Gottlieb groups of spheres

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Abstract

This paper takes up the systematic study of the Gottlieb groups $G_{n+k}(\mathbb{S}^n)$ of spheres for $k \leq 13$ by means of the classical homotopy theory methods. We fully determine the groups $G_{n+k}(\mathbb{S}^n)$ for $k \leq 13$ except for the 2-primary components in the cases: $k = 9, n = 53; k = 11, n = 115$. Especially, we show $[\iota_n, \eta_n^2\sigma_{n+2}] = 0$ if $n = 2^i - 7$ for $i \geq 4$.

Key words: EHP sequence, fibration, Gottlieb group, rotation group, Stiefel manifold, Toda bracket, Whitehead product
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Introduction

The Gottlieb groups $G_k(X)$ of a pointed space X have been defined by Gottlieb in [9] and [10]; first $G_1(X)$ and then $G_k(X)$ for all $k \geq 1$. The higher Gottlieb groups $G_k(X)$ are related in [10] to the existence of sectioning fibrations with fiber X. For instance, if $G_k(X)$ is trivial then there is a cross-section for every fibration over the $(k+1)$-sphere $\mathbb{S}^{k+1}$, with fiber X.

This paper grew out of our attempt to develop techniques in calculating $G_{n+k}(\mathbb{S}^n)$ for $k \leq 13$ and any $n \geq 1$. The composition methods developed by

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Toda [36] are the main tools used in the paper. Our calculations also deeply depend on the results of [13], [16] and [21].

Section 1 serves as background to the rest of the paper. Write ι_n for the homotopy class of the identity map of $\mathbb{S}^n$. Then, the homomorphism

$$P' \colon \pi_k(\mathbb{S}^n) \longrightarrow \pi_{k+n-1}(\mathbb{S}^n)$$

defined by $P'(\alpha) = [\iota_n, \alpha]$ for $\alpha \in \pi_k(\mathbb{S}^n)$ [11] leads to the formula $G_k(\mathbb{S}^n) = \operatorname{Ker} P'$, where $[-,-]$ denotes the Whitehead product. Let $SO(n)$ be the rotation group and $J \colon \pi_k(SO(n)) \to \pi_{n+k}(\mathbb{S}^n)$ be the J-homomorphism. We recall $P' = J \circ \Delta$ and so, we have

$$\operatorname{Ker}\{\Delta \colon \pi_k(\mathbb{S}^n) \to \pi_{k-1}(SO(n))\} \subset G_k(\mathbb{S}^n).$$

By use of this result and [13, Table 2], we can show the lower bounds of the 2-primary component of $G_{n+k}(\mathbb{S}^n)$ if $n \geq 13$ and $k \leq 11$.

Our main task is to consult first [11], [12], [20], [21], [35] and [36] about the order of $[\iota_n, \alpha]$ and then to determine some Whitehead products in unsettled cases as well. In light of Serre's result [33, Proposition IV.5], the p-primary component of $G_{2m+k}(\mathbb{S}^{2m})$ vanishes for any odd prime p, if $2m \geq k+1$.

Let EX be the suspension of a space X and denote by $E \colon \pi_k(X) \to \pi_{k+1}(EX)$ the suspension map. Write $\eta_2 \in \pi_3(\mathbb{S}^2)$, $\nu_4 \in \pi_7(\mathbb{S}^4)$ and $\sigma_8 \in \pi_{15}(\mathbb{S}^8)$ for the Hopf maps, respectively. We set $\eta_n = E^{n-2}\eta_2 \in \pi_{n+1}(\mathbb{S}^n)$ for $n \geq 2$, $\nu_n = E^{n-4}\nu_4 \in \pi_{n+3}(\mathbb{S}^n)$ for $n \geq 4$ and $\sigma_n = E^{n-8}\sigma_8 \in \pi_{n+7}(\mathbb{S}^n)$ for $n \geq 8$. Write $\eta_n^2 = \eta_n \circ \eta_{n+1}$, $\nu_n^2 = \nu_n \circ \nu_{n+3}$ and $\sigma_n^2 = \sigma_n \circ \sigma_{n+7}$. Section 2 is a description of $G_{n+k}(\mathbb{S}^n)$ for $k \leq 7$. To reach that for $G_{n+6}(\mathbb{S}^n)$, we make use of Theorem 2.2 partially extending the result of [17]: $[\iota_n, \nu_n^2] = 0$ *if and only if* $n \equiv 4, 5, 7 \pmod 8$ *or* $n = 2^i - 5$ *for* $i \geq 4$; for the proof of which Section 3 and Section 4 are devoted.

Section 5 devotes to proving Mahowald's result: $[\iota_{16s+7}, \sigma_{16s+7}] \neq 0$ for $s \geq 1$.

Section 6 takes up computations of $G_{n+k}(\mathbb{S}^n)$ for $8 \leq k \leq 13$. In a repeated use of [21], we have found out the triviality of the Whitehead product [23]:

$$[\iota_n, \eta_n^2\sigma_{n+2}] = 0, \text{ if } n = 2^i - 7 \ (i \geq 4),$$

which corrects thereby [21] for $n = 2^i - 7$.

1 Preliminaries on Gottlieb groups

Throughout this paper, spaces, maps and homotopies are based. We use the standard terminology and notations from the homotopy theory, mainly from [36]. We do not distinguish between a map and its homotopy class.

Let X be a connected space. The k-th *Gottlieb group* $G_k(X)$ of X is the subgroup of the k-th homotopy group $\pi_k(X)$ consisting of all elements which can be represented by a map $f\colon \mathbb{S}^k \to X$ such that $f \vee \mathrm{id}_X\colon \mathbb{S}^k \vee X \to X$ extends (up to homotopy) to a map $F\colon \mathbb{S}^k \times X \to X$. Define $P_k(X)$ to be the set of elements of $\pi_k(X)$ whose Whitehead product with all elements of all homotopy groups is zero. It turns out that $P_k(X)$ forms a subgroup of $\pi_k(X)$ and, by [10, Proposition 2.3], $G_k(X) \subseteq P_k(X)$. Recall from [18] that X is said to be a *G-space* (resp. *W-space*) if $\pi_k(X) = G_k(X)$ (resp. $\pi_k(X) = P_k(X)$) for all k.

Given $\alpha \in \pi_k(\mathbb{S}^n)$ for $k \geq 1$, we deduce that $\alpha \in G_k(\mathbb{S}^n)$ if and only if $[\iota_n, \alpha] = 0$. In other words, consider the map

$$P'\colon \pi_k(\mathbb{S}^n) \longrightarrow \pi_{k+n-1}(\mathbb{S}^n)$$

defined by $P'(\alpha) = [\iota_n, \alpha]$ for $\alpha \in \pi_k(\mathbb{S}^n)$. Then, this leads to the formula

$$G_k(\mathbb{S}^n) = \operatorname{Ker} P'.$$

Write now $\sharp$ for the order of a group or its any element. Then, from the above interpretation of Gottlieb groups of spheres, we obtain

(1.1) $G_k(\mathbb{S}^n) = (\sharp[\iota_n, \alpha])\pi_k(\mathbb{S}^n)$, if $\pi_k(\mathbb{S}^n)$ is a cyclic group with a generator α.

Since $\mathbb{S}^n$ is an H-space for $n = 3, 7$, we have

$$G_k(\mathbb{S}^n) = \pi_k(\mathbb{S}^n) \text{ for } k \geq 1, \text{ if } n = 3, 7.$$

We recall the following result from [12] and [42] needed in the sequel.

Lemma 1.1 (1) *If* $\xi \in \pi_m(X)$, $\eta \in \pi_n(X)$, $\alpha \in \pi_k(\mathbb{S}^m)$, $\beta \in \pi_l(\mathbb{S}^n)$ *and if* $[\xi, \eta] = 0$ *then* $[\xi \circ \alpha, \eta \circ \beta] = 0$.

(2) *Let* $\alpha \in \pi_{k+1}(X)$, $\beta \in \pi_{l+1}(X)$, $\gamma \in \pi_m(\mathbb{S}^k)$ *and* $\delta \in \pi_n(\mathbb{S}^l)$.

Then $[\alpha \circ E\gamma, \beta \circ E\delta] = [\alpha, \beta] \circ E(\gamma \wedge \delta)$.

(3) *If* $\alpha \in \pi_k(\mathbb{S}^2)$ *and* $\beta \in \pi_l(\mathbb{S}^2)$ *then* $[\alpha, \beta] = 0$ *unless* $k = l = 2$.

(4) $[\beta, \alpha] = (-1)^{(k+1)(l+1)}[\alpha, \beta]$ *for* $\alpha \in \pi_{k+1}(X)$ *and* $\beta \in \pi_{l+1}(X)$.

In particular, $2[\alpha, \alpha] = 0$ *for* $\alpha \in \pi_n(X)$ *if* n *is odd.*

(5) *If* $\alpha_1, \alpha_2 \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$ *and* $p \geq 1$, *then* $[\alpha_1 + \alpha_2, \beta] =$

$[\alpha_1, \beta] + [\alpha_2, \beta]$ *and* $[\beta, \alpha_1 + \alpha_2] = [\beta, \alpha_1] + [\beta, \alpha_2]$.

(6) $E[\alpha, \beta] = 0$ *for* $\alpha \in \pi_k(X)$ *and* $\beta \in \pi_l(X)$.

(7) *Let* $\alpha \in \pi_{n+1}(X)$. *If* n *is even,* $2[\alpha, \alpha] = 0$ *and* $[\alpha, [\alpha, \alpha]] = 0$. *If* n *is odd,* $3[\alpha, [\alpha, \alpha]] = 0$ *and all Whitehead products in* α *of weight* ≥ 4 *vanish.*

Let $G_k(X; p)$ and $\pi_k(X; p)$ be the p-primary components of $G_k(X)$ and $\pi_k(X)$ for a prime p, respectively. But for $X = \mathbb{S}^n$, recall the notation from [36]:

$$\pi_k^n = \begin{cases} \pi_n(\mathbb{S}^n), & \text{if } k = n; \\ E^{-1}\pi_{2n}(\mathbb{S}^{n+1}; 2), & \text{if } k = 2n - 1; \\ \pi_k(\mathbb{S}^n; 2), & \text{if } k \neq n, 2n - 1. \end{cases}$$

As it is well-known, $[\iota_n, \iota_n] = 0$ if and only if $n = 1, 3, 7$ and $\sharp[\iota_n, \iota_n] = 2$ for n odd and $n \neq 1, 3, 7$, and it is infinite provided n is even. Thus, we have reproved the result [10] that $G_n(\mathbb{S}^n) = \pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ for $n = 1, 3, 7$, $G_n(\mathbb{S}^n) = 2\pi_n(\mathbb{S}^n) \cong 2\mathbb{Z}$ for n odd and $n \neq 1, 3, 7$, and $G_n(\mathbb{S}^n) = 0$ for n even, where $\mathbb{Z}$ denotes the additive group of integers. It is easily obtained that $G_k(\mathbb{S}^n) = P_k(\mathbb{S}^n)$ for all k, n [18, Theorem I.9]. In other words, on the level of spheres the class of G-spaces coincides with that of W-spaces.

We show

Proposition 1.2 (1) $(2 + (-1)^n)[\iota_n, \iota_n] \in G_{2n-1}(\mathbb{S}^n)$. *In particular, the infinite direct summand of* $G_{4n-1}(\mathbb{S}^{2n})$ *is* $\{3[\iota_{2n}, \iota_{2n}]\}$ *unless* $n = 1, 2, 4$.

(2) *If* $k \geq 3$ *then* $G_k(\mathbb{S}^2) = \pi_k(\mathbb{S}^2)$.

(3) *If* n *is odd and* $n \neq 1, 3, 7$ *then* $2\pi_k(\mathbb{S}^n) \subset G_k(\mathbb{S}^n)$. *In particular,*

$G_k(\mathbb{S}^n; p) = \pi_k(\mathbb{S}^n; p)$ *for any odd prime* p *and* $k \geq 1$.

(4) $G_k(\mathbb{S}^n) = \pi_k(\mathbb{S}^n)$ *provided that* $E\colon \pi_{k+n-1}(\mathbb{S}^n) \to \pi_{k+n}(\mathbb{S}^{n+1})$ *is*

a monomorphism.

PROOF. By Lemma 1.1.(7), $[\iota_n,[\iota_n,\iota_n]] = 0$ for n odd. In light of [19, Theorem 1.2.2],

$$(1.2) \qquad \sharp[\iota_{2n},[\iota_{2n},\iota_{2n}]] = 3, \text{ if } n \geq 2.$$

Hence, (1) follows.

(2) follows from Lemma 1.1.(3) what it was shown in [8] as well.

By Lemma 1.1.(4);(5), $[2\iota_n,\iota_n] = 0$. So, by Lemma 1.1.(1), $[\iota_n, 2\alpha] = [2\iota_n,\alpha] = 0$ for $\alpha \in \pi_k(\mathbb{S}^n)$. This leads to (3).

(4) is a direct consequence of Lemma 1.1.(6). This completes the proof.

We note that $P' \colon \pi_k(\mathbb{S}^n) \longrightarrow \pi_{k+n-1}(\mathbb{S}^n)$ and the homomorphism

$$P \colon \pi_{k+n+1}(\mathbb{S}^{2n+1}) \longrightarrow \pi_{k+n-1}(\mathbb{S}^n) \ (k \leq 2n-2)$$

in the EHP sequence defined as the notation "Δ" in [36, Chapter II] are related as follows:

$$P' = P \circ E^{n+1} \text{ for } k \leq 2n-2.$$

Denote by $i_n(\mathbb{R}) \colon SO(n-1) \hookrightarrow SO(n)$ and $p_n(\mathbb{R}) \colon SO(n) \to \mathbb{S}^{n-1}$ the inclusion and projection maps, respectively. We use the following exact sequence induced from the fibration $SO(n+1) \overset{SO(n)}{\longrightarrow} \mathbb{S}^n$:

$$(\mathcal{SO}^n_k) \qquad \pi_{k+1}(\mathbb{S}^n) \xrightarrow{\Delta} \pi_k(SO(n)) \xrightarrow{i_*} \pi_k(SO(n+1)) \xrightarrow{p_*} \pi_k(\mathbb{S}^n) \longrightarrow \cdots,$$

where $i = i_{n+1}(\mathbb{R})$, $p = p_{n+1}(\mathbb{R})$ and $\Delta \colon \pi_k(\mathbb{S}^n) \to \pi_{k-1}(SO(n))$ the connecting map.

We recall, for the J-homomorphism $J \colon \pi_k(SO(n)) \to \pi_{n+k}(\mathbb{S}^n)$,

$$(1.3) \qquad P' = J \circ \Delta$$

and so,

$$(1.4) \qquad \mathrm{Ker}\{\Delta \colon \pi_k(\mathbb{S}^n) \to \pi_{k-1}(SO(n))\} \subset G_k(\mathbb{S}^n).$$

Denote by $V_{n,k}$ the Stiefel manifold consisting of k-frames in $\mathbb{R}^n$ for $k \leq n-1$. We consider the commutative diagram:

$$\begin{array}{ccc}
\pi_k(V_{n+1,1}) & \xrightarrow{i_*} & \pi_k(V_{2n,n}) \\
\Big\downarrow{=} & & \Big\downarrow{\Delta'} \\
\pi_k(\mathbb{S}^n) & \xrightarrow{\Delta} & \pi_{k-1}(SO(n)),
\end{array}$$

where $i\colon V_{n+1,1} \hookrightarrow V_{2n,n}$ is the inclusion and Δ' is the connecting map associated with the fibration $SO(2n) \overset{SO(n)}{\longrightarrow} V_{2n,n}$.

By [5, Theorem 2], Δ' is a split monomorphism if $k \le 2n-2$ and $n \ge 13$. So, we have $\sharp(\Delta\alpha) = \sharp(i_*\alpha)$ for $\alpha \in \pi_k(\mathbb{S}^n)$ if $k \le 2n-2$ and $n \ge 13$. Hence, by (1.4) and [13, Table 2], we obtain the following.

Proposition 1.3 *Let $n \ge 13$. Then, $G_{n+k}(\mathbb{S}^n) = \pi_{n+k}(\mathbb{S}^n)$ for $k = 1,2,8,9$ if $n \equiv 3 \pmod 4$; $G_{n+3}(\mathbb{S}^n;2) = \pi^n_{n+3}$ if $n \equiv 7 \pmod 8$; $G_{n+6}(\mathbb{S}^n) = \pi_{n+6}(\mathbb{S}^n)$ if $n \equiv 4,5,7 \pmod 8$; $G_{n+7}(\mathbb{S}^n;2) = \pi^n_{n+7}$ if $n \equiv 15 \pmod{16}$; $G_{n+10}(\mathbb{S}^n;2) = \pi^n_{n+10}$ if $n \equiv 2,3 \pmod 4$; $G_{n+11}(\mathbb{S}^n;2) = \pi^n_{n+11}$ if n is odd unless $n \equiv 115 \pmod{128}$.*

In virtue of [33, Proposition IV.5] ([36, (13.1)]), Serre's isomorphism

$$\text{(1.5)} \qquad \pi_{i-1}(\mathbb{S}^{2m-1};p) \oplus \pi_i(\mathbb{S}^{4m-1};p) \cong \pi_i(\mathbb{S}^{2m};p)$$

is given by the correspondence $(\alpha,\beta) \mapsto E\alpha + [\iota_{2m},\iota_{2m}] \circ \beta$.

By (1.5), the Freudenthal suspension theorem and the EHP sequence, we obtain

$$\text{(1.6)} \qquad G_{2n+k}(\mathbb{S}^{2n};p) = 0, \text{ if } p \text{ is an odd prime and } k \le 2n-1.$$

The notation $\pi_{n+m}(\mathbb{S}^n) = \{\alpha_n\}$ (resp. $\{\alpha(n)\}$) means that there exist some $k \ge 1$ and an element α_k (resp. $\alpha(k)$) $\in \pi_{k+m}(\mathbb{S}^k)$ satisfying $\alpha_n = E^{n-k}\alpha_k$ (resp. $\alpha(n) = E^{n-k}\alpha(k)$) for $n \ge k$. For the p-primary component with any prime p, the notation is available.

Hereafter, we omit the reference [36] unless otherwise stated. Now, we know that $\pi_{n+3}(\mathbb{S}^n;3) = \{\alpha_1(n)\} \cong \mathbb{Z}_3$ and $\pi_{n+7}(\mathbb{S}^n;3) = \{\alpha_2(n)\} \cong \mathbb{Z}_3$ for $n \ge 3$. We have the relations [36, (13.7), Lemma 13.8, Theorem 13.9]:

$$\text{(1.7)} \qquad \alpha_1(5)\alpha_1(8) = 0 \text{ and } \alpha_1(7)\alpha_2(10) = 0.$$

Write $\{-,-,-\}_n$ for the Toda bracket, where $n \ge 0$ and $\{-,-,-\} = \{-,-,-\}_0$. We recall that there exists the element $\beta_1(5) \in \pi_{15}(\mathbb{S}^5)$ satisfying $\beta_1(5) \in \{\alpha_1(5),\alpha_1(8),\alpha_1(11)\}_1$, $3\beta_1(5) = -\alpha_1(5)\alpha_2(8)$ and that $\pi_{n+10}(\mathbb{S}^n;3) = \{\beta_1(n)\} \cong \mathbb{Z}_9$ for $n = 5,6$ and $\cong \mathbb{Z}_3$ for $n \ge 7$.

Let $\Omega^2\mathbb{S}^{2m+1} = \Omega(\Omega\mathbb{S}^{2m+1})$ be the double loop space of $\mathbb{S}^{2m+1}$ and $Q_2^{2m-1} = \Omega(\Omega^2\mathbb{S}^{2m+1}, \mathbb{S}^{2m-1})$ the homotopy fiber of the canonical inclusion (the double suspension map) $i\colon \mathbb{S}^{2m-1} \to \Omega^2\mathbb{S}^{2m+1}$. Then, the (mod p) EHP sequence [39, (2.1.3)] or [36, (13.2)] is stated as follows:

$$(1.8)\quad \cdots \xrightarrow{E^2} \pi_{i+3}(\mathbb{S}^{2m+1}) \xrightarrow{H} \pi_i(Q_2^{2m-1}) \xrightarrow{P} \pi_i(\mathbb{S}^{2m-1}) \xrightarrow{E^2} \pi_{i+2}(\mathbb{S}^{2m+1}) \xrightarrow{H} \cdots .$$

By making use of [36, Corollary 13.2], we obtain the generators of the following groups which are all isomorphic to $\mathbb{Z}_3$:

(1.9) $\pi_{6m-3}(Q_2^{2m-1};3) = \{i(2m-1)\}$,
where $i_{2m-1}\colon \mathbb{S}^{6m-3} \hookrightarrow Q_2^{2m-1}$ is the inclusion;
$\pi_{6m}(Q_2^{2m-1};3) = \{a_1(2m-1)\}$ $(a_1(2m-1) = i(2m-1)\alpha_1(6m-3))$;
$\pi_{6m+4}(Q_2^{2m-1};3) = \{a_2(2m-1)\}$ $(a_2(2m-1) = i(2m-1)\alpha_2(6m-3))$;
$\pi_{6m+7}(Q_2^{2m-1};3) = \{b_1(2m-1)\}$ $(b_1(2m-1) = i(2m-1)\beta_1(6m-3))$.

The following result and its proof have been shown by Toda [40].

Theorem 1.4 *Let $n \geq 2$. Then, $[\iota_{2n}, [\iota_{2n}, \alpha_1(2n)]] \neq 0$ if and only if $n \neq 2$ and $2n \equiv 1 \pmod 3$.*

PROOF. First of all, observe that using the proof of [14, Corollary (5.9)], the formula

$$(1.10)\qquad [[\alpha,\beta],\gamma] \in E\pi_{6n-2}(X) \text{ for } \alpha,\beta,\gamma \in \pi_{2n}(X)$$

holds. By (1.2), (1.3) and (1.10), we obtain

$$(1.11)\qquad [\iota_{2n}, [\iota_{2n}, \iota_{2n}]] = J\Delta[\iota_{2n}, \iota_{2n}] \in E\pi_{6n-3}(\mathbb{S}^{2n-1};3).$$

By (1.8) and (1.9), $[\iota_{2n}, [\iota_{2n}, \iota_{2n}]] = \pm EP(i(2n-1))$. By the naturality [39, (2.1.5)], we obtain $[\iota_{2n}, [\iota_{2n}, \alpha_1(2n)]] = \pm EP(a_1(2n-1))$. By [39, (4.15), Proposition 4.4], $(n+1)a_1(2n-1) = HP(i(2n+1))$. So, $P(a_1(2n-1)) = \pm PHP(i(2n+1)) = 0$ if $2n \not\equiv 1 \pmod 3$. For the case $n=2$, the assertion is trivial.

Next, assume that $n \neq 2$ and $2n \equiv 1 \pmod 3$. Then, by [38, Theorem 10.3], there exists an element $v \in \pi_{6n-2}(\mathbb{S}^{2n-3})$ satisfying $H(v) = b_1(2n-5)$ and $E^2v = P(a_1(2n-1))$. Furthermore, by [38, Proposition 5.3.(ii)], we obtain $P(a_2(2n-3)) = 3v$. Hence, by the (mod 3) EHP sequence (1.8), we have $P(a_1(2n-1)) \neq 0$. This implies the sufficient condition and completes the proof.

We show

Proposition 1.5 (1) *Let* $3 \le n \le 27$. *Then,* $G_{4n+2}(\mathbb{S}^{2n};3) = 0$ *if* $n = 5, 8, 11, 14, 17, 20, 23, 26$ *and* $G_{4n+2}(\mathbb{S}^{2n};3) = \{[\iota_{2n}, \alpha_1(2n)]\} \cong \mathbb{Z}_3$ *otherwise.*
(2) *Let* $3 \le n \le 9$. *Then,* $G_{6n-2}(\mathbb{S}^{2n};3) = \{[\iota_{2n}, [\iota_{2n}, \iota_{2n}]]\} \cong \mathbb{Z}_3$ *for* $n = 3, 5, 9$,
$G_{22}(\mathbb{S}^8;3) = \{[\iota_8, [\iota_8, \iota_8]], [\iota_8, \alpha_2(8)]\} \cong (\mathbb{Z}_3)^2$,
$G_{34}(\mathbb{S}^{12};3) = \{[\iota_{12}, [\iota_{12}, \iota_{12}]], [\iota_{12}, \alpha_3'(12)]\} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_9$,
$G_{40}(\mathbb{S}^{14};3) = \{[\iota_{14}, [\iota_{14}, \iota_{14}]], [\iota_{14}, \alpha_1(14)\beta_1(17)]\} \cong (\mathbb{Z}_3)^2$ *and*
$G_{46}(\mathbb{S}^{16};3) = \{[\iota_{16}, [\iota_{16}, \iota_{16}]], [\iota_{16}, \alpha_4(16)]\} \cong (\mathbb{Z}_3)^2$.

PROOF. Notice that $G_{6n-2}(\mathbb{S}^{2n}) \ni [\iota_{2n}, [\iota_{2n}, \iota_{2n}]]$ by Lemma 1.1.(7).

The assertion is obtained from [39, pp. 60-1: Table], (1.5), (1.2), Theorem 1.4. We determine $\pi_{38}(\mathbb{S}^{18};3)$ and $\pi_{34}(\mathbb{S}^{12};3)$. The rest is similar.

(1) By [39, pp. 60-1: Table], $\pi_{n+20}(\mathbb{S}^n;3) = \{\beta_1^2(n)\} \cong \mathbb{Z}_3$ for $n \ge 5$. So, by (1.5), $\pi_{38}(\mathbb{S}^{18};3) = \{\beta_1^2(18), [\iota_{18}, \alpha_1(18)]\} \cong (\mathbb{Z}_3)^2$. Again, by (1.5), we get $[\iota_{18}, \beta_1^2(18)] \ne 0$. Hence, by Theorem 1.4, $G_{38}(\mathbb{S}^{18};3) = \{[\iota_{18}, \alpha_1(18)]\} \cong \mathbb{Z}_3$.

(2) By (1.5), $\pi_{34}(\mathbb{S}^{12};3) = E\pi_{23}(\mathbb{S}^{11};3) \oplus \{[\iota_{12}, \iota_{12}] \circ \alpha_3'(23)\}$. By [39, pp. 60-1: Table] and (1.11), $[\iota_{12}, [\iota_{12}, \iota_{12}]] \in E^3\pi_{31}(\mathbb{S}^9;3)$ and so, $[\iota_{12}, [\iota_{12}, \alpha_3'(12)]] \in E^3\pi_{42}(\mathbb{S}^9;3)$. Moreover, $\pi_{42}(\mathbb{S}^9;3) \cong \mathbb{Z}_3$ and $E^4 : \pi_{42}(\mathbb{S}^9;3) \to \pi_{45}(\mathbb{S}^{13};3) \cong \mathbb{Z}_9$ is injective. This implies $[\iota_{12}, [\iota_{12}, \alpha_3'(12)]] = 0$ and hence, the group $G_{34}(\mathbb{S}^{12};3)$ follows.

Remark 1.6 In virtue of (1.10) and Lemma 1.1.(2);(6), $[\iota_{2n}, [\iota_{2n}, [\iota_{2n}, \iota_{2n}]]] = [\iota_{2n}, \iota_{2n}] \circ E^{2n-1}[\iota_{2n}, [\iota_{2n}, \iota_{2n}]] = 0$.

2 Gottlieb groups of spheres with stems for $k \le 7$

According to [11], [12], [17], [20], [35] and [36], we know the following results:

(2.1) $\qquad [\iota_n, \eta_n] = 0$ if and only if $n \equiv 3 \pmod 4$ or $n = 2, 6$;

(2.2) $\qquad [\iota_n, \eta_n^2] = 0$ if and only if $n \equiv 2, 3 \pmod 4$ or $n = 5$.

Hence, (1.1) completely determines $G_{n+k}(\mathbb{S}^n)$ for $k = 1, 2$ overlaping with Proposition 1.3.

We recall that $\pi_6^3 = \{\nu'\} \cong \mathbb{Z}_4$, where $2\nu' = \eta_3^3$. Write ω for a generator of the J-image $J\pi_3(SO(3)) = \pi_6(\mathbb{S}^3) \cong \mathbb{Z}_{12}$ satisfying $\omega = \nu' - \alpha_1(3)$. We recall the relation $[\iota_4, \iota_4] = \pm(2\nu_4 - E\omega)$. By abuse of notation, ν_n represents a generator of π_{n+3}^n and $\pi_{n+3}(\mathbb{S}^n)$ for $n \ge 4$, respectively. Then, $\pi_7(\mathbb{S}^4) =$

$\{\nu_4, E\omega\} \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$, $\pi_{n+3}(\mathbb{S}^n) = \{\nu_n\} \cong \mathbb{Z}_{24}$ for $n \geq 5$. Here, we write up the relations:

(2.3) $$\eta_3^3 = 2\nu' \text{ and } \eta_n^3 = 4\nu_n \text{ for } n \geq 5.$$

By [36, (5.9-11), Proposition 5.11],

(2.4) $$\eta_3\nu_4 = \nu'\eta_6, \eta_5\nu_6 = 0, [\iota_4, \eta_4] = (E\nu')\eta_7, \\ [\iota_5, \iota_5] = \nu_5\eta_8, \nu_6\eta_9 = 0 \text{ and } \nu'\nu_6 = 0.$$

By [2, Corollary (7.4)],

(2.5) $$[\iota_4, \nu_4] = \pm 2\nu_4^2.$$

In light of Lemma 1.1.(2) and (2.4), we obtain

$$[\iota_4, E\nu'] = (2\nu_4 - E\nu') \circ 2\nu_7 = 4\nu_4^2.$$

So, we have $2E\nu' \in G_7(\mathbb{S}^4)$. Consequently, by Proposition 1.2.(1) and (1.6),

$$G_7(\mathbb{S}^4) = \{3[\iota_4, \iota_4], 2E\nu'\} \cong 3\mathbb{Z} \oplus \mathbb{Z}_2.$$

By Lemma 1.1.(2) and (2.4), we obtain

(2.6) $$[\iota_5, \nu_5] = 0.$$

We recall the relations [36, (7.1), (7.4), p. 64, Lemma 6.3]:

(2.7) $$\eta_7\sigma_8 = \sigma'\eta_{14} + \bar{\nu}_7 + \varepsilon_7,\ \varepsilon_3\eta_{11} = \eta_3\varepsilon_4,\ \eta_6\bar{\nu}_7 = \bar{\nu}_6\eta_{14} = \nu_6^3.$$

and

(2.8) $$[\iota_9, \iota_9] = \eta_9\sigma_{10} + \sigma_9\eta_{16};\ [\iota_9, \eta_9] = \eta_9^2\sigma_{11} + \sigma_9\eta_{16}^2.$$

By [36, Lemma 6.2],

$$[\iota_6, \nu_6] = \pm 2\bar{\nu}_6.$$

By [36, (7.19-20)],

(2.9) $$\sigma'\nu_{14} = x\nu_7\sigma_{10} \text{ and } [\iota_8, \nu_8] = 2\sigma_8\nu_{15} - x\nu_8\sigma_{11}\ (x : \text{odd}), 4\nu_9\sigma_{12} = 0.$$

By [36, (7.22), Theorem 7.6]

(2.10) $$[\iota_9, \nu_9] = \bar{\nu}_9\nu_{17}$$

and $\sharp[\iota_{10},\nu_{10}]=4$. In light of [17], [20], [21], [34], [35], [36], Proposition 1.2.(3) and (1.5), we know the following:

$$(2.11)\quad \sharp[\iota_n,\nu_n]=\begin{cases}1, & \text{if } n\equiv 7 \pmod 8 \text{ or } \ n=2^i-3 \text{ for } i\geq 3;\\ 2, & \text{if } n\equiv 1,3,5 \pmod 8 \text{ and } n\geq 9 \text{ and } n\neq 2^i-3;\\ 12, & \text{if } n\equiv 2 \pmod 4 \text{ and } n\geq 6 \text{ or } n=4,12;\\ 24, & \text{if } n\equiv 0 \pmod 4 \text{ and } n\geq 8 \text{ unless } n=12.\end{cases}$$

Thus, (1.1) leads to a complete description of $G_{n+3}(\mathbb{S}^n)$ for $n\geq 5$.

By [36, (7.20-1)],

$$(2.12)\qquad [\iota_{10},\eta_{10}]=2\sigma_{10}\nu_{17},\ [\iota_{11},\iota_{11}]=\sigma_{11}\nu_{18},\ \nu_{11}\sigma_{14}=0 \text{ and } \sigma_{12}\nu_{19}=0.$$

By (2.4), (2.5) and (2.6), we have $[\iota_4,\nu_4\eta_7]=[\iota_4,(E\nu')\eta_7]=[\iota_5,\nu_5\eta_8]=0$. Hence, by the group structures of $\pi_{n+k}(\mathbb{S}^n)$ for $k=4,5$ and Proposition 1.2.(1), we get

Proposition 2.1 *$G_{n+4}(\mathbb{S}^n)=\pi_{n+4}(\mathbb{S}^n)$; $G_{n+5}(\mathbb{S}^n)=\pi_{n+5}(\mathbb{S}^n)$ unless $n=6$ and $G_{11}(\mathbb{S}^6)=3\pi_{11}(\mathbb{S}^6)\cong 3\mathbb{Z}$.*

In the next two sections, we will prove the following result partially extending that of [17, Theorem 1.3].

Theorem 2.2 *$[\iota_n,\nu_n^2]=0$ if and only if $n\equiv 4,5,7 \pmod 8$ or $n=2^i-5$ for $i\geq 4$.*

We recall that $\pi_{10}(\mathbb{S}^4)=\{\nu_4^2,\alpha_1(4)\alpha_1(7),\nu_4\alpha_1(7)\}\cong\mathbb{Z}_8\oplus(\mathbb{Z}_3)^2$. By (2.5) and (1.7), we get that $[\iota_4,\nu_4\alpha_1(7)]=[\iota_4,\alpha_1(4)\alpha_1(7)]=0$. Recall from [36, Lemma 5.14] that $\pi^5_{12}=\{\sigma'''\}\cong\mathbb{Z}_2$, $\pi^6_{13}=\{\sigma''\}\cong\mathbb{Z}_4$ and $\pi^7_{14}=\{\sigma'\}\cong\mathbb{Z}_8$, where

$$(2.13)\qquad E\sigma'''=2\sigma'', E\sigma''=2\sigma' \text{ and } E^2\sigma'=2\sigma_9.$$

By [2, Corollary (7.4)], (2.4) and (2.13), we obtain

$$[\iota_5,\sigma''']=[\iota_5,\iota_5]\circ E^4\sigma'''=0, [\iota_6,\sigma'']=[\iota_6,\iota_6]\circ E^5\sigma''=4([\iota_6,\iota_6]\circ\sigma_{11})$$

and $2[\iota_6,\sigma'']\neq 0$. We recall the relation $[\iota_8,\iota_8]=\pm(2\sigma_8-E\sigma')$. In $\pi^8_{22}=\mathbb{Z}_{16}\{\sigma_8^2\}\oplus\mathbb{Z}_8\{(E\sigma')\sigma_{15}\}\oplus\mathbb{Z}_4\{\kappa_8\}$, we have $[\iota_8,E\sigma']=2[\iota_8,\iota_8]\sigma_{15}=\pm 2(2\sigma_8^2-(E\sigma')\sigma_{15})$ and in view of [2, Corollary (7.4)], we obtain $[\iota_8,\sigma_8]=[\iota_8,\iota_8]\circ\sigma_{15}=\pm(2\sigma_8^2-(E\sigma')\sigma_{15})$. We know that $\pi_{n+7}(\mathbb{S}^n;5)=\{\alpha_1'(n)\}\cong\mathbb{Z}_5$ for $n\geq 3$. Thus, by Propositions 1.2, 1.3 and Theorem 2.2, we obtain

Proposition 2.3 (1) *$G_{n+6}(\mathbb{S}^n)=\pi_{n+6}(\mathbb{S}^n)$ if $n\equiv 4,5,7 \pmod 8$ or $n=2^i-5$ and $G_{n+6}(\mathbb{S}^n)=0$ otherwise.*

(2) $G_{n+7}(\mathbb{S}^n) = 0$ *if* $n = 4, 6$, $G_{12}(\mathbb{S}^5) = \pi_{12}(\mathbb{S}^5)$ *and* $G_{15}(\mathbb{S}^8) = \{3[\iota_8, \iota_8], 4E\sigma'\} \cong 3\mathbb{Z} \oplus \mathbb{Z}_2$.

Let $H\colon \pi_k(\mathbb{S}^n) \to \pi_k(\mathbb{S}^{2n-1})$ be the Hopf homomorphism. Then, by [1] and [31, Proposition 4.5], there exists an element $\gamma \in \pi^{n-7}_{2n-8}$ satisfying

(2.14) $[\iota_n, \iota_n] = E^7\gamma$, if $n \equiv 7 \pmod 8$; $H\gamma = \sigma_{2n-15}$, if $n \equiv 7 \pmod{16}$ and $n \geq 23$.

Concerning (2.14), we obtain

Theorem 2.4 (Mahowald [23]) $[\iota_n, \sigma_n] \neq 0$, *if* $n \equiv 7 \pmod{16}$ *and* $n \geq 23$. *It desuspends seven dimensions whose Hopf invariant is* σ^2_{2n-15}.

In virtue of Theorem 6.1.(2), the first half of Theorem 2.4 is obtained and this will be proved in Section 5.

By abuse of notation, σ_n represents a generator of π^n_{n+7} and $\pi_{n+7}(\mathbb{S}^n)$ for $n \geq 9$, respectively.

By [36, (10.18), Theorem 10.5],

(2.15)
$$[\iota_9, \sigma_9] = \sigma_9(\bar{\nu}_{16} + \varepsilon_{16}) \neq 0$$

and

(2.16)
$$\sigma_{11}\bar{\nu}_{18} = \sigma_{11}\varepsilon_{18} = 0.$$

In view of [36, Theorem 12.16], $\sharp[\iota_{10}, \sigma_{10}] = 16$ and, by [36, Lemma 12.14],

(2.17)
$$[\iota_{11}, \sigma_{11}] = 0.$$

We know that $\sharp[\iota_{12}, \sigma_{12}] = 16$ [36, Lemma 12.19, Theorem 12.22] and $[\iota_{13}, \sigma_{13}] \neq 0$ [36, p. 166]. We also know that $\sharp[\iota_{14}, \sigma_{14}] = 16$ [26, p. 52], $[\iota_{15}, \sigma_{15}] = 0$ [24, Lemma 6.2], $\sharp[\iota_{16}, \sigma_{16}] = 16$ [24, p. 323], $[\iota_{17}, \sigma_{17}] \neq 0$ [25, p. 27] and $\sharp[\iota_{18}, \sigma_{18}] = 16$ [25, (5.36)]. By [32, p. 72: (7.23)], $[\iota_{19}, \sigma_{19}] \neq 0$. By [32, p. 142, Theorem 3.(b)], $\sharp[\iota_{20}, \sigma_{20}] = 16$. Hence, by combining the results of [20, Theorem (1.1.2c)], [21, Theorem C], [36, Theorem 10.3], Proposition 1.2.(3), (1.5) and Theorem 2.4, we obtain
(2.18)
$$\sharp[\iota_n, \sigma_n] = \begin{cases} 1, & \text{if } n = 11 \text{ or } n \equiv 15 \pmod{16}; \\ 2, & \text{if } n \text{ is odd and } n \geq 9 \text{ unless } n = 11 \text{ and } n \equiv 15 \pmod{16}; \\ 120, & \text{if } n = 8; \\ 240, & \text{if } n \text{ is even and } n \geq 10. \end{cases}$$

Whence, by means of (1.1), the group $G_{n+7}(\mathbb{S}^n)$ for $n \geq 9$ has been fully described as well.

3 Proof of Theorem 2.2, part I

Since $SO(n) \cong SO(n-1) \times \mathbb{S}^{n-1}$ for $n = 4, 8$, we get that

$$(3.1) \qquad \Delta\pi_{k+1}(\mathbb{S}^n) = 0, \text{ if } n = 3, 7.$$

By the exact sequence $(\mathcal{SO}^n_n)$ and the fact that $\pi_n(SO(n)) \cong \mathbb{Z}$ for $n \equiv 3 \pmod 4$ [16, pp. 161-2], we have

$$(3.2) \qquad \Delta\eta_n = 0, \text{ if } n \equiv 3 \pmod 4.$$

We recall the formula [16, Lemma 1]

$$(3.3) \qquad \Delta(\alpha \circ E\beta) = \Delta\alpha \circ \beta.$$

By (3.2) and (3.3),

$$(3.4) \qquad \Delta(\eta_n^2) = 0, \text{ if } n \equiv 3 \pmod 4.$$

Given elements $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ and $\beta \in \pi_{n+k}(SO(n+1))$ satisfying $p_{n+1}(\mathbb{R})\beta = \alpha$, then β is called a lift of α and we write

$$\beta = [\alpha].$$

For $m \leq n-1$, set $i_{m,n} = i_n(\mathbb{R}) \circ \cdots \circ i_{m+1}(\mathbb{R})$. We set $[\alpha]_n = i_{m,n*}[\alpha] \in \pi_k(SO(n))$, where $[\alpha] \in \pi_k(SO(m))$ is a lift of $\alpha \in \pi_k(\mathbb{S}^{m-1})$. Observe that $J[\iota_3] = \nu_4$ and $J[\iota_7] = \sigma_8$.

Next, we need

Lemma 3.1 *Let $n \equiv 3 \pmod 4$ and $n \geq 7$. Then,*

(1) $\{\Delta\iota_n, \eta_{n-1}, 2\iota_n\} = 0$*;*

(2) $\Delta(E\{\eta_{n-1}, 2\iota_n, \alpha\}) = 0$*, where $\alpha \in \pi_k(\mathbb{S}^n)$ is an element satisfying $2\iota_n \circ \alpha = 0$.*

PROOF. By [36, Proposition 1.4] and the fact that $2\pi_{n+1}(SO(n+1)) = 0$ [16, p. 161], we obtain

$$i_{n+1}(\mathbb{R}) \circ \{\Delta\iota_n, \eta_{n-1}, 2\iota_n\} = -\{i_{n+1}(\mathbb{R}), \Delta\iota_n, \eta_{n-1}\} \circ 2\iota_{n+1} \subset 2\pi_{n+1}(SO(n+1)) = 0.$$

It follows from $(\mathcal{SO}^n_{n+1})$ and (3.4) that $i_{n+1}(\mathbb{R})_*\colon \pi_{n+1}(SO(n)) \to \pi_{n+1}(SO(n+1))$ is a monomorphism. This leads to (1).

By (3.3) and (1), for any $\beta \in \{\eta_{n-1}, 2\iota_n, \alpha\}$, we obtain

$$\Delta(E\beta) \in \Delta\iota_n \circ \{\eta_{n-1}, 2\iota_n, \alpha\} = -\{\Delta\iota_n, \eta_{n-1}, 2\iota_n\} \circ E\alpha = 0.$$

This leads to (2) and completes the proof.

We recall that $\varepsilon_{n-1} \in \{\eta_{n-1}, 2\iota_n, \nu_n^2\}$ and $\mu_{n-1} \in \{\eta_{n-1}, 2\iota_n, E^{n-5}\sigma'''\}$ for $n \geq 5$. By (3.1) and Lemma 3.1.(2), we obtain

Example 3.2 $\Delta\varepsilon_n = 0$ *and* $\Delta\mu_n = 0$*, if* $n \equiv 3 \pmod 4$.

We show

Lemma 3.3 (1) $\Delta(\nu_n^2) = 0$*, if* $n \equiv 5 \pmod 8$;

(2) $\Delta(\nu_{4n}^2) = 0$*, if* n *is odd.*

PROOF. Since $\pi_7(SO(5)) \cong \mathbb{Z}$ [16, p. 162], $\Delta\colon \pi_8(\mathbb{S}^5) \to \pi_7(SO(5))$ is trivial and $\Delta\nu_5 = 0$. So, by (3.3), $\Delta(\nu_5^2) = 0$. Let now $n \equiv 5 \pmod 8$ and $n \geq 13$. We consider the exact sequence $(\mathcal{SO}^n_{n+5})$:

$$\pi_{n+6}(\mathbb{S}^n) \xrightarrow{\Delta} \pi_{n+5}(SO(n)) \xrightarrow{i_*} \pi_{n+5}(SO(n+1)) \to 0.$$

By [5, Theorem 2], we obtain

$$\pi_{n+5}(SO(n)) \cong \pi_{n+5}(SO) \oplus \pi_{n+6}(V_{n+8,8}).$$

In light of [13, Table 1], $\pi_{n+6}(V_{n+8,8}) \cong \mathbb{Z}_8$ and by [6], $\pi_{n+5}(SO) = 0$. So, $\pi_{n+5}(SO(n)) \cong \mathbb{Z}_8$. By [16, p. 161], $\pi_{n+5}(SO(n+1)) \cong \mathbb{Z}_8$. From the fact that $\pi_{n+6}(\mathbb{S}^n) = \{\nu_n^2\} \cong \mathbb{Z}_2$, we obtain $\Delta(\nu_n^2) = 0$, and hence (1) follows.

We obtain $\pi_9(SO(4)) \cong \pi_9(SO(3)) \oplus \pi_9(\mathbb{S}^3) \cong (\mathbb{Z}_3)^2$, and so $\Delta(\nu_4^2) = 0$. Let now $n \geq 3$. Then, we consider the exact sequence $(\mathcal{SO}^{4n}_{4n+5})$:

$$\pi_{4n+6}(\mathbb{S}^{4n}) \xrightarrow{\Delta} \pi_{4n+5}(SO(4n)) \xrightarrow{i_*} \pi_{4n+5}(SO(4n+1)) \to 0.$$

By [16, p. 161],

$$\pi_{4n+5}(SO(4n+1)) \cong \mathbb{Z}_2 \ (n \geq 2). \tag{3.5}$$

By [15, Theorem 1.(iii)], $\pi_{17}(SO(12)) = \{[\iota_7]_{12}\eta_7\mu_8\} \cong \mathbb{Z}_2$. Since $J([\iota_7]_{12}\eta_7\mu_8) = \sigma_{12}\eta_{19}\mu_{20} \neq 0$ in $\pi_{29}(\mathbb{S}^{12})$, we get that $\Delta(\nu_{12}^2) = 0$. Let n be odd and $n \geq 5$. In light of [5, Theorem 2],

$$\pi_{4n+5}(SO(4n)) \cong \pi_{4n+5}(SO) \oplus \pi_{4n+6}(V_{4n+8,8}).$$

By means of [6] and [13, Table 1], $\pi_{4n+5}(SO) \cong \mathbb{Z}_2$ and $\pi_{4n+6}(V_{4n+8,8}) = 0$. Hence, we obtain $\Delta(\nu_{4n}^2) = 0$ if n is odd with $n \geq 5$. This leads to (2) and completes the proof.

[17, Theorem 1.3] suggests the non-triviality of $[\iota_n, \nu_n^2]$ for $n \equiv 0, 1, 2, 3, 6 \pmod 8$ and $n \geq 6$ and [28, Proposition 3.4] gives an explicit proof of its non-triviality for $n \equiv 2 \pmod 4$ and $n \geq 6$.

By Lemma 1.1.(1) and (2.11), we have $[\iota_n, \nu_n^2] = 0$ if $n \equiv 7 \pmod 8$ or $n = 2^i - 3$ for $i \geq 3$. In virtue of Lemma 3.3 and (1.3), we get that

$$[\iota_n, \nu_n^2] = 0, \text{ if } n \equiv 5 \pmod 8 \tag{3.6}$$

and

$$[\iota_n, \nu_n^2] = 0, \text{ if } n \equiv 4 \pmod 8. \tag{3.7}$$

Let now $n \equiv 0 \pmod 4$ and $n \geq 8$. By [5, Theorem 2], [6] and [13, Table 1], $\pi_{2n+3}(SO(2n-2)) \cong \mathbb{Z} \oplus \mathbb{Z}_4$. In the exact sequence $(\mathcal{SO}_{2n+3}^{2n-3})$, the map $p_{2n-2}(\mathbb{R})_* \colon \pi_{2n+3}(SO(2n-2)) \to \pi_{2n+3}(\mathbb{S}^{2n-3})$ is an epimorphism by Lemma 3.3.(1). So, the direct summand $\mathbb{Z}_4$ of $\pi_{2n+3}(SO(2n-2))$ is generated by $[\nu_{2n-3}^2]$. By [16, p. 161], $\pi_{2n+3}(SO(2n+1)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $\pi_{2n+3}(SO(2n+2)) \cong \mathbb{Z}$. It follows from $(\mathcal{SO}_{2n+3}^{2n+1})$ that the direct summand $\mathbb{Z}_2$ of $\pi_{2n+3}(SO(2n+1))$ is generated by $\Delta\nu_{2n+1}$. By [16, p. 161], $\pi_{2n+3}(SO(2n+k-1)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ for $0 \leq k \leq 2$. Hence, by use of $(\mathcal{SO}_{2n+3}^{2n+k-1})$ for $-1 \leq k \leq 2$, $(i_{2n-2,2n+1})_* \colon \pi_{2n+3}(SO(2n-2)) \to \pi_{2n+3}(SO(2n+1))$ is an epimorphism and we get the relation

$$[\nu_{2n-3}^2]_{2n+1} = \Delta\nu_{2n+1}.$$

Thus, we conclude

Lemma 3.4 $E^3 J[\nu_{2n-3}^2] = [\iota_{2n+1}, \nu_{2n+1}]$, *if* $n \equiv 0 \pmod 4$ *and* $n \geq 8$.

Hereafter, we use often the EHP sequence of the following type:

$$(\mathcal{PE}_{n+k}^n) \quad \pi_{n+k+2}^{2n+1} \xrightarrow{P} \pi_{n+k}^n \xrightarrow{E} \pi_{n+k+1}^{n+1}.$$

It is well-known that

$$H[\iota_n, \iota_n] = 0 \text{ for } n \text{ odd, and } H[\iota_n, \iota_n] = \pm 2\iota_{2n-1} \text{ for } n \text{ even.}$$

So, by [36, Proposition 2.5], we obtain

(3.8) $$HP(E^3\gamma) = \pm(1+(-1)^n)E\gamma \text{ for } \gamma \in \pi_k^{2n-2}.$$

Suppose that $\Delta\alpha = 0$ for $\alpha \in \pi_k(\mathbb{S}^n)$. Then, by [41, pp. 214-5], we obtain

(3.9) $$H(J[\alpha]) = \pm E^{n+1}\alpha \text{ for } k \leq 2n.$$

Now, we show

I. $[\iota_n, \nu_n^2] \neq 0$ if $n \equiv 1 (\bmod\ 8)$ and $n \geq 9$.

In virtue of (2.10) and [36, Lemmas 9.2,10.1, Theorem 20.3], $[\iota_9, \nu_9^2] = \bar{\nu}_9\nu_{17}^2 \equiv 2\kappa_9 + 8a\sigma_9^2 \neq 0$ for $a \in \{0, 1\}$.

Let $n \equiv 0 \pmod 4$ and $n \geq 8$. By Lemma 3.4, $[\iota_{2n+1}, \nu_{2n+1}^2] = E^3(J[\nu_{2n-3}^2] \circ \nu_{4n+1})$. Suppose that $E^3(J[\nu_{2n-3}^2] \circ \nu_{4n+1}) = 0$. Then, by use of $(\mathcal{PE}_{4n+6}^{2n})$, we obtain $E^2(J[\nu_{2n-3}^2] \circ \nu_{4n+1}) = 8a[\iota_{2n}, \sigma_{2n}]$ for $a \in \{0, 1\}$. By means of [36, Proposition 11.11.(i)], there exists an element $\beta \in \pi_{4n+4}^{2n-2}$ such that $P(8\sigma_{4n+1}) = E^2\beta$ and $H\beta \in \{2\iota_{4n-5}, \eta_{4n-5}, 8\sigma_{4n-4}\}_2$. By [36, (1.15), Proposition 1.2.0);ii), Lemma 1.1] and the relation $2\eta_{4n-5} = 0$, we see that

$$\{2\iota_{4n-5}, \eta_{4n-5}, 8\sigma_{4n-4}\}_2 \subset \{2\iota_{4n-5}, \eta_{4n-5}, 8\sigma_{4n-4}\} \subset$$
$$\{2\iota_{4n-5}, 0, 4\sigma_{4n-4}\} = 2\iota_{4n-5} \circ \pi_{4n+4}^{4n-5} + \pi_{4n-3}^{4n-5} \circ 4\sigma_{4n-3} = 0.$$

So, there exists an element $\beta' \in \pi_{4n+3}^{2n-3}$ such that $\beta = E\beta'$. Hence, $E^2(J[\nu_{2n-3}^2] \circ \nu_{4n+1}) = aE^3\beta'$.

In virtue of Lemma 1.1.(1) and (2.1), $[\iota_{2n-1}, \eta_{2n-1}\sigma_{2n}] = 0$. In light of (1.3) and Example 3.2, $[\iota_{2n-1}, \varepsilon_{2n-1}] = 0$, and so $P\pi_{4n+7}^{4n-1} = 0$. Therefore, by $(\mathcal{PE}_{4n+5}^{2n-1})$, $E(J[\nu_{2n-3}^2] \circ \nu_{4n+1}) = aE^2\beta'$. Finally, by use of $(\mathcal{PE}_{4n+4}^{2n-2})$ and (3.9), we have a contradictory relation $\nu_{4n-5}^3 = 0$. Thus, we get $[\iota_{2n+1}, \nu_{2n+1}^2] = E^3(J[\nu_{2n-3}^2] \circ \nu_{4n+1}) \neq 0$.

We denote by $\mathbb{R}P^n$ the real n-dimensional projective space, by $\gamma_n\colon \mathbb{S}^n \to \mathbb{R}P^n$ the covering map and by $p_n'\colon \mathbb{R}P^n \to \mathbb{S}^n$ the collapsing map, respectively. Then, we can take $\Delta\iota_n = j \circ \gamma_{n-1}$, where $j\colon \mathbb{R}P^{n-1} \hookrightarrow SO(n)$ is the canonical embedding. Hence, by the relations $j \circ p_n(\mathbb{R}) = p_{n-1}'$ and $p_n' \circ \gamma_n = (1 + (-1)^{n+1})\iota_n$, we obtain

(3.10) $$p_n(\mathbb{R})(\Delta\iota_n) = (1 + (-1)^n)\iota_{n-1}.$$

Let $n \equiv 0 \pmod 8$ and $n \geq 8$. By use of $(\mathcal{SO}_{n+1}^{n-1})$ and [16, pp. 161-2], we get that $i_n(\mathbb{R})_*\colon \pi_{n+1}(SO(n-1)) \to \pi_{n+1}(SO(n))$ is a monomorphism. So, we obtain

(3.11) $$\Delta\nu_{n-1} = 0, \text{ if } n \equiv 0 \pmod 8 \text{ and } n \geq 8.$$

Hence, by Lemma 3.3.(2), ν_{n-1} and ν_{n-4}^2 are lifted to $[\nu_{n-1}] \in \pi_{n+2}(SO(n))$ and $[\nu_{n-4}^2] \in \pi_{n+2}(SO(n-3))$, respectively. We show the following

Lemma 3.5 *Let $n \equiv 0 \pmod 8$ and $n \geq 16$. Then,*

(1) $2[\nu_{n-1}] - \Delta\nu_n = x[\nu_{n-4}^2]_n$ *for odd x;*

(2) $\pi_{n+5}(SO(n+1)) = \{[\nu_{n-1}]_{n+1}\nu_{n+2}\} \cong \mathbb{Z}_2$.

PROOF. By use of $(\mathcal{SO}_{n+2}^{n-k})$ for $2 \leq k \leq 4$, Lemma 3.3 and [16, p. 161], we see that $(i_{n-3,n-1})_*\colon \pi_{n+2}(SO(n-3)) \to \pi_{n+2}(SO(n-1)) \cong \mathbb{Z}_8$ is an isomorphism and $\pi_{n+2}(SO(n-3)) = \{[\nu_{n-4}^2]\}$. In virtue of [16, p. 161], $\pi_{n+2}(SO(n+1)) \cong \mathbb{Z}_8$ and $\pi_{n+2}(SO(n)) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_8$. So, by $(\mathcal{SO}_{n+2}^{n-k})$ for $k = 0, 1$, we get $\pi_{n+2}(SO(n)) = \{\Delta\nu_n, [\nu_{n-1}]\}$. By (3.10), we obtain $p_n(\mathbb{R})(\Delta\nu_n) = 2\nu_{n-1}$, and hence $2[\nu_{n-1}] - \Delta\nu_n \in \mathrm{Im}\,\{i_n(\mathbb{R})_*\colon \pi_{n+2}(SO(n-1)) \to \pi_{n+2}(SO(n))\}$. Since $\sharp(2[\nu_{n-1}] - \Delta\nu_n) = 8$, we have the required relation of (1).

We consider the exact sequence $(\mathcal{SO}_{n+5}^n)$:

$$\pi_{n+6}(S^n) \xrightarrow{\Delta} \pi_{n+5}(SO(n)) \xrightarrow{i_*} \pi_{n+5}(SO(n+1)) \longrightarrow 0.$$

By (3.5), $\pi_{n+5}(SO(n+1)) \cong \mathbb{Z}_2$. In view of [5, Theorem 2], [6] and [13, Table 1], we obtain

(3.12) $$\pi_{n+5}(SO(n)) \cong (\mathbb{Z}_2)^2 \ (n \equiv 0 \pmod 8 \text{ and } n \geq 8).$$

By (3.11), ν_{n-1}^2 is lifted to $[\nu_{n-1}]\nu_{n+2}$. Consequently, we obtain $\pi_{n+5}(SO(n)) = \{\Delta(\nu_n^2), [\nu_{n-1}]\nu_{n+2}\}$ and $\pi_{n+5}(SO(n+1)) = \{[\nu_{n-1}]_{n+1}\nu_{n+2}\}$. This leads to (2) and completes the proof.

The relation in [36, Lemma 11.17] is regarded as the J-image of that in Lemma 3.5.(1).

Remark 3.6 The results in (3.2), (3.4), Lemma 3.3, Example 3.2 and (3.11) overlaps with [13, Table 2].

Now, we present a proof of the non-triviality of $[\iota_n, \nu_n^2]$ in the case $n \equiv 0$ (mod 8) and $n \geq 8$.

II. $[\iota_n, \nu_n^2] \neq 0$ if $n \equiv 0 \pmod 8$ and $n \geq 8$.

By (2.9) and [36, Theorem 7.7], $[\iota_8, \nu_8^2] = \nu_8\sigma_{11}\nu_{18} \neq 0$. Let $n \equiv 0 \pmod 8$ and $n \geq 16$. In light of (3.12), $\pi_{n+5}(SO(n)) \cong (\mathbb{Z}_2)^2$. So, by (3.3) and Lemma 3.5,

$$\Delta(\nu_n^2) = [\nu_{n-4}^2]_n\nu_{n+2}$$

and hence $[\iota_n, \nu_n^2] = E^3(J[\nu_{n-4}^2] \circ \nu_{2n-1})$.

Suppose that $E^3(J[\nu_{n-4}^2] \circ \nu_{2n-1}) = 0$. Then, $E^2(J[\nu_{n-4}^2] \circ \nu_{2n-1}) \in P\pi_{2n+6}^{2n-1} = \{[\iota_{n-1}, \sigma_{n-1}]\}$. By [36, Proposition 11.11.(ii)], it holds $P\pi_{2n+5}^{2n-3} \subset E^2\pi_{2n+1}^{n-4}$. So, by (2.14) and using $(\mathcal{PE}_{2n+4-k}^{n-1-k})$ for $k = 0, 1$, we get that

$$J[\nu_{n-4}^2] \circ \nu_{2n-1} - aE^5(\gamma\sigma_{2n-10}) - E\beta \in P\pi_{2n+4}^{2n-5}$$

for some $\beta \in \pi_{2n+1}^{n-4}$ and $a \in \{0, 1\}$. Hence, (3.8) and (3.9) imply a contradictory relation $\nu_{2n-7}^3 = 0$, and thus $[\iota_n, \nu_n^2] \neq 0$.

We note that Nomura [30] has a different proof of II.

4 Proof of Theorem 2.2, part II

Let $\omega_n(\mathbb{R}) \in \pi_{n-1}(O(n))$, $\omega_n(\mathbb{C}) \in \pi_{2n}(U(n))$ and $\omega_n(\mathbb{H}) \in \pi_{4n+2}(Sp(n))$ be the characteristic elements for the orthogonal $O(n)$, unitary $U(n)$ and symplectic $Sp(n)$ groups, respectively. We note that $\omega_n(\mathbb{R}) = \Delta\iota_n$ and $\sharp(\Delta\iota_n) = 2$ for odd $n \geq 9$.

Let $r_n\colon U(n) \to SO(2n)$ and $c_n\colon Sp(n) \to SU(2n)$ be the canonical maps, respectively. Set $i_n(\mathbb{C})\colon U(n-1) \hookrightarrow U(n)$ for the inclusion map. As it is well-known,

$$i_{2n+1}(\mathbb{R})r_n\omega_n(\mathbb{C}) = \omega_{2n+1}(\mathbb{R}) \text{ and } i_{2n+1}(\mathbb{C})c_n\omega_n(\mathbb{H}) = \omega_{2n+1}(\mathbb{C}).$$

Let

$$\tau'_{2n} = r_n\omega_n(\mathbb{C}) \in \pi_{2n}(SO(2n)) \text{ and } \bar{\tau}'_{4n} = r_{2n}c_n\omega_n(\mathbb{H}) \in \pi_{4n+2}(SO(4n)).$$

By use of the exact sequence $(\mathcal{SO}_{2n}^{2n})$ and [16, p. 161], we obtain the following:

$$i_{2n+1}(\mathbb{R})\tau'_{2n} = \Delta\iota_{2n+1} \text{ for } n \geq 4. \tag{4.1}$$

Let $n \equiv 2 \pmod 4$ and $n \geq 10$. Then, by use of $(\mathcal{SO}_n^n)$, (4.1) and [16, p. 161], we obtain

(4.2) $\pi_n(SO(n)) = \{\tau'_n\} \cong \mathbb{Z}_4$ and $2\tau'_n = \Delta\eta_n,$ if $n \equiv 2 \pmod 4$ and $n \geq 10$.

By the commutative diagram

$$\begin{array}{ccc} \pi_{4n+2}(U(2n)) & \xrightarrow{i_{2n+1}(\mathbb{C})_*} & \pi_{4n+2}(U(2n+1)) \\ \downarrow{\scriptstyle r_{2n*}} & & \downarrow{\scriptstyle r_{2n+1*}} \\ \pi_{4n+2}(SO(4n)) & \xrightarrow{i_{4n,4n+2*}} & \pi_{4n+2}(SO(4n+2)), \end{array}$$

we obtain

(4.3) $$(i_{4n,4n+2})\bar{\tau}'_{4n} = \tau'_{4n+2}.$$

It is well-known that

(4.4) $p_{2n}(\mathbb{R})\tau'_{2n} = (n-1)\eta_{2n-1}$ and $p_{4n}(\mathbb{R})\bar{\tau}'_{4n} = \pm(n+1)\nu_{4n-1}$ for $n \geq 2$.

By use of $(\mathcal{SO}^{4n+1}_{4n+2})$, (4.1), (4.3) and [16, p. 161], we obtain

(4.5) $$\Delta(\eta^2_{4n+1}) = 4i_{4n+1}(\mathbb{R})\bar{\tau}'_{4n}, \text{ if } n \geq 2.$$

So, by $(\mathcal{SO}^{4n}_{4n+2})$, (4.1) and (4.5), we have $\tau'_{4n}\eta^2_{4n} - 4\bar{\tau}'_{4n} \in \{\Delta\nu_{4n}\}$. Composing $p_{4n}(\mathbb{R})$ with this relation, using the fact that $\eta^3_{4n-1} = 12\nu_{4n-1}$ (2.3), (3.10) and (4.4),

$$\tau'_{4n}\eta^2_{4n} \equiv 4\bar{\tau}'_{4n} \pmod{2a\Delta\nu_{4n}}, \text{ for } a \text{ odd and } n \geq 2.$$

Set $\tau_{2n} = J\tau'_{2n} \in \pi_{4n}(\mathbb{S}^{2n})$ and $\bar{\tau}_{4n} = J\bar{\tau}'_{4n} \in \pi_{8n+2}(\mathbb{S}^{4n})$. Then, we note that

(4.6) $$E\tau_{2n} = [\iota_{2n+1}, \iota_{2n+1}], H\tau_{2n} = (n-1)\eta_{4n-1}$$

and

(4.7) $$E^3\bar{\tau}_{4n} = [\iota_{4n+3}, \iota_{4n+3}], H\bar{\tau}_{4n} = \pm(n+1)\nu_{8n-1}$$

By (4.5), we have

(4.8) $$[\iota_{4n+1}, \eta^2_{4n+1}] = 4E\bar{\tau}_{4n}.$$

Let ι_X be the identity class of a space X. Denote by $\mathrm{P}^n(2)$ the Moore space of type $(\mathbb{Z}_2, n-1)$ and by $i_n\colon \mathbb{S}^{n-1} \hookrightarrow \mathrm{P}^n(2)$, $p_n\colon \mathrm{P}^n(2) \to \mathbb{S}^n$ the inclusion and collapsing maps, respectively. We recall from [37, p. 307, Corollary] that

(4.9) $$2\iota_{\mathrm{P}^n(2)} = i_n\eta_{n-1}p_n, \text{ if } n \geq 3.$$

Let $\bar{\eta}_n \in [\mathrm{P}^{n+2}(2), \mathbb{S}^n] \cong \mathbb{Z}_4$ and $\tilde{\eta}_n \in \pi_{n+2}(\mathrm{P}^{n+1}(2)) \cong \mathbb{Z}_4$ for $n \geq 3$ be an extension and a coextension of η_n, respectively. We note that

(4.10) $$\bar{\eta}_n \in \{\eta_n, 2\iota_{n+1}, p_{n+1}\}, \text{ if } n \geq 3$$

and

(4.11) $$\tilde{\eta}_n \in \{i_{n+1}, 2\iota_n, \eta_n\}, \text{ if } n \geq 3.$$

We have

(4.12) $$2\bar{\eta}_n = \eta_n^2 p_{n+2} \text{ and } 2\tilde{\eta}_n = i_{n+1}\eta_n^2, \text{ if } n \geq 3.$$

We recall that $\bar{\eta}_n\tilde{\eta}_{n+1} = \pm E^{n-3}\nu'$ for $n \geq 3$. Furthermore, we recall that $\pi_{n+8}(\mathbb{S}^n) = \{\varepsilon_n\} \cong \mathbb{Z}_2$ for $3 \leq n \leq 5$ and $\varepsilon_3 \in \{\eta_3, E\nu', \nu_7\}$. We need

Lemma 4.1 $\varepsilon_n = \{\eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}, \nu_{n+4}\}_{n-5}$ *for* $n \geq 5$.

PROOF. By the fact that $\tilde{\eta}_7 \in \{i_8, 2\iota_7, \eta_7\}$ and [36, Propositon 1.4],

$$\tilde{\eta}_7 \circ \nu_9 \in \{i_8, 2\iota_7, \eta_7\} \circ \nu_9 = i_8 \circ \{2\iota_7, \eta_7, \nu_8\} \subset i_8 \circ \pi_{12}(\mathbb{S}^7) = 0.$$

So, by [36, Proposition 1.2.(ii)], we can take

$$\varepsilon_5 \in \{\eta_5, 2\nu_6, \nu_9\} = \{\eta_5, \bar{\eta}_6\tilde{\eta}_7, \nu_9\} = \{\eta_5\bar{\eta}_6, \tilde{\eta}_7, \nu_9\}$$

and

$$\varepsilon_n = E^{n-5}\varepsilon_5 \in E^{n-5}\{\eta_5\bar{\eta}_6, \tilde{\eta}_7, \nu_9\} \subset \{\eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}, \nu_{n+4}\}_{n-5} \text{ if } n \geq 5.$$

The indeterminacy of the bracket $\{\eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}, \nu_{n+4}\}$ is $\eta_n\bar{\eta}_{n+1} \circ \pi_{n+8}(P^{n+3}(2)) + \pi_{n+5}(\mathbb{S}^n) \circ \nu_{n+5}$. Since $\eta_{n+4}\nu_{n+5} = 0$ (2.4) and $\pi_{n+5}(\mathbb{S}^n) = \{\nu_n\eta_{n+3}^2\}$ if $n \geq 5$, we obtain $\pi_{n+5}(\mathbb{S}^n) \circ \nu_{n+5} = 0$. By use of the homotopy exact sequence of a pair $(P^{n+3}(2), S^{n+2})$, we obtain $\pi_{n+8}(P^{n+3}(2)) = \{i_{n+3}\nu_{n+2}^2\}$. So $\bar{\eta}_{n+1} \circ \pi_{n+8}(P^{n+3}(2)) = \{\eta_{n+1}\nu_{n+2}^2\} = 0$, and hence $\eta_n\bar{\eta}_{n+1} \circ \pi_{n+8}(P^{n+3}(2)) = 0$. Thus, the indeterminacy is trivial. This completes the proof.

Although the following result is directly obtained from [13, Table 2], we show

Theorem 4.2 $[\iota_n, \eta_n\varepsilon_{n+1}] = 0$ *if* $n \equiv 1 \pmod 8$ *and* $n \geq 9$.

PROOF. For $n = 9$, the assertion is obtained in [17, p. 336]. By [16, p. 161] and Lemma 3.5.(2), we get that

$$\pi_{n+3}(SO(n)) = 0$$

and

$$\pi_{n+4}(SO(n)) = \{[\nu_{n-2}]_n \nu_{n+1}\} \cong \mathbb{Z}_2.$$

We consider the exact sequence $(\mathcal{SO}^n_{n+1})$:

$$0 \longrightarrow \pi_{n+2}(\mathbb{S}^n) \xrightarrow{\Delta} \pi_{n+1}(SO(n)) \xrightarrow{i_*} \pi_{n+1}(SO(n+1)) \longrightarrow 0,$$

where $\pi_{n+1}(SO(n)) \cong \mathbb{Z}_8$ and $\pi_{n+1}(SO(n+1)) = \{\tau'_{n+1}\} \cong \mathbb{Z}_4$ (4.2). By (4.3), $i_n(\mathbb{R})\bar{\tau}'_{n-1}$ becomes a generator of $\pi_{n+1}(SO(n))$ and we have $4i_n(\mathbb{R})\bar{\tau}'_{n-1} = \Delta(\eta^2_n)$. Hence, we obtain $\Delta\eta_n \circ \eta_n\bar{\eta}_{n+1} = 0$ and we can define a Toda bracket $\{\Delta\eta_n, \eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}\}_{n-5} \subset \pi_{n+5}(SO(n))$. By [36, the second formula in Proposition 1.6 and Proposition 1.2.0)] and the relation $2(\eta_5\bar{\eta}_6) = 0$, we obtain

$$\begin{aligned} 2\{\Delta\eta_n, \eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}\}_{n-5} &= \{\Delta\eta_n, E^{n-5}(2(\eta_5\bar{\eta}_6)), E^{n-5}\tilde{\eta}_7\}_{n-5} \\ &= \Delta\eta_n \circ E^{n-5}\pi^5_{10} + [P^{n+4}(2), SO(n)] \circ \tilde{\eta}_{n+3}. \end{aligned}$$

Since $E^{n-5}\pi^5_{10} = \{E^{n-5}(\nu_5\eta^2_8)\} = 0$, we have $\Delta\eta_n \circ E^{n-5}\pi^5_{10} = 0$. By the fact that $\pi_{n+3}(SO(n)) = 0$ and the relation $\nu_{n+1}\eta_{n+4} = 0$, we obtain $[P^{n+4}(2), SO(n)] \circ \tilde{\eta}_{n+3} = \pi_{n+4}(SO(n)) \circ \eta_{n+4} = 0$. This implies

$$(*) \qquad 2\{\Delta\eta_n, \eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}\}_{n-5} = 0.$$

In virtue of [5, Theorem 2], [6] and [13, Table 1],

(4.13) $\pi_{n+4}(SO(n)) \cong \mathbb{Z}_{8d}$, where $d = 2$ or 1 according as
$n \equiv 2 \pmod 8$ and $n \geq 18$ or $n \equiv 6 \pmod 8$ and $n \geq 14$

and $\pi_{n+5}(SO(n)) \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_2$. By use of the exact sequence $(\mathcal{SO}^n_{n+5})$, we see that the direct summand $\mathbb{Z}_2$ is generated by $\Delta(\nu^2_n)$. So, by $(*)$, $\{\Delta\eta_n, \eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}\}_{n-5}$ contains possibly $\Delta(\nu^2_n)$ (mod $8\pi_{n+5}(SO(n))$. By Lemma 4.1 and [36, Proposition 1.4],

$$\Delta(\eta_n\varepsilon_{n+1}) = \Delta\eta_n \circ \varepsilon_n \in \{\Delta\eta_n, \eta_n\bar{\eta}_{n+1}, \tilde{\eta}_{n+2}\}_{n-5} \circ \nu_{n+4}.$$

Thus, we obtain $\Delta(\eta_n\varepsilon_{n+1}) = a\Delta(\nu^3_n)$ for $a \in \{0, 1\}$.

Suppose that $[\iota_n, \eta_n\varepsilon_{n+1}] \neq 0$. Then, $[\iota_n, \eta_n\varepsilon_{n+1}] = [\iota_n, \nu^3_n]$. On the other hand, by [31, Proposition 4.2], $[\iota_n, \eta_n\varepsilon_{n+1}] = b[\iota_n, \eta^2_n\sigma_{n+2}]$ for $b \in \{0, 1\}$. The assumption induces the equality $[\iota_n, \eta_n\varepsilon_{n+1}] = [\iota_n, \eta^2_n\sigma_{n+2}]$. Then, we have

$[\iota_n, \eta_n\varepsilon_{n+1}] = [\iota_n, \nu_n^3] + [\iota_n, \eta_n^2\sigma_{n+2}] = 2[\iota_n, \eta_n\varepsilon_{n+1}] = 0$. This completes the proof.

Since $\pi_{4n}(SO(4n)) \cong (\mathbb{Z}_2)^3$ or $(\mathbb{Z}_2)^2$, if $n \geq 2$ [16, p. 161], we obtain

(4.14) $$\sharp\tau'_{4n} = 2, \text{ if } n \geq 2.$$

Next, we show

Lemma 4.3 *If* $n \equiv 0, 1 \pmod 4$ *and* $n \geq 8$ *then* $[\iota_n, \alpha] \neq 0$ *for* $\alpha = \varepsilon_n, \bar{\nu}_n, \eta_n\sigma_{n+1}$ *and* μ_n.

PROOF. We show $[\iota_n, \varepsilon_n] \neq 0$. Let $n \equiv 0 \pmod 4$ and $n \geq 8$. By [36, Proposition 11.10.(i)], there exists an element $\beta \in \pi^{n-1}_{2n+6}$ such that $E\beta = [\iota_n, \varepsilon_n]$ and $H\beta = \eta_{2n-3}\varepsilon_{2n-2}$. Suppose that $[\iota_n, \varepsilon_n] = 0$. Then, by $(\mathcal{PE}^{n-1}_{2n+6})$, we have $\beta \in P\pi^{2n-1}_{2n+8}$. This induces a contradictory relation $\eta_{2n-3}\varepsilon_{2n-2} = 0$, and hence $[\iota_n, \varepsilon_n] \neq 0$. Next, consider the case $n \equiv 1 \pmod 4$ and $n \geq 9$. Then, by (4.6), $[\iota_n, \varepsilon_n] = E(\tau_{n-1}\varepsilon_{2n-2})$ and $H(\tau_{n-1}\varepsilon_{2n-2}) = \eta_{2n-3}\varepsilon_{2n-2}$. Suppose that $[\iota_n, \varepsilon_n] = 0$. Then, $(\mathcal{PE}^{n-1}_{2n+6})$, (3.8) and (4.6) lead to a contradictory relation $\eta_{2n-3}\varepsilon_{2n-2} = 0$, and so $[\iota_n, \varepsilon_n] \neq 0$. For other elements, the argument goes ahead similarly.

By (1.3) and Lemma 4.3, $\Delta\colon \pi_{n+8}(\mathbb{S}^n) \to \pi_{n+7}(SO(n))$ is a monomorphism, if $n \equiv 0, 1 \pmod 4$ and $n \geq 12$. So, by $(\mathcal{SO}^n_{n+8})$, we obtain the exact sequence

(4.15) $$\pi_{n+9}(\mathbb{S}^n) \xrightarrow{\Delta} \pi_{n+8}(SO(n)) \xrightarrow{i_*} \pi_{n+8}(SO(n+1)) \longrightarrow 0,$$
$$\text{if } n \equiv 0, 1 \pmod 4 \text{ and } n \geq 12.$$

By (2.9) and [36, Lemma 12.10],

(4.16) $$\sigma'\nu_{14}^3 = \eta_7\bar{\varepsilon}_8.$$

(4.16) and [36, Theorem 12.6] yield
$[\iota_8, \eta_8^2\sigma_{10}] = (E\sigma')(\eta_{15}\varepsilon_{16} + \nu_{15}^3) = \eta_8\bar{\varepsilon}_9 + E^2\zeta' \neq 0$.

By (2.8), (2.3) and (2.9), $[\iota_9, \eta_9^2\sigma_{11}] = (\eta_9^2\sigma_{11} + \sigma_9\eta_{16}^2) \circ (\eta_{18}\sigma_{19}) = 0$.

The formula (2.2) and [23, Theorem C] yield

$$(4.17) \qquad \sharp[\iota_n, \eta_n^2\sigma_{n+2}] = \begin{cases} 1, & \text{if } n \equiv 2,3 \pmod 4 \text{ and } n \geq 6; \\ 2, & \text{if } n \equiv 0 \pmod 4 \text{ and } n \geq 8 \end{cases}$$

and

$$(4.18) \qquad \sharp[\iota_n, \eta_n^2\sigma_{n+2}] = 2, \text{ if } n \equiv 1 \pmod 8 \text{ and } n \geq 17.$$

Now, we conclude

Proposition 4.4 $[\iota_n, \nu_n^3] = 0$ *if* $n \equiv 5 \pmod 8$ *and* $[\iota_n, \eta_n\varepsilon_{n+1}] = [\iota_n, \eta_n^2\sigma_{n+2}] = 0$ *provided* $n \equiv 5 \pmod 8$ *and* $n \geq 13$ *unless* $n \equiv 53 \pmod{64}$.

PROOF. By (3.3) and Lemma 3.3.(1), $\Delta(\nu_n^3) = 0$ if $n \equiv 5 \pmod 8$. So, the first assertion holds. In light of [24, (7.9)], the second assertion holds for $n = 13$. Let $n \equiv 5 \pmod 8$ and $n \geq 21$. We consider the exact sequence (4.15). By [5, Theorem 2], [6] and [13, Table 1], we see that

$$\pi_{n+8}(SO(n+1)) \cong \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_2, & \text{if } n \equiv 5 \pmod{32} \text{ and } n \geq 37; \\ (\mathbb{Z}_4)^2, & \text{if } n \equiv 21 \pmod{32}; \\ \mathbb{Z}_4, & \text{if } n \equiv 13 \pmod{16} \end{cases}$$

and

$$\pi_{n+8}(SO(n)) \cong \begin{cases} \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2, & \text{if } n \equiv 5 \pmod{32} \text{ and } n \geq 37; \\ (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2, & \text{if } n \equiv 21 \pmod{64}; \\ \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2, & \text{if } n \equiv 53 \pmod{64}; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2, & \text{if } n \equiv 13 \pmod{16}. \end{cases}$$

By (3.3) and (4.5), we obtain

$$\Delta(\eta_n^2\sigma_{n+2}) = 4i_n(\mathbb{R})\bar{\tau}'_{n-1}\sigma_{n+1}$$

and hence

$$\Delta(\eta_n^2\sigma_{n+2}) = \begin{cases} 0, & \text{if } n \not\equiv 53 \pmod{64}; \\ 4i_n(\mathbb{R})\bar{\tau}'_{n-1}\sigma_{n+1} \neq 0, & \text{if } n \equiv 53 \pmod{64}. \end{cases}$$

This leads to the second assertion and the proof is complete.

Next, we show the following

Lemma 4.5 *Let* $n \equiv 1 \pmod 4$ *and* $n \geq 5$. *Then* $E(\bar{\tau}_{2n-2}\nu^2_{4n-2}) = [\iota_{2n-1}, \bar{\nu}_{2n-1}]$ *if and only if* $[\iota_{2n+1}, \nu^2_{2n+1}] = 0$.

PROOF. By (4.7), $E^3(\bar{\tau}_{2n-2}\nu^2_{4n-2}) = [\iota_{2n+1}, \nu^2_{2n+1}]$ and this implies the necessary condition.

Suppose that $[\iota_{2n+1}, \nu^2_{2n+1}] = 0$. Then, by $(\mathcal{PE}^{2n}_{4n+6})$,

$$\pi^{4n+1}_{4n+8} \xrightarrow{P} \pi^{2n}_{4n+6} \xrightarrow{E} \pi^{2n+1}_{4n+7},$$

$E^2(\bar{\tau}_{2n-2}\nu^2_{4n-2}) \in P\pi^{4n+1}_{4n+8} \cong \mathbb{Z}_{16}$. We can set $E^2(\bar{\tau}_{2n-2}\nu^2_{4n-2}) = 8xP(\sigma_{4n+1})$ for $x \in \{0, 1\}$.

Apply [36, Proposition 11.11.(ii)] to the case $\alpha = 8\sigma_{4n-6}$, then there exists an element $\beta \in \pi^{2n-2}_{4n+4}$ such that

$$P(8\sigma_{4n+1}) = E^2\beta \quad \text{and} \quad H(\beta) \in \{\eta_{4n-5}, 2\iota_{4n-4}, 8\sigma_{4n-4}\}_2.$$

By [36, Lemma 6.5, Theorem 7.1] and (2.7),

$$\mu_{4n-5} \in \{\eta_{4n-5}, 2\iota_{4n-4}, 8\sigma_{4n-4}\}_2 \bmod \eta_{4n-5} \circ E^2\pi^{4n-6}_{4n+2} = \{\nu^3_{4n-5}, \eta_{4n-5}\varepsilon_{4n-4}\}.$$

So we obtain

$$H(\beta) = \mu_{4n-5} + y\nu^3_{4n-5} + z\eta_{4n-5}\varepsilon_{4n-4} \ (y, z \in \{0, 1\}).$$

By using $(\mathcal{PE}^{2n-1}_{4n+5})$ and the assumption,

$$E(\bar{\tau}_{2n-2}\nu^2_{4n-2}) - xE\beta \in P\pi^{4n-1}_{4n+7} = \{P(\bar{\nu}_{4n-1}), P(\varepsilon_{4n-1})\}.$$

By Lemma 4.1, $P(\bar{\nu}_{4n-1}) = E(\tau_{2n-2}\bar{\nu}_{4n-4})$ and $P(\varepsilon_{4n-1}) = E(\tau_{2n-2}\varepsilon_{4n-4})$. So, by using $(\mathcal{PE}^{2n-2}_{4n+4})$,

$$\bar{\tau}_{2n-2}\nu^2_{4n-2} - x\beta - a\tau_{2n-2}\bar{\nu}_{4n-4} - b\tau_{2n-2}\varepsilon_{4n-4} \in P\pi^{4n-3}_{4n+6} \ (a, b \in \{0, 1\}).$$

By applying $H\colon \pi^{2n-2}_{4n+5} \to \pi^{4n-5}_{4n+5}$ to the equation, by use of (4.6), (4.7) and (2.7), we obtain

$$\nu^3_{4n-5} - x(\mu_{4n-5} + y\nu^3_{4n-5} + z\eta_{4n-5}\varepsilon_{4n-4}) = a\nu^3_{4n-5} + b\eta_{4n-5}\varepsilon_{4n-4}.$$

Since $\mu_{4n-5}, \nu^3_{4n-5}, \eta_{4n-5}\varepsilon_{4n-4}$ generate π^{4n-5}_{4n+4} independently, we have $x = 0, a = 1$ and $b = 0$. Hence, $E(\bar{\tau}_{2n-2}\nu^2_{4n-2}) = E(\tau_{2n-2}\bar{\nu}_{4n-4})$. This completes the proof.

Since $\nu_n\eta_{n+3} = 0$ (2.4) and $\bar{\nu}_n\eta_{n+8} = \nu^3_n$ (2.7) for $n \geq 6$, Lemma 4.5 implies

Corollary 4.6 *If* $[\iota_{8n+3}, \nu^2_{8n+3}] = 0$, *then* $[\iota_{8n+1}, \nu^3_{8n+1}] = 0$.

Now, we show

III. $[\iota_n, \nu_n^2] = 0$ **if** $n = 2^i - 5$ $(i \geq 4)$**.**

We recall the Mahowald element $\eta'_i \in \pi^S_{2^i}(\mathbb{S}^0)$ [22, Theorem 1] for $i \geq 3$. We set $\eta'_{i-1,m} = \eta'_{i-1}$ on $\mathbb{S}^m$ for $m = 2^{i-1} - 2$ with $i \geq 4$, that is, $\eta'_{i-1,m} \in \pi_{2^{i-1}+m}(\mathbb{S}^m)$. It satisfies the relation $H(\eta'_{i-1,m}) = \nu_{2m-1}$. Then, the assertion follows directly from [3, Proposition] taking $\alpha = \beta = \eta'_{i-1,m}$.

Finally, we show

IV. $[\iota_n, \nu_n^2] \neq 0$ **if** $n \equiv 3 \pmod 8$ **and** $n \geq 19$ **unless** $n = 2^i - 5$**.**

By III and Corollary 4.6, we obtain

$$[\iota_n, \nu_n^3] = 0, \quad \text{if} \quad n = 2^i - 7 \ (i \geq 4).$$

Hence, from Theorem 4.2 and the relation $\eta_n^2\sigma_{n+2} = \nu_n^3 + \eta_n\varepsilon_{n+1}$,

$$[\iota_n, \eta_n^2\sigma_{n+2}] = 0, \ \text{if } n = 2^i - 7 \, (i \geq 4).$$

Let $n \equiv 1 \pmod 8$ and $n \geq 17$. Considering the exact sequence (4.15), in virtue of [5, Theorem 2], [6] and [13, Table 1], we obtain

$$\pi_{n+8}(SO(n)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8 \quad \text{and} \quad \pi_{n+8}(SO(n+1)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

By (4.8) and (4.18), we get the relation

$$4E(\bar{\tau}_{n-1}\sigma_{2n}) = [\iota_n, \eta_n^2\sigma_{n+2}] \neq 0.$$

Hence, by (4.18) and Theorem 4.2, we obtain

$$[\iota_n, \nu_n^3] = [\iota_n, \eta_n^2\sigma_{n+2}] \neq 0, \ \text{if } n \equiv 1 \pmod 8 \text{ and } n \geq 17 \text{ and } n \neq 2^i - 7.$$

Thus, by Corollary 4.6, we obtain the assertion.

We are in a position to assert that Mahowald's result [21, Table 2 for $\eta^2\rho_1$] should be stated as follows.

Theorem 4.7 *Let* $n \equiv 1 \pmod 8$ *and* $n \geq 9$*. Then* $[\iota_n, \eta_n^2\sigma_{n+2}] \neq 0$ *if and only if* $n \neq 2^i - 7$*.*

5 Proof of $[\iota_{16s+7}, \sigma_{16s+7}] \neq 0$ for $s \geq 1$

We give a proof of the first part of Theorem 2.4: $[\iota_{16s+7}, \sigma_{16s+7}] \neq 0$ for $s \geq 1$.

We recall from [36, pp. 95-6] the construction of the element $\kappa_7 \in \pi_{21}(\mathbb{S}^7)$. It is a representative of a Toda bracket

$$\{\nu_7, E\alpha, E^2\beta\}_1,$$

where $\alpha = \bar{\eta}_9 \in [\mathrm{P}^{11}(2), \mathbb{S}^9]$ is an extension of η_9 and $\beta = \tilde{\bar{\nu}}_9 \in \pi_{18}(\mathrm{P}^{10}(2))$ is a coextension of $\bar{\nu}_9$ satisfying $\alpha \circ E\beta = 0$. Furthermore, $\kappa_n = E^{n-7}\kappa_7$ for $n \geq 7$ and set $\tilde{\bar{\nu}}_n = E^{n-9}\tilde{\bar{\nu}}_9$ for $n \geq 9$. Then, we can take

$$\kappa_n \in \{\nu_n, \bar{\eta}_{n+3}, \tilde{\bar{\nu}}_{n+4}\} \text{ for } n \geq 7.$$

By [16, p. 161], $\pi_{n+4}(SO(n+k)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ for $k = 1, 2$ if $n \equiv 7 \pmod 8$. And, by $(\mathcal{SO}^{n+2}_{n+4})$, the direct summand $\mathbb{Z}_2$ of $\pi_{n+4}(SO(n+2))$ is generated by $\Delta\nu_{n+2}$. So, the non-triviality of $[\nu_n]\eta_{n+3} \in \pi_{n+4}(SO(n+1))$ induces the relation $i_{n+2}(\mathbb{R})_*([\nu_n]\eta_{n+3}) = \Delta\nu_{n+2}$. Because of the fact that $[\iota_{n+2}, \nu^2_{n+2}] \neq 0$, this induces a contradictory relation $0 = \Delta\nu^2_{n+2} \neq 0$. Hence, we obtain

$$[\nu_n]\eta_{n+3} = 0, \text{ if } n \equiv 7 \pmod 8.$$

Next, by [16, p. 161],

$$\{[\nu_n], \eta_{n+3}, 2\iota_{n+4}\} \subset \pi_{n+5}(SO(n+1)) = 0, \text{ if } n \equiv 7 \pmod 8.$$

So, by (4.10), we have $[\nu_n]\bar{\eta}_{n+3} \in \{[\nu_n], \eta_{n+3}, 2\iota_{n+4}\} \circ p_{n+5} = 0$ and hence we can define a lift of κ_n for $n \equiv 7 \pmod 8$, as follows:

$$[\kappa_n] \in \{[\nu_n], \bar{\eta}_{n+3}, \tilde{\bar{\nu}}_{n+4}\} \subset \pi_{n+14}(SO(n+1)) \text{ for } n \equiv 7 \pmod 8.$$

Let $n \equiv 7 \pmod 8$ and $n \geq 15$. By use of $(\mathcal{SO}^{n-k}_{n-4})$ for $k = 3, 4$, $(\mathcal{SO}^{n-l}_{n-3})$ for $l = 2, 3, 5$, $(\mathcal{SO}^{n-m}_{n-2})$ for $2 \leq m \leq 5$ and [16, p. 161], we obtain

$$\pi_{n-4}(SO(n-4)) = \{\beta\} \cong \mathbb{Z};\ \pi_{n-4}(SO(n-3)) = \{i_{n-3}(\mathbb{R})\beta, \Delta\iota_{n-3}\} \cong (\mathbb{Z})^2;$$

$$\pi_{n-3}(SO(n-4)) = \{[\eta^2_{n-5}]\} \cong \mathbb{Z}_2;\ \pi_{n-3}(SO(n-3)) = \{[\eta_{n-4}], \Delta\eta_{n-3}\} \cong (\mathbb{Z}_2)^2;$$

$$\pi_{n-2}(SO(n-4)) = \{[\eta^2_{n-5}]\eta_{n-3}, \Delta\nu_{n-4}\} \cong (\mathbb{Z}_2)^2;$$

$$\pi_{n-2}(SO(n-3)) = \{[\eta_{n-4}]\eta_{n-3}, \Delta\eta^2_{n-3}\} \cong (\mathbb{Z}_2)^2;\ \pi_{n-2}(SO(n-2)) = \{\Delta\eta_{n-2}\} \cong \mathbb{Z}_2,$$

where β is a generator of $\pi_{n-4}(SO(n-4))$ and

$$\Delta\eta_{n-3} = [\eta^2_{n-5}]_{n-3}. \tag{5.1}$$

We need

$$\{p_n(\mathbb{R}), i_n(\mathbb{R}), \Delta\iota_{n-1}\} \ni \iota_{n-1} \pmod{2\iota_{n-1}} \text{ for } n \geq 9. \tag{5.2}$$

By the same reason as (3.1), we obtain $\Delta(\bar{\eta}_3) = 0 \in [\mathrm{P}^4(2), SO(3)]$. Let $n \equiv 7 \pmod 8$ and $n \geq 15$. Then, by Lemma 3.1.(1) and (4.10), we obtain

$$\Delta(\bar{\eta}_{n-4}) = \Delta\iota_{n-4} \circ \bar{\eta}_{n-5} \in -\{\Delta\iota_{n-4}, \eta_{n-5}, 2\iota_{n-4}\} \circ p_{n-3} = 0.$$

So, $\bar{\eta}_{n-4}$ is lifted to $[\bar{\eta}_{n-4}] \in [\mathrm{P}^{n-2}(2), SO(n-3)]$ for $n \equiv 7 \pmod 8$. We set $[\bar{\eta}_{n-4}] \circ i_{n-2} = [\eta_{n-4}]$, which is a lift of η_{n-4}. By (5.1) and (5.2), we get

(5.3) $[\eta_{n-4}] \in \{i_{n-3}(\mathbb{R}), \Delta\iota_{n-4}, \eta_{n-5}\}$ (mod $i_{n-3}(\mathbb{R}) \circ \pi_{n-3}(SO(n-4))$ $+ \pi_{n-4}(SO(n-3)) \circ \eta_{n-4} = \{\Delta\eta_{n-3}\}$) for $n \equiv 7 \pmod 8$ and $n \geq 15$.

By use of the cofiber sequence $\mathbb{S}^{n-3} \xrightarrow{i_{n-2}} \mathrm{P}^{n-2}(2) \xrightarrow{p_{n-2}} \mathbb{S}^{n-2}$ and the relation $[\bar{\eta}_{n-4}] \circ i_{n-2} = [\eta_{n-4}]$, we obtain

(5.4) $\overline{[\eta_{n-4}]} \equiv [\bar{\eta}_{n-4}]$ (mod $\pi_{n-2}(SO(n-3)) \circ p_{n-2} = 2[P^{n-2}(2), SO(n-3)]$).

We show

Lemma 5.1 *Let* $n \equiv 7 \pmod 8$ *and* $n \geq 15$*. Then,*

(1) $\overline{[\eta_{n-4}]} \in \{i_{n-3}(\mathbb{R}), \Delta\iota_{n-4}, \bar{\eta}_{n-5}\}$ (mod $\{\Delta(\bar{\eta}_{n-3})\} + K$), *where* $K = i_{n-3}(\mathbb{R})_*[\mathrm{P}^{n-2}(2), SO(n-4)] + \pi_{n-4}(SO(n-3)) \circ \bar{\eta}_{n-4}$;

(2) $i_{n-2}(\mathbb{R})_* K \subset \{(\Delta\eta_{n-2})p_{n-2}\}$.

PROOF. By (4.9), (5.4) and (5.3), we have (1).

We see that $[\mathrm{P}^{n-2}(2), SO(n-4)] = \{\overline{[\eta^2_{n-5}]}, (\Delta\nu_{n-4})p_{n-2}\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, where $\overline{[\eta^2_{n-5}]}$ is an extension of $[\eta^2_{n-5}]$ and $2\overline{[\eta^2_{n-5}]} = [\eta^2_{n-5}]\eta_{n-3}p_{n-2}$. Hence, by (5.1),

$$i_{n-4,n-2*}\overline{[\eta^2_{n-5}]} \in i_{n-2}(\mathbb{R}) \circ \{\Delta\eta_{n-3}, 2\iota_{n-3}, p_{n-3}\} =$$

$$-\{i_{n-2}(\mathbb{R}), \Delta\eta_{n-3}, 2\iota_{n-3}\} \circ p_{n-2}.$$

Since $\{i_{n-2}(\mathbb{R}), \Delta\eta_{n-3}, 2\iota_{n-3}\} \subset \pi_{n-2}(SO(n-2)) = \{\Delta\eta_{n-2}\}$, we have $i_{n-4,n-2*}[\mathrm{P}^{n-2}(2), SO(n-4)] \subset \{(\Delta\eta_{n-2})p_{n-2}\}$.

From the relation $p_{n-3}(\mathbb{R})\beta = 0$, we obtain $\beta\eta_{n-4} = 0 \in \pi_{n-3}(SO(n-4))$. So, by (4.10), we have $\beta\bar{\eta}_{n-4} \in \{\beta, \eta_{n-4}, 2\iota_{n-3}\} \circ p_{n-2} \subset \pi_{n-2}(SO(n-2)) \circ p_{n-2}$. Hence,we obtain $i_{n-2}(\mathbb{R})_*(\pi_{n-4}(SO(n-3)) \circ \bar{\eta}_{n-4}) \subset \{(\Delta\eta_{n-2})p_{n-2}\}$. This leads to (2) and completes the proof.

We show

Lemma 5.2 $[\kappa_{n-8}]_{n-1} = \Delta\bar{\nu}_{n-1}$ *if* $n \equiv 7 \pmod 8$ *and* $n \geq 15$.

PROOF. By use of $(\mathcal{SO}^{n-7+k}_{n-5})$ for $0 \leq k \leq 3$ and [16, p. 161], we have $[\nu_{n-8}]_{n-4} = \Delta\iota_{n-4}$, and so

$$[\kappa_{n-8}]_{n-1} \in (i_{n-4,n-1})_*\{\Delta\iota_{n-4}, \bar{\eta}_{n-5}, \tilde{\bar{\nu}}_{n-4}\}.$$

By (5.4) and Lemma 5.1, we obtain

$$\begin{aligned} i_{n-3}(\mathbb{R})_*\{\Delta\iota_{n-4}, \bar{\eta}_{n-5}, \tilde{\bar{\nu}}_{n-4}\} &= -\{i_{n-3}(\mathbb{R}), \Delta\iota_{n-4}, \bar{\eta}_{n-5}\} \circ \tilde{\bar{\nu}}_{n-3} \\ &\equiv \overline{[\eta_{n-4}]} \circ \tilde{\bar{\nu}}_{n-3} \in \{[\eta_{n-4}], 2\iota_{n-3}, \bar{\nu}_{n-3}\} \\ (\text{mod } [\eta_{n-4}] \circ \pi_{n+6}(\mathbb{S}^{n-3}) &+ \pi_{n-2}(SO(n-3)) \circ \bar{\nu}_{n-2} + K \circ \tilde{\bar{\nu}}_{n-3}). \end{aligned}$$

By Lemma 5.1 and (3.6), $i_{n-2}(\mathbb{R})_*(K \circ \tilde{\bar{\nu}}_{n-3}) \subset \{\Delta\eta_{n-2}\} \circ \bar{\nu}_{n-3} = \{\Delta\nu^3_{n-2}\} = 0$. From the relation $[\eta_{n-4}]_{n-2} = \Delta\iota_{n-2}$, we see that

$$[\kappa_{n-8}]_{n-2} \in \{\Delta\iota_{n-2}, 2\iota_{n-3}, \bar{\nu}_{n-3}\} \ (\text{mod } \Delta\pi_{n+7}(\mathbb{S}^{n-2}))$$

and

$$\begin{aligned} [\kappa_{n-8}]_{n-1} &\in -i_{n-1}(\mathbb{R}) \circ \{\Delta\iota_{n-2}, 2\iota_{n-3}, \bar{\nu}_{n-3}\} \\ &= \{i_{n-1}(\mathbb{R}), \Delta\iota_{n-2}, 2\iota_{n-3}\} \circ \bar{\nu}_{n-2}. \end{aligned}$$

Since $\{i_{n-1}(\mathbb{R}), \Delta\iota_{n-2}, 2\iota_{n-3}\} \equiv \Delta\iota_{n-1}$ (mod $2\Delta\iota_{n-1}$) by (5.2), we have

$$\{i_{n-1}(\mathbb{R}), \Delta\iota_{n-2}, 2\iota_{n-3}\} \circ \bar{\nu}_{n-2} = \Delta\bar{\nu}_{n-1}.$$

This completes the proof.

Hereafter, we fix $n = 16s + 7 \geq 23$. Suppose that $E^7(\gamma\sigma_{2n-8}) = [\iota_n, \sigma_n] = 0$, where γ is the element in (2.14). Then, by $(\mathcal{PE}^{n-1}_{2n+5})$ and Lemma 5.2, $E^6(\gamma\sigma_{2n-8}) \in \{[\iota_{n-1}, \bar{\nu}_{n-1}] = E^6J[\kappa_{n-7}], [\iota_{n-1}, \eta_{n-1}\sigma_n]\}$.

By [29, p. 382: Table], there exists an element $\delta \in \pi^{n-8}_{2n-10}$ such that

$$(5.5) \qquad [\iota_{n-1}, \eta_{n-1}] = E^7\delta \text{ and } H\delta = \sigma_{2n-17}$$

and so, $[\iota_{n-1}, \eta_{n-1}\sigma_n]$ desuspends until we reach seven dimensions. Hence, in the sequel argument, it suffices to consider $E^6(\gamma\sigma_{2n-8}) = aE^6J[\kappa_{n-7}]$ for $a \in \{0, 1\}$. By $(\mathcal{PE}^{n-2}_{2n+4})$, we have

$$E^5(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}]) \in P\pi^{2n-3}_{2n+6}.$$

By Lemma 4.3 and Proposition 4.4, $P\mu_{2n-3} \neq 0$ and $P(\nu^3_{2n-3}) = 0$. By [29, p. 383: Table], $[\iota_{n-2}, \eta^2_{n-2}]$ and $[\iota_{n-2}, \eta^2_{n-2}\sigma_n]$ desuspend until 7 dimensions. Hence, for $x \in \{0, 1\}$, we have

$$E^5(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}]) = xP\mu_{2n-3}.$$

By [36, Proposition 11.10.(ii)], there exists an element $\beta \in \pi^{n-3}_{2n+3}$ such that $P\mu_{2n-3} = E\beta$ and $H\beta = \eta_{2n-7}\mu_{2n-6}$. Then, by $(\mathcal{PE}^{n-3}_{2n+3})$, we have

$$E^4(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}]) - x\beta \in P\pi^{2n-5}_{2n+5}.$$

This induces the relation $x\eta_{2n-7}\mu_{2n-6} = 0$. Hence, $x = 0$ and we can set

$$E^4(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}]) = yP(\eta_{2n-5}\mu_{2n-4}) \text{ for } y \in \{0, 1\}.$$

By [36, Proposition 11.10.(i)], there exists an element $\beta' \in \pi^{n-4}_{2n+2}$ such that $P(\eta_{2n-5}\mu_{2n-4}) = E\beta'$ and $H\beta' = \eta^2_{2n-9}\mu_{2n-7}$. So, we have

$$E^3(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}]) - y\beta' \in P\pi^{2n-7}_{2n+4}.$$

This leads to the relation $y\eta^2_{2n-9}\mu_{2n-7} = 0$, and hence $y = 0$. Therefore, by (4.7), we obtain

$$E^3(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}] - b\bar{\tau}_{n-7}\zeta_{2n-12}) = 0 \ (b \in \{0, 1\}).$$

By $(\mathcal{PE}^{n-5-k}_{2n+1-k})$ for $k = 0, 1$ and 2, we have

$$E^2(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}] - b\bar{\tau}_{n-7}\zeta_{2n-12}) \in P\pi^{2n-9}_{2n+3} = 0$$

$$E(\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}] - b\bar{\tau}_{n-7}\zeta_{2n-12}) \in P\pi^{2n-11}_{2n+2} = 0$$

and

$$\gamma\sigma_{2n-8} - aJ[\kappa_{n-7}] - b\bar{\tau}_{n-7}\zeta_{2n-12} \in P\pi^{2n-13}_{2n+1}.$$

By (4.7) and [36, Lemma 9.2, Theorem 10.3], $H(\bar{\tau}_{n-7}\zeta_{2n-12}) = \pm(\frac{n-3}{4})\nu_{2n-15}\zeta_{2n-12} = \pm 2(n-3)\sigma^2_{2n-15} = 0$. Then, the last relation induces the contradictory relation $\sigma^2_{2n-15} = a\kappa_{2n-15}$. Thus, we obtain the non-triviality of $[\iota_n, \sigma_n]$ if $n \equiv 7 \pmod{16}$ and $n \geq 23$.

By Lemma 5.2, we have $[\iota_n, \bar{\nu}_n] = E^6J[\kappa_{n-7}]$ if $n \equiv 6 \pmod 8$ and $n \geq 14$. By the parallel arguments to the above, we obtain

Corollary 5.3 $[\iota_n, \bar{\nu}_n] \neq 0$, *if* $n \equiv 6 \pmod 8$ *and* $n \geq 14$.

6 Gottlieb groups of spheres with stems for $8 \leq k \leq 13$

By [36, Theorems 7.1,7.4,7.6, p. 186: Table], $\pi_{n+8}(\mathbb{S}^n) = \{\varepsilon_n\} \cong \mathbb{Z}_2$ for $n = 4, 5$ and $[\iota_4, \varepsilon_4] = (E\nu')\varepsilon_7 \neq 0$, $[\iota_5, \varepsilon_5] = \nu_5\eta_8\varepsilon_9 \neq 0$.

We recall $\pi_{14}(\mathbb{S}^6) = \{\bar{\nu}_6, \varepsilon_6, [\iota_6, \alpha_1(6)]\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2$. By [36, (7.27)],

$$(6.1) \qquad [\iota_6, \bar{\nu}_6] = [\iota_6, \varepsilon_6] = 0.$$

So, we obtain $G_{14}(\mathbb{S}^6;2) = \pi^6_{14}$. By Proposition 1.5.(1), $G_{14}(\mathbb{S}^6;3) = \pi_{14}(\mathbb{S}^6;3)$. This shows $G_{14}(\mathbb{S}^6) = \pi_{14}(\mathbb{S}^6)$.

We recall $\pi_{16}(\mathbb{S}^8) = \{\sigma_8\eta_{15}, (E\sigma')\eta_{15}, \bar{\nu}_8, \varepsilon_8\} \cong (\mathbb{Z}_2)^4$ and $\pi_{17}(\mathbb{S}^9) = \{\sigma_9\eta_{16}, \bar{\nu}_9, \varepsilon_9\} \cong (\mathbb{Z}_2)^3$. We have $[\iota_8, \sigma_8\eta_{15}] = (E\sigma')\sigma_{15}\eta_{22} = (E\sigma')(\bar{\nu}_{15}+\varepsilon_{15}) = [\iota_8, \bar{\nu}_8] + [\iota_8, \varepsilon_8]$. By (2.15) and [36, Theorem 12.6], $[\iota_9, \sigma_9\eta_{16}] = \sigma_9(\nu^3_{16} + \eta_{16}\varepsilon_{17}) \neq 0$. So, obtain $G_{16}(\mathbb{S}^8) = \{(E\sigma')\eta_{15}, \sigma_8\eta_{15} + \bar{\nu}_8 + \varepsilon_8\} \cong (\mathbb{Z}_2)^2$ and $G_{17}(\mathbb{S}^9) = \{[\iota_9, \iota_9]\} \cong \mathbb{Z}_2$. Hence, by Lemma 4.3, we get that

$$G_{n+8}(\mathbb{S}^n) = 0, \quad \text{if} \quad n \equiv 0,1 \pmod 4 \text{ and } n \geq 4 \text{ unless } n = 8,9.$$

Since $\pi_{27}(\mathbb{S}^{10}) \to \pi_{28}(\mathbb{S}^{11})$ is a monomorphism [36, (12.21)], we obtain

$$G_{18}(\mathbb{S}^{10}) = \pi_{18}(\mathbb{S}^{10}).$$

Let $n \equiv 3 \pmod 4$ and $n \geq 11$. Then, by Lemma 1.1.(1) and (2.1), $[\iota_n, \eta_n\sigma_{n+1}] = 0$. In virtue of (1.3) and Example 3.2, we obtain $[\iota_n, \varepsilon_n] = 0$. Thus, as it is expected in Proposition 1.3,

$$G_{n+8}(\mathbb{S}^n) = \pi_{n+8}(\mathbb{S}^n), \quad \text{if} \quad n \equiv 3 \pmod 4.$$

By Lemma 4.3 and [21, Theorem C],
(6.2)

$$\sharp[\iota_n, \eta_n\sigma_{n+1}] = \begin{cases} 2, & \text{if } n \equiv 0,1,2,4,5 \pmod 8 \text{ and } n \geq 8 \text{ unless } n = 10; \\ 1, & \text{if } n \equiv 3 \pmod 4 \text{ and } n \geq 7. \end{cases}$$

Here we recall from [4, p. 137, Corollary 1.6] and [7, p. 48: Theorem], the following

Theorem 6.1 (Barratt-Jones-Feder-Gitler-Lam-Mahowald) *Let β's generate the J-image in the s-stem and assume $3s-2 \leq 2n$. Then,*

(1) $[\iota_n, \beta] = 0$, *provided n and s satisfy $3 \leq \nu_2(n+s+2) \leq \phi(s)$;*

(2) $[\iota_n, \beta] \neq 0$ *provided n and s satisfy $\nu_2(n+s+2) \geq \phi(s)+1 \geq 3$, but $n+s+2 \neq 2^{\phi(s)+1}$.*

Here $\nu_2(m)$ is the exponent of 2 in the factorization of m and $\phi(s)$ denotes the number of integers in the closed interval $[1,s]$ which are congruent to $0,1,2$ or 4 modulo 8.

By use of Theorem 6.1, we obtain
(6.3)

$$\sharp[\iota_n, \eta_n\sigma_{n+1}] = \begin{cases} 2, & \text{if } n \equiv 22 \pmod{32} \text{ and } n \geq 54; \\ 1, & \text{if } n \equiv 14 \pmod{16} \text{ or } n \equiv 6 \pmod{32} \text{ and } n \geq 14 \end{cases}$$

and
(6.4)

$$\sharp[\iota_n, \eta_n^2\sigma_{n+1}] = \begin{cases} 2, & \text{if } n \equiv 53 \pmod{64} \text{ and } n \geq 117; \\ 1, & \text{if } n \equiv 13 \pmod{16},\ 5 \pmod{32} \text{ or } 21 \pmod{64} \text{ and } n \geq 13. \end{cases}$$

Now, we show

Lemma 6.2 (1) *Let* $n \equiv 2 \pmod 8$ *and* $n \geq 18$*. Then,* $\Delta\varepsilon_n = 0$.

(2) *Let* $n \equiv 6 \pmod 8$ *and* $n \geq 14$*. Then,* $\Delta\varepsilon_n = \pm 2[\nu_{n-2}^2]_n\nu_{n+4}$.

PROOF. Although (1) is directly obtained by [13, Table 2], we give a different proof.

Let $n \equiv 2 \pmod 4$ and $n \geq 18$. Then, by the fact that $\pi_{n+1}(SO(n)) \cong \mathbb{Z}$ [16, p. 161], we have $\tau'_n\eta_n = 0$. So, by (3.3), (4.12) and (4.2), we obtain

$$\Delta(\eta_n\bar{\eta}_{n+1}) = 2\tau'_n \circ \bar{\eta}_n = \tau'_n \circ \eta_n^2 p_{n+2} = 0.$$

Therefore, by Lemma 4.1, we get

$$\Delta\varepsilon_n = \Delta\iota_n \circ \varepsilon_{n-1} = \Delta\iota_n \circ \{\eta_{n-1}\bar{\eta}_n, \tilde{\eta}_{n+1}, \nu_{n+3}\} = -\{\Delta\iota_n, \eta_{n-1}\bar{\eta}_n, \tilde{\eta}_{n+1}\} \circ \nu_{n+4}.$$

We have

$$\{\Delta\iota_n, \eta_{n-1}\bar{\eta}_n, \tilde{\eta}_{n+1}\} \subset \pi_{n+4}(SO(n)).$$

Noting the relation $4\tilde{\eta}_{n+1} = 0$, we obtain

$$4\{\Delta\iota_n, \eta_{n-1}\bar{\eta}_n, \tilde{\eta}_{n+1}\} = -\Delta\iota_n \circ \{\eta_{n-1}\bar{\eta}_n, \tilde{\eta}_{n+1}, 4\iota_{n+3}\} \subset -\Delta\iota_n \circ \pi_{n+4}(\mathbb{S}^{n-1}) = 0.$$

This induces $\Delta\varepsilon_n \in (2d)(\pi_{n+4}(SO(n)) \circ \nu_{n+4})$, where d is the number in (4.13). Since $4\pi_{n+7}(SO(n)) = 0$ by [5, Theorem 2], [6] and [13, Table 1], we obtain (1).

Let $n \equiv 6 \pmod 8$ and $n \geq 14$. By the exact sequences $(\mathcal{SO}_{n+4}^{n+k})$ for $k = -2, -1$ and Lemma 3.3 we get that $i_n(\mathbb{R})_* \colon \pi_{n+4}(SO(n-1)) \to \pi_{n+4}(SO(n))$ is an isomorphism and $\pi_{n+4}(SO(n-1)) = \{[\nu_{n-2}^2]\} \cong \mathbb{Z}_8$.

By [13, Table 2], $\Delta\varepsilon_n \neq 0$ for $n \equiv 6 \pmod 8$ and $n \geq 14$. Hence, (2) follows and the proof is complete.

Now, by Lemma 6.2.(1) and (6.2),

$$[\iota_n, \varepsilon_n] = 0 \text{ and } [\iota_n, \bar{\nu}_n] = [\iota_n, \eta_n\sigma_{n+1}] \neq 0, \text{ if } n \equiv 2 \pmod 8 \text{ and } n \geq 18.$$

Whence, we conclude that

$$G_{n+8}(\mathbb{S}^n) = \{\varepsilon_n\} \cong \mathbb{Z}_2, \text{ if } n \equiv 2 \pmod 8 \text{ and } n \geq 18.$$

We show $[\iota_n, \varepsilon_n] \neq 0$ if $n \equiv 22 \pmod{32}$ and $n \geq 22$. By (5.5), there exists an element $\delta \in \pi^{n-7}_{2n-8}$ such that $[\iota_n, \eta_n] = E^7\delta$ and $H\delta = \sigma_{2n-15}$. Hence, by Lemma 5.2, $[\iota_n, \varepsilon_n] = E^6(J[\kappa_{n-7}] + E(\delta\sigma_{2n-7}))$. Suppose that $[\iota_n, \varepsilon_n] = 0$. Then, by the parallel argument to that in the proof the non-triviality of $[\iota_{n+1}, \sigma_{n+1}]$, we get a contradiction.

By [24, (7.13)], $\mathrm{Ker}\{P\colon \pi_{37}(\mathbb{S}^{29}) \to \pi_{35}(\mathbb{S}^{14})\} = \{\eta_{14}\sigma_{15}\}$ and hence, $G_{22}(\mathbb{S}^{14}) = \{\eta_{14}\sigma_{15}\} \cong \mathbb{Z}_2$. By [32, p. 134: (7.29)], $\mathrm{Ker}\{P\colon \pi^{45}_{53} \to \pi^{22}_{51}\} = \{\eta_{45}\sigma_{46}\}$ and hence, $G_{30}(\mathbb{S}^{22}) = \{\eta_{22}\sigma_{23}\} \cong \mathbb{Z}_2$. Thus, we have shown

Proposition 6.3 *The group $G_{n+8}(\mathbb{S}^n)$ is equal to the following: 0 if $n \equiv 0, 1 \pmod 4$ and $n \geq 4$ unless $n = 8, 9$ or $n \equiv 22 \pmod{32}$ and $n \geq 54$; $\pi_{n+8}(\mathbb{S}^n)$ if $n = 6, 10$ or $n \equiv 3 \pmod 4$; $\{\varepsilon_n\} \cong \mathbb{Z}_2$, if $n \equiv 2 \pmod 8$ and $n \geq 18$. Moreover, $G_{n+8}(\mathbb{S}^n) = \{\eta_n\sigma_{n+1}\} \cong \mathbb{Z}_2$ if $n = 22$, $n \equiv 14 \pmod{16}$ or $n \equiv 6 \pmod{32}$ with $n \geq 14$; $G_{16}(\mathbb{S}^8) = \{(E\sigma')\eta_{15}, \sigma_8\eta_{15} + \bar{\nu}_8 + \varepsilon_8\} \cong (\mathbb{Z}_2)^2$ and $G_{17}(\mathbb{S}^9) = \{[\iota_9, \iota_9]\} \cong \mathbb{Z}_2$.*

By [36, Theorem 7.6],

$$[\iota_4, \mu_4] = (E\nu')\mu_7 \neq 0. \tag{6.5}$$

We have $[\iota_5, \mu_5] = \nu_5\eta_8\mu_9 \neq 0$ [36, Theorem 7.7].

By [36, (10.6)],

$$[\iota_6, \mu_6] = 0. \tag{6.6}$$

We have $[\iota_8, \mu_8] = (E\sigma')\mu_{15} \neq 0$ [36, Theorem 12.6] and $[\iota_9, \mu_9] = \eta_9\mu_{10}\sigma_{19} + \sigma_9\eta_{16}\mu_{17} \neq 0$ [36, (12.21), Theorem 12.7].

We recall the relations (2.8) and [36, Proposition 3.1, Lemma 12.12]: $\sigma_{10}\eta_{17} = \eta_{10}\sigma_{11}$, $\sigma_{11}\mu_{18} = \mu_{11}\sigma_{20}$ and $4\zeta_9\sigma_{20} = 8\sigma_9\zeta_{16} = 0$. By these relations, (2.8) and (6.13), $[\iota_9, \eta_9\mu_{10}] = (\eta^2_9\sigma_{11} + \sigma_9\eta^2_{16})\mu_{18} = 4\zeta_9\sigma_{20} + 4\sigma_9\zeta_{16} = 4\sigma_9\zeta_{16} \neq 0$. That is,

$$[\iota_9, \eta_9\mu_{10}] = 4\sigma_9\zeta_{16} \neq 0. \tag{6.7}$$

Making use of the EHP sequence $(\mathcal{PE}^9_{17})$, by [36, Theorem 12.8] and (6.7), we have

$$\sharp(\sigma_{10}\zeta_{17}) = 4.$$

So, by [36, (12.25)],

$$(6.8) \qquad [\iota_{10}, \mu_{10}] = 2\sigma_{10}\zeta_{17} \neq 0.$$

By Example 3.2, $[\iota_{11}, \mu_{11}] = 0$. We have $[\iota_{12}, \mu_{12}] \neq 0$ [36, Lemma 16.2] and $[\iota_{13}, \mu_{13}] \neq 0$ [24, p. 309]. By [24, pp. 321-2], $[\iota_{14}, \mu_{14}] \neq 0$. By [32, p. 140: (8.31), Theorem 3.(b)], $[\iota_{22}, \mu_{22}] \neq 0$. Hence, by Lemma 4.3 and [21, Theorem C],

$$(6.9) \qquad \sharp[\iota_n, \mu_n] = \begin{cases} 1, & \text{if } n = 6 \text{ or } n \equiv 3 \pmod 4; \\ 2, & \text{if } n \equiv 0, 1, 2 \pmod 4 \text{ and } n \geq 4 \text{ unless } n = 6. \end{cases}$$

We have $[\iota_4, \eta_4\mu_5] = (E\nu')\eta_7\mu_8 \neq 0$ and $[\iota_5, \eta_5\mu_6] = \nu_5\eta_8^2\mu_{10} = 4\nu_5\zeta_8 = 0$ (6.13), [36, Theorem 10.3]. That is,

$$(6.10) \qquad [\iota_5, \eta_5\mu_6] = 0.$$

By (2.1) and (4.2), $[\iota_n, \eta_n\mu_{n+1}] = 0$ for $n = 6, 10$ and 11. By [36, Theorem 12.7],

$$(6.11) \qquad [\iota_8, \eta_8\mu_9] = (E\sigma')\eta_{15}\mu_{16} \neq 0$$

and $[\iota_{11}, \eta_{11}\mu_{11}] = 0$ (2.1). By [24, (7.8)], $[\iota_{12}, \eta_{12}\mu_{13}] \neq 0$. By [24, p. 321], $[\iota_{13}, \eta_{13}\mu_{14}] = 8\rho_{13}\sigma_{28} \neq 0$. By [32, p. 139: (8.27)], $[\iota_{21}, \eta_{21}\mu_{22}] \neq 0$. Hence, by [21, Theorem C],

$$(6.12) \qquad \sharp[\iota_n, \eta_n\mu_{n+1}] = \begin{cases} 1, & \text{if } n = 5 \text{ or } n \equiv 2, 3 \pmod 4; \\ 2, & \text{if } n \equiv 0, 1 \pmod 4 \text{ and } n \geq 4 \text{ unless } n = 5. \end{cases}$$

We recall $\pi_{15}(\mathbb{S}^6) = \{\nu_6^3, \mu_6, \eta_6\varepsilon_7\} \cong (\mathbb{Z}_2)^3$. Since $[\iota_6, \eta_6] = 0$ and $\nu_6^3 = \eta_6\bar{\nu}_7$ (2.7), we have $[\iota_6, \nu_6^3] = [\iota_6, \eta_6\varepsilon_7] = 0$. So, by (6.6), we obtain $G_{15}(\mathbb{S}^6) = \pi_{15}(\mathbb{S}^6)$.

Next, we recall $\pi_{19}(\mathbb{S}^{10}) = \{[\iota_{10}, \iota_{10}], \nu_{10}^3, \mu_{10}, \eta_{10}\varepsilon_{11}\} \cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3$. By (4.2) and (2.7), $[\iota_{10}, \nu_{10}^3] = [\iota_{10}, \eta_{10}\varepsilon_{11}] = 0$. So, by (6.8), $G_{19}(\mathbb{S}^{10}) = \{3[\iota_{10}, \iota_{10}], \nu_{10}^3, \eta_{10}\varepsilon_{11}\} \cong 3\mathbb{Z} \oplus (\mathbb{Z}_2)^2$.

Let $n \equiv 2 \pmod 4$ and $n \geq 14$. Then, by (4.2),

$$[\iota_n, \eta_n^2\sigma_{n+2}] = [\iota_n, \eta_n\varepsilon_{n+1}] = 0.$$

By (6.9), $[\iota_n, \mu_n] \neq 0$. Whence, we obtain

$$G_{n+9}(\mathbb{S}^n) = \{\nu_n^3, \eta_n\varepsilon_{n+1}\} \cong (\mathbb{Z}_2)^2, \text{ if } n \equiv 2 \pmod 4 \text{ and } n \geq 14.$$

Let $n \equiv 3 \pmod 4$ and $n \geq 11$. Then, by (2.1) and Example 3.2,

$$G_{n+9}(\mathbb{S}^n) = \pi_{n+9}(\mathbb{S}^n), \text{ if } n \equiv 3 \pmod 4.$$

We recall $\pi_{13}(\mathbb{S}^4) = \{\nu_4^3, \mu_4, \eta_4\varepsilon_5\} \cong (\mathbb{Z}_2)^3$. We have $[\iota_4, \nu_4^3] = 2\nu_4^2 \circ \nu_{10}^2 = 0$ and $[\iota_4, \eta_4\varepsilon_5] = (E\nu')\eta_7\varepsilon_8 \neq 0$ [36, Theorem 7.6]. So, by (6.5), $G_{13}(\mathbb{S}^4) = \{\nu_4^3\} \cong \mathbb{Z}_2$.

Let now $n \equiv 4 \pmod 8$ and $n \geq 12$. By Lemma 1.1.(1) and (3.7), we have $[\iota_n, \nu_n^3] = 0$. In light of (6.9) and (4.17), $[\iota_n, \eta_n\varepsilon_{n+1}] = [\iota_n, \eta_n^2\sigma_{n+2}] \neq 0$ and $[\iota_n, \mu_n] \neq 0$. Suppose that $P(\alpha_{2n+1} + \mu_{2n+1}) = 0$ for $\alpha_{2n+1} = \eta_{2n+1}\varepsilon_{2n+2}$ or $\eta_{2n+1}^2\sigma_{2n+3}$. By [36, Proposition 11.10.(i)], there exists an element $\beta \in \pi_{2n+7}^{n-1}$ satisfying $E\beta = 0$ and $H\beta = \eta_{2n-3}(\alpha_{2n-2} + \mu_{2n-2}) = \eta_{2n-3}\mu_{2n-2}$. On the other hand, $(\mathcal{PE}_{2n+7}^{n-1})$ implies a contradictory relation $\beta \in P\pi_{2n+9}^{2n-1} = 0$. So, $[\iota_n, \alpha_n] \neq [\iota_n, \mu_n]$ and hence

$$G_{n+9}(\mathbb{S}^n) = \{\nu_n^3\} \cong \mathbb{Z}_2, \quad \text{if} \quad n \equiv 4 \pmod 8.$$

By (2.7), (2.8) and (2.16), $[\iota_9, \nu_9^3] = (\eta_9^2\sigma_{11} + \sigma_9\eta_{16}^2) \circ \bar{\nu}_{18} = 0$. By (2.15) and (2.12), $[\iota_9, \sigma_9\eta_{16}^2] = \sigma_9(\sigma_{16}\eta_{23}^3) = 4\sigma_9^2\nu_{23} = 0$. So, we obtain $G_{18}(\mathbb{S}^9) = \{\sigma_9\eta_{16}^2, \nu_9^3, \eta_9\varepsilon_{10}\} \cong (\mathbb{Z}_2)^3$. Let now $n \equiv 1 \pmod 8$ and $n \geq 17$. By (6.9), $[\iota_n, \mu_n] \neq 0$ and by (4.2), $[\iota_n, \eta_n\varepsilon_{n+1}] = 0$. In light of IV, $[\iota_n, \nu_n^3] = 0$ if $n = 2^i - 7$ for $i \geq 4$ and $[\iota_n, \nu_n^3] = [\iota_n, \eta_n^2\sigma_{n+2}] \neq 0$ if $n \equiv 1 \pmod 8$ and $n \geq 17$ and $n \neq 2^i - 7$. We show $[\iota_n, \eta_n^2\sigma_{n+2}] \neq [\iota_n, \mu_n]$. Suppose otherwise. Then, by [36, Proposition 11.10.(ii)], there is an element $\beta \in \pi_{2n+7}^{n-1}$ such that $E\beta = P(\eta_{2n+1}^2\sigma_{2n+2} + \mu_{2n+1}) = 0$ and $H\beta = \eta_{2n-3}\mu_{2n-2}$. On the other hand, by $(\mathcal{PE}_{2n+7}^{n-1})$ and (3.8), $H\beta = 0$, and so we get the assertion. Hence, we obtain

$$G_{n+9}(\mathbb{S}^n) = \begin{cases} \{\eta_n\varepsilon_{n+1}\} \cong \mathbb{Z}_2, \text{ if } n \equiv 1 \pmod 8 \text{ and } n \geq 17 \text{ and } n \neq 2^i - 7; \\ \{\eta_n\varepsilon_{n+1}, \nu_n^3\} \cong (\mathbb{Z}_2)^2, \text{ if } n = 2^i - 7 \ (i \geq 5). \end{cases}$$

By (2.4) and [36, (7.10)], $[\iota_5, \eta_5\varepsilon_6] = \nu_5\eta_8^2\varepsilon_{10} = 4\nu_5^2\sigma_{11} = 0$. So, we obtain $G_{14}(\mathbb{S}^5) = \{\nu_5^3, \eta_5\varepsilon_6\} \cong (\mathbb{Z}_2)^2$. Let $n \equiv 5 \pmod 8$ and $n \geq 13$. By Proposition 4.4 and (6.9), $\nu_n^3 \in G_{n+9}(\mathbb{S}^n)$ and $\mu_n \notin G_{n+9}(\mathbb{S}^n)$. Furthermore, by Proposition 4.4, $\eta_n\varepsilon_{n+1} \in G_{n+9}(\mathbb{S}^n)$ unless $n \equiv 53 \pmod{64}$. So, we obtain

$$G_{n+9}(\mathbb{S}^n) = \{\nu_n^3, \eta_n\varepsilon_{n+1}\} \cong (\mathbb{Z}_2)^2, \text{ if } n \equiv 5 \pmod 8 \text{ and } n \not\equiv 53 \pmod{64}.$$

At the end, we use the following:

$$\zeta_n \in \{2\iota_n, \eta_n, \alpha_{n+1}\}_2 \pmod{2\zeta_n} \text{ for } \alpha_{n+1} = \eta_{n+1}^2\sigma_{n+3} \text{ or } \eta_{n+1}\varepsilon_{n+2}, \text{ if } n \geq 11.$$

Let $n \equiv 0 \pmod 8$ and $n \geq 16$. By [36, Proposition 11.11.(i)], there exists an element $\beta \in \pi^{n-2}_{2n+6}$ such that $[\iota_n, \alpha_n] = E^2\beta$ and $H\beta \in \{2\iota_{2n-5}, \eta_{2n-5}, \alpha_{2n-4}\}_2 \ni \zeta_{2n-5} \pmod{2\zeta_{2n-5}}$. Suppose that $[\iota_n,\ \alpha_n] = 0$. Then, $(\mathcal{PE}^{n-1}_{2n+7})$ induces a relation $E\beta \in P\pi^{2n-1}_{2n+9} = 0$. By $(\mathcal{PE}^{n-2}_{2n+6})$ and (3.8), we have a contradictory relation $\zeta_{2n-5} \in 2\pi^{2n-5}_{2n+6}$. Whence, we get that $[\iota_n, \alpha_n] \neq 0$. In light of (6.9) and (6.12), we know $[\iota_n, \mu_n] \neq 0$ and $[\iota_n, \mu_n]\eta_{2n+8} \neq 0$. This implies that $[\iota_n, \alpha_n] \neq [\iota_n, \mu_n]$ and $[\iota_n, \nu^3_n] \neq [\iota_n, \mu_n]$.

By (2.9) and (4.16), $[\iota_8, \nu^3_8] = (E\sigma')\nu^3_{15} = \eta_8\bar{\varepsilon}_9$ and $[\iota_8, \sigma_8\eta^2_{16}] = (E\sigma')\sigma_{15}\eta^2_{22} = (E\sigma')(\eta_{15}\varepsilon_{16} + \nu^3_{15}) = [\iota_8, \eta_8\varepsilon_9] + [\iota_8, \nu^3_8]$. We have $[\iota_8, (E\sigma')\eta^2_{15}] = 0$. So, we obtain $G_{17}(\mathbb{S}^8) = \{(E\sigma')\eta^2_{15}, \sigma_8\eta^2_{15} + \nu^3_8 + \eta_8\varepsilon_9\} \cong (\mathbb{Z}_2)^2$. By [32, p. 71], $\mathrm{Ker}\{P\colon \pi^{33}_{42} \to \pi^{16}_{40}\} = 0$ and hence, $G_{25}(\mathbb{S}^{16}) = 0$.

By [36, (7.14)],

$$2\zeta_5 = \pm E^2\mu' \text{ and } 4\zeta_n = \eta^2_n\mu_{n+2} \text{ for } n \geq 5. \tag{6.13}$$

Let $n \equiv 2 \pmod 4$ and $n \geq 6$. By (6.13), Lemma 1.1.(1) and (2.2), $4[\iota_n, \zeta_n] = 0$. So, by the relation $H[\iota_n, \zeta_n] = \pm 2\zeta_{2n-1}$, we obtain

$$\sharp[\iota_n, \zeta_n] = 4, \text{ if } n \equiv 2 \pmod 4 \text{ and } n \geq 6. \tag{6.14}$$

By [29, 4.14], there exists an element $\tau_1 \in \pi^{n-6}_{2n+2}$ such that

$$[\iota_n, \nu^3_n] = E^6\tau_1,\ H\tau_1 = \eta_{2n-13}\kappa_{2n-12}, \text{ if } n \equiv 0 \pmod 8 \text{ and } n \geq 16.$$

Suppose that $[\iota_n, \nu^3_n] = 0$. Then, by $(\mathcal{PE}^{n-1}_{2n+7})$, we have $E^5\tau_1 = 0$. So, by $(\mathcal{PE}^{n-2}_{2n+6})$, we have $E^4\tau_1 \in P\pi^{2n-3}_{2n+8} = \{[\iota_{n-2}, \zeta_{n-2}]\}$. By applying $H\colon \pi^{n-2}_{2n+6} \to \pi^{2n-5}_{2n+6}$ to this relation and by (6.14), we obtain $E^4\tau_1 = 4a[\iota_{n-2}, \zeta_{n-2}] = 0$ for $a \in \{0, 1\}$. By the fact that $\pi^{2n-5}_{2n+7} = \pi^{2n-7}_{2n+6} = 0$, we obtain $E^2\tau_1 = 0$. Hence, by $(\mathcal{PE}^{n-5}_{2n+3})$ and (4.7), we have

$$E\tau_1 \in P\pi^{2n-9}_{2n+5} = E^3\bar{\tau}_{n-8} \circ \{\sigma^2_{2n-11}, \kappa_{2n-11}\}.$$

By $(\mathcal{PE}^{n-6}_{2n+2})$, we obtain

$$\tau_1 + E^2(b\bar{\tau}_{n-8}\sigma^2_{2n-14} + b\bar{\tau}_{n-8}\kappa_{2n-14}) \in P\pi^{2n-11}_{2n+4} \text{ with } b, c \in \{0, 1\}.$$

This induces a contradictory relation $\eta_{2n-13}\kappa_{2n-12} \in 2\pi^{2n-13}_{2n+2}$. Thus, we conclude that

$$[\iota_n, \nu^3_n] \neq 0, \text{ if } n \equiv 0 \pmod 8 \text{ and } n \geq 16.$$

Summing the above, we get

Proposition 6.4 *The group $G_{n+9}(\mathbb{S}^n)$ is equal to the following: $\pi_{n+9}(\mathbb{S}^n)$ if $n = 6$ or $n \equiv 3 \pmod 4$; $\{\nu^3_n, \eta_n\varepsilon_{n+1}\} \cong (\mathbb{Z}_2)^2$ if $n \equiv 2 \pmod 4$ and $n \geq 14$, $n = 2^i - 7$ for $i \geq 5$ or $n \equiv 5 \pmod 8$ unless $n \equiv 53 \pmod{64}$; $\{\nu^3_n\} \cong \mathbb{Z}_2$ if*

$n \equiv 4 \pmod 8$ *or* $53 \pmod{64}$ *and* $n \geq 117$; $\{\eta_n\varepsilon_{n+1}\} \cong \mathbb{Z}_2$ *if* $n \equiv 1 \pmod 8$ *and* $n \geq 17$ *and* $n \neq 2^i - 7$; 0 *if* $n \equiv 0 \pmod 8$ *and* $n \geq 16$. *Moreover,* $G_{17}(\mathbb{S}^8) = \{(E\sigma')\eta_{15}^2, \sigma_8\eta_{15}^2 + \nu_8^3 + \eta_8\varepsilon_9\} \cong (\mathbb{Z}_2)^2$, $G_{18}(\mathbb{S}^9) = \{\sigma_9\eta_{16}^2, \nu_9^3, \eta_9\varepsilon_{10}\} \cong (\mathbb{Z}_2)^3$ *and* $G_{19}(\mathbb{S}^{10}) = \{3[\iota_{10}, \iota_{10}], \nu_{10}^3, \eta_{10}\varepsilon_{11}\} \cong 3\mathbb{Z} \oplus (\mathbb{Z}_2)^2$.

By (1.1), Propositions 1.2.(3), 1.3, (1.6) and (6.12), we can determine $G_{n+10}(\mathbb{S}^n)$ for $n \geq 12$.

We have $G_{14}(\mathbb{S}^4; 5) = \pi_{14}(\mathbb{S}^4; 5) \cong \mathbb{Z}_5$ and $G_{14}(\mathbb{S}^4; 3) = \pi_{14}(\mathbb{S}^4; 3) \cong (\mathbb{Z}_3)^2$ by (1.7).

By [36, Theorem 7.3], $\pi_{14}^4 = \{\nu_4\sigma', E\varepsilon', \eta_4\mu_5\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$. We have $[\iota_4, \nu_4\sigma'] = 2\nu_4^2E^3\sigma'$ and $[\iota_4, E\varepsilon'] = 2\nu_4E^4\varepsilon' - E(\nu'E^3\varepsilon')$. By the definition of ε' [36, p. 58], we obtain

$$\nu'E^3\varepsilon' \in \nu' \circ -\{2\nu_6, 2\nu_9, \nu_{12}\} = \{\nu', 2\nu_6, 2\nu_9\} \circ \nu_{13}$$

$$= 2\{\nu', \nu_6, 2\nu_9\} \circ \nu_{13} \ni 2\varepsilon'\nu_{13} \pmod{\nu'\sigma''\nu_{13}}.$$

By the relations $2\varepsilon' = \eta_3^2\varepsilon_5$ [36, Lemma 6.6] and $\varepsilon_4\nu_{12} = P(\bar{\nu}_9)$ [36, (7.13)], we obtain $2\varepsilon'\nu_{13} = 0$. By (2.3), (2.13) and [36, (7.4)], $E(\nu'\sigma'') = \eta_4^3\sigma' = \eta_4^2 \circ 4\bar{\nu}_6 = 0$ and so, we obtain $\nu'\sigma'' = 0$, $\nu'\sigma''\nu_{13} = 0$. This implies $\nu'E^3\varepsilon' = 0$. By [36, (7.10), (7.16)], $\nu_5E\sigma' = 2(\nu_5\sigma_8) = \pm E^2\varepsilon'$. Therefore, we conclude that $\nu_4\sigma' \pm E\varepsilon' \in G_{14}(\mathbb{S}^4)$. We also obtain $2E\varepsilon' \in G_{14}(\mathbb{S}^4)$, because $[\iota_4, 2E\varepsilon'] = 4(\nu_4E^4\varepsilon') = 0$. By (2.6) and (6.10), $G_{15}(\mathbb{S}^5) = \pi_{15}(\mathbb{S}^5)$.

We recall the following:

$$\pi_{16}(\mathbb{S}^6) = \{\nu_6\sigma_9, \eta_6\mu_7, \beta_1(6)\} \cong \mathbb{Z}_{72} \oplus \mathbb{Z}_2,$$

$$\pi_{18}(\mathbb{S}^8) = \{\sigma_8\nu_{15}, \nu_8\sigma_{11}, \eta_8\mu_9, \sigma_8\alpha_1(15), \beta_1(8)\} \cong (\mathbb{Z}_{24})^2 \oplus \mathbb{Z}_2,$$
$$\pi_{19}^9 = \{\sigma_9\nu_{16}, \eta_9\mu_{10}\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2,$$
$$\pi_{20}^{10} = \{\sigma_{10}\nu_{17}, \eta_{10}\mu_{11}\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2, \ \pi_{21}^{11} = \{\sigma_{11}\nu_{18}, \eta_{11}\mu_{12}\} \cong (\mathbb{Z}_2)^2.$$

The order $\sharp[\iota_6, \beta_1(6)] = \sharp[\iota_6, \iota_6] \circ \beta_1(11) = 3$. By (2.1), $[\iota_6, \eta_6\mu_7] = 0$. By (2.12), $[\iota_6, \nu_6\sigma_9] = [\iota_6, \iota_6](\nu_{11}\sigma_{14}) = 0$. This yields $G_{16}(\mathbb{S}^6) = 3\pi_{16}(\mathbb{S}^6)$.

It holds that $[\iota_8, \beta_1(8)] \neq 0$ and $[\iota_8, \sigma_8\alpha_1(15)] = [\iota_8, \iota_8](\alpha_2(15)\alpha_1(22)) = 0$ (1.7). By (2.12), $[\iota_8, \sigma_8\nu_{15}] = [\iota_8, \nu_8\sigma_{11}] = 0$. Hence, by (6.11), we get that $G_{18}(\mathbb{S}^8) = \{\sigma_8\nu_{15}, \nu_8\sigma_{11}, \sigma_8\alpha_1(15)\} \cong (\mathbb{Z}_8)^2 \oplus \mathbb{Z}_3$.

We have $[\iota_9, \sigma_9\nu_{16}] = 0$. So, by (6.7) and Proposition 1.2.(3), $G_{19}(\mathbb{S}^9) = \{\sigma_9\nu_{16}, \beta_1(9)\} \cong \mathbb{Z}_{24}$.

We obtain $[\iota_{10}, \sigma_{10}\nu_{17}] = 0$ by (2.12), $[\iota_{10}, \eta_{10}\mu_{11}] = 0$ by (4.2) and hence, $G_{20}(\mathbb{S}^{10}) = \pi_{20}^{10}$.

By (2.1) and (2.17), $[\iota_{11}, \eta_{11}\mu_{12}] = [\iota_{11}, \sigma_{11}\nu_{18}] = 0$. This yields $G_{21}(\mathbb{S}^{11}) = \pi_{21}(\mathbb{S}^{11})$.

Therefore, we conclude that

$$G_{n+10}(\mathbb{S}^n) = \begin{cases} \{\nu_4\sigma' \pm E\varepsilon', 2E\varepsilon', \alpha_1(4)\alpha_2(7), & \\ \nu_4\alpha_2(7), \nu_4\alpha'_1(7)\}, & \text{if } n = 4; \\ \pi_{15}(\mathbb{S}^5), & \text{if } n = 5; \\ \pi^6_{16} \oplus \{3\beta_1(6)\}, & \text{if } n = 6; \\ \{\sigma_8\nu_{15}, \nu_8\sigma_{11}, \sigma_8\alpha_1(15)\}, & \text{if } n = 8; \\ \{\sigma_9\nu_{16}, \beta_1(9)\}, & \text{if } n = 9; \\ \pi^{10}_{20} = \{\sigma_{10}\nu_{17}, \eta_{10}\mu_{11}\}, & \text{if } n = 10; \\ \pi_{21}(\mathbb{S}^{11}), & \text{if } n = 11. \end{cases}$$

Thus, by summing up the above results, we get

Proposition 6.5 *The group $G_{n+10}(\mathbb{S}^n)$ is isomorphic to the following: $\mathbb{Z}_{120} \oplus \mathbb{Z}_6$, $\mathbb{Z}_{72} \oplus \mathbb{Z}_2$, $\mathbb{Z}_{24} \oplus \mathbb{Z}_2$, $\mathbb{Z}_{24} \oplus \mathbb{Z}_8$, $\mathbb{Z}_{24}$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ according as $n = 4, 5, 6, 8, 9, 10, 11$. Furthermore, $G_{n+10}(\mathbb{S}^n)$ is isomorphic to the group: 0 if $n \equiv 0 \pmod 4$ and $n \geq 12$; $\mathbb{Z}_2$ if $n \equiv 2 \pmod 4$ and $n \geq 14$; $\mathbb{Z}_3$ if $n \equiv 1 \pmod 4$ and $n \geq 13$ and $\mathbb{Z}_6$ if $n \equiv 3 \pmod 4$ and $n \geq 15$.*

We recall that $\pi_{n+11}(\mathbb{S}^n; 3) = \{\alpha_3(n)\} \cong \mathbb{Z}_3$ for $n = 3, 4$ and that $\pi_{n+11}(\mathbb{S}^n; 3) = \{\alpha'_3(n)\} \cong \mathbb{Z}_9$ for $n \geq 5$, where $3\alpha'_3(n) = \alpha_3(n)$ for $n \geq 5$.

By [36, (10.14)], $[\iota_5, \zeta_5] = 0$. By (6.14), $\sharp[\iota_6, \zeta_6] = \sharp[\iota_{10}, \zeta_{10}] = 4$. By [36, Theorem 12.8, Lemma 12.12], $\sharp[\iota_8, \zeta_8] = 8$. By [36, (12.22)], $E\colon \pi^9_{28} \to \pi^{10}_{29}$ is an isomorphism, and so $[\iota_9, \zeta_9] = 0$. By [24, pp. 307, 320], $[\iota_{11}, \zeta_{11}] = 0$ and $\sharp[\iota_{12}, \zeta_{12}] = 8$. By [25, (3.10)], $[\iota_{13}, \zeta_{13}] = 0$. By summing up these results, $\sharp[\iota_n, \zeta_n] = 1, 4, 8, 1, 4, 1, 8, 1$ according as $n = 5, 6, 8, 9, 10, 11, 12, 13$.

By (6.13), we have $[\iota_4, E\mu'] = 4\nu_4\zeta_7 \neq 0$. By [36, (7.12)], $[\iota_4, \varepsilon_4\nu_{12}] = 0$. We note that $[\iota_6, \bar{\nu}_6] = 0$ (6.1) and $[\iota_n, \bar{\nu}_n\nu_{n+8}] = 0$ for $n = 8, 9$ by (2.10). Hence, by the group structure of π^n_{n+11} [36, Theorem 7.4], we obtain $G_{n+11}(\mathbb{S}^n; 2)$ for

$5 \leq n \leq 12$. Summing up, we obtain

$$G_{n+11}(\mathbb{S}^n) = \begin{cases} \{\nu_4\sigma'\eta_{14}, \nu_4\bar{\nu}_7, \nu_4\varepsilon_7, \\ 2E\mu', \varepsilon_4\nu_{12}, (E\nu')\varepsilon_7\}, & \text{if } n = 4; \\ \pi_{16}(\mathbb{S}^5), & \text{if } n = 5; \\ \{4\zeta_6, \bar{\nu}_6\nu_{14}\}, & \text{if } n = 6; \\ \{\bar{\nu}_8\nu_{16}\}, & \text{if } n = 8; \\ \pi_{20}(\mathbb{S}^9), & \text{if } n = 9; \\ 4\pi^{10}_{21}, & \text{if } n = 10; \\ \pi_{22}(\mathbb{S}^{11}), & \text{if } n = 11; \\ \{3[\iota_{12}, \iota_{12}]\}, & \text{if } n = 12. \end{cases}$$

By abuse of notations, ζ_n for $n \geq 5$ represents a generator of the direct summands $\mathbb{Z}_8$ of π^n_{n+11} and $\mathbb{Z}_{504}$ of $\pi_{n+11}(\mathbb{S}^n)$, respectively.

We already know $[\iota_5, \zeta_5] = 0$ and $\sharp[\iota_{12}, \zeta_{12}] = 8$. By [32, p. 139: (8.24)], $\sharp[\iota_{20}, \zeta_{20}] = 8$. Hence, by [21, Theorem C], Proposition 1.2.(3), (1.6), Theorem 6.1 and (6.14), we obtain

$$\sharp[\iota_n, \zeta_n] = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 2 \text{ and } n \geq 5 \text{ unless } n \equiv 115 \pmod{128}; \\ 2, & \text{if } n \equiv 115 \pmod{128} \text{ and } n \geq 243; \\ 252, & \text{if } n \equiv 2 \pmod 4 \text{ and } n \geq 6; \\ 504, & \text{if } n \equiv 0 \pmod 4 \text{ and } n \geq 8. \end{cases}$$

Thus, by summing up the above results, we get

Proposition 6.6 *The group $G_{n+11}(\mathbb{S}^n)$ is isomorphic to the following: $(\mathbb{Z}_2)^6$, $\mathbb{Z}_{504}\oplus(\mathbb{Z}_2)^2$, $\mathbb{Z}_2\oplus\mathbb{Z}_4$, $\mathbb{Z}_2$, $\mathbb{Z}_{504}\oplus\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_{504}$, $3\mathbb{Z}$ according as $n = 4, 5, 6, 8, 9, 10, 11, 12$. Furthermore, $G_{n+11}(\mathbb{S}^n)$ is isomorphic to the group: $\mathbb{Z}_{504}$ if $n \equiv 1 \pmod 2$ and $n \geq 13$ unless $n \equiv 115 \pmod{128}$; $\mathbb{Z}_{252}$ if $n \equiv 115 \pmod{128}$ and $n \geq 243$; $\mathbb{Z}_2$ if $n \equiv 2 \pmod 4$ and $n \geq 14$ and 0 if $n \equiv 0 \pmod 4$ and $n \geq 16$.*

By use of [36, Theorem 7.6, p. 187: Table], we obtain $G_{n+12}(\mathbb{S}^n) = \pi_{n+12}(\mathbb{S}^n)$ for $n \leq 9$.

We recall $\pi_{22}(\mathbb{S}^{10}) = \{[\iota_{10}, \nu_{10}]\} \cong \mathbb{Z}_{12}$. By Proposition 1.5.(1), $G_{22}(\mathbb{S}^{10}; 3) = 0$ and hence, $G_{22}(\mathbb{S}^{10}) = \pi^{10}_{22}$. By [24, (7.7)], $G_{23}(\mathbb{S}^{11}) = \pi_{23}(\mathbb{S}^{11})$. By [36, (7.30)] and [25, (4.29)], we obtain $G_{n+12}(\mathbb{S}^n) = \pi_{n+12}(\mathbb{S}^n)$ for $n = 12$ and 13. Summing up, we obtain

$$G_{n+12}(\mathbb{S}^n) = \pi_{n+12}(\mathbb{S}^n) \text{ unless } n = 10 \text{ and } G_{22}(\mathbb{S}^{10}) = \pi_{22}^{10}.$$

By use of [36, Theorem 7.7, pp. 187-8: Table], we obtain $G_{n+13}(\mathbb{S}^n)$. In particular, we need the relations: $[\iota_{11}, \theta'] = 0$ and $[\iota_{12}, \theta] = 0$ for $\theta' \in \pi_{23}^{11}$ and $\theta \in \pi_{24}^{12}$. We show the case $n = 4$. We recall

$$\pi_{17}(\mathbb{S}^4) = \{\nu_4^2\sigma_{10}, \nu_4\eta_7\mu_8, (E\nu')\eta_7\mu_8, \nu_4\beta_1(7), \alpha_1(4)\beta_1(7)\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2.$$

We have $G_{17}(\mathbb{S}^4; 2) = \pi_{17}^4$. We see that $[\iota_4, \nu_4\beta_1(7)] = \pm 2\nu_4\alpha_1(7)\beta_1(10)$ and $[\iota_4, \alpha_1(4)\beta_1(7)] = \pm(2\nu_4 + \alpha_1(4))(\alpha_1(7)\beta_1(10))$. By making use of the exact sequence in [36, Proposition 13.3], we have $\pi_{19}(\mathbb{S}^3; 3) = \{\alpha_1(3)\alpha_1(6)\beta_1(9)\} \cong \mathbb{Z}_3$. So, $[\iota_4, \nu_4\beta_1(7)]$ and $[\iota_4, \alpha_1(4)\beta_1(7)]$ generate the group $\pi_{20}(\mathbb{S}^4; 3) \cong (\mathbb{Z}_3)^2$ and hence, $G_{17}(\mathbb{S}^4; 3) = 0$.

Summing up, we obtain

$$G_{n+13}(\mathbb{S}^n) = \begin{cases} \pi_{n+13}(\mathbb{S}^n), & \text{if } n \text{ is odd or } n = 2; \\ \pi_{n+13}^n, & \text{if } n \text{ is even unless } n = 2, 14; \\ \{3[\iota_{14}, \iota_{14}]\} \cong 3\mathbb{Z}, & \text{if } n = 14. \end{cases}$$

We close the paper with the two types of tables.

First, the table of the order of $[\iota_n, \alpha]$, where $\alpha \in \pi_{n+k}^n$ for $n \geq k+2$, $k \leq 11$ and $n \equiv r \pmod 8$ with $0 \leq r \leq 7$, given except as otherwise noted. This corrects the table in [27, the second page], where $m \equiv n\ (k)$ indicates $m \equiv n \pmod k$ and symbols in italic stress irregular cases.

Table of the order of $[\iota_n, \alpha]$, I.

$\alpha \backslash r$	0	1	2	3	4	5	6	7
η	2	2	2	1	2	2	2, $\neq 6$ 1, $= 6$	1
η^2	2	2	1	1	2	2, $\neq 5$ 1, $= 5$	1	1
ν	8	2	4	2	8, $\neq 12$ 4, $= 12$	2, $\neq 2^i - 3$ 1, $= 2^i - 3$	4	1
ν^2	2	2	2	2, $\neq 2^i - 5$ 1, $= 2^i - 5$	1	1	2	1
σ	16	2	16	2, $\neq 11$ 1, $= 11$	16	2	16	2, $7(16)$ 1, $15(16)$
$\eta\sigma$	2	2	2, $\neq 10$ 1, $= 10$	1	2	2	2, $\equiv 22(32)$ ≥ 54 1, *otherwise*	1
ε	2	2	1	1	2	2	2	1
$\bar{\nu}$	2	2	2, $\neq 10$ 1, $= 10$	1	2	2	2	1
$\eta^2\sigma$	2	2, $\neq 2^i - 7$ 1, $= 2^i - 7$	1	1	2	2, $\equiv 53(64)$ ≥ 117 1, $\not\equiv 53(64)$	1	1
$\eta\varepsilon$	2	1	1	1	2	2, $\equiv 53(64)$ ≥ 117 1, $\not\equiv 53(64)$	1	1
ν^3	2	2, $\neq 2^i - 7$ 1, $= 2^i - 7$	1	1	1	1	1	1
μ	2	2	2	1	2	2	2	1
$\eta\mu$	2	2	1	1	2	2	1	1
ζ	8	1	4	2, $\equiv 115(128)$ ≥ 243 1, $\not\equiv 115(128)$	8	1	4	1

The next three tables of $G_{n+k}(\mathbb{S}^n)$ for $1 \leq k \leq 13$ and $2 \leq n \leq 26$ are given by compiling our results. Like in [36, Chapter XIV], an integer n indicates the cyclic group $\mathbb{Z}_n$ of order n, the symbol ∞ the infinite cyclic group $\mathbb{Z}$, the symbol + the direct sum of groups and $(2)^k$ the direct sum of k-copies of $\mathbb{Z}_2$.

Table of $G_{n+k}(\mathbb{S}^n)$, II.

$G_{n+k}(\mathbb{S}^n)$	n=2	n=3	n=4	n=5	n=6	n=7	n=8
k=1	∞	2	0	0	2	2	0
k=2	2	2	0	2	2	2	0
k=3	2	12	$3\infty + 2$	24	2	24	0
k=4	12	2	$(2)^2$	2	0	0	0
k=5	2	2	$(2)^2$	2	3∞	0	0
k=6	2	3	$24 + 3$	2	0	2	0
k=7	3	15	0	30	0	120	$3\infty + 2$
k=8	15	2	0	0	$24 + 2$	$(2)^3$	$(2)^2$
k=9	2	$(2)^2$	2	$(2)^2$	$(2)^3$	$(2)^4$	$(2)^2$
k=10	$(2)^2$	$12 + 2$	$120 + 6$	$72 + 2$	$24 + 2$	$24 + 2$	$24 + 8$
k=11	$12 + 2$	$84 + (2)^2$	$(2)^6$	$504 + (2)^2$	$4 + 2$	$504 + 2$	2
k=12	$84 + (2)^2$	$(2)^2$	$(2)^6$	$(2)^3$	240	0	0
k=13	$(2)^2$	6	$8 + (2)^2$	$6 + 2$	2	6	$(2)^2$

Table of $G_{n+k}(\mathbb{S}^n)$, III.

$G_{n+k}(\mathbb{S}^n)$	n=9	n=10	n=11	n=12	n=13	n=14	n=15	n=16	n=17
k=1	0	0	2	0	0	0	2	0	0
k=2	0	2	2	0	0	2	2	0	0
k=3	12	2	12	2	24	2	24	0	12
k=4	0	0	0	0	0	0	0	0	0
k=5	0	0	0	0	0	0	0	0	0
k=6	0	0	2	2	2	0	2	0	0
k=7	120	0	240	0	120	0	240	0	120
k=8	2	$(2)^2$	$(2)^2$	0	0	2	$(2)^2$	0	0
k=9	$(2)^3$	$3\infty + (2)^2$	$(2)^3$	2	$(2)^2$	$(2)^2$	$(2)^3$	0	2
k=10	24	$4+2$	$6+2$	0	3	2	6	0	3
k=11	$504+2$	2	504	3∞	504	2	504	0	504
k=12	0	4	2	$(2)^2$	2	0	0	0	0
k=13	6	2	$6+2$	$(2)^2$	6	3∞	3	0	3

Table of $G_{n+k}(\mathbb{S}^n)$, IV.

$G_{n+k}(\mathbb{S}^n)$	n=18	n=19	n=20	n=21	n=22	n=23	n=24	n=25	n=26
k=1	0	2	0	0	0	2	0	0	0
k=2	2	2	0	0	2	2	0	0	2
k=3	2	12	0	12	2	24	0	12	2
k=4	0	0	0	0	0	0	0	0	0
k=5	0	0	0	0	0	0	0	0	0
k=6	0	0	2	2	0	2	0	0	0
k=7	0	120	0	120	0	120	0	120	0
k=8	2	$(2)^2$	0	0	2	$(2)^2$	0	0	2
k=9	$(2)^2$	$(2)^3$	2	$(2)^2$	$(2)^2$	$(2)^3$	0	2	$(2)^2$
k=10	2	6	0	3	2	6	0	3	2
k=11	2	504	0	504	2	504	0	504	2
k=12	0	0	0	0	0	0	0	0	0
k=13	0	3	0	3	0	3	0	3	0

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