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Equivalence Classes of Boundary Conditions in $SU(N)$ Gauge Theory on 2-Dimensional Orbifolds

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We study equivalence classes of boundary conditions in an $SU(N)$ gauge theory on six-dimensional space-time including two-dimensional orbifolds. For five types of two-dimensional orbifolds $S^1/Z_2 \times S^1/Z_2$ and T^2/Z_m ($m = 2, 3, 4, 6$), orbifold conditions and their gauge transformation properties are given and the equivalence relations among boundary conditions are derived. The classification of boundary conditions related to diagonal representatives is carried out using the equivalence relations.

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§1. Introduction

Grand unified theories on orbifolds have been considered phenomenologically since Higgs mass splitting was well realized by the orbifold-breaking mechanism.[1),2),**)] Various types of models have been constructed from a variety of ingredients such as gauge groups, representations of fields, extra dimensions and boundary conditions (BCs) for fields. The features of the first three ingredients have been studied intensively, but those of the last one are not yet fully understood with a few exceptions such as BCs on the orbifolds S^1/Z_2, T^2/Z_2 and T^2/Z_3.

The BCs for bulk fields are classified into equivalence classes using the gauge invariance. Several sets of BCs belong to the same equivalence class and describe the same physics if they are related to gauge transformations. Specifically, the symmetry of BCs is not necessarily the same as the physical symmetry. The physical symmetry is determined by the Hosotani mechanism after the rearrangement of gauge symmetry.[4)] Equivalence classes of BCs and dynamical gauge symmetry breaking have been studied for gauge theories on S^1/Z_2,[5),6),***)] T^2/Z_2[8)] and T^2/Z_3.[9)] It is interesting to study the equivalence classes of BCs for gauge theories on other orbifolds and to construct a phenomenologically viable model based on them.

In the present paper, we study equivalence classes of BCs in an $SU(N)$ gauge theory on six-dimensional space-time including two-dimensional orbifolds. For five types of two-dimensional orbifolds $S^1/Z_2 \times S^1/Z_2$ and T^2/Z_m ($m = 2, 3, 4, 6$), orbifold conditions and their gauge transformation properties are given and the equivalence relations among BCs are derived. The classification of BCs related to diagonal representatives is carried out using the equivalence relations.

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**) In four-dimensional heterotic string models, extra colored Higgs are projected by the Wilson line mechanism.[3)]

***) See Ref. 7) for the breakdown of gauge symmetry on S^1/Z_2 by the Hosotani mechanism.

In §2, general arguments are given for BCs in gauge theories on $S^1/Z_2 \times S^1/Z_2$. Equivalence classes of BCs are defined by the gauge invariance, and the classification of BCs is carried out using the equivalence relations among BCs. In §3, we study the equivalence classes of BCs and classify BCs related to diagonal representatives on T^2/Z_m ($m = 2, 3, 4, 6$). Section 4 is devoted to conclusions.

§2. $S^1/Z_2 \times S^1/Z_2$ orbifold and equivalence classes

2.1. *Boundary conditions*

We study an $SU(N)$ gauge theory defined on a six-dimensional space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$,*) where M^4 is the four-dimensional Minkowski space-time. An extra space $S^1/Z_2 \times S^1/Z_2$ is obtained by identifying points on $T^2 = S^1 \times S^1$ by their parity. Let x and $\vec{y} = (y_1, y_2)$ be coordinates of M^4 and $S^1/Z_2 \times S^1/Z_2$, respectively. On $S^1 \times S^1$, the points $\vec{y} + \vec{e}_1$ and $\vec{y} + \vec{e}_2$ are identified with point $\vec{y}$, where $\vec{e}_1$ and $\vec{e}_2$ are basis vectors, which we take as the unit vectors $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$.**) The orbifold $S^1/Z_2 \times S^1/Z_2$ is obtained by further identifying $(-y_1, y_2)$ and $(y_1, -y_2)$ with (y_1, y_2). The fixed lines or points on $S^1/Z_2 \times S^1/Z_2$ are lines or points that transform themselves under the Z_2 transformations $\vec{y} \to \theta_1\vec{y} = (-y_1, y_2)$, $\vec{y} \to \theta_2\vec{y} = (y_1, -y_2)$ or $\vec{y} \to \theta_3\vec{y} (= \theta_1\theta_2\vec{y} = \theta_2\theta_1\vec{y}) = (-y_1, -y_2)$. There are two fixed lines $(0, y_2)$ and $(1/2, y_2)$ for the first Z_2 transformation and two fixed lines $(y_1, 0)$ and $(y_1, 1/2)$ for the second Z_2 transformation. There are four fixed points $\vec{0} (= (0, 0))$, $\vec{e}_1/2$, $\vec{e}_2/2$ and $(\vec{e}_1 + \vec{e}_2)/2$ for the third Z_2 transformation. Around these lines and points, we define the following ten transformations:

$$\begin{aligned}
&s_{10} : \vec{y} \to \theta_1\vec{y}, \quad s_{11} : \vec{y} \to \theta_1\vec{y} + \vec{e}_1, \quad s_{20} : \vec{y} \to \theta_2\vec{y}, \quad s_{21} : \vec{y} \to \theta_2\vec{y} + \vec{e}_2, \\
&s_{30} : \vec{y} \to \theta_3\vec{y}, \quad s_{31} : \vec{y} \to \theta_3\vec{y} + \vec{e}_1, \quad s_{32} : \vec{y} \to \theta_3\vec{y} + \vec{e}_2, \\
&s_{33} : \vec{y} \to \theta_3\vec{y} + \vec{e}_1 + \vec{e}_2, \quad t_1 : \vec{y} \to \vec{y} + \vec{e}_1, \quad t_2 : \vec{y} \to \vec{y} + \vec{e}_2. \qquad (2\cdot1)
\end{aligned}$$

These satisfy the following relations:

$$\begin{aligned}
&s_{10}^2 = s_{11}^2 = s_{20}^2 = s_{21}^2 = s_{30}^2 = s_{31}^2 = s_{32}^2 = s_{33}^2 = I, \\
&s_{11} = t_1 s_{10}, \quad s_{21} = t_2 s_{20}, \quad t_1 t_2 = t_2 t_1, \\
&s_{30} = s_{10}s_{20} = s_{20}s_{10}, \quad s_{31} = s_{11}s_{20} = s_{20}s_{11}, \\
&s_{32} = s_{10}s_{21} = s_{21}s_{10}, \quad s_{33} = s_{11}s_{21} = s_{21}s_{11}, \qquad (2\cdot2)
\end{aligned}$$

where I is the identity operation. On $S^1/Z_2 \times S^1/Z_2$, point $\vec{y}$ is identified by $\vec{y} + \vec{e}_i$ ($i = 1, 2$) and $\theta_j\vec{y}$ ($j = 1, 2, 3$), but not all six-dimensional bulk fields necessarily take identical values at these points. Under the requirement that the Lagrangian density should be single-valued on $M^4 \times S^1/Z_2 \times S^1/Z_2$, the following BCs for gauge field $A_M(x, \vec{y})$ are allowed:

$$s_{10} : \ A_M(x, \theta_1\vec{y}) = \kappa^{10}_{[M]} P_{10} A_M(x, \vec{y}) P_{10}^\dagger,$$

*) The models on $M^4 \times S^1/Z_2 \times S^1/Z_2$ were studied in Ref. 10).

**) For the estimation of physical quantities, we use physical sizes such as $|\vec{e}_1| = 2\pi R_1$ and $|\vec{e}_2| = 2\pi R_2$. We use integral multiples of the unit vectors as basis vectors for other orbifolds.

$$\text{for } \kappa^{10}_{[\mu]} = 1,\ \kappa^{10}_{[y_1]} = -1,\ \kappa^{10}_{[y_2]} = 1, \tag{2·3}$$

$$s_{11}:\ A_M(x, \theta_1\vec{y} + \vec{e}_1) = \kappa^{11}_{[M]} P_{11} A_M(x, \vec{y}) P^\dagger_{11},$$
$$\text{for } \kappa^{11}_{[\mu]} = 1,\ \kappa^{11}_{[y_1]} = -1,\ \kappa^{11}_{[y_2]} = 1, \tag{2·4}$$

$$s_{20}:\ A_M(x, \theta_2\vec{y}) = \kappa^{20}_{[M]} P_{20} A_M(x, \vec{y}) P^\dagger_{20},$$
$$\text{for } \kappa^{20}_{[\mu]} = 1,\ \kappa^{20}_{[y_1]} = 1,\ \kappa^{20}_{[y_2]} = -1, \tag{2·5}$$

$$s_{21}:\ A_M(x, \theta_2\vec{y} + \vec{e}_2) = \kappa^{21}_{[M]} P_{21} A_M(x, \vec{y}) P^\dagger_{21},$$
$$\text{for } \kappa^{21}_{[\mu]} = 1,\ \kappa^{21}_{[y_1]} = 1,\ \kappa^{21}_{[y_2]} = -1, \tag{2·6}$$

$$s_{30}:\ A_M(x, \theta_3\vec{y}) = \kappa^{30}_{[M]} P_{30} A_M(x, \vec{y}) P^\dagger_{30},$$
$$\text{for } \kappa^{30}_{[\mu]} = 1,\ \kappa^{30}_{[y_1]} = -1,\ \kappa^{30}_{[y_2]} = -1, \tag{2·7}$$

$$s_{31}:\ A_M(x, \theta_3\vec{y} + \vec{e}_1) = \kappa^{31}_{[M]} P_{31} A_M(x, \vec{y}) P^\dagger_{31},$$
$$\text{for } \kappa^{31}_{[\mu]} = 1,\ \kappa^{31}_{[y_1]} = -1,\ \kappa^{31}_{[y_2]} = -1, \tag{2·8}$$

$$s_{32}:\ A_M(x, \theta_3\vec{y} + \vec{e}_2) = \kappa^{32}_{[M]} P_{32} A_M(x, \vec{y}) P^\dagger_{32},$$
$$\text{for } \kappa^{32}_{[\mu]} = 1,\ \kappa^{32}_{[y_1]} = -1,\ \kappa^{32}_{[y_2]} = -1, \tag{2·9}$$

$$s_{33}:\ A_M(x, \theta_3\vec{y} + \vec{e}_1 + \vec{e}_2) = \kappa^{33}_{[M]} P_{33} A_M(x, \vec{y}) P^\dagger_{33},$$
$$\text{for } \kappa^{33}_{[\mu]} = 1,\ \kappa^{33}_{[y_1]} = -1,\ \kappa^{33}_{[y_2]} = -1, \tag{2·10}$$

$$t_1:\ A_M(x, \vec{y} + \vec{e}_1) = U_1 A_M(x, \vec{y}) U^\dagger_1, \tag{2·11}$$

$$t_2:\ A_M(x, \vec{y} + \vec{e}_2) = U_2 A_M(x, \vec{y}) U^\dagger_2, \tag{2·12}$$

where P_{10}, P_{11}, P_{20}, P_{21}, P_{30}, P_{31}, P_{32}, P_{33}, U_1 and U_2 are $N \times N$ matrices. Here, we take matrices with constant elements to define the BCs for the bulk fields for simplicity. The counterparts of Eq. (2·2) are given by

$$\begin{aligned}
&P^2_{10} = P^2_{11} = P^2_{20} = P^2_{21} = P^2_{30} = P^2_{31} = P^2_{32} = P^2_{33} = I, \\
&P_{11} = U_1 P_{10}, \quad P_{21} = U_2 P_{20}, \quad U_1 U_2 = U_2 U_1, \\
&P_{30} = P_{10} P_{20} = P_{20} P_{10}, \quad P_{31} = P_{11} P_{20} = P_{20} P_{11}, \\
&P_{32} = P_{10} P_{21} = P_{21} P_{10}, \quad P_{33} = P_{11} P_{21} = P_{21} P_{11},
\end{aligned} \tag{2·13}$$

where I is the $N \times N$ unit matrix. Then the BCs in $SU(N)$ gauge theories on $S^1/Z_2 \times S^1/Z_2$ are specified with $(P_{10}, P_{11}, P_{20}, P_{21}, P_{30}, P_{31}, P_{32}, P_{33}, U_1, U_2)$. Because any four of these matrices are mutually independent, we choose four unitary and Hermitian matrices P_{10}, P_{11}, P_{20} and P_{21} as independent matrices and often refer to them simply as *BCs*.

2.2. *Gauge invariance and equivalence class*

Given the BCs $(P_{10}, P_{11}, P_{20}, P_{21})$, there still remains residual gauge invariance. Under the gauge transformation with the transformation function $\Omega(x, \vec{y})$, A_M is transformed as

$$A_M \to A'_M = \Omega A_M \Omega^\dagger - \frac{i}{g} \Omega \partial_M \Omega^\dagger, \tag{2·14}$$

where A'_M satisfies, instead of Eqs. (2·3) – (2·6),

$$s_{10}: \ A'_M(x,\theta_1\vec{y}) = \kappa^{10}_{[M]}\left(P'_{10}A'_M(x,\vec{y})P'^{\dagger}_{10} - \frac{i}{g}P'_{10}\partial_M P'^{\dagger}_{10}\right), \tag{2·15}$$

$$s_{11}: \ A'_M(x,\theta_1\vec{y}+\vec{e}_1) = \kappa^{11}_{[M]}\left(P'_{11}A'_M(x,\vec{y})P'^{\dagger}_{11} - \frac{i}{g}P'_{11}\partial_M P'^{\dagger}_{11}\right), \tag{2·16}$$

$$s_{20}: \ A'_M(x,\theta_2\vec{y}) = \kappa^{20}_{[M]}\left(P'_{20}A'_M(x,\vec{y})P'^{\dagger}_{20} - \frac{i}{g}P'_{20}\partial_M P'^{\dagger}_{20}\right), \tag{2·17}$$

$$s_{21}: \ A'_M(x,\theta_2\vec{y}+\vec{e}_2) = \kappa^{21}_{[M]}\left(P'_{21}A'_M(x,\vec{y})P'^{\dagger}_{21} - \frac{i}{g}P'_{21}\partial_M P'^{\dagger}_{21}\right). \tag{2·18}$$

Here P'_{10}, P'_{11}, P'_{20} and P'_{21} are given by

$$\begin{aligned} P'_{10}(\vec{y}) &= \Omega(x,\theta_1\vec{y})P_{10}\Omega^{\dagger}(x,\vec{y}), \quad P'_{11}(\vec{y}) = \Omega(x,\theta_1\vec{y}+\vec{e}_1)P_{11}\Omega^{\dagger}(x,\vec{y}), \\ P'_{20}(\vec{y}) &= \Omega(x,\theta_2\vec{y})P_{20}\Omega^{\dagger}(x,\vec{y}), \quad P'_{21}(\vec{y}) = \Omega(x,\theta_2\vec{y}+\vec{e}_2)P_{21}\Omega^{\dagger}(x,\vec{y}). \end{aligned} \tag{2·19}$$

Theories with different BCs should be equivalent with regard to physical content if they are connected by gauge transformations. The key observation is that the physics should not depend on the gauge chosen. The equivalence is guaranteed in the Hosotani mechanism[4)] and the two sets of BCs are equivalent:

$$(P_{10},P_{11},P_{20},P_{21}) \sim (P'_{10}(\vec{y}),P'_{11}(\vec{y}),P'_{20}(\vec{y}),P'_{21}(\vec{y})). \tag{2·20}$$

The corresponding relations for P'_{10}, P'_{11}, P'_{20} and P'_{21} are given by

$$\begin{aligned} &P'_{10}(\vec{y})P'_{10}(\theta_1\vec{y}) = P'_{10}(\theta_1\vec{y})P'_{10}(\vec{y}) = I, \\ &P'_{11}(\vec{y})P'_{11}(\theta_1\vec{y}+\vec{e}_1) = P'_{11}(\theta_1\vec{y}+\vec{e}_1)P'_{11}(\vec{y}) = I, \\ &P'_{20}(\vec{y})P'_{20}(\theta_2\vec{y}) = P'_{20}(\theta_2\vec{y})P'_{20}(\vec{y}) = I, \\ &P'_{21}(\vec{y})P'_{21}(\theta_2\vec{y}+\vec{e}_2) = P'_{21}(\theta_2\vec{y}+\vec{e}_2)P'_{21}(\vec{y}) = I. \end{aligned} \tag{2·21}$$

In the case that P'_{10}, P'_{11}, P'_{20} and P'_{21} are independent of $\vec{y}$, the above relations reduce to the usual ones, $P'^2_{10} = P'^2_{11} = P'^2_{20} = P'^2_{21} = I$. The equivalence relation (2·20) defines equivalence classes of the BCs.

We illustrate the change of BCs under a singular gauge transformation using an $SU(2)$ gauge theory with the gauge transformation function defined by

$$\Omega(\vec{y}) = \exp[i\alpha(a\tau_1+b\tau_2)y_1 + i\beta(a\tau_1+b\tau_2)y_2]\ , \quad (\alpha,\beta,a,b\in\mathbb{R}) \tag{2·22}$$

where τ_k $(k=1,2,3)$ are Pauli matrices. When we take $(P_{10},P_{11}) = (\tau_3,\tau_3)$, they are transformed as

$$P'_{10} = \Omega(x,\theta_1\vec{y})P_{10}\Omega^{\dagger}(x,\vec{y}) = \exp[i\beta(a\tau_1+b\tau_2)y_2]\tau_3, \tag{2·23}$$

$$P'_{11} = \Omega(x,\theta_1\vec{y}+\vec{e}_1)P_{11}\Omega^{\dagger}(x,\vec{y}) = \exp[i\alpha(a\tau_1+b\tau_2) + i\beta(a\tau_1+b\tau_2)y_2]\tau_3. \tag{2·24}$$

P'_{10} becomes diagonal with $\beta=0$ and then P'_{10} and P'_{11} take the following form:

$$P'_{10} = \tau_3, \tag{2·25}$$

$$\begin{aligned} P'_{11} &= \exp[i(a\tau_1+b\tau_2)]\tau_3 \\ &= \left(I\cos\sqrt{a^2+b^2} + i\frac{a\tau_1+b\tau_2}{\sqrt{a^2+b^2}}\sin\sqrt{a^2+b^2}\right)\tau_3, \end{aligned} \tag{2·26}$$

where we set $\alpha = 1$ and I is the 2×2 unit matrix. P'_{11} also becomes the diagonal form $(-1)^n \tau_3$ when $\sqrt{a^2+b^2} = n\pi$ for an integer n. To obtain a diagonal representative for both (P_{20}, P_{21}) and (P'_{20}, P'_{21}) with the gauge transformation function $\Omega(\vec{y}) = \exp[i(a\tau_1 + b\tau_2)y_1]$, P_{20} and P_{21} should be I or $-I$. In this way, we obtain the following equivalence relation:

$$(\tau_3, \tau_3, \eta_{20} I, \eta_{21} I) \sim (\tau_3, e^{i(a\tau_1+b\tau_2)}\tau_3, \eta_{20} I, \eta_{21} I), \tag{2·27}$$

where η_{20} and η_{21} are 1 or -1. In the same way, we obtain the following equivalence relation:

$$(\eta_{10} I, \eta_{11} I, \tau_3, \tau_3) \sim (\eta_{10} I, \eta_{11} I, \tau_3, e^{i(a\tau_1+b\tau_2)}\tau_3), \tag{2·28}$$

where η_{10} and η_{11} are 1 or -1. The equivalence relations between diagonal representatives are given by

$$(\tau_3, \tau_3, \eta_{20} I, \eta_{21} I) \sim (\tau_3, -\tau_3, \eta_{20} I, \eta_{21} I), \tag{2·29}$$

$$(\eta_{10} I, \eta_{11} I, \tau_3, \tau_3) \sim (\eta_{10} I, \eta_{11} I, \tau_3, -\tau_3). \tag{2·30}$$

2.3. *Classification of boundary conditions*

We classify BCs for bulk fields on the orbifold $S^1/Z_2 \times S^1/Z_2$.

First we show that all BCs are specified by diagonal matrices for the $SU(2)$ gauge group. (1) In the case that P_{10}, P_{11} and P_{20} are the 2×2 unit matrix I up to a sign factor, P_{21} can be diagonalized by a global $SU(2)$ transformation. (2) In the case that P_{10} and P_{11} are I up to a sign factor and P_{20} has a nondiagonal form, we derive $P_{20} = \pm\tau_3$ after a global $SU(2)$ transformation. Then $P_{21} = \pm\tau_3 \exp[i(a\tau_1 + b\tau_2)]$ is allowed, but we obtain $P_{21} = \pm\tau_3$ by the gauge transformation with $\Omega(x, \vec{y}) = \exp[i(a\tau_1 + b\tau_2)y_2]$. (3) In the case that P_{10} is $\pm I$ and P_{11} has a nondiagonal form, we derive $P_{11} = \pm\tau_3$ after a global $SU(2)$ transformation. We obtain $P_{20} = \pm I$ or $\pm\tau_3$ and $P_{21} = \pm I$ or $\pm\tau_3$ using the relations $P_{11}P_{20} = P_{20}P_{11}$ and $P_{11}P_{21} = P_{21}P_{11}$, respectively. (4) In the case that P_{10} has a nondiagonal form, we derive $P_{10} = \pm\tau_3$ after a global $SU(2)$ transformation. We obtain $P_{20} = \pm I$ or $\pm\tau_3$ and $P_{21} = \pm I$ or $\pm\tau_3$ using the relations $P_{10}P_{20} = P_{20}P_{10}$ and $P_{10}P_{21} = P_{21}P_{10}$, respectively. If P_{20} or P_{21} is $\pm\tau_3$, $P_{11} = \pm I$ or $\pm\tau_3$ using the relations $P_{11}P_{20} = P_{20}P_{11}$ and $P_{11}P_{21} = P_{21}P_{11}$. If both P_{20} and P_{21} are $\pm I$, $P_{11} = \pm\tau_3 \exp[i(a\tau_1 + b\tau_2)]$ is allowed. In this case, we obtain $P_{11} = \pm\tau_3$ after the gauge transformation with $\Omega(x, \vec{y}) = \exp[i(a\tau_1 + b\tau_2)y_1]$.

In a similar way, all BCs for the $SU(N)$ gauge group are made of those specified by diagonal matrices after suitable global unitary transformations and local gauge transformations on $S^1/Z_2 \times S^1/Z_2$. We here sketch the proof of this. P_{10} and P_{11} can be diagonalized by a global unitary transformation and a local gauge transformation using the same argument as that for the case of S^1/Z_2.[6] Note that P_{10} and P_{11} remain $\vec{y}$-independent after the transformations as shown from the equivalent relation (2·27). From the relations $P_{10}P_{20} = P_{20}P_{10}$, $P_{10}P_{21} = P_{21}P_{10}$, $P_{11}P_{20} = P_{20}P_{11}$ and $P_{11}P_{21} = P_{21}P_{11}$, we find that P_{20} and P_{21} are block diagonal matrices and that P_{20} is diagonalized by a unitary matrix that belongs to a subgroup of $SU(N)$. Thus, P_{21}

can also be diagonalized by a suitable gauge transformation following the equivalent relation (2·28). In this way, we find that there is at least one diagonal representative of BCs in every equivalent class. The diagonals P_{10}, P_{11}, P_{20} and P_{21} in $SU(N)$ gauge theories are specified by sixteen nonnegative integers (p_k, q_k, r_k, s_k) $(k = 1, 2, 3, 4)$ such that

$$\begin{aligned}
P_{10} = \mathrm{diag}(&\overbrace{[+1]_{p_1}, [+1]_{p_2}, [+1]_{p_3}, [+1]_{p_4}}^{p}, \overbrace{[+1]_{q_1}, [+1]_{q_2}, [+1]_{q_3}, [+1]_{q_4}}^{q}, \\
&\overbrace{[-1]_{r_1}, [-1]_{r_2}, [-1]_{r_3}, [-1]_{r_4}}^{r}, \overbrace{[-1]_{s_1}, [-1]_{s_2}, [-1]_{s_3}, [-1]_{s_4}}^{N-p-q-r}), \\
P_{11} = \mathrm{diag}(&[+1]_{p_1}, [+1]_{p_2}, [+1]_{p_3}, [+1]_{p_4}, [-1]_{q_1}, [-1]_{q_2}, [-1]_{q_3}, [-1]_{q_4}, \\
&[+1]_{r_1}, [+1]_{r_2}, [+1]_{r_3}, [+1]_{r_4}, [-1]_{s_1}, [-1]_{s_2}, [-1]_{s_3}, [-1]_{s_4}), \\
P_{20} = \mathrm{diag}(&[+1]_{p_1}, [+1]_{p_2}, [-1]_{p_3}, [-1]_{p_4}, [+1]_{q_1}, [+1]_{q_2}, [-1]_{q_3}, [-1]_{q_4}, \\
&[+1]_{r_1}, [+1]_{r_2}, [-1]_{r_3}, [-1]_{r_4}, [+1]_{s_1}, [+1]_{s_2}, [-1]_{s_3}, [-1]_{s_4}), \\
P_{21} = \mathrm{diag}(&[+1]_{p_1}, [-1]_{p_2}, [+1]_{p_3}, [-1]_{p_4}, [+1]_{q_1}, [-1]_{q_2}, [+1]_{q_3}, [-1]_{q_4}, \\
&[+1]_{r_1}, [-1]_{r_2}, [+1]_{r_3}, [-1]_{r_4}, [+1]_{s_1}, [-1]_{s_2}, [+1]_{s_3}, [-1]_{s_4}), \qquad (2{\cdot}31)
\end{aligned}$$

where $N = \sum_{k=1}^{4}(p_k + q_k + r_k + s_k)$, $0 \le p_k, q_k, r_k, s_k \le N$ and $[+1]_{p_1}$ stands for

$$[+1]_{p_1} = \underbrace{+1, \cdots, +1}_{p_1}, \cdots, [-1]_{s_4} = \underbrace{-1, \cdots, -1}_{s_4}. \qquad (2{\cdot}32)$$

Then the symmetry of BC becomes

$$\begin{aligned}
SU(N) \;\longrightarrow\; & SU(p_1) \times \cdots \times SU(p_4) \times SU(q_1) \times \cdots \times SU(q_4) \\
& \times SU(r_1) \times \cdots \times SU(r_4) \times SU(s_1) \times \cdots \times SU(s_4) \times U(1)^{15-l}, \qquad (2{\cdot}33)
\end{aligned}$$

where l is the number of $SU(0)$ and $SU(1)$ in $SU(p_1) \times \cdots \times SU(s_4)$. Here and hereafter $SU(0)$ has no meaning and $SU(1)$ unconventionally stands for $U(1)$. We refer to BCs specified by diagonal matrices as *diagonal BCs* and denote the above BC (2·31) as $[p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4; s_1, s_2, s_3, s_4]$. Note that the symmetry of a BC is not necessarily identical to the physical symmetry.

Using the relations (2·27) and (2·28), we can derive the following equivalence relations:

$$\begin{aligned}
&[p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4; s_1, s_2, s_3, s_4] \\
&\sim [p_1 - 1, p_2, p_3, p_4; q_1 + 1, q_2, q_3, q_4; r_1 + 1, r_2, r_3, r_4; s_1 - 1, s_2, s_3, s_4] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } p_1, s_1 \ge 1, \\
&\sim [p_1 + 1, p_2, p_3, p_4; q_1 - 1, q_2, q_3, q_4; r_1 - 1, r_2, r_3, r_4; s_1 + 1, s_2, s_3, s_4] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } q_1, r_1 \ge 1, \\
&\dots \\
&\sim [p_1, p_2, p_3, p_4 - 1; q_1, q_2, q_3, q_4 + 1; r_1, r_2, r_3, r_4 + 1; s_1, s_2, s_3, s_4 - 1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } p_4, s_4 \ge 1,
\end{aligned}$$

$$
\begin{aligned}
&\sim [p_1, p_2, p_3, p_4+1; q_1, q_2, q_3, q_4-1; r_1, r_2, r_3, r_4-1; s_1, s_2, s_3, s_4+1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } q_4, r_4 \geq 1, \\
&\sim [p_1-1, p_2+1, p_3+1, p_4-1; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4; s_1, s_2, s_3, s_4] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } p_1, p_4 \geq 1, \\
&\sim [p_1+1, p_2-1, p_3-1, p_4+1; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4; s_1, s_2, s_3, s_4] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } p_2, p_3 \geq 1, \\
&\cdots \\
&\sim [p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4; s_1-1, s_2+1, s_3+1, s_4-1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } s_1, s_4 \geq 1, \\
&\sim [p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4; r_1, r_2, r_3, r_4; s_1+1, s_2-1, s_3-1, s_4+1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } s_2, s_3 \geq 1. \qquad (2{\cdot}34)
\end{aligned}
$$

Hence, the number of equivalence classes of BCs is ${}_{N+15}C_{15} - 8 \cdot {}_{N+13}C_{15}$.

When the BCs for bulk fields are given, mode expansions are carried out and the one-loop effective potential for Wilson line phases is calculated using the standard method.[4)–9)] From the minimum of effective potential, the physical symmetry and mass spectrum are obtained for each model. We do not carry out these calculations since our purpose is not to study the dynamics of models but to classify the BCs.

As a comment, we can extend our argument to the case with the orbifold $S^1/Z_2 \times \cdots \times S^1/Z_2$. In this case, diagonal BCs are specified by 4^k integers and the number of equivalence classes is ${}_{N+4^k-1}C_{4^k-1} - 4^{k-1}k \cdot {}_{N+4^k-3}C_{4^k-1}$, where k is the number of S^1/Z_2.

§3. T^2/Z_m orbifold and equivalence classes

In this section, we study $SU(N)$ gauge theory on $M^4 \times T^2/Z_m$, where $m = 2, 3, 4$ and 6. We discuss equivalence classes of BCs and obtain the number of BCs related to diagonal representatives for each orbifold.

3.1. *T^2/Z_2 orbifold*

Here we study $SU(N)$ gauge theory on $M^4 \times T^2/Z_2$.*) Let z be the complex coordinate of T^2/Z_2. Here, T^2 is constructed using the $SU(2) \times SU(2)(\simeq SO(4))$ lattice. On T^2, the points $z + e_1$ and $z + e_2$ are identified with the point z where e_1 and e_2 are basis vectors and we take $e_1 = 1$ and $e_2 = i$. The orbifold T^2/Z_2 is obtained by further identifying $-z$ with z. The resultant space is the area depicted in Fig. 1. The fixed points

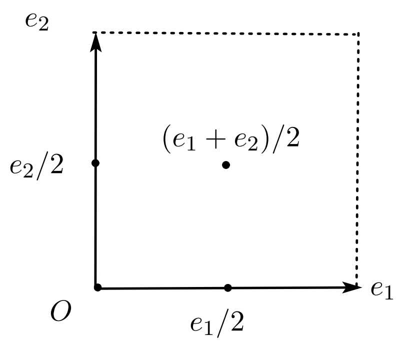

Fig. 1. Orbifold T^2/Z_2.

*) Equivalence classes of BCs and dynamical gauge symmetry breaking were studied for $SU(2)$ gauge theory on T^2/Z_2 in Ref. 8).

$z_{\rm fp}$ for the Z_2 transformation $z \to \theta z = -z$ satisfy

$$z_{\rm fp} = \theta z_{\rm fp} + ne_1 + ne_2, \tag{3·1}$$

where m and n are integers that characterize fixed points. There are four points: 0, $e_1/2$, $e_2/2$ and $(e_1 + e_2)/2$. Around these points, we define the following six transformations:

$$\begin{aligned} &s_0 : z \to -z, \quad s_1 : z \to -z + e_1, \quad s_2 : z \to -z + e_2, \\ &s_3 : z \to -z + e_1 + e_2, \quad t_1 : z \to z + e_1, \quad t_2 : z \to z + e_2. \end{aligned} \tag{3·2}$$

These satisfy the following relations:

$$\begin{aligned} &s_0^2 = s_1^2 = s_2^2 = s_3^2 = I, \ s_1 = t_1 s_0, \ s_2 = t_2 s_0, \\ &s_3 = t_1 t_2 s_0 = s_1 s_0 s_2 = s_2 s_0 s_1, \ t_1 t_2 = t_2 t_1. \end{aligned} \tag{3·3}$$

The BCs of bulk fields are specified by matrices $(P_0, P_1, P_2, P_3, U_1, U_2)$ satisfying the relations

$$\begin{aligned} &P_0^2 = P_1^2 = P_2^2 = P_3^2 = I, \ P_1 = U_1 P_0, \ P_2 = U_2 P_0, \\ &P_3 = U_1 U_2 P_0 = P_1 P_0 P_2 = P_2 P_0 P_1, \ U_1 U_2 = U_2 U_1. \end{aligned} \tag{3·4}$$

Because any three of these matrices are mutually independent, we choose three unitary and Hermitian matrices P_0, P_1 and P_2.

Given the BCs (P_0, P_1, P_2), there is still residual gauge invariance. Under the gauge transformation with $\Omega(x, z, \bar{z})$, P_0, P_1 and P_2 are transformed as

$$\begin{aligned} P_0'(z, \bar{z}) &= \Omega(x, -z, -\bar{z}) P_0 \Omega^\dagger(x, z, \bar{z}), \\ P_1'(z, \bar{z}) &= \Omega(x, -z + e_1, -\bar{z} + \bar{e}_1) P_1 \Omega^\dagger(x, z, \bar{z}), \\ P_2'(z, \bar{z}) &= \Omega(x, -z + e_2, -\bar{z} + \bar{e}_2) P_2 \Omega^\dagger(x, z, \bar{z}). \end{aligned} \tag{3·5}$$

These BCs should be equivalent:

$$(P_0, P_1, P_2) \sim (P_0'(z, \bar{z}), P_1'(z, \bar{z}), P_2'(z, \bar{z})). \tag{3·6}$$

This equivalence relation defines equivalence classes of the BCs. Let us consider an $SU(2)$ gauge theory with the gauge transformation function defined by

$$\Omega(z, \bar{z}) = \exp\left[i\alpha(a\tau_1 + b\tau_2)z + i\bar{\alpha}(a\tau_1 + b\tau_2)\bar{z}\right], \tag{3·7}$$

where a and b are real numbers. When we take $(P_0, P_1, P_2) = (\tau_3, \tau_3, \tau_3)$, they are transformed as

$$(\tau_3, \tau_3, \tau_3) \to (\tau_3, e^{2i{\rm Re}\alpha(a\tau_1 + b\tau_2)}\tau_3, e^{-2i{\rm Im}\alpha(a\tau_1 + b\tau_2)}\tau_3). \tag{3·8}$$

In this way, we obtain the following equivalence relations among the diagonal representatives:

$$(\tau_3, \tau_3, \tau_3) \sim (\tau_3, \tau_3, -\tau_3) \sim (\tau_3, -\tau_3, \tau_3,) \sim (\tau_3, -\tau_3, -\tau_3). \tag{3·9}$$

We classify BCs for fields on the orbifold T^2/Z_2. It is shown that all BCs are specified by diagonal matrices for the $SU(2)$ gauge group. As shown in Ref. 8), the 2×2 matrices that satisfy (3·4) are given by $(P_0, P_1, P_2) = (I, I, I)$, (I, I, τ_3), (I, τ_3, I), (I, τ_3, τ_3) and $(\tau_3, \tau_3 e^{2i\mathrm{Re}\alpha(a\tau_1+b\tau_2)}, \tau_3 e^{-2i\mathrm{Im}\alpha(a\tau_1+b\tau_2)})$ up to a sign factor for each component using a global $SU(2)$ transformation. In the case that $(P_0, P_1, P_2) = (\tau_3, \tau_3 e^{2i\mathrm{Re}\alpha(a\tau_1+b\tau_2)}, \tau_3 e^{-2i\mathrm{Im}\alpha(a\tau_1+b\tau_2)})$, we obtain $(P_0, P_1, P_2) = (\tau_3, \tau_3, \tau_3)$ up to a sign factor for each component after the gauge transformation with $\Omega(z, \bar{z})$ given by (3·7).

For the $SU(N)$ $(N \neq 2, 3)$ gauge group, there are BCs specified by matrices that cannot be diagonalized simultaneously by global unitary transformations and local gauge transformations. Here we give an example. The following set of 4×4 matrices satisfy (3·4):

$$P_0 = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}, \quad P_1 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}, \quad P_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_2 e^{i\zeta\tau_1} & \tau_1 \\ \tau_1 & \tau_2 e^{i\zeta\tau_1} \end{pmatrix}, \tag{3·10}$$

where ζ is an arbitrary real number. In the case with $\zeta \neq n\pi$ $(n \in \mathbb{Z})$, these matrices cannot be transformed into diagonal ones simultaneously. These BCs have no symmetry because there are no 4×4 traceless diagonal matrices simultaneously commutable with the above matrices. Not every component in an $SU(N)$ multiplet necessarily becomes a simultaneous eigenstate of Z_2 parities if the BCs contain off-diagonal elements. As an example, we consider the $SU(N)$ gauge field $A_M = A_M^a T^a$, which satisfies the BC for the Z_2 transformation $z \to -z + e_2$,

$$A_M(x, -z + e_2, -\bar{z} + \bar{e}_2) = \kappa_{[M]} P_2 A_M(x, z, \bar{z}) P_2^\dagger \ , \tag{3·11}$$

where $\kappa_{[\mu]} = 1$, $\kappa_{[z]} = -1$ and $\kappa_{[\bar{z}]} = -1$. In terms of components A_M^a, the above BC (3·11) is written as

$$\begin{aligned} A_M^a(x, -z + e_2, -\bar{z} + \bar{e}_2) &= \sum_b C^a_{[M]b} A_M^b(x, z, \bar{z}) \ , \\ C^a_{[M]b} &\equiv 2\kappa_{[M]} \mathrm{Tr}(P_2 T^b P_2^\dagger T^a) \ , \end{aligned} \tag{3·12}$$

where we use the relation $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$. The A_M^a do not have a definite Z_2 parity for the components, that become mixed with others through the mixing matrices $C^a_{[M]b}$. In this way, the rank of gauge group can be reduced using BCs including off-diagonal elements, which would be useful for the model-building of grand unification or gauge-Higgs unification.

Hereafter, we classify BCs specified by diagonal matrices. The diagonals P_0, P_1 and P_2 are specified by eight non-negative integers (p_i, q_i, r_i, s_i) $(i = 1, 2)$ such that

$$\begin{aligned} P_0 &= \mathrm{diag}(\overbrace{[+1]_{p_1}, [+1]_{p_2}}^{p}, \overbrace{[+1]_{q_1}, [+1]_{q_2}}^{q}, \overbrace{[-1]_{r_1}, [-1]_{r_2}}^{r}, \overbrace{[-1]_{s_1}, [-1]_{s_2}}^{N-p-q-r}), \\ P_1 &= \mathrm{diag}([+1]_{p_1}, [+1]_{p_2}, [-1]_{q_1}, [-1]_{q_2}, [+1]_{r_1}, [+1]_{r_2}, [-1]_{s_1}, [-1]_{s_2}), \\ P_2 &= \mathrm{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{q_1}, [-1]_{q_2}, [+1]_{r_1}, [-1]_{r_2}, [+1]_{s_1}, [-1]_{s_2}), \end{aligned} \tag{3·13}$$

where $0 \le p_i, q_i, r_i, s_i \le N$. Then the symmetry of BC becomes

$$\begin{aligned} SU(N) \longrightarrow\ & SU(p_1) \times SU(p_2) \times SU(q_1) \times SU(q_2) \\ & \times SU(r_1) \times SU(r_2) \times SU(s_1) \times SU(s_2) \times U(1)^{7-l}. \end{aligned} \tag{3·14}$$

We denote the above BC as $[p_1, p_2; q_1, q_2; r_1, r_2; s_1, s_2]$. Using (3·9), we can derive the following relations in $SU(N)$ gauge theory:

$$\begin{aligned} & [p_1, p_2; q_1, q_2; r_1, r_2; s_1, s_2] \\ & \sim [p_1 - 1, p_2 + 1; q_1, q_2; r_1, r_2; s_1 + 1, s_2 - 1], \\ & \sim [p_1 - 1, p_2; q_1 + 1, q_2; r_1, r_2 + 1; s_1, s_2 - 1], \\ & \sim [p_1 - 1, p_2; q_1, q_2 + 1; r_1 + 1, r_2; s_1, s_2 - 1], \qquad \text{for } p_1, s_2 \ge 1, && (3·15) \\ & \sim [p_1 + 1, p_2 - 1; q_1, q_2; r_1, r_2; s_1 - 1, s_2 + 1], \\ & \sim [p_1, p_2 - 1; q_1 + 1, q_2; r_1, r_2 + 1; s_1 - 1, s_2], \\ & \sim [p_1, p_2 - 1; q_1, q_2 + 1; r_1 + 1, r_2; s_1 - 1, s_2], \qquad \text{for } p_2, s_1 \ge 1, && (3·16) \\ & \sim [p_1 + 1, p_2; q_1 - 1, q_2; r_1, r_2 - 1; s_1, s_2 + 1], \\ & \sim [p_1, p_2 + 1; q_1 - 1, q_2; r_1, r_2 - 1; s_1 + 1, s_2], \\ & \sim [p_1, p_2; q_1 - 1, q_2 + 1; r_1 + 1, r_2 - 1; s_1, s_2], \qquad \text{for } q_1, r_2 \ge 1, && (3·17) \\ & \sim [p_1 + 1, p_2; q_1, q_2 - 1; r_1 - 1, r_2; s_1, s_2 + 1], \\ & \sim [p_1, p_2 + 1; q_1, q_2 - 1; r_1 - 1, r_2; s_1 + 1, s_2], \\ & \sim [p_1, p_2; q_1 + 1, q_2 - 1; r_1 - 1, r_2 + 1; s_1, s_2], \qquad \text{for } q_2, r_1 \ge 1. && (3·18) \end{aligned}$$

Hence, the number of equivalence classes of BCs related to diagonal representatives is ${}_{N+7}C_7 - 3 \cdot {}_{N+5}C_7$.

3.2. T^2/Z_3 *orbifold*

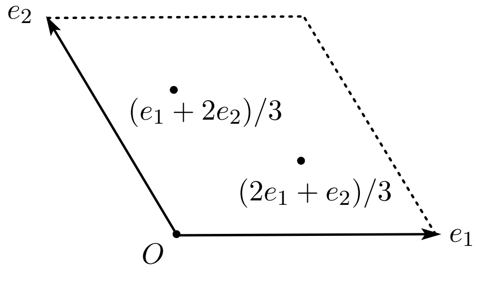

Fig. 2. Orbifold T^2/Z_3.

The BCs for $SU(N)$ gauge theory on $M^4 \times T^2/Z_3$ were studied in Ref. 9).*) For completeness, we explain the results briefly in this subsection. Let z be the coordinate of T^2/Z_3. Here, T^2 is constructed using the $SU(3)$ lattice whose basis vectors are given by $e_1 = 1$ and $e_2 = e^{2\pi i/3} \equiv \omega$. The orbifold T^2/Z_3 is obtained by further identifying ωz with z. The resultant space is the area depicted in Fig. 2. The fixed points for the Z_3 transformation $z \to \omega z$ are $z = 0$, $(2e_1 + e_2)/3$ and $(e_1 + 2e_2)/3$. Around these points, we define five transformations:

$$s_0 : z \to \omega z, \quad s_1 : z \to \omega z + e_1, \quad s_2 : z \to \omega z + e_1 + e_2,$$

*) The six-dimensional extension of the Z_3 orbifold was initially introduced into the construction of four-dimensional heterotic string models.[11)] The models on T^2/Z_3 have been utilized in the search for the origin of three families for quarks and leptons[12)] and the unification of gauge, Higgs and families.[13)]

$$t_1 : z \to z + e_1, \quad t_2 : z \to z + e_2. \tag{3·19}$$

Among the above operations, the following relations hold:

$$\begin{aligned} &s_0^3 = s_1^3 = s_2^3 = s_0 s_1 s_2 = s_1 s_2 s_0 = s_2 s_0 s_1 = I, \\ &s_1 = t_1 s_0, \quad s_2 = t_2 t_1 s_0, \quad t_1 t_2 = t_2 t_1. \end{aligned} \tag{3·20}$$

The BCs of bulk fields are specified by matrices $(\Theta_0, \Theta_1, \Theta_2, \Xi_1, \Xi_2)$ satisfying the relations

$$\begin{aligned} &\Theta_0^3 = \Theta_1^3 = \Theta_2^3 = \Theta_0\Theta_1\Theta_2 = \Theta_1\Theta_2\Theta_0 = \Theta_2\Theta_0\Theta_1 = I, \\ &\Theta_1 = \Xi_1\Theta_0, \quad \Theta_2 = \Xi_2\Xi_1\Theta_0, \quad \Xi_1\Xi_2 = \Xi_2\Xi_1. \end{aligned} \tag{3·21}$$

Because these matrices are pairwise independent, we choose unitary matrices Θ_0 and Θ_1.

Given the BCs (Θ_0, Θ_1), there is still remains residual gauge invariance. Under the gauge transformation $\Omega(x, z, \bar{z})$, Θ_0 and Θ_1 are transformed as

$$\begin{aligned} &\Theta_0'(z, \bar{z}) = \Omega(x, \omega z, \bar{\omega}\bar{z})\Theta_0\Omega^\dagger(x, z, \bar{z}), \\ &\Theta_1'(z, \bar{z}) = \Omega(x, \omega z + 1, \bar{\omega}\bar{z} + 1)\Theta_1\Omega^\dagger(x, z, \bar{z}). \end{aligned} \tag{3·22}$$

These BCs should be equivalent:

$$(\Theta_0, \Theta_1) \sim (\Theta_0'(z, \bar{z}), \Theta_1'(z, \bar{z})). \tag{3·23}$$

This equivalence relation defines equivalence classes of the BCs.

Let us consider an $SU(3)$ gauge theory with the gauge transformation function defined by

$$\Omega(z, \bar{z}) = \exp\left[ia\left(Y_+^1 z + Y_-^1 \bar{z}\right)\right], \tag{3·24}$$

where a is a real number and Y_+^1 and Y_-^1 are given by

$$Y_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y_-^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3·25}$$

When we take $(\Theta_0, \Theta_1) = (X, X)$ where X is given by

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}, \tag{3·26}$$

Θ_0 and Θ_1 are transformed as

$$(\Theta_0, \Theta_1) = (X, X) \to (X, e^{ia(Y_+^1 + Y_-^1)}X). \tag{3·27}$$

In this way, we obtain the following equivalence relations among the diagonal representatives:

$$(X, X) \sim (X, \bar{\omega}X) \sim (X, \omega X), \tag{3·28}$$

where we use the relation

$$\exp[iaY] = \frac{1}{3}(e^{2ia} + 2e^{-ia})I + \frac{1}{3}(e^{2ia} - e^{-ia})Y. \tag{3·29}$$

Here, I is the 3×3 unit matrix and $Y = Y_+^1 + Y_-^1$.

There are BCs specified by matrices that cannot be diagonalized simultaneously by global unitary transformations and local gauge transformations. For example, the following set of 3×3 matrices cannot be diagonalized simultaneously by global unitary transformations and local gauge transformations:

$$\Theta_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Theta_1 = \begin{pmatrix} 0 & e^{ia} & 0 \\ 0 & 0 & e^{ib} \\ e^{ic} & 0 & 0 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0 & 0 & e^{-ic} \\ e^{-ia} & 0 & 0 \\ 0 & e^{-ib} & 0 \end{pmatrix}, \tag{3·30}$$

where a, b and c are arbitrary real numbers satisfying $a + b + c = 2n\pi$ $(n \in \mathbb{Z})$. The above BC (3·30) satisfies (3·21). These BCs have no symmetry because there are no 3×3 traceless diagonal matrices commutable with (Θ_1, Θ_2) given in (3·30). The BCs specified by $N \times N$ matrices including off-diagonal elements can be constructed in the form that the above set of 3×3 matrices or their transposes are contained as submatrices.

We classify the BCs specified by diagonal matrices for simplicity. The diagonal $N \times N$ matrices (Θ_0, Θ_1) are specified by nine nonnegative integers (p_j, q_j, r_j) $(j = 1, 2, 3)$ such that

$$\begin{aligned}
\Theta_0 &= \mathrm{diag}\,(\overbrace{[1]_{p_1}, [1]_{p_2}, [1]_{p_3}}^{p}, \overbrace{[\omega]_{q_1}, [\omega]_{q_2}, [\omega]_{q_3}}^{q}, \overbrace{[\bar{\omega}]_{r_1}, [\bar{\omega}]_{r_2}, [\bar{\omega}]_{r_3}}^{r=N-p-q}), \\
\Theta_1 &= \mathrm{diag}([1]_{p_1}, [\omega]_{p_2}, [\bar{\omega}]_{p_3}, [1]_{q_1}, [\omega]_{q_2}, [\bar{\omega}]_{q_3}, [1]_{r_1}, [\omega]_{r_2}, [\bar{\omega}]_{r_3}),
\end{aligned} \tag{3·31}$$

where $0 \le p, q, r \le N$. Then the symmetry of BC becomes

$$\begin{aligned}
SU(N) \longrightarrow & SU(p_1) \times SU(p_2) \times SU(p_3) \times SU(q_1) \times SU(q_2) \\
& \times SU(q_3) \times SU(r_1) \times SU(r_2) \times SU(r_3) \times U(1)^{8-l}.
\end{aligned} \tag{3·32}$$

We denote the above BC as $[p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3]$. Using (3·28), we can derive the following equivalence relations in $SU(N)$ gauge theory:

$$\begin{aligned}
&[p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3] \\
&\sim [p_1 - 1, p_2 + 1, p_3; q_1, q_2 - 1, q_3 + 1; r_1 + 1, r_2, r_3 - 1], \\
&\sim [p_1 - 1, p_2, p_3 + 1; q_1 + 1, q_2 - 1, q_3; r_1, r_2 + 1, r_3 - 1], \quad \text{for } p_1, q_2, r_3 \ge 1, && (3·33) \\
&\sim [p_1 + 1, p_2 - 1, p_3; q_1, q_2 + 1, q_3 - 1; r_1 - 1, r_2, r_3 + 1], \\
&\sim [p_1, p_2 - 1, p_3 + 1; q_1 + 1, q_2, q_3 - 1; r_1 - 1, r_2 + 1, r_3], \quad \text{for } p_2, q_3, r_1 \ge 1, && (3·34) \\
&\sim [p_1, p_2 + 1, p_3 - 1; q_1 - 1, q_2, q_3 + 1; r_1 + 1, r_2 - 1, r_3], \\
&\sim [p_1 + 1, p_2, p_3 - 1; q_1 - 1, q_2 + 1, q_3; r_1, r_2 - 1, r_3 + 1], \quad \text{for } p_3, q_1, r_2 \ge 1. && (3·35)
\end{aligned}$$

Hence, the number of equivalence classes of BCs related to diagonal representatives is ${}_{N+8}C_8 - 2 \cdot {}_{N+5}C_8$.

3.3. T^2/Z_4 orbifold

Here we study $SU(N)$ gauge theory on $M^4 \times T^2/Z_4$. Let z be the coordinate of T^2/Z_4. Here, T^2 is constructed using the $SU(2)\times SU(2)(\simeq SO(4))$ lattice whose basis vectors are e_1 and e_2. The orbifold T^2/Z_4 is obtained by further identifying iz and $-z$ with z. The resultant space is the area depicted in Fig. 3. Then the fixed points on T^2/Z_4 are $z = 0$ and $(e_1 + e_2)/2$ for the Z_4 transformation $z \to iz$, and $z = 0$, $e_1/2$, $e_2/2$ and $(e_1+e_2)/2$ are the fixed points for the Z_2 transformation $z \to -z$. Around these points, we define eight transformations:

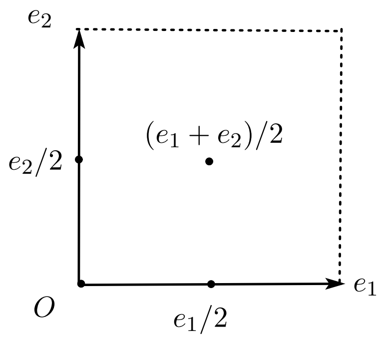

Fig. 3. Orbifold T^2/Z_4.

$$
\begin{aligned}
&s_0 : z \to iz, \quad s_1 : z \to iz + e_1, \quad s_{20} : z \to -z, \\
&s_{21} : z \to -z + e_1, \quad s_{22} : z \to -z + e_2, \quad s_{23} : z \to -z + e_1 + e_2, \\
&t_1 : z \to z + e_1, \quad t_2 : z \to z + e_2.
\end{aligned} \tag{3·36}
$$

These satisfy the following relations:

$$
\begin{aligned}
&s_0^4 = s_1^4 = s_{20}^2 = s_{21}^2 = s_{22}^2 = s_{23}^2 = I, \quad s_1 = t_1 s_0, \quad s_{21} = t_1 s_{20}, \\
&s_{22} = t_2 s_{20}, \quad s_{20} = s_0^2, \quad s_{21} = s_1 s_0, \quad s_{22} = s_0 s_1, \\
&s_{23} = t_1 t_2 s_{20} = s_{21} s_{20} s_{22} = s_{22} s_{20} s_{21}, \quad t_1 t_2 = t_2 t_1.
\end{aligned} \tag{3·37}
$$

The BCs of bulk fields are specified by matrices $(Q_0, Q_1, P_0, P_1, P_2, P_3, U_1, U_2)$ satisfying the relations:

$$
\begin{aligned}
&Q_0^4 = Q_1^4 = P_0^2 = P_1^2 = P_2^2 = P_3^2 = I, \ Q_1 = U_1 Q_0, \ P_1 = U_1 P_0, \\
&P_2 = U_2 P_0, \ P_0 = Q_0^2, \ P_1 = Q_1 Q_0, \ P_2 = Q_0 Q_1, \\
&P_3 = U_1 U_2 P_0 = P_1 P_0 P_2 = P_2 P_0 P_1, \ U_1 U_2 = U_2 U_1,
\end{aligned} \tag{3·38}
$$

where Q_m $(m = 0, 1)$ are unitary matrices and P_n $(n = 0, 1, 2, 3)$ are unitary and Hermitian matrices. Because these matrices are pairwise independent, we choose matrices Q_0 and P_1.

Given the BCs (Q_0, P_1), there still remains residual gauge invariance. Under the gauge transformation $\Omega(x, z, \bar{z})$, Q_0 and P_1 are transformed as

$$
\begin{aligned}
&Q_0'(z, \bar{z}) = \Omega(x, iz, -i\bar{z}) Q_0 \Omega^\dagger(x, z, \bar{z}), \\
&P_1'(z, \bar{z}) = \Omega(x, -z + e_1, -\bar{z} + \bar{e}_1) P_1 \Omega^\dagger(x, z, \bar{z}).
\end{aligned} \tag{3·39}
$$

These BCs should be equivalent:

$$
(Q_0, P_1) \sim (Q_0'(z, \bar{z}), P_1'(z, \bar{z})). \tag{3·40}
$$

This equivalence relation defines equivalence classes of the BCs. Let us consider an $SU(4)$ gauge theory with the gauge transformation function defined by

$$\Omega(z,\bar{z}) = \exp\{ia(Y_+^1 z + Y_-^1 \bar{z})\}, \tag{3·41}$$

where a is a real number and Y_+^1 and Y_-^1 are given by

$$Y_+^1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad Y_-^1 = \begin{pmatrix} 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{3·42}$$

When we take $(Q_0, P_1) = (X, X^2)$, where X is given by

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tag{3·43}$$

Q_0 and P_1 are transformed as

$$(Q_0, P_1) = (X, X^2) \to (X, e^{ia(Y_+^1 + Y_-^1)} X^2). \tag{3·44}$$

In this way, we obtain the following equivalence relation between the diagonal representatives:

$$(X, X^2) \sim (X, -X^2), \tag{3·45}$$

where we use the relation

$$\exp[iaY] = I\cos(\sqrt{2}a) + \frac{i}{\sqrt{2}} Y \sin(\sqrt{2}a). \tag{3·46}$$

Here I is the 4×4 unit matrix and $Y = Y_+^1 + Y_-^1$.

There are BCs specified by matrices that cannot be diagonalized simultaneously by global unitary transformations and local gauge transformations. For example, the following 4×4 matrices cannot be diagonalized simultaneously by global unitary transformations and local gauge transformations:

$$Q_0 = \begin{pmatrix} 0 & e^{ia} & 0 & 0 \\ 0 & 0 & e^{ib} & 0 \\ 0 & 0 & 0 & e^{ic} \\ e^{id} & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3·47}$$

where a, b, c and d are arbitrary real numbers satisfying $a+b+c+d = 2n\pi$ ($n \in \mathbb{Z}$). The above BC (3·47) satisfies (3·38). These BCs have no symmetry because there are no 4×4 traceless diagonal matrices commutable with Q_0 given in (3·47). The BCs specified by $N \times N$ matrices including off-diagonal elements can be constructed in the form that the above set of 4×4 matrices or their transposes are contained as submatrices.

We classify the BCs specified by diagonal matrices for simplicity. The diagonal matrices Q_0 and P_1 are specified by eight nonnegative integers (p_i, q_i, r_i, s_i) $(i = 1, 2)$ such that

$$\begin{aligned} Q_0 &= \mathrm{diag}(\overbrace{[+1]_{p_1}, [+1]_{p_2}}^{p}, \overbrace{[+i]_{q_1}, [+i]_{q_2}}^{q}, \overbrace{[-1]_{r_1}, [-1]_{r_2}}^{r}, \overbrace{[-i]_{s_1}, [-i]_{s_2}}^{s=N-p-q-r}), \\ P_1 &= \mathrm{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{q_1}, [-1]_{q_2}, [+1]_{r_1}, [-1]_{r_2}, [+1]_{s_1}, [-1]_{s_2}), \end{aligned} \tag{3·48}$$

where $0 \le p_i, q_i, r_i, s_i \le N$ $(i = 1, 2)$. Then the symmetry of BC becomes

$$\begin{aligned} SU(N) \ \longrightarrow \ & SU(p_1) \times SU(p_2) \times SU(q_1) \times SU(q_2) \\ & \times SU(r_1) \times SU(r_2) \times SU(s_1) \times SU(s_2) \times U^{7-l}. \end{aligned} \tag{3·49}$$

We denote the above BC as $[p_1, p_2; q_1, q_2; r_1, r_2;\ s_1, s_2]$. Using (3·45), we can derive the following equivalence relations in $SU(N)$ gauge theory:

$$\begin{aligned} & [p_1, p_2; q_1, q_2; r_1, r_2; s_1, s_2;] \\ & \quad \sim [p_1 - 1, p_2 + 1; q_1 + 1, q_2 - 1; r_1 - 1, r_2 + 1; s_1 + 1, s_2 - 1] \\ & \qquad\qquad \text{for } p_1, q_2, r_1, s_2 \ge 1, \\ & \quad \sim [p_1 + 1, p_2 - 1; q_1 - 1, q_2 + 1; r_1 + 1, r_2 - 1; s_1 - 1, s_2 + 1] \\ & \qquad\qquad \text{for } p_2, q_1, r_2, s_1 \ge 1. \end{aligned} \tag{3·50}$$

Hence, the number of equivalence classes of BCs including diagonal representatives is ${}_{N+7}C_7 - {}_{N+3}C_7$.

3.4. T^2/Z_6 *orbifold*

Here we study $SU(N)$ gauge theory on $M^4 \times T^2/Z_6$. Let z be the coordinate of T^2/Z_6. Here, T^2 is constructed using the G_2 lattice whose basis vectors are $e_1 = 1$ and $e_2 = (-3 + i\sqrt{3})/2$. The orbifold T^2/Z_6 is obtained by further identifying ρz with z, where $\rho^6 = 1$. The resultant space is the area depicted in Fig. 4. The Z_6 transformation $z \to \rho z$ is a rotation of $\pi/3$ around the origin and the basis vectors are transformed as $\rho e_1 = 2e_1 + e_2$, $\rho e_2 = -3e_1 - e_2$. Then the fixed points on T^2/Z_6 are $z = 0$ for $z \to \rho z$, $z = 0$; $e_2/3$ and $e_2/3$ for $z \to \rho^2 z$; and $z = 0$, $e_1/2$, $e_2/2$ and $(e_1 + e_2)/2$ for $z \to \rho^3 z$, and around these points we define ten transformations:

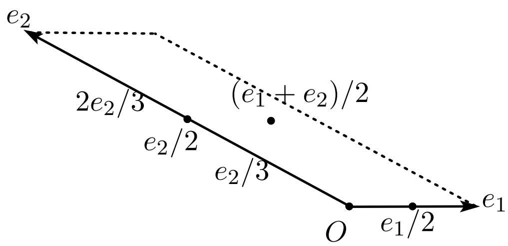

Fig. 4. Orbifold T^2/Z_6.

$$\begin{aligned} & s_0 : z \to \rho z, \quad s_{10} : z \to \rho^2 z, \quad s_{11} : z \to \rho^2 z + e_1 + e_2, \quad s_{12} : z \to \rho^2 z + 2e_1 + 2e_2, \\ & s_{20} : z \to \rho^3 z, \quad s_{21} : z \to \rho^3 z + e_1, \quad s_{22} : z \to \rho^3 z + e_2, \quad s_{23} : z \to \rho^3 z + e_1 + e_2, \\ & t_1 : z \to z + e_1, \quad t_2 : z \to z + e_2. \end{aligned} \tag{3·51}$$

These satisfy the following relations:

$$
\begin{aligned}
&s_0^6 = s_{10}^3 = s_{11}^3 = s_{12}^3 = s_{20}^2 = s_{21}^2 = s_{22}^2 = s_{23}^2 = I,\ s_{11} = t_1 t_2 s_{10},\ s_{12} = t_1^2 t_2^2 s_{10},\\
&s_{21} = t_1 s_{20},\ s_{22} = t_2 s_{20},\ s_{23} = t_1 t_2 s_{20} = s_{21} s_{20} s_{22} = s_{22} s_{20} s_{21} = s_{11} s_0,\\
&s_{10} = s_0^2,\ s_{20} = s_0^3,\ t_1 t_2 = t_2 t_1. \qquad (3{\cdot}52)
\end{aligned}
$$

The BCs of bulk fields are specified by matrices $(\Theta_0, \Theta_{10}, \Theta_{11}, \Theta_{12}, \Theta_{20}, \Theta_{21}, \Theta_{22}, \Theta_{23}, \Xi_1, \Xi_2)$ satisfying the relations

$$
\begin{aligned}
&\Theta_0^6 = \Theta_{10}^3 = \Theta_{11}^3 = \Theta_{12}^3 = \Theta_{20}^2 = \Theta_{21}^2 = \Theta_{22}^2 = \Theta_{23}^2 = I,\\
&\Theta_{11} = \Xi_1 \Xi_2 \Theta_{10},\ \Theta_{12} = \Xi_1^2 \Xi_2^2 \Theta_{10},\ \Theta_{21} = \Xi_1 \Theta_{20},\ \Theta_{22} = \Xi_2 \Theta_{20},\\
&\Theta_{23} = \Xi_1 \Xi_2 \Theta_{20} = \Theta_{21} \Theta_{20} \Theta_{22} = \Theta_{22} \Theta_{20} \Theta_{21} = \Theta_{11} \Theta_0,\\
&\Theta_{10} = \Theta_0^2,\ \Theta_{20} = \Theta_0^3,\ \Theta_{11} = \Theta_{23} \Theta_{20} \Theta_{10},\ \Theta_{12} = \Theta_{23} \Theta_{20} \Theta_{23} \Theta_{20} \Theta_{10},\\
&\Xi_1 \Xi_2 = \Xi_2 \Xi_1. \qquad (3{\cdot}53)
\end{aligned}
$$

Because any three of these matrices are mutually independent, we choose unitary matrices Θ_0, Θ_{21} and Θ_{22}.

Given the BCs $(\Theta_0, \Theta_{21}, \Theta_{22})$, there still remains residual gauge invariance. Under a gauge transformation $\Omega(x, z, \bar{z})$, Θ_0, Θ_{21} and Θ_{22} are transformed as

$$
\begin{aligned}
\Theta_0'(z, \bar{z}) &= \Omega(x, \rho z, \bar{\rho} \bar{z}) \Theta_0 \Omega^\dagger(x, z, \bar{z}),\\
\Theta_{21}'(z, \bar{z}) &= \Omega(x, \rho^3 z + e_1, \bar{\rho}^3 \bar{z} + \bar{e}_1) \Theta_{21} \Omega^\dagger(x, z, \bar{z}),\\
\Theta_{22}'(z, \bar{z}) &= \Omega(x, \rho^3 z + e_2, \bar{\rho}^3 \bar{z} + \bar{e}_2) \Theta_{22} \Omega^\dagger(x, z, \bar{z}). \qquad (3{\cdot}54)
\end{aligned}
$$

These BCs should be equivalent:

$$(\Theta_0, \Theta_{21}, \Theta_{22}) \sim (\Theta_0'(z, \bar{z}), \Theta_{21}'(z, \bar{z}), \Theta_{22}'(z, \bar{z})). \qquad (3{\cdot}55)$$

This equivalence relation defines equivalence classes of the BCs. There are no equivalence relations between diagonal representatives. To illustrate this, let us consider an $SU(6)$ gauge theory with the gauge transformation function defined by

$$\Omega(z, \bar{z}) = \exp\{ia(Y_+^1 z + Y_-^1 \bar{z})\}, \qquad (3{\cdot}56)$$

where a is a real number and Y_+^1 and $Y_-^1 = (Y_+^1)^\dagger$ are 6×6 matrices. When the diagonal matrix Θ_0 is transformed into the diagonal matrix Θ_0' under the gauge transformation, Θ_0, Θ_0' and Y_+^1 are determined by

$$
\Theta_0 = \Theta_0' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^5 \end{pmatrix}, \quad Y_+^1 = \begin{pmatrix} 0 & b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6 \\ b_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3{\cdot}57)
$$

up to an overall factor of ρ^k for Θ_0 and Θ_0'. Here b_i $(i = 1, \cdots, 6)$ are arbitrary complex numbers. It is shown that the diagonal Θ_{21} cannot be transformed into a different diagonal form. All diagonal representatives are independent of each other. The diagonal matrices Θ_0, Θ_{21} and Θ_{22} for $SU(N)$ gauge theories are specified by twenty-four nonnegative integers and the number of equivalence classes of BCs related to diagonal representatives is ${}_{N+23}C_{23}$.

§4. Conclusions

We have studied equivalence classes of BCs in an $SU(N)$ gauge theory on six-dimensional space-time including two-dimensional orbifolds. For five types of two-dimensional orbifolds $S^1/Z_2 \times S^1/Z_2$ and T^2/Z_m $(m = 2, 3, 4, 6)$, orbifold conditions and their gauge transformation properties have been given and the equivalence relations among boundary conditions have been derived. We have classified equivalence classes of BCs related to diagonal representatives for each orbifold. There are BCs specified by matrices that cannot be diagonalized simultaneously by global unitary transformations and local gauge transformations on T^2/Z_m. Not every component in an $SU(N)$ multiplet necessarily becomes a simultaneous eigenstate for Z_m transformations if BCs contain off-diagonal elements. The rank of gauge group can be reduced using these BCs, which should be useful for the model-building of grand unification or gauge-Higgs unification.

If the BCs for bulk fields are given, mode expansions are carried out and the one-loop effective potential for Wilson line phases is calculated using the standard method. From the minimum of effective potential, the physical symmetry and mass spectrum are obtained for each model. It is crucial to study dynamical gauge symmetry breaking and mass generation in a realistic model including fermions. It is also important to construct a phenomenologically viable model realizing gauge-Higgs unification[14)] and/or family unification[15)] based on them. The local grand unification can be realized by taking nontrivial BCs.[*)] It is interesting to study the phenomenological aspects of such models. We hope to further study these subjects in the near future.

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*) The 'local' gauge groups at fixed points were realized on T^2/Z_2 in Ref. 16). String-derived orbifold grand unification theories were studied in Refs. 17) and 18).

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