

ON THE EXISTENCE OF EMBEDDINGS INTO MODULES OF FINITE HOMOLOGICAL DIMENSIONS

RYO TAKAHASHI, SIAMAK YASSEMI, AND YUJI YOSHINO

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ABSTRACT. Let R be a commutative Noetherian local ring. We show that R is Gorenstein if and only if every finitely generated R -module can be embedded in a finitely generated R -module of finite projective dimension. This extends a result of Auslander and Bridger to rings of higher Krull dimension, and it also improves a result due to Foxby where the ring is assumed to be Cohen-Macaulay.

1. INTRODUCTION

Throughout this paper, let R be a commutative Noetherian local ring. All R -modules in this paper are assumed to be finitely generated.

In [1, Proposition 2.6 (a) and (d)] Auslander and Bridger proved the following.

Theorem 1.1 (Auslander-Bridger). *The following are equivalent:*

- (1) R is quasi-Frobenius (i.e. Gorenstein with Krull dimension zero).
- (2) Every R -module can be embedded in a free R -module.

On the other hand, in [4, Theorem 2] Foxby showed the following.

Theorem 1.2 (Foxby). *The following are equivalent:*

- (1) R is Gorenstein.
- (2) R is Cohen-Macaulay, and every R -module can be embedded in an R -module of finite projective dimension.

For an R -module C we denote by $\text{add}_R C$ the class of R -modules which are direct summands of finite direct sums of copies of C . The C -dimension of an R -module X , $C\text{-dim}_R X$, is defined as the infimum of nonnegative integers n such that there exists an exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_0 \rightarrow X \rightarrow 0$$

of R -modules with $C_i \in \text{add}_R C$ for all $0 \leq i \leq n$.

In this paper, we prove the following theorem. This result removes from Theorem 1.2 the assumption that R is Cohen-Macaulay, and it extends Theorem 1.1 to

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rings of higher Krull dimension. It should be noted that our proof of this result is different from Foxby’s proof for the special case $C = R$.

Theorem 1.3. *Let R be a commutative Noetherian local ring with residue field k . Let C be a semidualizing R -module of depth t . Then the following are equivalent:*

- (1) C is dualizing.
- (2) Every R -module can be embedded in an R -module of finite C -dimension.
- (3) The R -module $\text{Tr}\Omega^t k \otimes_R C$ can be embedded in an R -module of finite C -dimension. (Here $\text{Tr}\Omega^t k$ denotes the transpose of the t -th syzygy of the R -module k .)

Moreover, if one of these three conditions holds, then R is Cohen-Macaulay.

2. PROOF OF THEOREM 1.3 AND ITS APPLICATIONS

First of all, we recall the definition of a semidualizing module.

Definition 2.1. An R -module C is called *semidualizing* if the natural homomorphism $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Note that a dualizing module is nothing but a semidualizing module of finite injective dimension. Another typical example of a semidualizing module is a free module of rank one. Recently a considerable number of authors have studied semidualizing modules and have obtained many results concerning these modules.

We denote by \mathfrak{m} the maximal ideal of R and by k the residue field of R . To prove our main theorem, we establish two lemmas.

Lemma 2.2. *Let C be a semidualizing R -module. Let $g : M \rightarrow X$ be an injective homomorphism of R -modules with $C\text{-dim}_R X < \infty$. If $\text{Ext}_R^i(M, C) = 0$ for any $1 \leq i \leq C\text{-dim}_R X$, then the natural map $\lambda_M : M \rightarrow \text{Hom}_R(\text{Hom}_R(M, C), C)$ is injective.*

Proof. First of all we prove that M can be embedded in a module C_0 in $\text{add}_R C$. For this we set $n = C\text{-dim}_R X$. If $n = 0$, then this is obvious from the assumption, since $X \in \text{add}_R C$. If $n > 0$, then there exists an exact sequence

$$0 \rightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} C_0 \xrightarrow{d_0} X \rightarrow 0$$

with $C_i \in \text{add}_R C$ for $0 \leq i \leq n$. Putting $X_i = \text{Im } d_i$, we have exact sequences

$$0 \rightarrow X_{i+1} \rightarrow C_i \rightarrow X_i \rightarrow 0 \quad (0 \leq i \leq n - 1).$$

Then we have $\text{Ext}_R^1(M, X_1) = 0$, since there are isomorphisms $\text{Ext}_R^1(M, X_1) \cong \text{Ext}_R^2(M, X_2) \cong \dots \cong \text{Ext}_R^n(M, X_n) \cong \text{Ext}_R^n(M, C_n) = 0$. Hence $\text{Hom}_R(M, d_0) : \text{Hom}_R(M, C_0) \rightarrow \text{Hom}_R(M, X)$ is surjective. This implies that the homomorphism $g \in \text{Hom}_R(M, X)$ is lifted to $f \in \text{Hom}_R(M, C_0)$, i.e. $d_0 \cdot f = g$. Since g is injective, f is injective as well. Therefore M has an embedding f into C_0 .

To prove that λ_M is injective, we note that λ_{C_0} is an isomorphism because of $C_0 \in \text{add}_R C$. Since there is an injective homomorphism $f : M \rightarrow C_0$, the following commutative diagram forces λ_M to be injective:

$$\begin{array}{ccc} M & \xrightarrow{f} & C_0 \\ \lambda_M \downarrow & & \lambda_{C_0} \downarrow \cong \\ \text{Hom}_R(\text{Hom}_R(M, C), C) & \xrightarrow{\text{Hom}_R(\text{Hom}_R(f, C), C)} & \text{Hom}_R(\text{Hom}_R(C_0, C), C). \quad \square \end{array}$$

Lemma 2.3. *Let C be a semidualizing R -module and let M be an R -module. Assume that M is free on the punctured spectrum of R . Then there is an isomorphism*

$$\text{Ext}_R^i(M, R) \cong \text{Ext}_R^i(M \otimes_R C, C)$$

for each integer $i \leq \text{depth}_R C$.

Proof. Set $t = \text{depth}_R C$. Since C is semidualizing, we have a spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(\text{Tor}_q^R(M, C), C) \Rightarrow \text{Ext}_R^{p+q}(M, R).$$

Note by assumption that the R -module $\text{Tor}_q^R(M, C)$ has finite length for $q > 0$. By [2, Proposition 1.2.10(e)] we have $E_2^{p,q} = 0$ if $p < t$ and $q > 0$. Hence

$$\text{Ext}_R^i(M \otimes_R C, C) = E_2^{i,0} \cong \text{Ext}_R^i(M, R)$$

for $i \leq t$. □

Let M be an R -module. Take a free resolution

$$F_\bullet = (\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \rightarrow 0)$$

of M . Then for a nonnegative integer n we define the n -th syzygy of M by the image of d_n and denote it by $\Omega_R^n M$ or simply by $\Omega^n M$. We also define the (Auslander) transpose of M by the cokernel of the map $\text{Hom}_R(d_1, R) : \text{Hom}_R(F_0, R) \rightarrow \text{Hom}_R(F_1, R)$ and denote it by $\text{Tr}_R M$ or simply by $\text{Tr } M$. Note that the n th syzygy and the transpose of M are uniquely determined up to free summand. Note also that they commute with localization; namely, for every prime ideal \mathfrak{p} of R there are isomorphisms $(\Omega_R^n M)_{\mathfrak{p}} \cong \Omega_{R_{\mathfrak{p}}}^n M_{\mathfrak{p}}$ and $(\text{Tr}_R M)_{\mathfrak{p}} \cong \text{Tr}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ up to free summand.

Recall that for a positive integer n an R -module is called n -torsionfree if

$$\text{Ext}_R^i(M, R) = 0$$

for all $1 \leq i \leq n$. Now we can prove our main theorem.

Proof of Theorem 1.3. (1) \Rightarrow (2): By virtue of [6, Theorem (3.11)], the local ring R is Cohen-Macaulay. Now assertion (2) follows from [4, Theorem 1].

(2) \Rightarrow (3): This implication is obvious.

(3) \Rightarrow (1): We denote by $(-)^{\dagger}$ the C -dual functor $\text{Hom}_R(-, C)$. Put $t = \text{depth}_R C$ and set $M = \text{Tr } \Omega^t k$. Then we have $\text{depth } R = t$ by [5]. Since

$$\text{grade}_R \text{Ext}_R^i(k, R) \geq i - 1$$

for $1 \leq i \leq t$, the module $\Omega^t k$ is t -torsionfree by [1, Proposition (2.26)]. Hence $\text{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq t$. As M is free on the punctured spectrum of R , Lemma 2.3 implies $\text{Ext}_R^i(M \otimes_R C, C) = 0$ for $1 \leq i \leq t$. By assumption (3), the module $M \otimes_R C$ has an embedding into a module X with $C\text{-dim}_R X < \infty$. According to [7, Lemma 4.3], we have $C\text{-dim}_R X \leq t$. Lemma 2.2 shows that the natural map $\lambda_{M \otimes_R C} : M \otimes_R C \rightarrow (M \otimes_R C)^{\dagger\dagger}$ is injective. On the other hand, since there are natural isomorphisms

$$\begin{aligned} (M \otimes_R C)^{\dagger\dagger} &= \text{Hom}_R(\text{Hom}_R(M \otimes_R C, C), C) \cong \text{Hom}_R(\text{Hom}_R(M, \text{Hom}_R(C, C)), C) \\ &\cong \text{Hom}_R(\text{Hom}_R(M, R), C), \end{aligned}$$

we see from [1, Proposition (2.6)(a)] that

$$\begin{aligned} \text{Ker } \lambda_{M \otimes_R C} &\cong \text{Ext}_R^1(\text{Tr } M, C) \cong \text{Ext}_R^1(\Omega^t k, C) \\ &\cong \text{Ext}_R^{t+1}(k, C). \end{aligned}$$

Thus we obtain $\text{Ext}_R^{t+1}(k, C) = 0$. By [3, Theorem (1.1)], the R -module C must have finite injective dimension.

As we observed in the proof of the implication (1) \Rightarrow (2), assertion (1) implies that R is Cohen-Macaulay. Thus the last assertion follows. \square

Now we give applications of our main theorem. Letting $C = R$ in Theorem 1.3, we obtain the following result. This improves Theorem 1.2 and extends Theorem 1.1.

Corollary 2.4. *The following are equivalent:*

- (1) R is Gorenstein.
- (2) Every R -module can be embedded in an R -module of finite projective dimension.

Combining Corollary 2.4 with [4, Theorem 1], we have the following.

Corollary 2.5. *If every finitely generated R -module can be embedded in a finitely generated R -module of finite projective dimension, then every finitely generated R -module can be embedded in a finitely generated R -module of finite injective dimension.*

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REFERENCES

- [1] M. AUSLANDER; M. BRIDGER, Stable module theory. Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR0269685 (42:4580)
- [2] W. BRUNS; J. HERZOG, Cohen-Macaulay rings. Revised edition. Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1998. MR1251956 (95h:13020)
- [3] R. FOSSUM; H.-B. FOXBY; P. GRIFFITH; I. REITEN, Minimal injective resolutions with applications to dualizing modules and Gorenstein modules. *Inst. Hautes Études Sci. Publ. Math.* **45** (1975), 193–215. MR0396529 (53:392)
- [4] H.-B. FOXBY, Embedding of modules over Gorenstein rings. *Proc. Amer. Math. Soc.* **36** (1972), 336–340. MR0309930 (46:9034)
- [5] E. S. GOLOD, G-dimension and generalized perfect ideals (Russian). Algebraic geometry and its applications. *Trudy Mat. Inst. Steklov.* **165** (1984), 62–66. MR752933 (85m:13011)
- [6] R. Y. SHARP, Gorenstein modules. *Math. Z.* **115** (1970), 117–139. MR0263801 (41:8401)
- [7] R. TAKAHASHI, A new approximation theory which unifies spherical and Cohen-Macaulay approximations. *J. Pure Appl. Algebra* **208** (2007), no. 2, 617–634. MR2277700 (2007h:13016)

DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, SHINSHU UNIVERSITY, 3-1-1 ASAHI, MATSUMOTO, NAGANO 390-8621, JAPAN

E-mail address: takahasi@math.shinshu-u.ac.jp

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEHRAN, P. O. BOX 13145-448, TEHRAN, IRAN – AND – SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P. O. BOX 19395-5746, TEHRAN, IRAN

E-mail address: yassemi@ipm.ir

GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY, OKAYAMA UNIVERSITY, OKAYAMA 700-8530, JAPAN

E-mail address: yoshino@math.okayama-u.ac.jp