes \varepsilon_{y_0 1}, \\ \varepsilon_{(x_0, y_0)\bar{t}} &= \sum_{s \in S} \varepsilon_{x_0 s} \otimes \varepsilon_{y_0 t} = I_X \otimes \varepsilon_{y_0 t}. \end{aligned}$$

The structure of Terwilliger algebras of wreath products of association schemes was studied in [2]. We will give a generalization of their results.

4.1. Central primitive idempotents

Let (X, S) and (Y, T) be association schemes. Fix $x_0 \in X$ and $y_0 \in Y$ and consider the wreath product $(X \times Y, S \wr T)$. In the rest of this section, we assume that (Y, T) is a thin scheme or a class-one scheme. Set $F^{(t)} = (x_0, y_0)\bar{t} = (X, y_0t)$ and $U^{(t)} = (S \wr T)_{(x_0, y_0)\bar{t}}$ for $t \in T$. If (Y, T) is a thin scheme, then $(F^{(t)}, U^{(t)})$ is naturally isomorphic to (X, S) for every $t \in T$. If (Y, T) is a class-one scheme, then $(F^{(1)}, U^{(1)})$ is naturally isomorphic to (X, S) and $(F^{(t)}, U^{(t)})$ is isomorphic to the wreath product of (X, S) by the class-one scheme (Y', T') where $Y' = Y - \{y_0\}$. So, for both cases, $F^{(t)}$ induces an association scheme for every $t \in T$.

For $\chi \in \text{Irr}(\mathcal{T}(U^{(1)})) \setminus \{1_{\mathcal{T}(U^{(1)})}\}$, set

$$\tilde{e}_\chi = e_\chi \otimes \varepsilon_{\{y_0\}} \in \mathcal{T}(S \wr T).$$

Then clearly \tilde{e}_χ is an idempotent of $\mathcal{T}(S \wr T)$.

For $t \in T \setminus \{1\}$ and $\varphi \in \text{Irr}(\mathcal{A}(U^{(t)})) \setminus \{1_{\mathcal{A}(U^{(t)})}\}$, we will determine an idempotent. If (Y, T) is thin, then we put $\bar{e}_\varphi = e_\varphi \otimes \varepsilon_{y_0t}$ and this is an idempotent of $\mathcal{T}(S \wr T)$. Suppose that (Y, T) is a class-one scheme. Then $U^{(t)}$ is a wreath product of S and a class-one scheme. By [3] and [5], we can determine the set of all central primitive idempotents of $U^{(t)}$. It is given by

$$\{e_\mu \otimes \varepsilon_{y_0t} \mid \mu \in \text{Irr}(\mathcal{A}(S)) \setminus \{1_{\mathcal{A}(S)}\}\} \cup \{|X|^{-1} J_X \otimes e_\nu \mid \nu \in \text{Irr}(\mathcal{A}(T'))\}.$$

Naturally, we can see that they are idempotents of $\mathcal{T}(S \wr T)$. So we can define an idempotent \bar{e}_φ of $\mathcal{T}(S \wr T)$ for $\varphi \in \text{Irr}(\mathcal{A}(U^{(t)})) \setminus \{1_{\mathcal{A}(U^{(t)})}\}$.

We will show the following theorem.

Theorem 4.1. *Let (X, S) and (Y, T) be association schemes. Suppose that (Y, T) is a thin scheme or a class-one scheme. Fix $x_0 \in X$ and $y_0 \in Y$, and consider the wreath product $(X \times Y, S \wr T)$. Then*

$$\begin{aligned} & \{e_1\} \cup \{\tilde{e}_\chi \mid \chi \in \text{Irr}(\mathcal{T}(U^{(1)})) \setminus \{1_{\mathcal{T}(U^{(1)})}\}\} \\ & \cup \bigcup_{t \in T \setminus \{1\}} \{\bar{e}_\varphi \mid \varphi \in \text{Irr}(\mathcal{A}(U^{(t)})) \setminus \{1_{\mathcal{A}(U^{(t)})}\}\} \end{aligned}$$

is the set of all central primitive idempotents of $\mathcal{T}(X \times Y, S \wr T, (x_0, y_0))$.

To prove Theorem 4.1, we need some lemmas.

Lemma 4.2. *For $\chi \in \text{Irr}(\mathcal{T}(U^{(1)})) \setminus \{1_{\mathcal{T}(U^{(1)})}\}$, \tilde{e}_χ is a central idempotent of $\mathcal{T}(S \wr T)$.*

Proof. It is enough to show that $\tilde{e}_\chi = e_\chi \otimes \varepsilon_{\{y_0\}}$ commutes with $\sigma_s \otimes I_Y$, $J_X \otimes \sigma_t$ ($t \neq 1$), $\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}$ and $I_X \otimes \varepsilon_{y_0t}$ ($t \neq 1$).

By the form of $e_{1_{\mathcal{T}(S)}}$, we have $J_X = e_{1_{\mathcal{T}(S)}} J_X = J_X e_{1_{\mathcal{T}(S)}}$. So we have $e_\chi J_X = J_X e_\chi = 0$. So we can see that

$$(e_\chi \otimes \varepsilon_{\{y_0\}})(J_X \otimes \sigma_t) = (J_X \otimes \sigma_t)(e_\chi \otimes \varepsilon_{\{y_0\}}) = 0.$$

For $t \in T \setminus \{1\}$, since $\varepsilon_{\{y_0\}} \varepsilon_{y_0t} = \varepsilon_{y_0t} \varepsilon_{\{y_0\}} = 0$, we have

$$(e_\chi \otimes \varepsilon_{\{y_0\}})(I_X \otimes \varepsilon_{y_0t}) = (I_X \otimes \varepsilon_{y_0t})(e_\chi \otimes \varepsilon_{\{y_0\}}) = 0.$$

Since e_χ is a central element of $\mathcal{T}(S)$, \tilde{e}_χ commutes with $\sigma_s \otimes I_Y$ and $\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}$. \square

Lemma 4.3. For $t \in T \setminus \{1\}$ and $\varphi \in \text{Irr}(\mathcal{A}(U^{(t)})) \setminus \{1_{\mathcal{A}(U^{(t)})}\}$, \bar{e}_φ is a central idempotent of $\mathcal{T}(S \wr T)$.

Proof. It is enough to show that \bar{e}_φ commutes with $\sigma_s \otimes I_Y$, $J_X \otimes \sigma_u$ ($u \in T \setminus \{1\}$), $\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}$, and $I_X \otimes \varepsilon_{y_0u}$ ($u \in T \setminus \{1\}$).

Suppose that (Y, T) is thin. In this case, $\bar{e}_\varphi = e_\varphi \otimes \varepsilon_{y_0t}$. For $s \in S$, $\bar{e}_\varphi(\sigma_s \otimes I_Y) = \sum_{u \in T} \bar{e}_\varphi(\sigma_s \otimes \varepsilon_{y_0u}) = \bar{e}_\varphi(\sigma_s \otimes \varepsilon_{y_0t})$. Since $\sigma_s \otimes \varepsilon_{y_0t} \in \mathcal{A}(U^{(t)})$, we have $\bar{e}_\varphi(\sigma_s \otimes I_Y) = (\sigma_s \otimes I_Y) \bar{e}_\varphi$. Since $t \neq 1$, we have $\bar{e}_\varphi(\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}) = (\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}) \bar{e}_\varphi = 0$. Since $e_1 = |X|^{-1} J_X$ and $e_1 e_\varphi = e_\varphi e_1 = 0$, we have $e_\varphi J_X = J_X e_\varphi = 0$. So we can see that

$$(J_X \otimes \sigma_u)(e_\varphi \otimes \varepsilon_{y_0t}) = (e_\varphi \otimes \varepsilon_{y_0t})(J_X \otimes \sigma_u) = 0.$$

The last commutativity $\bar{e}_\varphi(I_X \otimes \varepsilon_{y_0u}) = (I_X \otimes \varepsilon_{y_0u}) \bar{e}_\varphi$ is trivial.

Suppose that (Y, T) is class-one scheme. Then

$$\bar{e}_\varphi = \begin{cases} e_\varphi \otimes \varepsilon_{y_0t} & \text{if } \varphi \in \text{Irr}(\mathcal{A}(S)) \setminus \{1\}, \\ |X|^{-1} J_X \otimes e_\varphi & \text{if } \varphi \in \text{Irr}(\mathcal{A}(T)) \setminus \{1\}. \end{cases}$$

If \bar{e}_φ is $e_\varphi \otimes \varepsilon_{y_0t}$, then the same argument as thin case will work. Next, we show that $\bar{e}_\varphi = |X|^{-1} J_X \otimes e_\varphi$ is central. Easily we can see that

$$\begin{aligned} \bar{e}_\varphi(\sigma_s \otimes I_Y) &= n_s \bar{e}_\varphi = (\sigma_s \otimes I_Y) \bar{e}_\varphi, \\ \bar{e}_\varphi(\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}) &= (\varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}}) \bar{e}_\varphi = 0, \\ \bar{e}_\varphi(I_X \otimes \varepsilon_{y_0t}) &= (I_X \otimes \varepsilon_{y_0t}) \bar{e}_\varphi = \bar{e}_\varphi. \end{aligned}$$

To show that $\bar{e}_\varphi(J_X \otimes \sigma_t) = (J_X \otimes \sigma_t)\bar{e}_\varphi$, it is enough to show that $e_\varphi \sigma_t = \sigma_t e_\varphi$. Now

$$e_\varphi \sigma_t = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \begin{array}{c} e_\varphi \\ \\ \\ \end{array} \right) \left(\begin{array}{c|ccc} 0 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & & \\ 1 & & & \end{array} \begin{array}{c} A \\ \\ \\ \end{array} \right) = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \begin{array}{c} e_\varphi A \\ \\ \\ \end{array} \right),$$

because the all-one vector is an eigenvector of e_φ with the eigenvalue zero. Since $e_\varphi A = A e_\varphi$, we can conclude $e_\varphi \sigma_t = \sigma_t e_\varphi$. Therefore \bar{e}_φ is a central idempotent of $\mathcal{T}(S \wr T)$. \square

Lemma 4.4. *For $\chi \in \text{Irr}(\mathcal{T}(U^{(1)})) \setminus \{1_{\mathcal{T}(U^{(1)})}\}$, \tilde{e}_χ is a central primitive idempotent of $\mathcal{T}(S \wr T)$. For $t \in T \setminus \{1\}$ and $\varphi \in \text{Irr}(\mathcal{A}(U^{(t)})) \setminus \{1_{\mathcal{A}(U^{(t)})}\}$, \bar{e}_φ is a central primitive idempotent of $\mathcal{T}(S \wr T)$.*

Proof. It is enough to show that \tilde{e}_χ , and \bar{e}_φ are primitive.

First, we prove \tilde{e}_χ to be primitive. The map $\pi : \mathcal{T}(S \wr T) \rightarrow \tilde{e}_\chi \mathcal{T}(S \wr T)$ is a projection. Now $\tilde{e}_\chi \mathcal{T}(S \wr T)$ is naturally isomorphic to $e_\chi \mathcal{T}(S)$. e_χ is a central primitive idempotent of $\mathcal{T}(S)$. So the map $f : \mathcal{T}(S \wr T) \rightarrow M_{\chi(1)}(\mathbb{C})$ is an epimorphism. Therefore \tilde{e}_χ is primitive.

By the same argument, \bar{e}_φ is primitive. \square

Lemma 4.5. *The sum of central primitive idempotents in Theorem 4.1 is the identity elements.*

Proof. e_1 is trivial idempotent of $\mathcal{T}(S \wr T)$. So,

$$e_1 = \sum_{s \in S} n_s^{-1} \varepsilon_{x_0 s} J_X \varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}} J_Y \varepsilon_{\{y_0\}} + \sum_{t \in T \setminus \{1\}} |X|^{-1} J_X \otimes n_t^{-1} \varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t}$$

Then, $\sum_{s \in S} n_s^{-1} \varepsilon_{x_0 s} J_X \varepsilon_{x_0 s}$ is the trivial idempotent of $\mathcal{T}(S)$.

If (Y, T) is the thin scheme, then $n_t = 1$ (for all $t \in T$) and $\varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t}$ is

the trivial idempotent of $\mathcal{A}(U^{(t)})$.

$$\begin{aligned}
& e_1 + \sum_{\chi \in \text{Irr}(\mathcal{T}(U^{(1)}) \setminus \{1\})} \tilde{e}_\chi + \sum_{t \in T \setminus \{1\}} \sum_{\varphi \in \text{Irr}(\mathcal{A}(U^{(t)}) \setminus \{1\})} \bar{e}_\varphi \\
= & \left(\sum_{s \in S} n_s^{-1} \varepsilon_{x_0 s} J_X \varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}} J_Y \varepsilon_{\{y_0\}} + \sum_{t \in T \setminus \{1\}} |X|^{-1} J_X \otimes n_t^{-1} \varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t} \right) \\
& + \sum_{\chi \in \text{Irr}(\mathcal{T}(U^{(1)}) \setminus \{1\})} e_\chi \otimes \varepsilon_{\{y_0\}} + \sum_{t \in T \setminus \{1\}} \sum_{\varphi \in \text{Irr}(\mathcal{A}(U^{(t)}) \setminus \{1\})} e_\varphi \otimes \varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t} \\
= & \sum_{\chi \in \text{Irr}(\mathcal{T}(U^{(1)}))} e_\chi \otimes \varepsilon_{\{y_0\}} + \sum_{t \in T \setminus \{1\}} \sum_{\varphi \in \text{Irr}(\mathcal{A}(U^{(t)}))} e_\varphi \otimes \varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t} \\
= & I_X \otimes \varepsilon_{\{y_0\}} + \sum_{t \in T \setminus \{1\}} I_X \otimes \varepsilon_{y_0 t} \\
= & I_X \otimes I_Y = I_{X \times Y}
\end{aligned}$$

If (Y, T) is the class-one scheme, then

$$\bar{e}_\varphi = \begin{cases} e_\varphi \otimes \varepsilon_{y_0 t} & (\varphi \in \text{Irr}(\mathcal{A}(S)) \setminus \{1\}) \\ |X|^{-1} J_X \otimes e_\varphi & (\varphi \in \text{Irr}(\mathcal{A}(T')) \setminus \{1\}) \end{cases}$$

and $n_t^{-1} \varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t}$ is the trivial idempotent of $\mathcal{A}(U^t)$.

$$\begin{aligned}
& e_1 + \sum_{\chi \in \text{Irr}(\mathcal{T}(U^{(1)}) \setminus \{1\})} \tilde{e}_\chi + \sum_{\varphi \in \text{Irr}(\mathcal{A}(U^{(t)}) \setminus \{1\})} \bar{e}_\varphi \\
= & \left(\sum_{s \in S} n_s^{-1} \varepsilon_{x_0 s} J_X \varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}} J_Y \varepsilon_{\{y_0\}} + \sum_{t \in T \setminus \{1\}} |X|^{-1} J_X \otimes n_t^{-1} \varepsilon_{y_0 t} J_Y \varepsilon_{y_0 t} \right) \\
& + \sum_{\chi \in \text{Irr}(\mathcal{T}(U^{(1)}) \setminus \{1\})} e_\chi \otimes \varepsilon_{\{y_0\}} + \sum_{\varphi \in \text{Irr}(\mathcal{A}(S)) \setminus \{1\}} e_\varphi \otimes \varepsilon_{y_0 t} + \sum_{\varphi \in \text{Irr}(\mathcal{A}(T')) \setminus \{1\}} |X|^{-1} J_X \otimes e_\varphi \\
= & \sum_{\chi \in \text{Irr}(\mathcal{T}(U^{(1)}))} e_\chi \otimes \varepsilon_{\{y_0\}} + \sum_{\varphi \in \text{Irr}(\mathcal{A}(S)) \setminus \{1\}} e_\varphi \otimes \varepsilon_{y_0 t} + |X|^{-1} J_X \otimes \varepsilon_{y_0 t} \\
= & I_X \otimes \varepsilon_{\{y_0\}} + \sum_{\varphi \in \text{Irr}(\mathcal{A}(S))} e_\varphi \otimes \varepsilon_{y_0 t} \\
= & I_X \otimes I_Y = I_{X \times Y}
\end{aligned}$$

□

Now, combining Lemma 4.2, Lemma 4.3, Lemma 4.4, and Lemma 4.5, the proof of Theorem 4.1 is completed. Finally, we state the degrees of irreducible representation as a corollary. This is clear by Lemma 4.4.

Corollary 4.6. *The degree of the trivial character of $\mathcal{T}(S)$ is $|S \wr T| = |S| + |T| - 1$. For $\chi \in \text{Irr}(\mathcal{T}(U^{(1)})) \setminus \{1_{\mathcal{T}(U^{(1)})}\}$, the degree of the corresponding character is $\chi(1)$. For $\varphi \in \text{Irr}(\mathcal{A}(U^{(t)})) \setminus \{1_{\mathcal{A}(U^{(t)})}\}$, the degree of the corresponding character is $\varphi(1)$.*

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