COHEN-MACAULAY MODULES AND HOLONOMIC MODULES OVER FILTERED RINGS

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Abstract. We study Gorenstein dimension and grade of a module M over a filtered ring whose assosiated graded ring is a commutative Noetherian ring. An equality or an inequality between these invariants of a filtered module and its associated graded module is the most valuable property for an investigation of filtered rings. We prove an inequality $\text{G-dim}M \leq \text{G-dim}\,\text{gr}M$ and an equality $\text{grade}M = \text{grade}\,\text{gr}M$, whenever Gorenstein dimension of $\text{gr}M$ is finite (Theorems 2.3 and 2.8). We would say that the use of G-dimension adds a new viewpoint for studying filtered rings and modules. We apply these results to a filtered ring with a Cohen-Macaulay or Gorenstein associated graded ring and study a Cohen-Macaulay, perfect or holonomic module.

1. Introduction

Homological theory of filtered (non-commutative) rings grew in studying , among others, D-modules, i.e., rings of differential operators (cf. [4], [17] etc.). The use of an invariant 'grade' is a core of the theory for Auslander regular or Gorenstein filtered rings ([4], [5], [6], [7], [14]). In particular, its invariance under forming associated graded modules is essential. Using Gorenstein dimension ([1], [9]), we extend the class of rings for which the invariance holds.

Let Λ be a left and right Noetherian ring. Let $\text{mod}\Lambda$ (respectively, $\text{mod}\Lambda^{\text{op}}$) be the category of all finitely generated left (respectively, right) Λ-modules. We denote the stable category by $\underline{\text{mod}}\Lambda$, the syzygy functor by $\Omega : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda$, and the transpose functor by $\text{Tr} : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda^{\text{op}}$ (see [2], Chapter 4, §1 or [1], Chapter 2, §1). For $M \in \text{mod}\Lambda$, we put $M^* := \text{Hom}_\Lambda(M, \Lambda) \in \text{mod}\Lambda^{\text{op}}$.

Gorenstein dimension, one of the most valuable invariants of the homological study of rings and modules, is introduced in [1]. A Λ-module M is said to have *Gorenstein dimension zero*, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\text{Ext}^k_\Lambda(M, \Lambda) = \text{Ext}^k_{\Lambda^{\text{op}}}(M^*, \Lambda) = 0$ for $k > 0$. It follows from [1], Proposition 3.8 that G-dim $M = 0$ if and only if $\text{Ext}^k_\Lambda(M, \Lambda) = \text{Ext}^k_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda) = 0$ for $k > 0$. For a positive integer k, M is said to have *Gorenstein dimension less than or equal to* k, denoted by G-dim $M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with G-dim $G_i = 0$ for $0 \leq i \leq k$. We have that G-dim $M \leq k$ if and only if G-dim $\Omega^k M = 0$ by [1], Theorem 3.13. It is also proved in [1] that if G-dim $M < \infty$ then G-dim $M = \sup\{k : \text{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$. In the following, we abbreviate 'Gorenstein dimension' to G-dimension.

We define another important invariant 'grade'. Let $M \in \text{mod}\Lambda$. We put $\text{grade}_\Lambda M := \inf\{k : \text{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$.

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In this paper we study G-dimension and grade of a filtered module over a filtered ring whose assiciated graded ring is commutative and Noetherian and apply the results to a filtered ring with a Gorenstein or Cohen-Macaulay associated graded ring.

In section two, we study G-dimension and grade of modules over a filtered ring. As usual, we analyze them by using the properties of assiciated graded modules. We start from studying G-dimension. When an associated graded ring $\mathrm{gr}\Lambda$ of a filtered ring Λ is commutative and Noetherian, a filtered Λ-module M whose associated graded module $\mathrm{gr}M$ has finite G-dimension has also finite G-dimension and an inequality $\text{G-dim}M \leq \text{G-dim}\,\mathrm{gr}M$ holds true (Theorem 2.3). However, it is open whether an equality holds or not in general. As for G-dimension zero, we show that if $\text{G-dim}\,\mathrm{gr}M = 0$, then $\text{G-dim}M = 0$ and the converse holds whenever some additional conditions for M are assumed (Theorem 2.5). Assume further that $\mathrm{gr}\Lambda$ is a *local ring with the condition (P) (see Appendix), then 'Auslander-Bridger formula' holds for a filtered module M such that $\mathrm{gr}M$ has finite G-dimension and $\text{G-dim}M = \text{G-dim}\,\mathrm{gr}M$: $\text{G-dim}M + {}^*\mathrm{depth}\,\mathrm{gr}M = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda$ (Proposition 2.6).
To handle grade in the literatures, a kind of 'finitary' condition over a ring such as 'regularity' or 'Gorensteiness' is setted ([7], §5 and [14], Chapter Ⅲ, §2, 2.5). We find out that only the finiteness of G-dimension of $\mathrm{gr}M$ implies $\mathrm{grade}M = \mathrm{grade}\,\mathrm{gr}M$ for a filtered module with a good filtration (Theorem 2.8). Suppose that $\mathrm{gr}\Lambda$ is Gorenstein. Then all finite $\mathrm{gr}\Lambda$-modules have finite G-dimension. Thus all filtered modules with a good filtration satisfy the equality. Since regularity implies Gorensteiness, our results also cover regular filtered rings.

In section three, we apply the results obtained in the previous section to Cohen-Macaulay modules over filtered rings with a Cohen-Macaulay associated graded ring and holonomic modules over Gorenstein filtered rings. When $\mathrm{gr}\Lambda$ is a Cohen-Macaulay *local ring with the condition (P), we define Cohen-Macaulay filtered modules and see that they are perfect. Then they satisfy a duality (Theorem 3.2). Moreover, assume that Λ is Gorenstein. Then injective dimension of Λ is finite, say d, so that we can define a holonomic module. A filtered module M with a good filtration is holonomic, if $\mathrm{grade}M = d$. We generalize some results in [14], Chapter Ⅲ, §4 and give a characterization of a holonomic module M by a property of $\mathrm{Min}(\mathrm{gr}M)$. An example of a filtered (non-regular) Gorenstein ring is given in 3.8.

The summary of commutative graded Noetherian rings, especially, *local rings are stated in Appendix.

2. Gorenstein dimension and grade for modules over filtered Noetherian rings

Let Λ be a ring. A family $\mathcal{F} = \{\mathcal{F}_p\Lambda : p \in \mathbb{N}\}$ of additive subgroups of Λ is called a *filtration* of Λ, if

(i) $1 \in \mathcal{F}_0\Lambda$,

(ii) $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$,

(iii) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$,

(iv) $\Lambda = \cup_{p\in\mathbb{N}}\mathcal{F}_p\Lambda$.

A pair $(\Lambda, \mathcal{F})$ is called a *filtered ring*. In the following, a ring Λ is always a filtered ring for some filtration $\mathcal{F}$, so that we only say that Λ is a filtered ring.

Let $\sigma_p : \mathcal{F}_p\Lambda \to \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ be a natural homomorphism. Put

$$\mathrm{gr}\Lambda = \mathrm{gr}_{\mathcal{F}}\Lambda := \bigoplus_{p=0}^{\infty} \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda \quad (\mathcal{F}_{-1}\Lambda = 0).$$

Then $\mathrm{gr}\Lambda$ is a graded ring with multiplication

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab), \quad a \in \mathcal{F}_p\Lambda,\ b \in \mathcal{F}_q\Lambda.$$

We always assume that $\mathrm{gr}\Lambda$ is a commutative Noetherian ring. Therefore, Λ is a right and left Noetherian ring. Our main objective is to study Λ by relating G-dimension and grade of $\mathrm{mod}\Lambda$ and those of $\mathrm{mod}(\mathrm{gr}\Lambda)$. Sometimes we assume further that $\mathrm{gr}\Lambda$ is a *local ring with the condition (P) (see Appendix).

Let M be a (left) Λ-module. A family $\mathcal{F} = \{\mathcal{F}_pM : p \in \mathbb{Z}\}$ of additive subgroups of M is called a *filtration* of M, if

(i) $\mathcal{F}_pM \subset \mathcal{F}_{p+1}M$,

(ii) $\mathcal{F}_{-p}M = 0$ for $p >> 0$,

(iii) $(\mathcal{F}_p\Lambda)(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}M$,

(iv) $M = \cup_{p\in\mathbb{Z}}\mathcal{F}_pM$.

A pair $(M, \mathcal{F})$ is called a *filtered* Λ-module. Similar to Λ, we sometimes abbreviate and say that M is a filtered module. Let $\tau_p : \mathcal{F}_pM \to \mathcal{F}_pM/\mathcal{F}_{p-1}M$ be a natural homomorphism. Put

$$\mathrm{gr}M = \mathrm{gr}_{\mathcal{F}}M := \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_pM/\mathcal{F}_{p-1}M.$$

Then $\mathrm{gr}M$ is a graded $\mathrm{gr}\Lambda$-module by

$$\sigma_p(a)\tau_q(x) = \tau_{p+q}(ax), \quad a \in \mathcal{F}_p\Lambda,\ x \in \mathcal{F}_qM.$$

As for filtered rings and module, the reader is referred to [14] or [20]. We only state here some definitions and facts. For a filtered module $(M, \mathcal{F})$, we call $\mathcal{F}$ to be a *good filtration*, if there exist $p_k \in \mathbb{Z}$ and $m_k \in M$ $(1 \le k \le r)$ such that

$$\mathcal{F}_pM = \sum_{k=1}^{r} (\mathcal{F}_{p-p_k}\Lambda)m_k$$

for all $p \in \mathbb{Z}$. Then the following three conditions are equivalent ([14], Chapter I, 5.2 and [20], Chapter D, IV.3)

(a) M has a good filtration.

(b) $\mathrm{gr}_{\mathcal{F}}M$ is a finite $\mathrm{gr}\Lambda$-module for a filtration $\mathcal{F}$.

(c) M is a finitely generated Λ-module.

Therefore, we only consider a good filtration for a finitely generated Λ-module M, so that $\mathrm{gr}M$ is a finite $\mathrm{gr}\Lambda$-module.

Let $M,\ N$ be filtered Λ-modules. A Λ-homomorphism $f : M \to N$ is called a *filtered homomorphism*, if $f(\mathcal{F}_pM) \subset \mathcal{F}_pN$ for all $p \in \mathbb{Z}$. Further, f is called *strict*, if $f(\mathcal{F}_pM) = \mathrm{Im}f \cap \mathcal{F}_pN$ for all $p \in \mathbb{Z}$. If M' is a submodule of M, then $\{M' \cap \mathcal{F}_pM : p \in \mathbb{Z}\}$, respectively $\{\mathcal{F}_pM + M'/M' : p \in \mathbb{Z}\}$ is a good filtration on M', respectively M/M'. We call them induced filtration on M' or M/M' and note that the canonical homomorphisms $M' \hookrightarrow M$ and $M \to M/M'$ are strict.

For a filtered homomorphism $f : M \to N$, we define a map $f_p : \mathcal{F}_pM/\mathcal{F}_{p-1}M \to \mathcal{F}_pN/\mathcal{F}_{p-1}N$ by $f_p(\tau_p(x)) = \tau_p(f(x))$ for $x \in \mathcal{F}_pM$. Then we define a $\mathrm{gr}\Lambda$-homomorphism

$$\mathrm{gr}f : \mathrm{gr}M = \oplus\mathcal{F}_pM/\mathcal{F}_{p-1}M \longrightarrow \mathrm{gr}N = \oplus\mathcal{F}_pN/\mathcal{F}_{p-1}N$$

by $\mathrm{gr}f := \oplus f_p$, so that $\mathrm{gr}f(\tau_p(x)) = \tau_p(f(x))$ for $x \in \mathcal{F}_pM$. It is easily seen that $\mathrm{gr}fg = (\mathrm{gr}f)(\mathrm{gr}g)$ for filtered homomorphisms $f : M \to N$ and $g : K \to M$.

For a filtered module M, an exact sequence

$$\cdots \longrightarrow F_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

is called a *filtered free resolution* of M, if all F_i are filtered free Λ-modules and all homomorphisms are strict filtered homomorphisms. We can always constract such a resolution with all F_i of finite rank for a finitely generated Λ-module (see [20], Chapter D, IV).

Let $M,\ N$ be filtered Λ-modules. We put, for $p \in \mathbb{Z}$,

$$\mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) = \{f \in \mathrm{Hom}_\Lambda(M,N) : f(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}N \text{ for all } q \in \mathbb{Z}\}$$

Then we have an ascending chain

$$\cdots \subset \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) \subset \mathcal{F}_{p+1}\mathrm{Hom}_\Lambda(M,N) \subset \cdots$$

of additive subgroups of $\mathrm{Hom}_\Lambda(M,N)$. Set

$$\mathrm{gr}\,\mathrm{Hom}_\Lambda(M,N) := \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)/\mathcal{F}_{p-1}\mathrm{Hom}_\Lambda(M,N)$$

Define an additive homomorphism

$$\varphi = \varphi(M,N) : \mathrm{gr}\,\mathrm{Hom}_\Lambda(M,N) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}N),\ \ \varphi(\tau_p(f))(\tau_q(x)) = \tau_{p+q}(f(x))$$

for $f \in \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N),\ x \in \mathcal{F}_qM$, where

$$\tau_p : \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) \longrightarrow \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)/\mathcal{F}_{p-1}\mathrm{Hom}_\Lambda(M,N)$$

is a natural homomorphism for every $p \in \mathbb{Z}$. When M is a filtered module with a good filtration, the following facts hold (see [14], Chapter I, 6.9 or [20], Chapter D, VI.6):

(1) $\mathrm{Hom}_\Lambda(M,N) = \cup_{p\in\mathbb{Z}}\mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)$.

(2) $\mathcal{F}_{-p}\mathrm{Hom}_\Lambda(M,N) = 0$ for $p >> 0$.

(3) φ is injective. Moreover, if M is a filtered free module, then it is bijective.

(4) When $N = \Lambda$, an additive group $\mathrm{Hom}_\Lambda(M,\Lambda)$ is a filtered Λ^{op}-module with a good filtration $\mathcal{F} := \{\mathcal{F}_p\mathrm{Hom}_\Lambda(M,\Lambda) : p \in \mathbb{Z}\}$ and φ is a $\mathrm{gr}\Lambda$-homomorphism.

Let $M \xrightarrow{f} N \xrightarrow{g} K$ be an exact sequence of filtered modules and filtered homomorphisms. Then $\mathrm{gr}M \overset{\mathrm{gr}f}{\to} \mathrm{gr}N \overset{\mathrm{gr}g}{\to} \mathrm{gr}K$ is exact (in $\mathrm{mod}\,\mathrm{gr}\Lambda$) if and only if f and g are strict (see [14], Chapter I, 4.2.4 or [20], Chapter D, Ⅲ.3).

The following proposition is well-known.

2.1. Proposition. *Let M be a filtered Λ-module with a good filtation. Then* $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M,\Lambda)$ *is a subfactor of* $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ *for* $i \geq 0$.

Proof. See [4], Chapter 2, 6.10 or [14], Chapter Ⅲ, 2.2.4. □

When G-dim gr$M = 0$, the functor Tr commutes with associated gradation.

2.2. Lemma. *Let M be a filtered Λ-module with a good filtation. Then there exists an epimorphism* $\alpha : \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) \to \mathrm{gr}(\mathrm{Tr}_\Lambda M)$.

Moreover, if G-dim gr$M = 0$ *or* grade gr$M > 1$, *then α is an isomorphism.*

Proof. Take a filtered free resolution of M:

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0.$$

By definition, we have an exact sequence

$$F_0^* \xrightarrow{f_1^*} F_1^* \xrightarrow{g} \mathrm{Tr}_\Lambda M = \mathrm{Cok} f_1^* \longrightarrow 0,$$

where g is a canonical epimorphism. Let $\mathrm{Tr}_\Lambda M$ be equipped with the induced filtration by g. Then g is a strict filtered epimorphism. Let us consider the following diagrams in mod grΛ with the commutative squares and all the φ's isomorphisms:

$$(1)\quad \begin{array}{ccccccc} \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_0, \mathrm{gr}\Lambda) & \xrightarrow{(\mathrm{gr}f_1)^*} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_1, \mathrm{gr}\Lambda) & \longrightarrow & \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) & \longrightarrow & 0(\text{exact}) \\ \varphi\uparrow & & \varphi\uparrow & & & & \\ \mathrm{gr}F_0^* & \xrightarrow{\mathrm{gr}(f_1^*)} & \mathrm{gr}F_1^* & \xrightarrow{\mathrm{gr}g} & \mathrm{gr}(\mathrm{Tr}_\Lambda M) & \longrightarrow & 0 \end{array}$$

$$(2)\quad \begin{array}{ccccc} \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_0, \mathrm{gr}\Lambda) & \xrightarrow{(\mathrm{gr}f_1)^*} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_1, \mathrm{gr}\Lambda) & \xrightarrow{(\mathrm{gr}f_2)^*} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_2, \mathrm{gr}\Lambda) \\ \varphi\uparrow & & \varphi\uparrow & & \varphi\uparrow \\ \mathrm{gr}F_0^* & \xrightarrow{\mathrm{gr}(f_1^*)} & \mathrm{gr}F_1^* & \xrightarrow{\mathrm{gr}(f_2^*)} & \mathrm{gr}F_2^* \end{array}$$

Since the induced sequence $\cdots \to \mathrm{gr}F_1 \xrightarrow{\mathrm{gr}f_1} \mathrm{gr}F_0 \to \mathrm{gr}M \to 0$ is a free resolution of gr$M$, the first row of (1) is exact. Since g is strict, gr$g$ is surjective. Hence there exists a graded epimorphism $\alpha : \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) \to \mathrm{gr}(\mathrm{Tr}_\Lambda M)$. By assumption, we see that $\mathrm{Ext}^1_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = 0$, so that the first row of (2) is exact. There exists a filtered homomorphism $h : \mathrm{Tr}_\Lambda M \to F_2^*$ such that $f_2^* = h \circ g$. Since $\mathrm{gr}f_2^* = \mathrm{gr}h \circ \mathrm{gr}g$, we have $\mathrm{Im}\,\mathrm{gr}f_1^* \subset \mathrm{Ker}\,\mathrm{gr}g \subset \mathrm{Ker}\,\mathrm{gr}f_2^*$. The exactness of the second row of (2) implies $\mathrm{Im}\,\mathrm{gr}f_1^* = \mathrm{Ker}\,\mathrm{gr}f_2^*$. Thus $\mathrm{Im}\,\mathrm{gr}f_1^* = \mathrm{Ker}\,\mathrm{gr}g$, hence the second row of (1) is also exact, which implies that α is an isomorphism. □

2.3. Theorem. *Let M be a filtered Λ-module with a good filtration such that* gr$M$ *is of finite G-dimension. Then G-dim$M \leq$ G-dim gr$M$.*

Proof. We show that if G-dim gr$M = k < \infty$, then G-dim$M \leq k$. Let $k = 0$. Assume that G-dim gr$M = 0$. For $i > 0$, since $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda)$ is a subfactor of $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$, we have $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$. Hence $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$. By Lemma 2.2, $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}\,\mathrm{Tr}_\Lambda M, \mathrm{gr}\Lambda) \cong \mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M), \mathrm{gr}\Lambda) = 0$ for $i > 0$. Hence $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}_\Lambda M, \Lambda) = 0$ as above. Thus G-dim$M = 0$.

Let $k > 0$. Since $\mathrm{gr}(\Omega^k M)$ and $\Omega^k(\mathrm{gr}M)$ are stably isomorphic(see [10], p.226 for the definition), the following holds:

$$\text{G-dim}\,\mathrm{gr}M \leq k \Leftrightarrow \text{G-dim}\,\Omega^k(\mathrm{gr}M) = 0 \Leftrightarrow \text{G-dim}\,\mathrm{gr}(\Omega^k M) = 0.$$

Thus the statement holds by the case of $k = 0$. □

2.4. Corollary. *Assume that* grΛ *is a *local ring with the condition (P). If* grΛ *is Gorenstein, then* $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq {}^*\mathrm{depth}\,\mathrm{gr}\Lambda$.

Proof. Let M be a finitely generated Λ-module. Then M is a filtered module with a good filtration. Then $\text{G-dim}\,\text{gr}M < \infty$ by Theorem A.9. Hence

$$\text{G-dim}M \leq \text{G-dim}\,\text{gr}M = {}^*\text{depth}\,\text{gr}\Lambda - {}^*\text{depth}\,\text{gr}M \leq {}^*\text{depth}\,\text{gr}\Lambda.$$

Therefore, $\text{Ext}^i_\Lambda(M,\Lambda) = 0$ for all $i > {}^*\text{depth}\,\text{gr}\Lambda$, so that $\text{id}_\Lambda\Lambda < \infty$. Similarly, we have $\text{id}_{\Lambda^{\text{op}}}\Lambda < \infty$. Thus $\text{id}_\Lambda\Lambda = \text{id}_{\Lambda^{\text{op}}}\Lambda \leq {}^*\text{depth}\,\text{gr}\Lambda$. □

Thanks to Corollary 2.4, we call a filtered ring Λ a "*Gorenstein filtered ring*", if $\text{gr}\Lambda$ is a Gorenstein *local ring with the condition (P).

We give a necessary and sufficient condition when $\text{G-dim}\,\text{gr}M = 0$.

2.5. Theorem. *Let M be a filtered Λ-module with a good filtration. Then the following (1) and (2) are equivalent.*

(1) $\text{G-dim}\,\text{gr}M = 0$.

(2) (2.1) $\text{G-dim}M = 0$.

(2.2) Suppose that $\cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$ *is a filtered free resolution of* M, *then all* f_i^* $(i > 0)$ *are strict.*

(2.2) Suppose that* $\cdots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M^* \to 0$ *is a filtered free resolution of* M^*, *then all* g_i^* $(i > 0)$ *are strict.*

(2.3) A canonical map $\theta : M \to M^{**}$ *is strict.*

Moreover, under the above conditions, $\varphi_M : \text{gr}M^* \to (\text{gr}M)^*$ *and* $\varphi_{M^*} : \text{gr}M^{**} \to (\text{gr}M^*)^*$ *are isomorphisms, where* $\varphi_M = \varphi(M,\Lambda)$, $\varphi_{M^*} = \varphi(M^*,\Lambda)$.

Proof. (1) ⇒ (2): It follows from Theorem 2.3 that $\text{G-dim}M = 0$. ¿From a filtered free resolution of M in (2.2), we get an exact sequence

$$0 \longrightarrow M^* \xrightarrow{f_0^*} F_0^* \xrightarrow{f_1^*} F_1^* \longrightarrow \cdots$$

This exact sequence and an exact sequence in $\text{mod}\,\text{gr}\Lambda$:

$$\cdots \longrightarrow \text{gr}F_1 \longrightarrow \text{gr}F_0 \longrightarrow \text{gr}M \longrightarrow 0$$

induced from a resolution in (2.2) give the following commutative diagram

$$(*)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \text{gr}M^* & \xrightarrow{\text{gr}(f_0^*)} & \text{gr}F_0^* & \xrightarrow{\text{gr}(f_1^*)} & \text{gr}F_1^* & \longrightarrow & \cdots \\ & & \varphi\downarrow & & \varphi_0\downarrow & & \varphi_1\downarrow & & \\ 0 & \longrightarrow & (\text{gr}M)^* & \longrightarrow & (\text{gr}F_0)^* & \longrightarrow & (\text{gr}F_1)^* & \longrightarrow & \cdots \end{array}$$

where $\varphi = \varphi(M,\Lambda)$, $\varphi_i = \varphi(F_i,\Lambda)$. Since $\text{G-dim}\,\text{gr}M = 0$, the second row is exact. For $i \geq 0$, φ_i are isomorphisms. Thus a sequence

$$\text{gr}F_0^* \xrightarrow{\text{gr}(f_1^*)} \text{gr}F_1^* \xrightarrow{\text{gr}(f_2^*)} \text{gr}F_2^* \longrightarrow \cdots$$

is exact, and so $f_1^*,\ f_2^*,\cdots$ are strict. Hence (2.2) holds. Since f_0 is a strict filtered epimorphism, f_0^* is a strict filtered monomorphism. Thus the first row of $(*)$ is exact. Therefore, $\varphi : \text{gr}M^* \to (\text{gr}M)^*$ is an isomorphism. Since $\text{G-dim}(\text{gr}M)^* = 0$, we have $\text{G-dim}\,\text{gr}M^* = 0$. Hence (2.2*) holds and φ_{M^*} is an isomorphism.

Let $\eta : \mathrm{gr}M \to (\mathrm{gr}M)^{**}$ be a canonical homomorphism. Consider the commutative diagram

$$(**) \qquad \begin{array}{ccc} \mathrm{gr}M & \xrightarrow{\mathrm{gr}\theta} & \mathrm{gr}M^{**} \\ \eta \downarrow & & \downarrow \varphi_{M^*} \\ (\mathrm{gr}M)^{**} & \xrightarrow{\varphi_M^*} & (\mathrm{gr}M^*)^*. \end{array}$$

Since η, φ_M^*, φ_{M^*} are isomorphisms, $\mathrm{gr}\theta$ is also an isomorphism. Thus θ is strict.

(2) $\Rightarrow$ (1): By (2.1) and (2.2), the first row of the diagram $(*)$ is exact. Thus the second row of $(*)$ is exact, so that $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = 0$ for $i > 0$ and $(\mathrm{gr}M)^* \cong \mathrm{gr}M^*$. Since $\text{G-dim}M^* = 0$, using the diagram $(*)$ obtained from (2.2^*), we can show that $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M^*, \mathrm{gr}\Lambda) = 0$ for $i > 0$ and $(\mathrm{gr}M^*)^* \cong \mathrm{gr}M^{**}$. Thus we have $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}((\mathrm{gr}M)^*, \mathrm{gr}\Lambda) = 0$ for $i > 0$. By (2.3) and the above argument, the maps $\mathrm{gr}\theta$, φ_M^* and φ_{M^*} are isomorphisms in the diagram $(**)$, so that η is an isomorphism. Thus $\text{G-dim}\,\mathrm{gr}M = 0$. □

Let $\mathrm{fil}\Lambda$ be a category of all filtered Λ-modules with a good filtration and filtered homomorphisms. Let $\mathcal{G}$ be a subcategory of $\mathrm{fil}\Lambda$ consisting of all filtered modules M whose associated graded module $\mathrm{gr}M$ has finite G-dimension. It holds from Theorem 2.3 that a module in $\mathcal{G}$ has finite G-dimension. We further put a subcategory $\mathcal{G}_e$ of $\mathcal{G}$

$$\mathcal{G}_e := \{M \in \mathcal{G} : \text{G-dim}M = \text{G-dim}\,\mathrm{gr}M \text{ for some good filtration of } M\}.$$

2.6. Proposition. *Assume that* $\mathrm{gr}\Lambda$ *is a* **local ring with the condition (P). Let* $M \in \mathcal{G}_e$. *Then the following equality holds.*

$$\text{G-dim}M + {}^*\mathrm{depth}\,\mathrm{gr}M = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda.$$

Proof. The statement follows from Theorem A.8. □

2.7. Remarks. (i) It is interesting to know when $\mathcal{G}_e = \mathcal{G}$. If this is true, then we see that $\text{G-dim}M = 0$ if and only if $\text{G-dim}\,\mathrm{gr}M = 0$ for $M \in \mathcal{G}$. Hence the condition (2.2), (2.2^*), (2.3) in Theorem 2.5 are superfluous.

(ii) Suppose that $0 \to M' \to M \to M'' \to 0$ is a strict exact sequence of $\mathrm{fil}\Lambda$. Then the followings are easy consequence of [9], Corollary 1.2.9 (b).
If $M', M'' \in \mathcal{G}_e$ and $\text{G-dim}M' > \text{G-dim}M''$, then $M \in \mathcal{G}_e$.
If $M, M'' \in \mathcal{G}_e$ and $\text{G-dim}M > \text{G-dim}M''$, then $M' \in \mathcal{G}_e$.

We shall study the another valuable invariant 'grade'. Its nicest feature that an equation $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ holds for a good filtered Λ-module M is proved when $\mathrm{gr}\Lambda$ is regular (see e.g. [14]). We prove this equation under 'module-wise' conditions by which we can apply this equation fairly wide classes of filtered rings.

2.8. Theorem. *Let* Λ *be a filtered ring such that* $\mathrm{gr}\Lambda$ *is a commutative Noetherian ring and* M *a filtered* Λ*-module with a good filtration. Assume that* $\mathrm{gr}M$ *has finite G-dimension. Then an equality* $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ *holds.*

Proof. Put $s = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$. In order to show that $\mathrm{grade}_\Lambda M = s$, we must prove:

(i) $\mathrm{Ext}^s_\Lambda(M, \Lambda) \neq 0$,

(ii) $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i < s$.

2.8.1. (cf. [14], Chapter Ⅲ, §1) Let $\cdots \to F_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$ be a filtered free resolution of M. Applying $(-)^*$ to it, we get a complex

$$F_\bullet : 0 \to F_0^* \xrightarrow{f_1^*} \cdots \to F_{i-2}^* \xrightarrow{f_{i-1}^*} F_{i-1}^* \xrightarrow{f_i^*} F_i^* \to \cdots$$

with each F_i^* filtered free and f_i^* a filtered homomorphism. We put, for $p, r, i \in \mathbb{N}$,

$$Z_p^r(i) := (f_i^*)^{-1}(\mathcal{F}_{p-r}F_i^*) \cap \mathcal{F}_pF_{i-1}^*, \quad Z_p^\infty(i) := \mathrm{Ker} f_i^* \cap \mathcal{F}_pF_{i-1}^*,$$

$$B_p^r(i) := f_{i-1}^*(\mathcal{F}_{p+r-1}F_{i-2}^*) \cap \mathcal{F}_pF_{i-1}^*, \quad B_p^\infty(i) := \mathrm{Im} f_{i-1}^* \cap \mathcal{F}_pF_{i-1}^*$$

Then the following sequence of inclusions holds:

$$Z_p^0(i) \supset Z_p^1(i) \supset \cdots \supset Z_p^\infty(i) \supset B_p^\infty(i) \supset \cdots \supset B_p^1(i) \supset B_p^0(i).$$

We put

$$E_p^r(i) := \frac{Z_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*}{B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*}, \quad E_i^r := \bigoplus_p E_p^r(i).$$

Then E_i^r is a $\mathrm{gr}\Lambda$-module for $r, i \geq 0$. When $r = 0$, we have

$$E_i^0 = \bigoplus_p \frac{(f_i^*)^{-1}(\mathcal{F}_pF_i^*) \cap \mathcal{F}_pF_{i-1}^* + \mathcal{F}_{p-1}F_{i-1}^*}{f_{i-1}^*(\mathcal{F}_{p-1}F_{i-2}^*) \cap \mathcal{F}_pF_{i-1}^* + \mathcal{F}_{p-1}F_{i-1}^*} = \bigoplus_p \frac{\mathcal{F}_pF_{i-1}^*}{\mathcal{F}_{p-1}F_{i-1}^*} = \mathrm{gr}F_{i-1}^*.$$

Hence we get a complex

$$E_\bullet^0 : 0 \to \mathrm{gr}F_0^* \to \cdots \to \mathrm{gr}F_i^* \to \cdots$$

which is an associated graded complex of $F_\bullet$. We show, for $r \geq 1$, that $\{E_i^r\}_{i \geq 0}$ also gives a complex $E_\bullet^r$. To do so, we define morphisms. By computation, it holds that

$$E_p^r(i) = \frac{Z_p^r(i)}{B_p^r(i) + Z_{p-1}^{r-1}(i)}, \quad f_i^*(Z_p^r(i)) = \mathcal{F}_{p-r}F_i^* \cap f_i^*(\mathcal{F}_pF_{i-1}^*) = B_{p-r}^{r+1}(i+1).$$

Thus the following hold:

(1) $f_i^*(Z_p^r(i)) = B_{p-r}^{r+1}(i+1) \subset Z_{p-r}^r(i+1)$,

(2) $f_i^*(B_p^r(i)) = 0$ and $f_i^*(Z_{p-1}^{r-1}(i)) = B_{p-r}^r(i+1)$.

We can show that f_i^* induces a map $\tilde{f}_p^r(i) : E_p^r(i) \to E_{p-r}^r(i+1)$, by

$$\tilde{f}_p^r(i)(x + B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*) = f_i^*(x) + B_{p-r}^r(i+1) + Z_{p-r-1}^{r-1}(i+1) \ (x \in Z_p^r(i)).$$

Hence $\tilde{f}_p^r(i) (p \in \mathbb{N})$ give a graded $\mathrm{gr}\Lambda$-homomorphism

$$\tilde{f}_i^r : E_i^r = \bigoplus_p E_p^r(i) \longrightarrow E_{i+1}^r = \bigoplus_p E_p^r(i+1)$$

of degree $-r$. It is easily seen that $E_\bullet^r : \cdots \to E_i^r \xrightarrow{\tilde{f}_i^r} E_{i+1}^r \to \cdots$ is a complex.

2.8.2. Lemma. (cf. [14], p.130 (6)) *Under the above notation, we have* $H^i(E_\bullet^r) \cong E_i^{r+1}$.

Proof. We show

$$H(E_{p+r}^r(i-1) \xrightarrow{f} E_p^r(i) \xrightarrow{g} E_{p-r}^r(i+1)) \cong E_p^{r+1}(i),$$

where we put $f := \tilde{f}_{p+r}^r(i-1)$, $g := \tilde{f}_p^r(i)$. Using (1) and (2), we can show that

$$x + B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^* \in \mathrm{Ker} g \iff x \in (f_i^*)^{-1}(B_{p-r}^r(i+1) + \mathcal{F}_{p-r-1}F_i^*).$$

Thus we get

$$\mathrm{Ker}g = \frac{(Z_p^r(i) \cap (f_i^*)^{-1}(B_{p-r}^r(i+1) + \mathcal{F}_{p-r-1}F_i^*) + \mathcal{F}_{p-1}F_{i-1}^*}{B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*}.$$

Further, we have

$$\mathrm{Im}f = \frac{f_{i-1}^*(Z_{p+r}^r(i-1)) + \mathcal{F}_{p-1}F_{i-1}^*}{B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*}.$$

Hence the desired homology is

$$\begin{aligned}
\frac{\mathrm{Ker}g}{\mathrm{Im}f} &= \frac{(Z_p^r(i) \cap (f_i^*)^{-1}(B_{p-r}^r(i+1) + \mathcal{F}_{p-r-1}F_i^*) + \mathcal{F}_{p-1}F_{i-1}^*}{f_{i-1}^*(Z_{p+r}^r(i-1)) + \mathcal{F}_{p-1}F_{i-1}^*} \\
&= \frac{Z_{p-1}^{r-1}(i) + Z_p^r(i) \cap (f_i^*)^{-1}(\mathcal{F}_{p-r-1}F_i^*) + \mathcal{F}_{p-1}F_{i-1}^*}{f_{i-1}^*(Z_{p+r}^r(i-1)) + \mathcal{F}_{p-1}F_{i-1}^*} \\
&= \frac{Z_{p-1}^{r-1}(i) + Z_p^{r+1}(i) + \mathcal{F}_{p-1}F_{i-1}^*}{f_{i-1}^*(Z_{p+r}^r(i-1)) + \mathcal{F}_{p-1}F_{i-1}^*} \\
&= \frac{Z_p^{r+1}(i) + \mathcal{F}_{p-1}F_{i-1}^*}{B_p^{r+1}(i) + \mathcal{F}_{p-1}F_{i-1}^*} \\
&= E_p^{r+1}(i),
\end{aligned}$$

where (2) (respectively, (1)) is used to show the second (respectively, fourth) equality. □

2.8.3. Corollary. *Assume that* $E_{i-1}^1 = 0$. *Then we have* $E_{i-1}^r = 0$ *for* $r \geq 1$ *and there exists an exact sequence*

$$0 \to E_i^{r+1} \to E_i^r \to E_{i+1}^r$$

of $\mathrm{gr}\Lambda$*-modules for each* $r \geq 1$.

Proof. The first assertion directly follows from Lemma 2.8.2. Then the complex $E_\bullet^r$ yields an exact sequence $0 \to H^i(E_\bullet^r) \to E_i^r \to E_{i+1}^r$. Since $H^i(E_\bullet^r) \cong E_i^{r+1}$ by lemma 2.8.2, we get the desired exact sequence. □

2.8.4. We will show in this subsection that $E_{s+1}^r \neq 0$.

Condider the following commutative diagram

$$\begin{array}{ccccccccc}
E_\bullet^0 = \mathrm{gr}(F_\bullet^*): & 0 & \to & \mathrm{gr}F_0^* & \to & \cdots & \to & \mathrm{gr}F_i^* & \to & \cdots \\
& & & \wr\| & & & & \wr\| & & \\
& 0 & \to & (\mathrm{gr}F_0)^* & \to & \cdots & \to & (\mathrm{gr}F_i)^* & \to & \cdots,
\end{array}$$

where rows are complexes and the second row is obtained by applying $\mathrm{Hom}_{\mathrm{gr}\Lambda}(-, \mathrm{gr}\Lambda)$ to a free resolution $\cdots \to \mathrm{gr}F_1 \to \mathrm{gr}F_0 \to \mathrm{gr}M \to 0$ of $\mathrm{gr}M$. Hence an isomorphism $E_{i+1}^1 \cong \mathrm{Ext}_{\mathrm{gr}\Lambda}^i(\mathrm{gr}M, \mathrm{gr}\Lambda)$ holds by Lemma 2.8.2. (Note that $E_i^0 \cong \mathrm{gr}F_{i-1}^*$.)

By assumption, we can apply A.15 to $\mathrm{gr}M$ and get the fact that grade $\mathrm{Ext}_{\mathrm{gr}\Lambda}^s(\mathrm{gr}M, \mathrm{gr}\Lambda) = s$. Hence it holds that grade$E_{s+1}^1 = s$ and $E_{i+1}^1 = 0$ for $i < s$. By Corollary 2.8.3, we get an exact sequence of $\mathrm{gr}\Lambda$-modules

$$(3) \quad 0 \to E_{s+1}^{r+1} \to E_{s+1}^r \xrightarrow{\varphi} E_{s+2}^r.$$

By Lemma 2.8.2, E_{s+2}^r is a subfactor of E_{s+2}^{r-1} for $r \geq 1$. Thus every $\mathrm{gr}\Lambda$-submodule U of E_{s+2}^r is also a subfactor of $E_{s+2}^1 = \mathrm{Ext}_{\mathrm{gr}\Lambda}^{s+1}(\mathrm{gr}M, \mathrm{gr}\Lambda)$, so that there exist $\mathrm{gr}\Lambda$-submodules $X, Y \subset \mathrm{Ext}_{\mathrm{gr}\Lambda}^{s+1}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ such that $U \cong X/Y$. Since grade$X \geq s+1$ and grade$Y \geq s+1$

by A.14, it holds that grade$U \geq s+1$. Therefore, grade(Imφ_r) $\geq s+1$ for $r \geq 1$. Consider the exact sequence induced from (3):

$$0 \to E^{r+1}_{s+1} \to E^r_{s+1} \to \mathrm{Im}\varphi_r \to 0.$$

Assume that grade$E^r_{s+1} = s$. Then grade$E^{r+1}_{s+1} = s$ holds. Hence grade$E^r_{s+1} = s$ holds for all $r \geq 1$ by induction. Especially, $E^r_{s+1} \neq 0$ holds for all $r \geq 1$.

2.8.5. Lemma *There is an isomorphism* $E^r_{i+1} \cong \mathrm{gr}(\mathrm{Ext}^i_\Lambda(M, \Lambda))$ *for* $i \geq 0$ *and* $r >> 0$.

Proof. Since the filtration $\mathcal{F}$ of Λ is Zariskian (see [14], Chapter I, §2, 2.4; §3, 3.3 and Chapter II, §2, 2.1, and Proposition 2.2.1), the lemma follows from [14], Chapter III, §2, Lemma 2.2.1(p. 150) and §1, Corollary 1.1.7(p. 133). □

2.8.6. We have shown that $E^r_{s+1} \neq 0$. Hence $\mathrm{Ext}^s_\Lambda(M, \Lambda) \neq 0$ by Lemma 2.8.5. Therefore, (i) holds.

Conversely, since grade gr$M = s$, we have $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = 0$ for $i < s$. Since $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda)$ is a subfactor of $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ by Proposition 2.1, we have $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i < s$. Therefore, $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i < s$, so that (ii) holds. This accomplishes the proof of 2.8. □

2.9. Remarks. (i) Let $M \in \mathcal{G}$. Then it follows from 2.3 and 2.8 that

$$\text{G-dim}\,\mathrm{gr}M \geq \text{G-dim}M \geq \mathrm{grade}M = \mathrm{grade}\,\mathrm{gr}M.$$

If gr$M$ is perfect, then above inequalities are equalities. Hence $M \in \mathcal{G}_e$.

(ii) Let $M \in \mathcal{G}_e$ with G-dim$M = d$. Then every syzygy $\Omega^i M$ of M is also in $\mathcal{G}_e$. For, as $\mathrm{gr}(\Omega^i M)$ and $\Omega^i(\mathrm{gr}M)$ are stably isomorphic, we see that G-dim $\Omega^i M$ = G-dim $\mathrm{gr}(\Omega^i M)$ = $\max\{0,\, d - i\}$.

Applying Theorem 2.8 to the case that grΛ is a Gorenstein ring, we get the following.

2.10. Corollary. *Let* Λ *be a filtered ring such that* grΛ *is a commutative Gorenstein ring and* M *a filtered* Λ*-module with a good filtration. Then the equality* $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ *holds.*

Proof. Since all the finitely generated grΛ-modules have finite G-dimension (see the proof of [1], Theorem 4.20), this follows from Theorem 2.8. □

2.11. Theorem. *Let* Λ *be a Gorenstein filtered ring. Let* M *be a filtered* Λ*-module with a good filtration. Then the following equality holds.*

$$\mathrm{grade}M + {}^*\mathrm{dim}\,\mathrm{gr}M = {}^*\mathrm{dim}\,\mathrm{gr}\Lambda = {}^*\mathrm{id}\,\mathrm{gr}\Lambda.$$

Proof. This follows from A.9, A.10, A.12 and 2.8. □

When Λ is a Gorenstein filtered ring, due to the above equality, we can define a holonomic module. Put ${}^*\mathrm{id}\,\mathrm{gr}\Lambda = n$ and $\mathrm{id}\Lambda = d$. Let M be a filtered Λ-module with a good filtration. Since grade$M \leq \mathrm{id}\Lambda = d$, we have $n - {}^*\mathrm{dim}\,\mathrm{gr}M \leq d$, hence

$${}^*\mathrm{dim}\,\mathrm{gr}M \geq n - d.$$

This inequality is a generalization of Bernstein's inequality for a Weyl algebra ([4]).

According to the case of Weyl algebras, we call a finitely generated filtered Λ-module M a *holonomic module*, if ${}^*\mathrm{dim}\,\mathrm{gr}M = n - d$.

3. Cohen-Macaulay modules and holonomic modules

Throughout this section, we assume that Λ is a filtered ring such that $\mathrm{gr}\Lambda$ is a Cohen-Macaulay *local ring with the condition (P) (cf. Appendix). Let M be a finitely generated filtered Λ-module such that $M \in \mathcal{G}$, i.e., $\text{G-dim}\,\mathrm{gr}M < \infty$. It follows from 2.3, 2.8, A.8 and A.12 that the following holds:

$$(1) \quad \text{G-dim}M + {}^*\mathrm{depth}\,\mathrm{gr}M \leq n$$

$$(2) \quad \mathrm{grade}M + {}^*\mathrm{dim}\,\mathrm{gr}M = n,$$

where we put $n := {}^*\mathrm{depth}\,\mathrm{gr}\Lambda = {}^*\mathrm{dim}\,\mathrm{gr}\Lambda$. We say that $M \in \mathcal{G}$ is a *Cohen-Macaulay Λ-module of codimension* k, if ${}^*\mathrm{depth}\,\mathrm{gr}M = {}^*\mathrm{dim}\,\mathrm{gr}M = n - k$. Then it is easily seen that if M is Cohen-Macaulay of codimension k then it is perfect of grade k, where, due to [1], Definition 4.34, we call M perfect if $\text{G-dim}M = \mathrm{grade}M$. Note also that M is Cohen-Macaulay if and only if $\mathrm{gr}M$ is a perfect $\mathrm{gr}\Lambda$-module by A.8 and A.12. We put

$$\mathcal{C}_k(\Lambda) := \{M \in \mathcal{G} : M \text{ is a Cohen-Macaulay } \Lambda\text{-module of codimension } k\}.$$

The following is an easy consequence of (1) and (2).

3.1. Proposition. *Let $M \in \mathcal{C}_k(\Lambda)$. Then $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $i \neq k$ $(i \geq 0)$.*

We slightly generalize [16], Lemma 2.7 and Theorem 2.8, and [15], as follows.

3.2. Theorem. *Let $M \in \mathcal{G}$.*

i) If $M \in \mathcal{C}_k(\Lambda)$, then $\mathrm{Ext}^k_\Lambda(M, \Lambda) \in \mathcal{C}_k(\Lambda^{\mathrm{op}})$.

ii) The functor $\mathrm{Ext}^k_\Lambda(-, \Lambda)$ induces a duality between the categories $\underline{\mathcal{C}}_k(\Lambda)$ and $\underline{\mathcal{C}}_k(\Lambda^{\mathrm{op}})$.

3.2.1. Lemma. *Let N be a finitely generated filtered Λ-module of* $\mathrm{grade}\,\mathrm{gr}N = s$. *If the G-dimension of $\mathrm{gr}N$ is finite, then we have an embedding $\mathrm{gr}(\mathrm{Ext}^s_\Lambda(N, \Lambda)) \hookrightarrow \mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda)$. Moreover, if $\mathrm{gr}N$ is perfect, then the embedding is an isomorphism.*

Proof. Let $\cdots \to F_1 \to F_0 \to N \to 0$ be a filtered free resolution of N. We use the notation of 2.8.1. It follows from 2.8.2 and 2.8 that

$$E^1_s \cong H^s(E^0_\bullet) \cong H^{s-1}(F_\bullet) = \mathrm{Ext}^{s-1}_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda) = 0,$$

where a complex $F_\bullet : 0 \to F^*_0 \to F^*_1 \to \cdots$ is as in 2.8.1. There exists an exact sequence

$$0 \to E^{r+1}_{s+1} \to E^r_{s+1} \to E^r_{s+2}$$

for all $r \geq 1$ by 2.8.3, so that $E^r_{s+1} \subset E^1_{s+1}$ for all $r \geq 1$. It follows from Lemma 2.8.5 that, for $r >> 0$,

$$E^r_{s+1} \cong \mathrm{gr}(\mathrm{Ext}^s_\Lambda(N, \Lambda)).$$

Thus, by 2.8.2, we get

$$\mathrm{gr}(\mathrm{Ext}^s_\Lambda(N, \Lambda)) \subset E^1_{s+1} \cong \mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda).$$

Assume further that $\mathrm{gr}N$ is perfect. Since E^r_{s+2} is a subfactor of $E^1_{s+2} \cong \mathrm{Ext}^{s+1}_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda) = 0$, we see $E^r_{s+2} = 0$, which shows that the embedding is an isomorphism. □

3.2.2. Proof of 3.2. i) Since $\mathrm{gr}M$ is perfect of grade k, it holds that $\mathrm{Ext}^k_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ is perfect of grade k by [1], Proposition 4.35 and its proof, and so $\mathrm{gr}\,\mathrm{Ext}^k_\Lambda(M, \Lambda)$ is perfect by Lemma 3.2.1. Hence $\mathrm{Ext}^k_\Lambda(M, \Lambda) \in \mathcal{C}_k(\Lambda^{\mathrm{op}})$. ii) Consider the exact sequence

$$0 \to \mathrm{Ext}^k_\Lambda(M, \Lambda) \to \mathrm{Tr}\Omega^{k-1}M \to \Omega\mathrm{Tr}\Omega^k M \to 0$$

(see, for example, the proof of [13], Lemma 2.1) and apply $(-)^*$ to it. Then we get a long exact sequence

$$\mathrm{Ext}^{k+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^k M, \Lambda) \to \mathrm{Ext}^{k}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^{k-1} M, \Lambda) \to \mathrm{Ext}^{k}_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda), \Lambda) \to \mathrm{Ext}^{k+2}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^k M, \Lambda).$$

Since G-dim$\mathrm{Tr}\Omega^k M = 0$ by assumption, the first and fourth terms of the above exact sequence vanishes. Hence $M \cong \mathrm{Ext}^{k}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^{k-1} M, \Lambda) \cong \mathrm{Ext}^{k}_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda), \Lambda)$ by [13], Lemma 2.5. Therefore, there is a natural isomorphism $M \cong \mathrm{Ext}^{k}_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda), \Lambda)$ for $M \in \mathcal{C}_k(\Lambda)$, which induces a duality between the categories $\underline{\mathcal{C}}_k(\Lambda)$ and $\underline{\mathcal{C}}_k(\Lambda^{\mathrm{op}})$. □

3.2.3. Remark. The proof 3.2.2 ii) only needs M to be perfect with grade$M = k$. Hence we see that if M is perfect of grade k then $M \cong \mathrm{Ext}^{k}_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda), \Lambda)$.

We shall study holonomic modules when Λ is a Gorenstein filtered ring, that is, grΛ is Gorenstein, and generalize the former theory which is under the assumption of regularity (cf. [14], Chapter III, §4). The assumption that Λ is Gorenstein implies that $\mathcal{G} = \mathrm{fil}\Lambda$ by A.9, where filΛ is the category of all finitely generated filtered (left) Λ-modules. We recall from Corollary 2.4 and the end of section two that $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda (= d)$ and $M \in \mathrm{fil}\Lambda$ is called holonomic, if ${}^*\dim \mathrm{gr} M = n - d$, where $n = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda = {}^*\dim \mathrm{gr}\Lambda = {}^*\mathrm{id}\,\mathrm{gr}\Lambda$. We see that if $M \in \mathcal{C}_d(\Lambda)$ then M is holonomic. We also note that M is holonomic if and only if grade$M = d$ (or grade gr$M = d$) if and only if M is perfect of grade d. We keep to assume Λ to be a Gorenstein filtered ring and $d = \mathrm{id}_\Lambda\Lambda$ in the rest of this section. According to [6], Theorem 3.9, if Λ is a Gorenstein filtered ring, then Λ satisfies 'Auslander condition' :

For every finitely generated Λ-module M and integer $k \geq 0$,
it holds that $\mathrm{grade}_{\Lambda^{\mathrm{op}}} N \geq k$ for all Λ^{op}-submodules $N \subset \mathrm{Ext}^k_\Lambda(M,\Lambda)$.

3.3. Proposition. *Let M be a finitely generated filtered Λ-module. Let M be holonomic and N a Λ-submodule of M. Then N, M/N are holonomic.*

Proof. It follows from [13], Lemma 2.11 (cf. also [6], Theorem 3.9) that grade$N \geq d$ and grade$M/N \geq d$, so that grade$N = d$ and grade$M/N = d$. □

3.4. Proposition. *A holonomic module is artinian. Therefore, it is of finite length.*

We use the following easy lemma for a proof.

3.4.1. Lemma. *Let M_i $(i = 0, 1, \cdots)$ be a module over a ring and $f_i : M_i \to M_{i+1}$ $(i = 0, 1, \cdots)$ is a homomorphism. Assume that M_0 is Noetherian and f_i $(i = 0, 1, \cdots)$ is surjective. Then there exists an interger m such that f_i is an isomorphism for all $i \geq m$.*

3.4.2. Proof of 3.4. Let M be a holonomic Λ-module and $M = M_0 \supset M_1 \supset \cdots$ a descending chain of Λ-submodules of M. Then M_i, M_{i-1}/M_i are holonomic $(i \geq 1)$, and so, from an exact sequence $0 \to M_i \to M_{i-1} \to M_{i-1}/M_i \to 0$, we get an exact sequence

$$0 \to \mathbb{E}(M_{i-1}/M_i) \to \mathbb{E}M_{i-1} \to \mathbb{E}M_i \to 0,$$

where we put $\mathbb{E}(-) = \mathrm{Ext}^d_\Lambda(-, \Lambda)$. By Lemma 3.4.1, there exists an integer m such that $\mathbb{E}M_{i-1} \to \mathbb{E}M_i$ is an isomorphism for $i \geq m+1$. Hence $\mathbb{E}(M_{i-1}/M_i) = 0$ for $i \geq m+1$. Hence $M_{i-1}/M_i = 0$ for $i \geq m+1$ by Remark 3.2.3, that is, $M_m = M_{m+1} = \cdots$. This completes the proof. □

We generalize [14], Chapter Ⅲ, 4.2.18 Theorem (p. 194), which characterizes a holonomic module by its associated graded module. We put $\mathrm{Min}(\mathrm{gr}M) = \{\mathfrak{p} : \mathfrak{p}$ is a minimal element of $\mathrm{Supp}(\mathrm{gr}M)\}$ for $M \in \mathrm{fil}\Lambda$.

3.5. Theorem. *Let $M \in \mathrm{fil}\Lambda$. Then the following are equivalent.*

(1) M is holonomic,

(2) $\mathrm{ht}\mathfrak{p} = d$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$.

A finitely generated module M over a two-sided Noetherian ring is called *pure*, if $\mathrm{grade}N = \mathrm{grade}M$ for all nonzero submodules N of M.

3.5.1. Lemma. *Let $M \in \mathrm{fil}\Lambda$. Then M is pure if and only if $\mathrm{gr}M$ is a pure $\mathrm{gr}\Lambda$-module under a suitable filtration on M.*

Proof. Let M be pure. Put $s = \mathrm{grade}M$ and $N := \mathrm{Ext}^s_\Lambda(M, \Lambda)$. Since Λ satisfies Auslander condition, it follows that $\mathrm{grade}N = s$ by [13], Lemma 2.8, so that $\mathrm{grade}\,\mathrm{gr}N = s$ by 2.8, hence $\mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda)$ is pure by [13], Proposition 2.13. By 3.2.1, we have $\mathrm{gr}\,\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda) \subset \mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda)$. Hence $\mathrm{gr}\,\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$ is a pure $\mathrm{gr}\Lambda$-module. By [13], Theorem 2.3, there exists an exact sequence

$$0 \to \mathrm{Ext}^{s+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^s M, \Lambda) \to M \to \mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda).$$

Since $\mathrm{grade}\,\mathrm{Ext}^{s+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^s M, \Lambda) \geq s+1$ by Auslander condition and M is pure, we see $\mathrm{Ext}^{s+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^s M, \Lambda) = 0$. Therefore, $M \subset \mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$. According to a filtration on M induced from that of $\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$, we get an inclusion $\mathrm{gr}M \subset \mathrm{gr}\,\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$, hence $\mathrm{gr}M$ is pure. The converse is obvious by Theorem 2.8. □

3.5.2. Lemma. *Let R be a commutative Gorenstein ring and M' a pure R-module. Then $\mathrm{grade}M' = \mathrm{dim}R_\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Min}(M')$.*

Proof. Since $R_\mathfrak{p}$ is a Gorenstein local ring, we have an equality $\mathrm{grade}M'_\mathfrak{p} + \mathrm{dim}M'_\mathfrak{p} = \mathrm{dim}R_\mathfrak{p}$ (cf. [11], Proposition 4.11). Since $\mathfrak{p}$ is minimal, $\mathrm{dim}M'_\mathfrak{p} = 0$, so that, $\mathrm{grade}M'_\mathfrak{p} = \mathrm{dim}R_\mathfrak{p}$.

Put $g = \mathrm{grade}M'_\mathfrak{p}$, $g' = \mathrm{grade}M'$. Since $\mathrm{Ext}^g_R(M', R)_\mathfrak{p} = \mathrm{Ext}^g_{R_\mathfrak{p}}(M'_\mathfrak{p}, R_\mathfrak{p}) \neq 0$, we have $\mathrm{Ext}^g_R(M', R) \neq 0$. Hence $g \geq g'$ holds. Suppose that $\mathrm{Ext}^k_R(\mathrm{Ext}^k_R(M', R), R) \neq 0$ for $k > g'$. Then there exists $N \subset M'$ such that $\mathrm{grade}N = k > g'$ by [13], Theorem 2.3, which contradicts the purity of M'. Hence $\mathrm{Ext}^k_R(\mathrm{Ext}^k_R(M', R), R) = 0$ for all $k > g'$. But by A.15, $\mathrm{grade}\,\mathrm{Ext}^g_{R_\mathfrak{p}}(M'_\mathfrak{p}, R_\mathfrak{p}) = g$. Therefore, we see $g \leq g'$, and so, $g = g'$. This completes the proof. □

3.5.3. proof of Theorem 3.5. Put $R = \mathrm{gr}\Lambda$.

(1)⇒(2): Assume that M is holonomic. Since M is pure by Proposition 3.3, $\mathrm{gr}M$ is pure by 3.5.1. Thus $d = \mathrm{grade}\,\mathrm{gr}M = \mathrm{dim}R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$ by 3.5.2. Therefore, $\mathrm{ht}\mathfrak{p} = d$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$.

(2)⇒(1): Put $I = [0 :_R \mathrm{gr}M]$. Since R is Cohen-Macaulay, we have $\mathrm{ht}I = \mathrm{grade}R/I$ by [8], Corollary 2.1.4. It follows from [8], Proposition 1.2.10(e) that $\mathrm{grade}R/I = \mathrm{grade}\,\mathrm{gr}M$. By assumption, $\mathrm{ht}I = d$, so that, $\mathrm{grade}\,\mathrm{gr}M = d$, that is, $\mathrm{grade}M = d$ by 2.8. Hence M is holonomic. □

A module having higher grade has a good property.

3.6. Proposition. *Let M be a finitely generated filtered Λ-module with* $\mathrm{grade}M = \ell$, *where $\ell = d-1$ or $d-2$. Then M is a perfect Λ-module if and only if there exists a finitely generated filtered Λ^{op}-module M' of grade ℓ with $M \cong \mathrm{Ext}^{\ell}_{\Lambda^{\mathrm{op}}}(M', \Lambda)$.*

Proof. Assume that $M \cong \mathrm{Ext}^{\ell}_{\Lambda^{\mathrm{op}}}(M', \Lambda)$ with $\mathrm{grade}M' = \ell$.

The case $\ell = d-1$: We see $\mathrm{grade}M = d-1$ by assumption. It follows that $\mathrm{grade}\,\mathrm{Ext}^d_\Lambda(M, \Lambda) = \mathrm{grade}\,\mathrm{Ext}^d_\Lambda(\mathrm{Ext}^{d-1}_\Lambda(M', \Lambda), \Lambda) \geq d+2$ by [13], Corollary 2.10. This shows that $\mathrm{Ext}^d_\Lambda(M, \Lambda) = 0$, that is, $\text{G-dim}M \leq d-1$. Hence $\text{G-dim}M = \mathrm{grade}M = d-1$, so that M is perfect.

The case $\ell = d-2$: It follows from the similar computations as the above case that $\mathrm{grade}\,\mathrm{Ext}^d_\Lambda(M, \Lambda) \geq d+2$ and $\mathrm{grade}\,\mathrm{Ext}^{d-1}_\Lambda(M, \Lambda) \geq d+1$. Hense $\mathrm{Ext}^d_\Lambda(M, \Lambda) = \mathrm{Ext}^{d-1}_\Lambda(M, \Lambda) = 0$, so that $\text{G-dim}M = \mathrm{grade}M = d-2$, i.e., M is perfect.

The converse follows from [15], Theorem 4. □

3.7. Following [14], Chapter III, 4.3, we call a filtered Λ-module M *geometrically pure* (*geo-pure* for short), if $\dim_{\mathrm{gr}\Lambda}\mathrm{gr}M = \dim(\mathrm{gr}\Lambda/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$. Then we have the following proposition which is a generalization of [14], Chapter III, 4.3.6 Corollary.

3.7.1. Proposition. *Let M be a finitely generated filtered Λ-module, and put* $\mathrm{Assh}\,\mathrm{gr}M := \{\, \mathfrak{p} \in \mathrm{Supp}\,\mathrm{gr}M \mid \dim \mathrm{gr}\Lambda/\mathfrak{p} = \dim \mathrm{gr}M \,\}$. *Then the following conditions are equivalent.*

(1) M is pure,

(2) M is geo-pure and $\mathrm{gr}M$ *has no embedded prime.*

(3) $\mathrm{Ass}\,\mathrm{gr}M = \mathrm{Assh}\,\mathrm{gr}M$

Proof. (1)⇒(2): Let M be pure. Then $\mathrm{gr}M$ is pure by 3.5.1. Take any $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$. Since $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{gr}M$, we have $\mathrm{gr}\Lambda/\mathfrak{p} \hookrightarrow \mathrm{gr}M$, so $\mathrm{grade}\,\mathrm{gr}\Lambda/\mathfrak{p} = \mathrm{grade}\,\mathrm{gr}M$. Using Theorem A.12, we have $\dim \mathrm{gr}\Lambda/\mathfrak{p} = \dim \mathrm{gr}M$. Hence M is geo-pure. Take any $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{gr}M$, then $\mathrm{gr}\Lambda/\mathfrak{p} \hookrightarrow \mathrm{gr}M$. Thus $\dim \mathrm{gr}\Lambda/\mathfrak{p} = \dim \mathrm{gr}M$, by A.12. Therefore, $\mathrm{Ass}\,\mathrm{gr}M = \mathrm{Min}\,\mathrm{gr}M$, i.e., $\mathrm{gr}M$ has no embedded primes.

(2)⇒(3): The former condition implies $\mathrm{Assh}\,\mathrm{gr}M = \mathrm{Min}\,\mathrm{gr}M$, and the latter one implies $\mathrm{Min}\,\mathrm{gr}M = \mathrm{Ass}\,\mathrm{gr}M$.

(3)⇒(1): By 3.5.1, it suffices to prove that $\mathrm{gr}M$ is pure. Let N be a $\mathrm{gr}\Lambda$-submodule of $\mathrm{gr}M$. Take any $\mathfrak{p} \in \mathrm{Ass}N$. Then $\mathrm{gr}\Lambda/\mathfrak{p} \hookrightarrow N$. Thus, by A.12 and assumption, we have $\mathrm{grade}\,\mathrm{gr}\Lambda/\mathfrak{p} = \mathrm{grade}\,\mathrm{gr}M$. By [13], Lemma 2.11, we have

$$\mathrm{grade}\,\mathrm{gr}M \leq \mathrm{grade}N \leq \mathrm{grade}\,\mathrm{gr}\Lambda/\mathfrak{p} = \mathrm{grade}\,\mathrm{gr}M.$$

Hence $\mathrm{grade}\,\mathrm{gr}M = \mathrm{grade}N$. This completes the proof. □

3.8. Example. We provide an example of a Gorenstein filtered ring Λ.
Let $R = k[[x^2, x^3]]$ be a subring of a formal power series ring $k[[x]]$, where k is a field of characteristic zero. Then $(R, \mathfrak{m})$ is a local Gorenstein (non-regular) ring of $\dim R = 1$, where $\mathfrak{m} = (x^2, x^3)$. Let a differential operator $T = x\partial$ with $\partial = d/dx$. Let Λ be a subring of the first Weyl algebra (see [4], [14]) generated by R and T. Then every element of Λ is written as $\Sigma a_i T^i$, $a_i \in R$. Note that $Tx^i = x^i T + ix^i$, $i \geq 2$. For $P = \Sigma a_i T^i \in \Lambda$, we put $\mathrm{ord}P = \max\{i : a_i \neq 0\}$, an order of P. Let $\mathcal{F}_i\Lambda := \{P \in \Lambda : \mathrm{ord}P \leq i\}$. Then $\{\mathcal{F}_i\Lambda\}$ is a filtration of Λ and $\mathrm{gr}\Lambda = R[t]$, where $t = \sigma_1(T)$. Thus $\mathrm{gr}\Lambda$ is Gorenstein *local of dimension 2. Note that $\mathfrak{m} + tR[t]$ is a unique *maximal ideal.

1) $\mathrm{id}\Lambda = 2$

Let $I := \Lambda T + \Lambda x^2$ be a left ideal of Λ. Then $I \neq \Lambda$. We put induced filtrations to I and Λ/I. , i.e.,

$$\mathcal{F}_i I = I \cap \mathcal{F}_i\Lambda, \quad \mathcal{F}_i(\Lambda/I) = (\mathcal{F}_i\Lambda + I)/I, \quad i \geq 0.$$

Then $0 \to I \to \Lambda \to \Lambda/I \to 0$ is a strict exact sequence. Hence $0 \to \mathrm{gr}I \to \mathrm{gr}\Lambda \to \mathrm{gr}(\Lambda/I) \to 0$ is exact. Since $\mathrm{gr}I$ contains t and x^2, $\mathrm{gr}(\Lambda/I) = \mathrm{gr}\Lambda/\mathrm{gr}I$ is an Artinian $\mathrm{gr}\Lambda$-module. Hence $\dim_{\mathrm{gr}\Lambda}\mathrm{gr}(\Lambda/I) = 0$. Thus $\mathrm{grade}\,\Lambda/I = 2$ by Theorem 2.11, and then $\mathrm{id}\Lambda = 2$ by Corollary 2.4. So Λ/I is holonomic.

2) $\mathrm{gl\,dim}\Lambda = \infty$

It is easily seen that $\mathrm{gr}(\Lambda/\Lambda\mathfrak{m}) \cong R/\mathfrak{m}[t]$, where a filtration of $\Lambda\mathfrak{m}$ is given by $\mathcal{F}_i(\Lambda\mathfrak{m}) = (\mathcal{F}_i\Lambda)\mathfrak{m}$. Thus $\mathrm{pd}_\Lambda\Lambda/\Lambda\mathfrak{m} = \infty$, and $\mathrm{gl\,dim}\Lambda = \infty$.

Appendix

In Appendix, we provide the fact about graded rings, especially *local rings .

1. Summary for *local rings

Let R be a commutative Noetherian ring. We gather some facts about a graded ring. For the detail, the reader is referred to [8], [12], and [20].

A ring R is called a *graded ring*, if

i) $R = \oplus_{i\in\mathbb{Z}} R_i$ as an additive group,

ii) $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$.

An R-module M is called a *graded module*, if

i) $M = \oplus_{i\in\mathbb{Z}} M_i$ as an additive groups,

ii) $R_i M_j \subset M_{i+j}$ for all $i,\ j \in \mathbb{Z}$.

An R-homomorphism $f : M \to N$ of graded modules is called a *graded homomorphism*, if $f(M_i) \subset N_i$ for all $i \in \mathbb{Z}$. All graded modules in $\mathrm{mod}R$ and all graded homomorphisms form the category of graded modules, which we denote by $\mathrm{mod}_0 R$.

A graded submodule of a graded ring R is called a *graded ideal*. For any ideal I of R, we denote by I^* the graded ideal generated by all homogeneous elements of I. A graded ideal $\mathfrak{m}$ of R is called **maximal*, if it is a maximal element of all proper graded ideals of R. We say that R is a **local* ring, if R has a unique *maximal ideal $\mathfrak{m}$. A *local ring R with the *maximal ideal $\mathfrak{m}$ is denoted by $(R, \mathfrak{m})$. The theory of *local ring is well developed and a lot of facts that hold for local rings also hold for *local rings (see [8] and [12]).

Let M be a finite R-module. For an ideal I, we denote I-depth of M by $\mathrm{depth}(I, M)$([18]). Let $(R, \mathfrak{m})$ be a *local ring and $M \in \mathrm{mod}R$. We put ${}^*\mathrm{depth}M := \mathrm{depth}(\mathfrak{m}, M)$. We shall use *depth as a substitute of depth for a local ring.

A graded module M over a graded ring R is called a **injective* module, if it is an injective object in $\mathrm{mod}_0 R$([8], §3.6). We denote by ${}^*\mathrm{id}M$ the *injective dimension of M. By definition, ${}^*\mathrm{id}M \leq k$ if and only if there exists a minimal *injective resolution

$$0 \to M \to {}^*E^0(M) \to \cdots \to {}^*E^k(M) \to 0.$$

It is easily seen that ${}^*\mathrm{id}M \leq k$ if and only if $\mathrm{Ext}^i_R(N, M) = 0$ for all $i > k$ and all $N \in \mathrm{mod}_0 R$.

Let $(R, \mathfrak{m})$ be a *local ring. Consider the following condition.

(P) There exists an element of positive degree in $R - \mathfrak{p}$
for any graded prime ideal $\mathfrak{p} \neq \mathfrak{m}$

A positively graded ring satisfies the condition (P). The other examples are seen in [20], Chapter B, Ⅲ, 3.2.

The following is known.

A.1. Proposition. *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P). Then, for every graded ideal $\mathfrak{a}$ and every set of graded prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$, there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$, whenever all homogeneous elements of $\mathfrak{a}$ are contained in $\cup_{i=1}^n \mathfrak{p}_i$.*

Proof. See [19], Lemma 2. □

Using Proposition A.1, the following is proved as the local case.

A.2. Proposition. *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P). Let M be a finite graded module with ${}^*\mathrm{depth} M = t$. Then there exists an M-sequence $x_1, \cdots, x_t$ consisting of homogeneous elements in $\mathfrak{m}$.*

We note the following graded version of Nakayama's Lemma.

A.3. Lemma. *Let $(R, \mathfrak{m})$ be a *local ring and M a finite graded R-module. If $\mathfrak{m}M = M$, then $M = 0$.*

In the following, we assume that $(R, \mathfrak{m})$ is a *local ring with the condition (P).

A.4. Lemma. *Let M, N be the non-zero finite graded R-module with ${}^*\mathrm{depth} N = 0$. Then $\mathrm{Hom}_R(M, N) \neq 0$.*

Proof. It is well-known, so we omit the proof. □

A.5. Corollary. *Assume that ${}^*\mathrm{depth} R = 0$. Let M be a finite graded R-module. Then $M^* = 0$ implies $M = 0$.*

We state the graded version of [1], 4.11-13 in the following A.6-A.8.

A.6. Proposition. *Assume that ${}^*\mathrm{depth} R = 0$. Let M be a finite graded R-module. Then $\text{G-dim} M < \infty$ if and only if $\text{G-dim} M = 0$.*

Proof. It suffices to prove that $\text{G-dim} M < \infty$ implies $\text{G-dim} M = 0$.

Suppose that $\text{G-dim} M \leq 1$. We have an exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with $\text{G-dim} L_i = 0$ $(i = 0, 1)$. Hence we have an exact sequence

$$0 \to M^* \to L_0^* \to L_1^* \to \mathrm{Ext}_R^1(M, R) \to 0$$

and $\mathrm{Ext}_R^i(M, R) = 0$ for $i > 1$. By this sequence, we have an exact sequence

$$0 \to \mathrm{Ext}_R^1(M, R)^* \to L_1 \to L_0,$$

where $L_1 \to L_0$ is monic. Thus $\mathrm{Ext}_R^1(M, R)^* = 0$, and so $\mathrm{Ext}_R^1(M, R) = 0$ by A.5 Corollary.

Suppose that $\text{G-dim} M \leq n$. Let $0 \to L_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} L_0 \to M \to 0$ be exact with $\text{G-dim} L_i = 0$ $(0 \leq i \leq n)$. Since $\text{G-dim}(\mathrm{Im} f_{n-1}) \leq 1$, we have $\text{G-dim}(\mathrm{Im} f_{n-1}) = 0$ by the above argument. Repeating this process, we get $\text{G-dim} M = 0$. □

We want to generalize [1], Theorem 4.13 (b) to the graded case. The proof of it needs a part of [1], Proposition 4.12. Thus we adapt this proposition as follows.

A.7. Proposition. *Assume that ${}^*\mathrm{depth} R = t$. Let M be a finite graded R-module with $\text{G-dim} M < \infty$. Then the following are equivalent.*

(1) G-dim$M = 0$.

(2) *depth$M \geq {}^*$depth$R$.

(3) *depth$M = {}^*$depth$R$.

Proof. (1) ⇒ (2): Let $x_1, \cdots, x_i$ be a homogeneous regular sequence in $\mathfrak{m}$. We show that $x_1, \cdots, x_i$ is an M-sequence by induction on i. Let $i = 1$. Since $M \cong M^{**}$ is torsionfree, x_1 is M-regular.

Suppose that $i > 1$ and the assertion holds for $i-1$. Then $x_1, \cdots, x_{i-1}$ is an M-sequence. Put $I = (x_1, \cdots, x_{i-1})$, $\overline{R} = R/I$, $\overline{M} = M/IM$. Then $(\overline{R}, \mathfrak{m}/I)$ is a *local ring with the condition (P). By [1], Lemma 4.9, G-dim$_{\overline{R}}\overline{M}$ = G-dim$_R M = 0$. Since $\overline{x}_i \in \overline{R}$ is a regular element, $\overline{x}_i$ is $\overline{M}$-regular, hence $x_1, \cdots, x_i$ is an M-sequence. Therefore, *depth$M \geq {}^*$depth$R$.

(2) ⇒ (1): By assumption, it suffices to prove that $\mathrm{Ext}^i_R(M, R) = 0$ for $i > 0$. We show the assertion by induction on $t = {}^*$depth$R$.

Let $t = 0$. Then G-dim$M = 0$ by Proposition A.6

Let $t > 0$. Then *depth$M \geq {}^*$depth$R \geq 1$. We take a homogeneous element $x \in \mathfrak{m}$ which is R and M-regular. Then, by [8], 1.2.10 (d),

$$ {}^*\mathrm{depth}_{R/xR} M/xM = {}^*\mathrm{depth}_R M - 1 \geq {}^*\mathrm{depth} R - 1 = {}^*\mathrm{depth} R/xR. $$

Hence we have $\mathrm{Ext}^i_{R/xR}(M/xM, R/xR) = 0$ for $i > 0$ by induction. This gives $\mathrm{Ext}^i_R(M, R/xR) = 0$ for $i > 0$. From an exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$, we get an exact sequence

$$ \mathrm{Ext}^i_R(M, R) \xrightarrow{x} \mathrm{Ext}^i_R(M, R) \to \mathrm{Ext}^i_R(M, R/xR) = 0. $$

By Nakayama's Lemma, it holds that $\mathrm{Ext}^i_R(M, R) = 0$ for $i > 0$.

(2) ⇒ (3): When *depth$R = 0$, we have G-dim$M = 0$ by Proposition A.6. Since *depth$R = 0$, we have an exact sequence $0 \to R/\mathfrak{m} \to R$ which gives an exact sequence

$$ 0 \to \mathrm{Hom}_R(M^*, R/\mathfrak{m}) \to M^{**} \cong M. $$

Since $M^* \neq 0$, we have $\mathrm{Hom}_R(M^*, R/\mathfrak{m}) \neq 0$. Since $\mathfrak{m}\mathrm{Hom}_R(M^*, R/\mathfrak{m}) = 0$, we see that $\mathfrak{m}$ has no M-regular element, so that *depth$M = 0$. Thus (3) holds.

Let *depth$R > 0$. We have *depth$M \geq {}^*$depth$R \geq 1$, so that there is a homogeneous element $x \in \mathfrak{m}$ which is R and M-regular. By [1], Lemma 4.9, we have G-dim$_{R/xR}M/xM < \infty$. We have

$$ {}^*\mathrm{depth}_{R/xR} M/xM = {}^*\mathrm{depth}_R M - 1 \geq {}^*\mathrm{depth} R - 1 = {}^*\mathrm{depth} R/xR. $$

Hence, by induction on *depth$R$, we have *depth$_{R/xR}M/xM = {}^*$depthR/xR, and then *depth$M = {}^*$depth$R$.

Since (3) ⇒ (2) is obvious, we accomplish the proof. □

A.8. Theorem. *Let M be a finite graded R-module with* G-dim$M < \infty$. *Then we have an equality*

$$ \text{G-dim}M + {}^*\mathrm{depth}M = {}^*\mathrm{depth}R $$

Proof. We state the proof which is an adaptation of [1]. If G-dim$M = 0$, we are done by the previous proposition. Suppose that G-dim$M = n > 0$ and the equation holds for $n-1$. Let $0 \to K \to F \to M \to 0$ be exact with F graded free and K a graded module. Since G-dim$K = n-1$, we have G-dim$K + {}^*$depth$K = {}^*$depth$R$ by induction. Suppose that *depth$M \geq {}^*$depth$F = {}^*$depth$R$. Then G-dim$M = 0$ holds by the previous proposition. This contradicts to G-dim$M > 0$. Hence *depth$M <$ *depth$F$, so *depth$K =$ *depth$M + 1$ by, e.g., [8], 1.2.9. Therefore, $n +$ *depth$M =$ *depth$R$. □

Let M be a finite graded R-module. Then the similar argument to [1], 4.14 and 4.15 shows that G-dim$M \leq n$ if and only if G-dim$M_{\mathfrak{p}} \leq n$ for all graded prime (respectively, graded maximal) ideals $\mathfrak{p}$ of R. Note that all the prime ideals in Ass$M$ are graded ideals (e.g. [8], Lemma 1.5.6). Thus, in *local case, we have that G-dim$M \leq n$ if and only if G-dim$M_{\mathfrak{m}} \leq n$. Thus we give the following characterization of Gorensteiness.

A.9. Theorem. *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P). Then the following are equivalent.*

(1) R is Gorenstein.

(2) Every finite graded R-module has finite G-dimension.

Under these equivalent conditions, the equality *id$R =$ *depth$R$ holds.

Proof. (1) ⇒ (2): Since $R_{\mathfrak{m}}$ is Gorenstein, we have G-dim$M_{\mathfrak{m}} < \infty$, hence G-dim$M < \infty$ by above.

(2) ⇒ (1): Let $t =$ *depth$R$. Take any finite graded R-module M. Since G-dim$M = t -$ *depth$M \leq t$ by Theorem A.8, we have that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > t$. Hence *id$R \leq t$. It holds from [8], Theorem 3.6.5 or [20], Chapter B, III.1.7 that id$R \leq$ *id$R + 1 \leq t + 1$. Hence R is Gorenstein.

The second statement follows from the similar argument to the local case (cf. [8],Theorem 3.1.17). We note that 'the residue field' in the local case should be replaced by 'the unique graded simple module $R/\mathfrak{m}$' in *local case and the use of the graded version of Bass's Lemma (see e.g. [20], Chapter B, III.1.9) is effective. □

Let $(R, \mathfrak{m})$ be a *local ring. Then one of the following cases occurs ([12], §1 or [8], §1.5):

A. $R/\mathfrak{m}$ is a field,

B. $R/\mathfrak{m} \cong k[t, t^{-1}]$, where k is a field and t is a homogeneous element of positive degree and transcendental over k.

We put *dim$R :=$ ht$\mathfrak{m}$ the *dimension of a *local ring $(R, \mathfrak{m})$. Note that *dim$R$ equals the supremum of all numbers h such that there exists a chain of graded prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h$ in R [8]. Let M be a finite graded R-module. It is easily seen that $[0 :_R M]$ is a graded ideal. Thus we put *dim$M :=$ *dim$R/[0 :_R M]$.

A.10. Lemma. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with the condition (P) and* dim$R = n$*, and M a finite graded R-module. Then we have*

$$^*\mathrm{dim}R = {}^*\mathrm{depth}R = \begin{cases} n & \textit{for} \quad \textit{Case A}, \\ n-1 & \textit{for} \quad \textit{Case B}. \end{cases}$$

$$^*\mathrm{dim}M = \begin{cases} \mathrm{dim}M & \textit{for} \quad \textit{Case A}, \\ \mathrm{dim}M - 1 & \textit{for} \quad \textit{Case B}. \end{cases}$$

Moreover, assume that R is Gorenstein, then id$R =$ dim$R = n$*, where* id$R$ *stands for the injective dimension of R.*

Proof. Case A. Let $\mathfrak{n}$ be a maximal ideal with ht $\mathfrak{n} = n$. If $\mathfrak{n} = \mathfrak{m}$, then ht $\mathfrak{m} = n$. Suppose that $\mathfrak{n}$ is not equal to $\mathfrak{m}$. Then $\mathfrak{n}$ is not graded, so ht $\mathfrak{n}/\mathfrak{n}^* = 1$. Since $R_\mathfrak{n}$ is Cohen-Macaulay,

$$\text{ht } \mathfrak{n}^* R_\mathfrak{n} + \text{dim} R_\mathfrak{n}/\mathfrak{n}^* R_\mathfrak{n} = \text{dim} R_\mathfrak{n} = n$$

([18], Theorem 17.4). Hence ht $\mathfrak{n}^* R_\mathfrak{n} = n-1$, so ht $\mathfrak{n}^* = n-1$. Thus ht $\mathfrak{m} \geq$ ht $\mathfrak{n}^* + 1 = n$, so that ht $\mathfrak{m} = n$. Therefore,

$$^*\text{depth} R = \text{depth} R_\mathfrak{m} = \text{dim} R_\mathfrak{m} = \text{ht } \mathfrak{m} = n.$$

Case B. Let $\mathfrak{n}$ be the same as in Case A. Since $\mathfrak{n}$ is not graded, we have ht $\mathfrak{n}^* = n-1$ by the similar way to Case A. By assumption, we have that $\mathfrak{m} \supset \mathfrak{n}^*$ and $\mathfrak{m}$ is not maximal, so $\mathfrak{m} = \mathfrak{n}^*$. Therefore, ht $\mathfrak{m} = n-1$, hence we get *depth$R = n-1$ by the similar way to Case A.

The equality concerning *dim$M$ follows from the fact that cases A and B are preserved modulo $[0 :_R M]$.

The latter statement is proved in [3] more generally. □

A.11. Lemma *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with the condition (P) and x a homogeneous element in $\mathfrak{m}$. If x is regular, then* dimR/xR = dim$R - 1$.

Proof. The well-known induction argument works due to A.10 Lemma. □

A.12. Theorem *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with the condition (P) and M a finite graded R-module. Then*

$$\text{grade} M + \text{dim} M = \text{dim} R$$

Proof. We follow the proof of [11], Proposition 4.11. Put $n = \text{dim} R$. We prove the statement by induction on n. Suppose that dim$M = n$ and take $\mathfrak{p} \in$ Supp$M$ with dim$R/\mathfrak{p} = n$. Then dim$R_\mathfrak{p} = 0$, so that depth$R_\mathfrak{p} = 0$. Thus $\mathfrak{p} R_\mathfrak{p} \in$ Ass$R_\mathfrak{p}$. Hence $\text{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \neq 0$ implies $\text{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$. Thus $\text{Hom}_R(M, R) \neq 0$, i.e., grade$M = 0$.

When $n = 0$, we have dim$M = 0$. Then the equality holds by above. Let $n > 0$. Then we can assume dim$M < n$. Since dim$R/\mathfrak{p} = n$ for any minimal prime ideal $\mathfrak{p}$ of R, it holds from the assumption that $[0 :_R M] \not\subset \mathfrak{p}$ for any minimal prime ideal $\mathfrak{p}$ of R. Thus $[0 :_R M] \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in$ Ass$R$. Since $[0 :_R M]$ is a graded ideal, $[0 :_R M]$ contains a homogeneous regular element x by A.1 Proposition. We have that $\text{Ext}^i_R(M, R) \cong \text{Ext}^{i-1}_{R/xR}(M, R/xR)$ for $i \geq 0$. Thus $\text{grade}_{R/xR} M = \text{grade}_R M - 1$. By Lemma A.11 and induction, we get $\text{dim}_{R/xR} M + \text{grade}_{R/xR} M = n-1$, hence $\text{dim}_R M + \text{grade}_R M - 1 = n-1$, which gives the desired equality. □

We state a characterization of a Cohen-Macaulay graded module over a *local ring by means of the *depth and *dimension.

A.13. Theorem *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P) and $M \in \text{mod}_0 R$. Then M is Cohen-Macaulay if and only if* *depth$M$ = *dim$M$.

Proof. Put $I = [0 :_R M]$ and $\overline{R} = R/I$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$. Then we have that $^*\text{dim} M = \text{dim} \overline{R}_{\overline{\mathfrak{m}}} = \text{dim} R_\mathfrak{m}/[0 :_{R_\mathfrak{m}} M_\mathfrak{m}] = \text{dim} M_\mathfrak{m}$. It holds from [19] or [20], Chapter B, Theorem III.2.1 that M is Cohen-Macaulay if and only if $M_\mathfrak{m}$ is Cohen-Macaulay. Look at the following inequalities

$$^*\text{depth} M = \text{depth}(\mathfrak{m}, M) \leq \text{depth} M_\mathfrak{m} \leq \text{dim} M_\mathfrak{m} = {}^*\text{dim} M.$$

If ${}^*\mathrm{depth}M = {}^*\mathrm{dim}M$, then $M_{\mathfrak{m}}$ is Cohen-Macaulay by above. Conversely, suppose M to be Cohen-Macaulay. Then $\mathrm{depth}(\mathfrak{m}, M) = \mathrm{depth}M_{\mathfrak{m}}$ holds by [18], Theorem 17.3. Thus we get ${}^*\mathrm{depth}M = {}^*\mathrm{dim}M$ from the above inequalities. □

A.14. Lemma. ([1], Proposition 4.16) *Let R be a commutative Noetherian ring and X a finite R-module with* $\text{G-dim}X < \infty$. *Then* $\mathrm{grade}U \geq i$ *for all* $i > 0$ *and all R-submodules U of* $\mathrm{Ext}^i_R(X, R)$.

Proof. Let $\mathfrak{p} \in \mathrm{Supp}U$. Then $\mathrm{Ext}^i_{R_\mathfrak{p}}(X_\mathfrak{p}, R_\mathfrak{p}) \neq 0$. Hence $\text{G-dim}_{R_\mathfrak{p}} X_\mathfrak{p} \geq i$. By Auslander-Bridger formula ([1], Theorem 4.13 (b) or [9], Theorem 1.4.8), it follows that

$$\mathrm{depth}R_\mathfrak{p} = \mathrm{depth}X_\mathfrak{p} + \text{G-dim}_{R_\mathfrak{p}} X_\mathfrak{p} \geq \text{G-dim}_{R_\mathfrak{p}} X_\mathfrak{p} \geq i.$$

Hence $\mathrm{grade}U = \min\{\mathrm{depth}R_\mathfrak{p} : \mathfrak{p} \in \mathrm{Supp}U\} \geq i$ by [1], Corollary 4.6. □

A.15. Lemma. *Let R be a commutative Noetherian ring and X a finite R-module of grade s. Assume* $\text{G-dim}X$ *to be finite. Then the equality* $\mathrm{grade}\,\mathrm{Ext}^s_R(X, R) = s$ *holds true.*

Proof. When $s = 0$, that is, $X^* \neq 0$, then $X^{***} \neq 0$. Hence $X^{**} \neq 0$.

We assume that $s > 0$. By A.14, it holds that $\mathrm{grade}\,\mathrm{Ext}^s_R(X, R) \geq s$. The converse inequality follows from [13], Lemma 4.4 (Its proof contains trivial misprints : in the last line of p.182, X^*_n should be read $(\Omega^n X)^*$ and three places in line 3-5 of p.183 should be read similarly). Hence we get the desired equality. □

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