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Infinitely many shape invariant potentials and cubic identities of the Laguerre and Jacobi polynomials

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Abstract

We provide analytic proofs for the shape invariance of the recently discovered (Odake and Sasaki, Phys. Lett. **B679** (2009) 414-417) two families of infinitely many exactly solvable one-dimensional quantum mechanical potentials. These potentials are obtained by deforming the well-known radial oscillator potential or the Darboux-Pöschl-Teller potential by a degree ℓ ($\ell = 1, 2, \ldots$) eigenpolynomial. The shape invariance conditions are attributed to new polynomial identities of degree 3ℓ involving cubic products of the Laguerre or Jacobi polynomials. These identities are proved elementarily by combining simple identities.

1 Introduction

In a previous Letter [1], two sets of infinitely many shape invariant [2] potentials were derived by deforming the radial oscillator potential [3, 4] and the Darboux-Pöschl-Teller (DPT) potential [5, 6] in terms of a polynomial eigenfunction of degree ℓ ($\ell = 1, 2, \ldots$). As the main part of the eigenfunctions of these exactly solvable quantum mechanical systems, the exceptional (X_ℓ) Laguerre and Jacobi polynomials were obtained [1]. The lowest ($\ell = 1$) examples, the X_1 Laguerre and Jacobi polynomials, are equivalent to those introduced in the pioneering work of Gomez-Ullate et al [7, 8] within the Sturm-Liouville theory. The reformulation in the framework of quantum mechanics and shape-invariant potentials was done by Quesne

et al [9, 10]. By construction these new orthogonal polynomials satisfy a second order dif-
ferential equation (the Schrödinger equation) without contradicting Bochner's theorem [11],
since they start at degree ℓ, $(\ell = 1, 2, \ldots)$ instead of the degree zero constant term.

Here we present analytical proofs of the main assertion that these deformed potentials are
indeed shape invariant, which could not be given in the Letter [1] due to the lack of space.
For both the deformed radial oscillator potential and deformed trigonometric/hyperbolic
DPT potentials, the condition for shape invariance for each ℓ is satisfied if a certain identity
involving cubic products of the Laguerre or Jacobi polynomials holds. To the best of our
knowledge, these infinitely many identities were not presented before. We show that these
identities are derived by combining several elementary relations among the Laguerre or Jacobi
polynomials.

This paper is organised as follows. In section two the general setting of shape invariance
and the deformation of a potential in terms of a polynomial eigenfunction is recapitulated
together with many preparatory materials. In section three we show that the condition
for shape invariance for each ℓ is attributed to a certain polynomial identity of degree 3ℓ
involving cubic products of the Laguerre or Jacobi polynomials of various parameters, (3.4)
and (3.11). Then we show that these identities are simply derived by combining simple
relations among the Laguerre polynomials (3.5)–(3.6) and Jacobi polynomials (3.12)–(3.13).
The final section is for a summary and comments on related results to be published in a
forthcoming paper.

2 General setting

As in [1], let us start with a generic one-dimensional quantum mechanical system in a
factorised form:

$$\mathcal{H} = \mathcal{A}^\dagger \mathcal{A} = p^2 + U(x), \quad p = -i\partial_x, \qquad U(x) \stackrel{\text{def}}{=} \left(\partial_x w(x)\right)^2 + \partial_x^2 w(x), \tag{2.1}$$

$$\mathcal{A} \stackrel{\text{def}}{=} \partial_x - \partial_x w(x), \quad \mathcal{A}^\dagger = -\partial_x - \partial_x w(x). \tag{2.2}$$

Here we call a real and smooth function $w(x)$ a *prepotential*. It parametrises the groundstate
wavefunction $\phi_0(x)$, which has *no node* and can be chosen real and positive, $\phi_0(x) = e^{w(x)}$.
It is trivial to verify $\mathcal{A}\phi_0(x) = 0$ and $\mathcal{H}\phi_0(x) = 0$.

Shape invariance is realised by specific dependence of the potential, or the prepotential
on a set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$, to be denoted by $w(x; \boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda})$, etc.

The shape invariance condition to be discussed in this paper is written simply:

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{2.3}$$

$$\text{or} \quad \left(\partial_x w(x; \boldsymbol{\lambda})\right)^2 - \partial_x^2 w(x; \boldsymbol{\lambda}) = \left(\partial_x w(x; \boldsymbol{\lambda} + \boldsymbol{\delta})\right)^2 + \partial_x^2 w(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{2.4}$$

in which $\boldsymbol{\delta}$ is a certain shift of the parameters. Then the entire set of discrete eigenvalues and the corresponding eigenfunctions of $\mathcal{H} = \mathcal{H}(\boldsymbol{\lambda})$,

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}), \qquad n = 0, 1, 2, \ldots, \tag{2.5}$$

$$\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x; \boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}), \tag{2.6}$$

is determined algebraically [2, 4, 12, 13]:

$$\mathcal{E}_n(\boldsymbol{\lambda}) = \sum_{k=0}^{n-1} \mathcal{E}_1(\boldsymbol{\lambda} + k\boldsymbol{\delta}), \tag{2.7}$$

$$\phi_n(x; \boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda})^\dagger \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \cdots \mathcal{A}(\boldsymbol{\lambda} + (n-1)\boldsymbol{\delta})^\dagger \times e^{w(x; \boldsymbol{\lambda} + n\boldsymbol{\delta})}. \tag{2.8}$$

The polynomial eigenfunction $P_n(\eta(x); \boldsymbol{\lambda})$, which is the Laguerre or Jacobi polynomial in $\eta(x)$, satisfies

$$-\partial_x^2 P_n(\eta(x); \boldsymbol{\lambda}) - 2\partial_x w(x; \boldsymbol{\lambda})\partial_x P_n(\eta(x); \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}). \tag{2.9}$$

Here $\eta(x)$ is a function of x called the *sinusoidal coordinate* [14].

In [1] a shape invariant prepotential $w(x; \boldsymbol{\lambda}) = w_0(x; \boldsymbol{\lambda})$ is *deformed* by a polynomial eigenfunction ξ_ℓ of its Hamiltonian to produce another shape invariant prepotential w_ℓ ($\ell = 1, 2, \ldots.$):

$$w_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} w_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) + \log \frac{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x); \boldsymbol{\lambda})}, \tag{2.10}$$

in which ξ_ℓ is related to the polynomial eigenfunction P_n above (2.6). It should be noted that the normalisation of the polynomial $\xi_\ell(x; \boldsymbol{\lambda})$ is irrelevant to the deformation. The $\ell = 0$ case corresponds to the original system. The ℓ-th Hamiltonian and eigenfunctions, etc are given by

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \partial_x - \partial_x w_\ell(x; \boldsymbol{\lambda}), \quad \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = -\partial_x - \partial_x w_\ell(x; \boldsymbol{\lambda}), \tag{2.11}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \mathcal{A}_\ell(\boldsymbol{\lambda}), \tag{2.12}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})\phi_{\ell,n}(x; \boldsymbol{\lambda}), \tag{2.13}$$

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$$\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \psi_\ell(x;\boldsymbol{\lambda})P_{\ell,n}(\eta(x);\boldsymbol{\lambda}), \quad \psi_\ell(x;\boldsymbol{\lambda}) \overset{\text{def}}{=} \frac{e^{w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}. \tag{2.14}$$

The orthogonality of the eigenfunctions of the Hamiltonian $\mathcal{H}_\ell(\boldsymbol{\lambda})$ reads

$$\int \psi_\ell(x;\boldsymbol{\lambda})^2 P_{\ell,n}(\eta(x);\boldsymbol{\lambda})P_{\ell,m}(\eta(x);\boldsymbol{\lambda})dx = h_{\ell,n}(\boldsymbol{\lambda})\delta_{nm}, \quad h_{\ell,n}(\boldsymbol{\lambda}) > 0, \tag{2.15}$$

in which $\psi_\ell(x;\boldsymbol{\lambda})^2$ is the orthogonality measure and $P_{\ell,n}(x;\boldsymbol{\lambda})$ is the n-th member of the *exceptional (X_ℓ) orthogonal polynomial*. It is expressed in terms of $\xi_\ell(x)$'s and $P_n(x)$'s as shown in (2.22), (2.27) and (2.34).

In this paper we will demonstrate that the deformed prepotential w_ℓ ($\ell = 1, 2, \ldots$) actually satisfies the shape invariance condition

$$\Delta_\ell(x;\boldsymbol{\lambda}) = 0, \tag{2.16}$$

in which $\Delta_\ell(x;\boldsymbol{\lambda})$ is defined by

$$\Delta_\ell(x;\boldsymbol{\lambda}) \overset{\text{def}}{=} \left(\partial_x w_\ell(x;\boldsymbol{\lambda})\right)^2 - \partial_x^2 w_\ell(x;\boldsymbol{\lambda}) - \left(\partial_x w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\right)^2 - \partial_x^2 w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) - \mathcal{E}_1(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{2.17}$$

for the three cases, the radial oscillator and the trigonometric/hyperbolic DPT discussed in [1]. The proof consists of two steps. Firstly in section three we transform, by utilising the differential equations for $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ (2.41) etc, the shape invariance condition (2.17) into an identity involving products of three Laguerre or Jacobi polynomials of various parameters, (3.4) and (3.11). The trigonometric and hyperbolic DPT lead to the same identities. Secondly these cubic identities are proved by combining simple identities among the Laguerre or Jacobi polynomials of neighbouring degrees n, $n-1$ and parameters α, $\alpha \pm 1$, β, $\beta \pm 1$.

Here we show various data necessary for the proof. They are recapitulated from [1].

1 radial oscillator:

$$\boldsymbol{\lambda} \overset{\text{def}}{=} g, \quad \boldsymbol{\delta} = 1, \quad g > 0, \tag{2.18}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4n, \quad \eta(x) \overset{\text{def}}{=} x^2, \quad 0 < x < \infty, \tag{2.19}$$

$$\phi_0(x;\boldsymbol{\lambda}) \overset{\text{def}}{=} e^{-\frac{x^2}{2}}x^g \Leftrightarrow w_0(x;\boldsymbol{\lambda}) \overset{\text{def}}{=} -\frac{x^2}{2} + g\log x, \tag{2.20}$$

$$P_n(x;\boldsymbol{\lambda}) \overset{\text{def}}{=} L_n^{(g-\frac{1}{2})}(x), \quad \xi_\ell(x;\boldsymbol{\lambda}) \overset{\text{def}}{=} L_\ell^{(g+\ell-\frac{3}{2})}(-x), \tag{2.21}$$

$$P_{\ell,n}(x;\boldsymbol{\lambda}) = \xi_\ell(x;g+1)P_n(x;g+\ell) - \xi_{\ell-1}(x;g+2)P_{n-1}(x;g+\ell). \tag{2.22}$$

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2 trigonometric DPT:

$$\boldsymbol{\lambda} \stackrel{\text{def}}{=} (g,h), \quad \boldsymbol{\delta} = (1,1), \quad h > g > 0, \tag{2.23}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(n+g+h), \quad \eta(x) \stackrel{\text{def}}{=} \cos 2x, \quad 0 < x < \frac{\pi}{2}, \tag{2.24}$$

$$\phi_0(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (\sin x)^g (\cos x)^h \Leftrightarrow w_0(x;\boldsymbol{\lambda}) = g \log \sin x + h \log \cos x, \tag{2.25}$$

$$P_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n^{(g-\frac{1}{2},h-\frac{1}{2})}(x), \quad \xi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(-g-\ell-\frac{1}{2},h+\ell-\frac{3}{2})}(x), \tag{2.26}$$

$$P_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} a_{\ell,n}(x;\boldsymbol{\lambda})P_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + b_{\ell,n}(x;\boldsymbol{\lambda})P_{n-1}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{2.27}$$

$$a_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_\ell(x;g+1,h+1) + \frac{2n(-g+h+\ell-1)\,\xi_{\ell-1}(x;g,h+2)}{(-g+h+2\ell-2)(g+h+2n+2\ell-1)}$$
$$- \frac{n(2h+4\ell-3)\,\xi_{\ell-2}(x;g+1,h+3)}{(2g+2n+1)(-g+h+2\ell-2)}, \tag{2.28}$$

$$b_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(-g+h+\ell-1)(2g+2n+2\ell-1)}{(2g+2n+1)(g+h+2n+2\ell-1)}\,\xi_{\ell-1}(x;g,h+2). \tag{2.29}$$

3 hyperbolic DPT:

$$\boldsymbol{\lambda} \stackrel{\text{def}}{=} (g,h), \quad \boldsymbol{\delta} = (1,-1), \quad h > g > 0, \quad \ell < n_B \stackrel{\text{def}}{=} [\tfrac{1}{2}(h-g)]', \tag{2.30}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(h-g-n), \quad \eta(x) = \cosh 2x, \quad 0 < x < \infty, \tag{2.31}$$

$$\phi_0(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (\sinh x)^g (\cosh x)^{-h} \Leftrightarrow w_0(x;\boldsymbol{\lambda}) = g \log \sinh x - h \log \cosh x, \tag{2.32}$$

$$P_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n^{(g-\frac{1}{2},-h-\frac{1}{2})}(x), \quad \xi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(-g-\ell-\frac{1}{2},-h+\ell-\frac{3}{2})}(x), \tag{2.33}$$

$$P_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} a_{\ell,n}(x;\boldsymbol{\lambda})P_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + b_{\ell,n}(x;\boldsymbol{\lambda})P_{n-1}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{2.34}$$

$$a_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_\ell(x;g+1,h-1) + \frac{2n(-g-h+\ell-1)\,\xi_{\ell-1}(x;g,h-2)}{(-g-h+2\ell-2)(g-h+2n+2\ell-1)}$$
$$- \frac{n(-2h+4\ell-3)\,\xi_{\ell-2}(x;g+1,h-3)}{(2g+2n+1)(-g-h+2\ell-2)}, \tag{2.35}$$

$$b_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(-g-h+\ell-1)(2g+2n+2\ell-1)}{(2g+2n+1)(g-h+2n+2\ell-1)}\,\xi_{\ell-1}(x;g,h-2). \tag{2.36}$$

In (2.30) $[x]'$ denotes the greatest integer not equal or exceeding x. Here $L_n^{(\alpha)}(x)$ is the Laguerre polynomial and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial. The polynomial eigenfunction $P_{\ell,n}(x;\boldsymbol{\lambda})$ (2.22), (2.27) or (2.34) is a degree $\ell+n$ polynomial in x. In a future publication [15] we will present equivalent forms of the X_ℓ polynomials $P_{\ell,n}(x;\boldsymbol{\lambda})$ which appear much simpler than those given in (2.22), (2.27) and (2.34). Needless to say that the deforming polynomials $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ and $\xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})$ are of the same sign in the domain for all the three cases. In other words, the deforming polynomials do not have a zero in the domains

listed in (2.19), (2.24) and (2.31), respectively. To see this we use the expansion formula of the Laguerre and Jacobi polynomials:

$$L_n^{(\alpha)}(x) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} (\alpha + k + 1)_{n-k} x^k, \tag{2.37}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{1}{k!} \frac{(-n)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k. \tag{2.38}$$

For the radial oscillator we obtain

$$\xi_\ell(\eta(x); \boldsymbol{\lambda}) = \sum_{k=0}^{\ell} \frac{(g+\ell+k-\frac{1}{2})_{\ell-k}}{k!\,(\ell-k)!} x^{2k} > 0, \tag{2.39}$$

and for the trigonometric/hyperbolic DPT

$$(-1)^\ell \xi_\ell(\eta(x); \boldsymbol{\lambda}) = \begin{cases} \dfrac{(g+\frac{1}{2})_\ell}{\ell!} \displaystyle\sum_{k=0}^{\ell} \dfrac{(\ell-k+1)_k(h-g+\ell-1)_k}{k!\,(g+\ell-k+\frac{1}{2})_k} (\sin x)^{2k} > 0, \\[6pt] \dfrac{(g+\frac{1}{2})_\ell}{\ell!} \displaystyle\sum_{k=0}^{\ell} \dfrac{(\ell-k+1)_k(g+h+2-\ell-k)_k}{k!\,(g+\ell-k+\frac{1}{2})_k} (\sinh x)^{2k} > 0, \end{cases} \tag{2.40}$$

because each term in summation is positive. This guarantees the positive definiteness of the orthogonality measure $\psi_\ell(x; \boldsymbol{\lambda})^2$ (2.14) and the singularity free structure as well as the hermiticity (self-adjointness) of the Hamiltonian. The oscillation theorem for the one-dimensional quantum mechanical systems dictates that the n-th excited state polynomial eigenfunction $P_{\ell,n}(\eta(x); \boldsymbol{\lambda})$ has n zeros in the domain (2.19), (2.24) or (2.31), although it is a degree $\ell + n$ polynomial in η.

Let us note that the terms of the form of the complete square $\left(\partial_x \xi_\ell(\eta(x); \boldsymbol{\lambda})/\xi_\ell(\eta(x); \boldsymbol{\lambda})\right)^2$, $\left(\partial_x \xi_\ell(\eta(x); \boldsymbol{\lambda}+\boldsymbol{\delta})/\xi_\ell(\eta(x); \boldsymbol{\lambda}+\boldsymbol{\delta})\right)^2$ and $\left(\partial_x \xi_\ell(\eta(x); \boldsymbol{\lambda}+2\boldsymbol{\delta})/\xi_\ell(\eta(x); \boldsymbol{\lambda}+2\boldsymbol{\delta})\right)^2$, cancel out in $\Delta_\ell(x; \boldsymbol{\lambda})$ (2.17). Then we use the fact that, corresponding to (2.9), the deforming polynomial $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ also satisfies a second order linear differential equation

$$-\partial_x^2 \xi_\ell(\eta(x); \boldsymbol{\lambda}) - 2\partial_x \widetilde{w}_0(x; \boldsymbol{\lambda}, \ell)\partial_x \xi_\ell(\eta(x); \boldsymbol{\lambda}) = \widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\xi_\ell(\eta(x); \boldsymbol{\lambda}), \tag{2.41}$$

where $\widetilde{w}_0(x; \boldsymbol{\lambda}, \ell)$ and $\widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})$ are

$$\widetilde{w}_0(x; \boldsymbol{\lambda}, \ell) = \begin{cases} \frac{1}{2}x^2 + (g+\ell-1)\log x & : \text{radial osci.} \\ w_0(x; -g-\ell, h+\ell-1) & : \text{trig.DPT} \\ w_0(x; -g-\ell, h-\ell+1) & : \text{hyper.DPT}, \end{cases} \tag{2.42}$$

$$\widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda}) = \begin{cases} -4\ell & : \text{radial osci.} \\ \mathcal{E}_\ell(-g-\ell, h+\ell-1) & : \text{trig.DPT} \\ \mathcal{E}_\ell(-g-\ell, h-\ell+1) & : \text{hyper.DPT.} \end{cases} \tag{2.43}$$

Substituting (2.10) into (2.17) and using the shape invariance of the undeformed system $\Delta_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) = 0$ and (2.41), we obtain

$$\Delta_\ell(x; \boldsymbol{\lambda}) \times \frac{1}{2}\, \xi_\ell(\eta; \boldsymbol{\lambda})\, \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})$$

$$= \partial_x \widetilde{w}_0(x; \boldsymbol{\lambda} + 2\boldsymbol{\delta}, \ell)\, \partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda})\, \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})$$

$$\quad - \partial_x \widetilde{w}_0(x; \boldsymbol{\lambda}, \ell)\, \partial_x \xi_\ell(\eta; \boldsymbol{\lambda})\, \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})$$

$$\quad + \tfrac{1}{2}\big(\widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda} + 2\boldsymbol{\delta}) - \widetilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\big)\, \xi_\ell(\eta; \boldsymbol{\lambda})\, \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})$$

$$\quad + \partial_x w_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\big(\partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda}) - \partial_x \xi_\ell(\eta; \boldsymbol{\lambda})\, \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\big)\xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})$$

$$\quad - \partial_x w_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \boldsymbol{\delta})\big(\partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) - \partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})\big)\xi_\ell(\eta; \boldsymbol{\lambda})$$

$$\quad - \partial_x \xi_\ell(\eta; \boldsymbol{\lambda})\, \partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta}) + \partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\, \partial_x \xi_\ell(\eta; \boldsymbol{\lambda} + 2\boldsymbol{\delta})\, \xi_\ell(\eta; \boldsymbol{\lambda}), \tag{2.44}$$

where $\eta = \eta(x)$. It is essential that the r.h.s. of (2.44) is now a polynomial in $\eta(x)$, because $\partial_x \eta(x)\partial_x w_0$ and $\partial_x \eta(x)\partial_x \widetilde{w}_0$ and $(\partial_x \eta(x))^2$ are expressed by $\eta(x)$,

$$\partial_x \eta(x)\partial_x w_0(x; \boldsymbol{\lambda}) = 2 \times \begin{cases} g - \eta(x) & : \text{radial osci.} \\ -\big(g - h + (g + h)\eta(x)\big) & : \text{trig.DPT} \\ g + h + (g - h)\eta(x) & : \text{hyper.DPT,} \end{cases} \tag{2.45}$$

$$\partial_x \eta(x)\partial_x \widetilde{w}_0(x; \boldsymbol{\lambda}, \ell) = 2 \times \begin{cases} g + \ell - 1 + \eta(x) & : \text{radial osci.} \\ g + h + 2\ell - 1 + (g - h + 1)\eta(x) & : \text{trig.DPT} \\ -\big(g - h + 2\ell - 1 + (g + h + 1)\eta(x)\big) & : \text{hyper.DPT,} \end{cases} \tag{2.46}$$

$$\big(\partial_x \eta(x)\big)^2 = 4 \times \begin{cases} \eta(x) & : \text{radial osci.} \\ 1 - \eta(x)^2 & : \text{trig.DPT} \\ -\big(1 - \eta(x)^2\big) & : \text{hyper.DPT.} \end{cases} \tag{2.47}$$

3 Cubic identities

The conditions for the shape invariance (2.16) are shown to be equivalent to cubic identities involving the Laguerre or Jacobi polynomials.

3.1 radial oscillator

We fix ℓ and use a new parameter α instead of g,

$$\alpha \stackrel{\text{def}}{=} g + \ell - \tfrac{1}{2}. \tag{3.1}$$

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By using the forward shift relation for the Laguerre polynomial,

$$\partial_x L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \tag{3.2}$$

the polynomial ξ_ℓ and its derivative are expressed as

$$\xi_\ell(\eta; \boldsymbol{\lambda}) = L_\ell^{(\alpha-1)}(-\eta), \quad \partial_x \xi_\ell(\eta; \boldsymbol{\lambda}) = \partial_x \eta \, L_{\ell-1}^{(\alpha)}(-\eta), \tag{3.3}$$

where $\eta = \eta(x)$. After replacing $-\eta(x)$ with x and dividing by 4, the condition for the shape invariance (2.16) with (2.44), is transformed into the following polynomial identity of degree 3ℓ in x which contains products of three Laguerre polynomials of various parameters:

$$\begin{aligned}
0 = & - x L_{\ell-1}^{(\alpha+2)}(x) L_\ell^{(\alpha-1)}(x) L_\ell^{(\alpha)}(x) - \alpha L_{\ell-1}^{(\alpha)}(x) L_\ell^{(\alpha+1)}(x) L_\ell^{(\alpha)}(x) \\
& + (x + \alpha + 1) L_{\ell-1}^{(\alpha+1)}(x) L_\ell^{(\alpha+1)}(x) L_\ell^{(\alpha-1)}(x) \\
& + x L_{\ell-1}^{(\alpha)}(x) L_{\ell-1}^{(\alpha+1)}(x) L_\ell^{(\alpha+1)}(x) - x L_{\ell-1}^{(\alpha+1)}(x) L_{\ell-1}^{(\alpha+2)}(x) L_\ell^{(\alpha-1)}(x).
\end{aligned} \tag{3.4}$$

To the best of our knowledge, this identity has not been reported before. For $\ell = 0$ this identity is trivial, since $L_{-1}^{(\alpha)}(x) = 0$. For lower ℓ it can be easily verified by direct calculation.

Below we will prove the identity (3.4) for an arbitrary positive integer ℓ by combining a few elementary relations among the Laguerre polynomials of neighbouring degrees n and $n-1$ and neighbouring parameters α, $\alpha \pm 1$:

$$\text{Lemma (A)} \qquad L_n^{(\alpha-1)}(x) + L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha)}(x), \tag{3.5}$$

$$\text{Lemma (B)} \qquad x L_{n-1}^{(\alpha+1)}(x) - \alpha L_{n-1}^{(\alpha)}(x) = -n L_n^{(\alpha-1)}(x), \tag{3.6}$$

which can be verified elementarily based on the expansion formula of the Laguerre polynomial (2.37). The r.h.s. of the identity (3.4) can be written as

$$\begin{aligned}
\text{r.h.s. of (3.4)} = & -x L_{\ell-1}^{(\alpha+2)}(x) L_\ell^{(\alpha-1)}(x) \left\{ L_\ell^{(\alpha)}(x) + L_{\ell-1}^{(\alpha+1)}(x) \right\} \\
& + x L_{\ell-1}^{(\alpha+1)}(x) L_\ell^{(\alpha+1)}(x) \left\{ L_\ell^{(\alpha-1)}(x) + L_{\ell-1}^{(\alpha)}(x) \right\} \\
& + L_\ell^{(\alpha+1)}(x) \left((\alpha+1) L_{\ell-1}^{(\alpha+1)}(x) L_\ell^{(\alpha-1)}(x) - \alpha L_{\ell-1}^{(\alpha)}(x) L_\ell^{(\alpha)}(x) \right) \\
= & \; L_\ell^{(\alpha+1)}(x) \Big(L_\ell^{(\alpha)}(x) \left\{ x L_{\ell-1}^{(\alpha+1)}(x) - \alpha L_{\ell-1}^{(\alpha)}(x) \right\} \\
& + L_\ell^{(\alpha-1)}(x) \left\{ -x L_{\ell-1}^{(\alpha+2)}(x) + (\alpha+1) L_{\ell-1}^{(\alpha+1)}(x) \right\} \Big) \\
= & \; L_\ell^{(\alpha+1)}(x) \Big(L_\ell^{(\alpha)}(x) \left(-\ell L_\ell^{(\alpha-1)}(x) \right) + L_\ell^{(\alpha-1)}(x) \left(+\ell L_\ell^{(\alpha)}(x) \right) \Big) = 0. \tag{3.7}
\end{aligned}$$

Here we have used Lemma (A) in the first two curly brackets $\{\cdots\}$ and Lemma (B) in the next two curly brackets. This concludes the proof of the identity (3.4).

3.2   trigonometric DPT

We fix ℓ and use new parameters α and β instead of g and h,

$$\alpha \overset{\text{def}}{=} -g - \ell - \tfrac{1}{2}, \quad \beta \overset{\text{def}}{=} h + \ell - \tfrac{3}{2}. \tag{3.8}$$

By using the forward shift relation for the Jacobi polynomial,

$$\partial_x P_n^{(\alpha, \beta)}(x) = \tfrac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x), \tag{3.9}$$

the polynomial ξ_ℓ and its derivative are expressed as

$$\xi_\ell(\eta; \boldsymbol{\lambda}) = P_\ell^{(\alpha, \beta)}(\eta), \quad \partial_x \xi_\ell(\eta; \boldsymbol{\lambda}) = \partial_x \eta \, \tfrac{1}{2}(\ell + \alpha + \beta + 1) P_{\ell-1}^{(\alpha+1, \beta+1)}(\eta), \tag{3.10}$$

where $\eta = \eta(x)$. The condition for the shape invariance (2.16) with (2.44), after replacing $\eta(x)$ with x and dividing by $-(\ell + \alpha + \beta + 1)$, is simplified to a polynomial identity of degree 3ℓ, which contains products of three Jacobi polynomials of various parameters:

$$\begin{aligned}
0 = {} & 2(\alpha - 1)(1 + x) P_{\ell-1}^{(\alpha-1, \beta+3)}(x) P_\ell^{(\alpha, \beta)}(x) P_\ell^{(\alpha-1, \beta+1)}(x) \\
& + 2(\beta + 1)(1 - x) P_{\ell-1}^{(\alpha+1, \beta+1)}(x) P_\ell^{(\alpha-2, \beta+2)}(x) P_\ell^{(\alpha-1, \beta+1)}(x) \\
& - 2\big(\alpha(1 + x) + (\beta + 2)(1 - x)\big) P_{\ell-1}^{(\alpha, \beta+2)}(x) P_\ell^{(\alpha, \beta)}(x) P_\ell^{(\alpha-2, \beta+2)}(x) \\
& + (\ell + \alpha + \beta + 1)(1 - x^2) P_{\ell-1}^{(\alpha+1, \beta+1)}(x) P_{\ell-1}^{(\alpha, \beta+2)}(x) P_\ell^{(\alpha-2, \beta+2)}(x) \\
& - (\ell + \alpha + \beta + 1)(1 - x^2) P_{\ell-1}^{(\alpha, \beta+2)}(x) P_{\ell-1}^{(\alpha-1, \beta+3)}(x) P_\ell^{(\alpha, \beta)}(x). \tag{3.11}
\end{aligned}$$

To the best of our knowledge, this identity has not been reported before, either. For $\ell = 0$ this identity is trivial, since $P_{-1}^{(\alpha, \beta)}(x) = 0$. It is straightforward to verify this identity for lower ℓ by direct calculation.

Below we will prove the identity (3.11) for an arbitrary positive integer ℓ by combining a few elementary relations among the Jacobi polynomials of neighbouring degrees n and $n-1$ and neighbouring parameters $\alpha \pm 1, \beta \pm 1$:

Lemma (C)

$$2(\alpha - 1) P_n^{(\alpha-1, \beta)}(x) - (n + \alpha + \beta)(1 - x) P_{n-1}^{(\alpha, \beta+1)}(x) = 2(n + \alpha - 1) P_n^{(\alpha-2, \beta+1)}(x), \tag{3.12}$$

Lemma (D)

$$2(\beta + 1) P_n^{(\alpha-1, \beta+1)}(x) + (n + \alpha + \beta + 1)(1 + x) P_{n-1}^{(\alpha, \beta+2)}(x) = 2(n + \beta + 1) P_n^{(\alpha, \beta)}(x). \tag{3.13}$$

It is straightforward to demonstrate Lemma (C) by using the expansion formula for the Jacobi polynomials (2.38). By using the property $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, Lemma (D) is obtained from Lemma (C) with the replacements $x \to -x$, $\alpha \to \beta + 2$, $\beta \to \alpha - 1$. The r.h.s. of the identity (3.11) can be written as

$$
\begin{aligned}
\text{r.h.s. of (3.11)} = {} & (1+x)P_{\ell-1}^{(\alpha-1,\beta+3)}(x)P_\ell^{(\alpha,\beta)}(x) \\
& \times \left\{ 2(\alpha-1)P_\ell^{(\alpha-1,\beta+1)}(x) - (\ell+\alpha+\beta+1)(1-x)P_{\ell-1}^{(\alpha,\beta+2)}(x) \right\} \\
& + (1-x)P_{\ell-1}^{(\alpha+1,\beta+1)}(x)P_\ell^{(\alpha-2,\beta+2)}(x) \\
& \times \left\{ 2(\beta+1)P_\ell^{(\alpha-1,\beta+1)}(x) + (\ell+\alpha+\beta+1)(1+x)P_{\ell-1}^{(\alpha,\beta+2)}(x) \right\} \\
& - 2\big(\alpha(1+x) + (\beta+2)(1-x)\big) P_{\ell-1}^{(\alpha,\beta+2)}(x)P_\ell^{(\alpha,\beta)}(x)P_\ell^{(\alpha-2,\beta+2)}(x) \\
= {} & P_\ell^{(\alpha,\beta)}(x)P_\ell^{(\alpha-2,\beta+2)}(x) \\
& \times \Big((1+x)\big\{ 2(\ell+\alpha-1)P_{\ell-1}^{(\alpha-1,\beta+3)}(x) - 2\alpha P_{\ell-1}^{(\alpha,\beta+2)}(x) \big\} \\
& \qquad + (1-x)\big\{ 2(\ell+\beta+1)P_{\ell-1}^{(\alpha+1,\beta+1)}(x) - 2(\beta+2)P_{\ell-1}^{(\alpha,\beta+2)}(x) \big\} \Big) \\
= {} & P_\ell^{(\alpha,\beta)}(x)P_\ell^{(\alpha-2,\beta+2)}(x)P_{\ell-2}^{(\alpha+1,\beta+3)}(x) \\
& \times (\ell+\alpha+\beta+2)\big((1+x)(x-1) + (1-x)(x+1) \big) \\
= {} & 0. \tag{3.14}
\end{aligned}
$$

Here we have used Lemma (C),(D) in the curly brackets $\{\cdots\}$. This concludes the proof of the identity (3.11).

It is well known that the Laguerre polynomial is obtained from the Jacobi polynomial in the following limit:

$$
\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \tfrac{2x}{\beta}\right) = L_n^{(\alpha)}(x). \tag{3.15}
$$

When the same limit is applied, the Lemma (C)–(D) reduce to Lemma (A)–(B). Likewise the cubic identity for the shape invariance of X_ℓ Jacobi polynomial (3.11) reduces to that of the X_ℓ Laguerre polynomial (3.4), when divided by -4 and α is replaced by $\alpha + 1$.

3.3 hyperbolic DPT

Here we briefly remark that the shape invariance condition for the deformed hyperbolic DPT reduces, as expected, to the same identity as that for the trigonometric DPT (3.11) derived above. We fix ℓ and use new parameters α and β instead of g and h,

$$
\alpha \overset{\text{def}}{=} -g - \ell - \tfrac{1}{2}, \qquad \beta \overset{\text{def}}{=} -h + \ell - \tfrac{3}{2}. \tag{3.16}
$$

By using (3.9), the polynomial ξ_ℓ and its derivative are expressed just same as (3.10). The conditions for the shape invariance (2.16) with (2.44), after replacing $\eta(x)$ with x and dividing by $\ell + \alpha + \beta + 1$, is simplified to the same identity (3.11) as that for the trigonometric DPT.

4   Summary and Comments

Analytic proofs are provided for the shape invariance of the recently derived infinite family of potentials [1] obtained by deforming the radial oscillator potential and the trigonometric/hyperbolic Darboux-Pöschl-Teller potential [5, 6] by a degree ℓ ($\ell = 1, 2, \ldots$) eigenpolynomial. The shape invariance conditions are reduced to new polynomial identities of degree 3ℓ involving cubic products of the Laguerre (3.4) or Jacobi polynomials (3.11). Then these identities are proved elementarily by combining simple linear identities (3.5)–(3.6) among the Laguerre and (3.12)–(3.13) among the Jacobi, polynomials of neighbouring degrees $n, n - 1$ and of neighbouring parameters, $\alpha, \alpha \pm 1$ and $\beta, \beta \pm 1$. Even these linear identities seem not widely recognised.

The totality of the eigenvalues and the corresponding eigenfunctions together with the normalisation constants etc. of these infinite family of quantum mechanical systems are obtained via the Rodrigues type formulas (2.7) and (2.8) and reported as (17)–(21), (29)–(36) and (44)–(46) of [1]. In a future publication [15], we will present analytic derivation of various results reported in [1]. They include: derivation of equivalent but much simpler looking forms of the X_ℓ polynomials together with the normalisation constants, the verification of the actions of the forward and backward shift operators on the X_ℓ polynomials, Gram-Schmidt orthonormalisation for the algebraic construction of the X_ℓ polynomials, the analysis of the second order differential equations for the X_ℓ polynomials within the framework of the Fuchsian differential equations in the entire complex x-plane, etc. In these analysis, the linear identities Lemma (A)–(D) play important rôles. The forward and backward shift relations mentioned above (eqs.(49),(22),(37),(47) of [1]) are also reduced to polynomial identities involving cubic products of the Laguerre or Jacobi polynomials. We will provide proofs for them in the future publication [15], too.

Let us also mention that the same method, deformation in terms of a degree ℓ eigenpolynomial, applied to the discrete quantum mechanical Hamiltonians for the Wilson and Askey-Wilson polynomials produced two sets of infinitely many shape invariant systems together with exceptional (X_ℓ) Wilson and Askey-Wilson polynomials ($\ell = 1, 2, \ldots$) [16].

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Acknowledgements

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