Equivalence Classes of Boundary Conditions in Gauge Theory on Z_3 Orbifold

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We study equivalence classes of boundary conditions in a gauge theory on the orbifold T^2/Z_3. Orbifold conditions and those gauge transformation properties are given and the gauge equivalence is understood by the Hosotani mechanism. Mode expansions are carried out for six-dimensional Z_3 singlet fields and a Z_3 triplet field, and the one-loop effective potential for Wilson line phases is calculated.

§1. Introduction

The boundary conditions (BCs) to be imposed on the fields in the bulk are classified into the equivalence classes using the gauge invariance, in higher-dimensional gauge theories. Several sets of BCs belong to the same equivalence class and describe the same physics, if they are related to gauge transformations. Specifically, the symmetry of BCs is not necessarily the same as the physical symmetry. The physical symmetry is determined by the Hosotani mechanism after the rearrangement of gauge symmetry.[1)]

Grand unified theories on an orbifold have been attracted phenomenologically since Higgs mass splitting was well realized by the orbifold breaking mechanism.[2),3),**)] Equivalence classes of BCs and dynamical gauge symmetry breaking were studied for gauge theories on the orbifolds S^1/Z_2[5),6),***)] and T^2/Z_2.[8)] It is interesting to study equivalence classes of BCs and the Hosotani mechanism for gauge theories on other orbifolds and to construct a phenomenologically viable model based on them. The Z_3 orbifold T^2/Z_3 is a candidate and has been utilized in the search for the origin of three families[9)] and the unification of gauge, Higgs and family.[10),†)]

In the present paper, we study equivalence classes of BCs in a gauge theory on T^2/Z_3. Orbifold conditions and those gauge transformation properties are given and the gauge equivalence is understood by the Hosotani mechanism. Mode expansions are carried out for six-dimensional Z_3 singlet fields and a Z_3 triplet field, and the one-loop effective potential for Wilson line phases is calculated.

In §2, general arguments are given for BCs in gauge theories on T^2/Z_3, and equivalence classes of BCs are defined by the invariance under the gauge transforma-

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**) In four-dimensional heterotic string models, extra colored Higgs are projected by the Wilson line mechanism.[4)]

***) See Ref. 7) for the breakdown of gauge symmetry on S^1/Z_2 by the Hosotani mechanism.

†) The six-dimensional extension of Z_3 orbifold was initially introduced into the construction of four-dimensional heterotic string models.[11)]

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tion. In §3, mode expansions on six-dimensional fields are given and the classification of BCs for the $SU(N)$ gauge group is carried out with the aid of equivalence relations. The one-loop effective potential for Wilson line phases is calculated using an $SU(3)$ gauge theory. Section 4 is devoted to conclusions.

§2. Orbifold conditions and equivalence classes

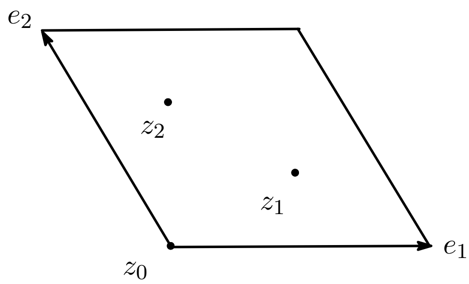

Fig. 1. Orbifold T^2/Z_3.

Let x and z be coordinates of M^4 and T^2/Z_3, respectively. T^2 is the two-dimensional torus whose basis vectors are $SU(3)$ root vectors, $e_1 = 1$ and $e_2 = e^{2\pi i/3} \equiv \omega$.*) On T^2, the point z is identified by $z + n_1 e_1 + n_2 e_2$, where n_1 and n_2 are integers. T^2/Z_3 is obtained by further identifying points on T^2 through a Z_3 rotation, i.e., z is identified with θz where $\theta^3 = 1$. The resultant space is the area depicted in Fig. 1, which contains the information on T^2.

2.1. *Boundary conditions*

The fixed points $z_{\rm fp}$ on T^2/Z_3 are points that transform themselves under the Z_3 transformation $z \to \theta z$ and satisfy

$$z_{\rm fp} = \theta z_{\rm fp} + n e_1 + m e_2, \tag{2·1}$$

where n and m are integers that characterize fixed points. There are three kinds of fixed points, namely,

$$\begin{aligned}
&z_0 = 0, \quad (n = m = 0)\\
&z_1 = \frac{1}{3}(2e_1 + e_2) = \frac{1}{\sqrt{3}} e^{\pi i/6}, \quad (n = 1, m = 0)\\
&z_2 = \frac{1}{3}(e_1 + 2e_2) = \frac{1}{\sqrt{3}} e^{\pi i/2}, \quad (n = m = 1)
\end{aligned} \tag{2·2}$$

where we take $\theta = \omega$. The Z_3 transformations around the fixed points z_0, z_1 and z_2 and shifts by e_1 and e_2 are defined by

$$\begin{aligned}
&s_0 : z \to \theta z = \omega z, \quad s_1 : z - z_1 \to \theta(z - z_1), \quad s_2 : z - z_2 \to \theta(z - z_2),\\
&t_1 : z \to z + e_1 = z + 1, \quad t_2 : z \to z + e_2 = z + \omega.
\end{aligned} \tag{2·3}$$

Using Eq. (2·2), the operations s_1 and s_2 are written as

$$\begin{aligned}
&s_1 : z \to \theta z + e_1 = \omega z + 1,\\
&s_2 : z \to \theta z + e_1 + e_2 = \omega z + 1 + \omega = \omega z - \bar{\omega},
\end{aligned} \tag{2·4}$$

*) We take the $SU(3)$ lattice as the unit lattice. On the estimation of physical quantities, we use physical sizes such as $e_1 = 2\pi R$ and $e_2 = 2\pi R\omega$.

where $\bar{\omega} = e^{-2\pi i/3} = e^{4\pi i/3}$ and we use the relation $1 + \omega + \bar{\omega} = 0$. Among the above operations, the following relations hold:

$$\begin{aligned} &s_0^3 = s_1^3 = s_2^3 = s_2 s_0 s_1 = s_0 s_1 s_2 = s_1 s_2 s_0 = I, \\ &s_1 = t_1 s_0, \quad s_2 = t_2 t_1 s_0, \quad t_1 t_2 = t_2 t_1, \end{aligned} \tag{2·5}$$

where I is the identity operation. s_2, t_1 and t_2 are not independent of s_0 and s_1.

On T^2/Z_3, the point z is identified by the points $z + e_1$, $z + e_2$ and θz, but all six-dimensional bulk fields do not necessarily take identical values at these points. Let the bulk field $\Phi(x, z, \bar{z})$ be a multiplet of some transformation group G and the Lagrangian density $\mathcal{L}$ be invariant under the transformation $\Phi(x, z, \bar{z}) \to \Phi'(x, z, \bar{z}) = T_\Phi \Phi(x, z, \bar{z})$ such that

$$\mathcal{L}(\Phi(x, z, \bar{z})) = \mathcal{L}(\Phi'(x, z, \bar{z})), \tag{2·6}$$

where T_Φ is a representation matrix of G on Φ. When we require $\mathcal{L}$ to be single-valued on $M^4 \times (T^2/Z_3)$, i.e.,

$$\begin{aligned} \mathcal{L}(\Phi(x, z, \bar{z})) &= \mathcal{L}(\Phi(x, z+1, \bar{z}+1)) = \mathcal{L}(\Phi(x, z+\omega, \bar{z}+\bar{\omega})) \\ &= \mathcal{L}(\Phi(x, \omega z, \bar{\omega}\bar{z})), \end{aligned} \tag{2·7}$$

the field can be identified such that[*)]

$$\begin{aligned} &\Phi(x, \omega z, \bar{\omega}\bar{z}) = T_\Phi[\hat{\Theta}_0]\Phi(x, z, \bar{z}), \quad \Phi(x, \omega z + 1, \bar{\omega}\bar{z} + 1) = T_\Phi[\hat{\Theta}_1]\Phi(x, z, \bar{z}), \\ &\Phi(x, \omega z + 1 + \omega, \bar{\omega}\bar{z} + 1 + \bar{\omega}) = T_\Phi[\hat{\Theta}_2]\Phi(x, z, \bar{z}), \\ &\Phi(x, z+1, \bar{z}+1) = T_\Phi[\hat{\Xi}_1]\Phi(x, z, \bar{z}), \quad \Phi(x, z+\omega, \bar{z}+\bar{\omega}) = T_\Phi[\hat{\Xi}_2]\Phi(x, z, \bar{z}), \end{aligned} \tag{2·8}$$

where $T_\Phi[\hat{\Theta}_0]$, $T_\Phi[\hat{\Theta}_1]$, $T_\Phi[\hat{\Theta}_2]$, $T_\Phi[\hat{\Xi}_1]$ and $T_\Phi[\hat{\Xi}_2]$ represent appropriate representation matrices, including an arbitrary Z_3 phase factor. The counterparts of Eq. (2·5) are given by

$$\begin{aligned} &T_\Phi[\hat{\Theta}_0]^3 = T_\Phi[\hat{\Theta}_1]^3 = T_\Phi[\hat{\Theta}_2]^3 = T_\Phi[\hat{\Theta}_2]T_\Phi[\hat{\Theta}_0]T_\Phi[\hat{\Theta}_1] = T_\Phi[\hat{\Theta}_0]T_\Phi[\hat{\Theta}_1]T_\Phi[\hat{\Theta}_2] \\ &\quad = T_\Phi[\hat{\Theta}_1]T_\Phi[\hat{\Theta}_2]T_\Phi[\hat{\Theta}_0] = I, \\ &T_\Phi[\hat{\Theta}_1] = T_\Phi[\hat{\Xi}_1]T_\Phi[\hat{\Theta}_0], \quad T_\Phi[\hat{\Theta}_2] = T_\Phi[\hat{\Xi}_2]T_\Phi[\hat{\Xi}_1]T_\Phi[\hat{\Theta}_0], \\ &T_\Phi[\hat{\Xi}_2]T_\Phi[\hat{\Xi}_1] = T_\Phi[\hat{\Xi}_1]T_\Phi[\hat{\Xi}_2], \end{aligned} \tag{2·9}$$

where I stands for the unit matrix. For instance, if Φ belongs to the fundamental representation of the $SU(N)$ gauge group and a singlet under Z_3 transformation, then $T_\Phi[\hat{\Theta}_0]\Phi$ is $\eta_0 \Theta_0 \Phi$, where Θ_0 is a $U(N)$ matrix, i.e., $\Theta_0^\dagger = \Theta_0^2 = \Theta_0^{-1}$, and η_0 is an intrinsic phase factor given by a qubic root. The same property applies to $T_\Phi[\hat{\Theta}_1]$ and $T_\Phi[\hat{\Theta}_2]$. By using Eq. (2·9), the representations of shifts are given by those of Z_3 rotations such that

$$\begin{aligned} T_\Phi[\hat{\Xi}_1] &= T_\Phi[\hat{\Theta}_1]T_\Phi[\hat{\Theta}_0]^\dagger = T_\Phi[\hat{\Theta}_1]T_\Phi[\hat{\Theta}_0]^2, \\ T_\Phi[\hat{\Xi}_2] &= T_\Phi[\hat{\Theta}_2]T_\Phi[\hat{\Theta}_1]^\dagger = T_\Phi[\hat{\Theta}_2]T_\Phi[\hat{\Theta}_1]^2. \end{aligned} \tag{2·10}$$

[*)] If fields and their superpartners yield different BCs, the Scherk-Schwarz mechanism can work.[12)]

Furthermore the representation of s_2 is given by other Z_3 rotations such that

$$T_\Phi[\hat{\Theta}_2] = T_\Phi[\hat{\Theta}_1]^\dagger T_\Phi[\hat{\Theta}_0]^\dagger = T_\Phi[\hat{\Theta}_1]^2 T_\Phi[\hat{\Theta}_0]^2. \tag{2·11}$$

Hereafter, we use two kinds of Z_3 rotations, s_0 and s_1, as independent operations.

Let G be a direct product of a gauge group and a 'flavor' group. The BCs imposed on the six-dimensional gauge field $A_M(x,z,\bar{z})$ are given by

$$\begin{aligned}
&A_\mu(x,\omega z,\bar{\omega}\bar{z}) = \Theta_0 A_\mu(x,z,\bar{z})\Theta_0^\dagger, \quad A_z(x,\omega z,\bar{\omega}\bar{z}) = \bar{\omega}\Theta_0 A_z(x,z,\bar{z})\Theta_0^\dagger,\\
&A_{\bar{z}}(x,\omega z,\bar{\omega}\bar{z}) = \omega\Theta_0 A_{\bar{z}}(x,z,\bar{z})\Theta_0^\dagger, \qquad (2·12)\\
&A_\mu(x,\omega z+1,\bar{\omega}\bar{z}+1) = \Theta_1 A_\mu(x,z,\bar{z})\Theta_1^\dagger,\\
&A_z(x,\omega z+1,\bar{\omega}\bar{z}+1) = \bar{\omega}\Theta_1 A_z(x,z,\bar{z})\Theta_1^\dagger,\\
&A_{\bar{z}}(x,\omega z+1,\bar{\omega}\bar{z}+1) = \omega\Theta_1 A_{\bar{z}}(x,z,\bar{z})\Theta_1^\dagger, \qquad (2·13)\\
&A_\mu(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) = \Theta_2 A_\mu(x,z,\bar{z})\Theta_2^\dagger,\\
&A_z(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) = \bar{\omega}\Theta_2 A_z(x,z,\bar{z})\Theta_2^\dagger,\\
&A_{\bar{z}}(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) = \omega\Theta_2 A_{\bar{z}}(x,z,\bar{z})\Theta_2^\dagger, \qquad (2·14)\\
&A_M(x,z+1,\bar{z}+1) = \Xi_1 A_M(x,z,\bar{z})\Xi_1^\dagger,\\
&A_M(x,z+\omega,\bar{z}+\bar{\omega}) = \Xi_2 A_M(x,z,\bar{z})\Xi_2^\dagger, \qquad (2·15)
\end{aligned}$$

where $(\Theta_0, \Theta_1, \Theta_2, \Xi_1, \Xi_2)$ are representation matrices of the gauge group (times $U(1)$s). These BCs are consistent with the gauge covariance of the derivative $D_M = \partial_M + igA_M(x,z,\bar{z})$, where g is a gauge coupling constant. For the bulk scalar field $\phi(x,z,\bar{z})$, which is a singlet under Z_3 transformation, BCs are given by

$$\begin{aligned}
&\phi(x,\omega z,\bar{\omega}\bar{z}) = T_\phi[\Theta_0]\phi(x,z,\bar{z}),\\
&\phi(x,\omega z+1,\bar{\omega}\bar{z}+1) = T_\phi[\Theta_1]\phi(x,z,\bar{z}),\\
&\phi(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) = T_\phi[\Theta_2]\phi(x,z,\bar{z}),\\
&\phi(x,z+1,\bar{z}+1) = T_\phi[\Xi_1]\phi(x,z,\bar{z}),\\
&\phi(x,z+\omega,\bar{z}+\bar{\omega}) = T_\phi[\Xi_2]\phi(x,z,\bar{z}). \qquad (2·16)
\end{aligned}$$

For a set of scalar fields $\phi^A(x,z,\bar{z})$ $(A=1,2,3)$ that form a triplet under Z_3 transformation, these BCs are given by

$$\begin{aligned}
&\phi^A(x,\omega z,\bar{\omega}\bar{z}) = T_{\phi^A}[\hat{\Theta}_0]\phi^A(x,z,\bar{z}) = \sum_{B=1}^{3} (\mathcal{X})^{\mathcal{A}}{}_B T_{\phi^B}[\Theta_0]\phi^B(x,z,\bar{z}),\\
&\phi^A(x,\omega z+1,\bar{\omega}\bar{z}+1)\\
&\qquad = T_{\phi^A}[\hat{\Theta}_1]\phi^A(x,z,\bar{z}) = \sum_{B=1}^{3} \left(e^{-2\pi i\gamma\mathcal{Y}}\mathcal{X}\right)^{\mathcal{A}}{}_B T_{\phi^B}[\Theta_1]\phi^B(x,z,\bar{z}),\\
&\phi^A(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega})\\
&\qquad = T_{\phi^A}[\hat{\Theta}_2]\phi^A(x,z,\bar{z}) = \sum_{B=1}^{3} \left(e^{2\pi i\gamma\mathcal{Y}_\omega}\mathcal{X}\right)^{\mathcal{A}}{}_B T_{\phi^B}[\Theta_2]\phi^B(x,z,\bar{z}),
\end{aligned}$$

$$\begin{aligned}
&\phi^A(x, z+1, \bar{z}+1) \\
&\qquad = T_{\phi^A}[\hat{\Xi}_1]\phi^A(x,z,\bar{z}) = \sum_{B=1}^{3} \left(e^{-2\pi i\gamma\mathcal{Y}}\right)^A{}_B T_{\phi^B}[\Xi_1]\phi^B(x,z,\bar{z}), \\
&\phi^A(x, z+\omega, \bar{z}+\bar{\omega}) \\
&\qquad = T_{\phi^A}[\hat{\Xi}_2]\phi^A(x,z,\bar{z}) = \sum_{B=1}^{3} \left(e^{-2\pi i\gamma\mathcal{Y}_{\bar{\omega}}}\right)^A{}_B T_{\phi^B}[\Xi_2]\phi^B(x,z,\bar{z}), \quad (2\cdot17)
\end{aligned}$$

where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Y}_\omega$ and $\mathcal{Y}_{\bar{\omega}}$ are 3×3 matrices and the parameter γ can take an arbitrary real value. Here, the cyclic group Z_3 is a discrete subgroup of the 'flavor' group. For the Z_3 singlet Dirac field $\psi(x,z,\bar{z})$ defined in the bulk, the gauge invariance of the kinetic energy term requires the following BCs:

$$\begin{aligned}
&\psi(x, \omega z, \bar{\omega}\bar{z}) = T_\psi[\Theta_0]S_0\psi(x,z,\bar{z}), \\
&\psi(x, \omega z+1, \bar{\omega}\bar{z}+1) = T_\psi[\Theta_1]S_1\psi(x,z,\bar{z}), \\
&\psi(x, \omega z+1+\omega, \bar{\omega}\bar{z}+1+\bar{\omega}) = T_\psi[\Theta_2]S_2\psi(x,z,\bar{z}), \\
&\psi(x, z+1, \bar{z}+1) = T_\psi[\Xi_1]S_1S_0^2\psi(x,z,\bar{z}), \\
&\psi(x, z+\omega, \bar{z}+\bar{\omega}) = T_\psi[\Xi_2]S_2S_1^2\psi(x,z,\bar{z}), \qquad (2\cdot18)
\end{aligned}$$

where S_i $(i=0,1,2)$ are 8×8 matrices acting on the Dirac spinor given by

$$S_i = I_{4\times4}\otimes\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} = \frac{1}{8}\left(\Gamma^z\Gamma^{\bar{z}} + \omega\Gamma^{\bar{z}}\Gamma^z\right). \qquad (2\cdot19)$$

Here, $I_{4\times4}$ is the 4×4 unit matrix, and arbitrary Z_3 phase factors are absorbed by the intrinsic ones η_i. We use the following representation for six-dimensional gamma matrices:

$$\begin{aligned}
&\Gamma^\mu = \gamma^\mu\otimes\sigma_3, \quad \Gamma^5 = I_{4\times4}\otimes\sigma_1, \quad \Gamma^6 = I_{4\times4}\otimes\sigma_2, \\
&\Gamma^z \equiv \Gamma^5 + i\Gamma^6 = 2I_{4\times4}\otimes\sigma_+, \quad \Gamma^{\bar{z}} \equiv \Gamma^5 - i\Gamma^6 = 2I_{4\times4}\otimes\sigma_-. \qquad (2\cdot20)
\end{aligned}$$

The following relations hold:

$$\Gamma^\mu S_i = S_i\Gamma^\mu, \quad \Gamma^z S_i = \omega S_i\Gamma^z, \quad \Gamma^{\bar{z}} S_i = \bar{\omega} S_i\Gamma^{\bar{z}}. \qquad (2\cdot21)$$

The BCs for a Z_3 triplet Dirac field are similarly given.

In this way, we find that BCs in gauge theories on T^2/Z_3 are specified by $(\Theta_0, \Theta_1, \gamma)$ and additional Z_3 phase factors.

2.2. *Residual gauge invariance and equivalence classes*

Given the BCs $(\Theta_0, \Theta_1, \Theta_2, \Xi_1, \Xi_2, \gamma)$, there still remains residual gauge invariance. Under gauge transformation with the transformation function $\Omega = \Omega(x,z,\bar{z})$, fields are transformed as

$$\begin{aligned}
&A_M \to A'_M = \Omega A_M\Omega^\dagger - \frac{i}{g}\Omega\partial_M\Omega^\dagger, \quad \phi\to\phi' = T_\phi[\Omega]\phi, \\
&\phi^A \to \phi'^A = T_{\phi^A}[\Omega]\phi^A, \quad \psi^A\to\psi'^A = T_{\psi^A}[\Omega]\psi^A, \qquad (2\cdot22)
\end{aligned}$$

where $A'_M(x,z,\bar{z})$ satisfies, instead of Eqs. (2·12) – (2·15),

$$\begin{aligned}
A'_\mu(x,\omega z,\bar{\omega}\bar{z}) &= \Theta'_0 A'_\mu(x,z,\bar{z})\Theta'^\dagger_0 - \frac{i}{g}\Theta'_0\partial_\mu\Theta'^\dagger_0, \\
A'_z(x,\omega z,\bar{\omega}\bar{z}) &= \bar{\omega}\left(\Theta'_0 A'_z(x,z,\bar{z})\Theta'^\dagger_0 - \frac{i}{g}\Theta'_0\partial_z\Theta'^\dagger_0\right), \\
A'_{\bar{z}}(x,\omega z,\bar{\omega}\bar{z}) &= \omega\left(\Theta'_0 A'_{\bar{z}}(x,z,\bar{z})\Theta'^\dagger_0 - \frac{i}{g}\Theta'_0\partial_{\bar{z}}\Theta'^\dagger_0\right), \qquad (2\cdot 23)\\
A'_\mu(x,\omega z+1,\bar{\omega}\bar{z}+1) &= \Theta'_1 A_\mu(x,z,\bar{z})\Theta'^\dagger_1 - \frac{i}{g}\Theta'_1\partial_\mu\Theta'^\dagger_1, \\
A'_z(x,\omega z+1,\bar{\omega}\bar{z}+1) &= \bar{\omega}\left(\Theta'_1 A'_z(x,z,\bar{z})\Theta'^\dagger_1 - \frac{i}{g}\Theta'_1\partial_z\Theta'^\dagger_1\right), \\
A'_{\bar{z}}(x,\omega z+1,\bar{\omega}\bar{z}+1) &= \omega\left(\Theta'_1 A'_{\bar{z}}(x,z,\bar{z})\Theta'^\dagger_1 - \frac{i}{g}\Theta'_1\partial_{\bar{z}}\Theta'^\dagger_1\right), \qquad (2\cdot 24)\\
A'_\mu(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) &= \Theta'_2 A'_\mu(x,z,\bar{z})\Theta'^\dagger_2 - \frac{i}{g}\Theta'_2\partial_\mu\Theta'^\dagger_2, \\
A'_z(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) &= \bar{\omega}\left(\Theta'_2 A'_z(x,z,\bar{z})\Theta'^\dagger_2 - \frac{i}{g}\Theta'_2\partial_z\Theta'^\dagger_2\right), \\
A'_{\bar{z}}(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) &= \omega\left(\Theta'_2 A'_{\bar{z}}(x,z,\bar{z})\Theta'^\dagger_2 - \frac{i}{g}\Theta'_2\partial_{\bar{z}}\Theta'^\dagger_2\right), \qquad (2\cdot 25)\\
A'_M(x,z+1,\bar{z}+1) &= \Xi'_1 A'_M(x,z,\bar{z})\Xi'^\dagger_1 - \frac{i}{g}\Xi'_1\partial_M\Xi'^\dagger_1, \\
A'_M(x,z+\omega,\bar{z}+\bar{\omega}) &= \Xi'_2 A_M(x,z,\bar{z})\Xi'^\dagger_2 - \frac{i}{g}\Xi'_2\partial_M\Xi'^\dagger_2. \qquad (2\cdot 26)
\end{aligned}$$

Here, Θ'_0, Θ'_1, Θ'_2, Ξ'_1 and Ξ'_2 are given by

$$\begin{aligned}
\Theta'_0 &= \Omega(x,\omega z,\bar{\omega}\bar{z})\Theta_0\Omega^\dagger(x,z,\bar{z}), \\
\Theta'_1 &= \Omega(x,\omega z+1,\bar{\omega}\bar{z}+1)\Theta_1\Omega^\dagger(x,z,\bar{z}), \\
\Theta'_2 &= \Omega(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega})\Theta_2\Omega^\dagger(x,z,\bar{z}), \\
\Xi'_1 &= \Omega(x,z+1,\bar{z}+1)\Xi_1\Omega^\dagger(x,z,\bar{z}), \\
\Xi'_2 &= \Omega(x,z+\omega,\bar{z}+\bar{\omega})\Xi_2\Omega^\dagger(x,z,\bar{z}). \qquad (2\cdot 27)
\end{aligned}$$

The scalar fields $\phi'(x,z,\bar{z})$ and $\phi'^A(x,z,\bar{z})$ and the Dirac fermion $\psi'(x,z,\bar{z})$ satisfy relations similar to Eqs. (2·16) – (2·18), where $(\Theta_0,\Theta_1,\Theta_2,\Xi_1,\Xi_2,\gamma)$ is replaced by $(\Theta'_0,\Theta'_1,\Theta'_2,\Xi'_1,\Xi'_2,\gamma)$.

The residual gauge invariance of the BCs is given by gauge transformations that preserve the given BCs, $\Theta'_0=\Theta_0$, $\Theta'_1=\Theta_1$, $\Theta'_2=\Theta_2$, $\Xi'_1=\Xi_1$ and $\Xi'_2=\Xi_2$:

$$\begin{aligned}
&\Omega(x,\omega z,\bar{\omega}\bar{z})\Theta_0 = \Theta_0\Omega(x,z,\bar{z}), \\
&\Omega(x,\omega z+1,\bar{\omega}\bar{z}+1)\Theta_1 = \Theta_1\Omega(x,z,\bar{z}), \\
&\Omega(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1\bar{\omega})\Theta_2 = \Theta_2\Omega(x,z,\bar{z}), \qquad (2\cdot 28)\\
&\Omega(x,z+1,\bar{z}+1)\Xi_1 = \Xi_1\Omega(x,z,\bar{z}), \\
&\Omega(x,z+\omega,\bar{z}+\bar{\omega})\Xi_2 = \Xi_2\Omega(x,z,\bar{z}). \qquad (2\cdot 29)
\end{aligned}$$

We refer to the residual gauge invariance of BCs as the gauge symmetry of BCs. The low-energy gauge symmetry of BCs is derived from the following relations that are independent of extradimensional coordinates:

$$\Omega(x)\Theta_0 = \Theta_0\Omega(x), \quad \Omega(x)\Theta_1 = \Theta_1\Omega(x), \quad \Omega(x)\Theta_2 = \Theta_2\Omega(x), \tag{2·30}$$

$$\Omega(x)\Xi_1 = \Xi_1\Omega(x), \quad \Omega(x)\Xi_2 = \Xi_2\Omega(x). \tag{2·31}$$

The symmetry is generated by generators that commute with Θ_0 and Θ_1.

Theories with different BCs should be equivalent in terms of physics content if they are connected by gauge transformations. The key observation is that physics should not depend on the gauge chosen. If $(\Theta_0', \Theta_1', \Theta_2', \Xi_1', \Xi_2')$ satisfies the conditions

$$\partial_M\Theta_0' = 0, \quad \partial_M\Theta_1' = 0, \quad \partial_M\Theta_2' = 0, \quad \partial_M\Xi_1' = 0, \quad \partial_M\Xi_2' = 0, \tag{2·32}$$

$$\Theta'^3_0 = \Theta'^3_1 = \Theta'^3_2 = \Theta_2'\Theta_0'\Theta_1' = \Theta_0'\Theta_1'\Theta_2' = \Theta_1'\Theta_2'\Theta_0' = I, \tag{2·33}$$

then the two sets of BCs are equivalent:

$$(\Theta_0', \Theta_1', \Theta_2', \Xi_1', \Xi_2') \sim (\Theta_0, \Theta_1, \Theta_2, \Xi_1, \Xi_2). \tag{2·34}$$

The equivalence relation (2·34) defines equivalence classes of BCs. Here, we illustrate the change of BCs under a singular gauge transformation. Let us consider an $SU(3)$ gauge theory with $(\Theta_0, \Theta_1, \Theta_2, \Xi_1, \Xi_2) = (X, X, X, I, I)$. Here, X and I are given by

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2·35}$$

We carry out the gauge transformation defined by

$$\Omega = \exp\left(ia\left(Y_+^1 z + Y_-^1 \bar{z}\right)\right), \tag{2·36}$$

where a is a real number, and Y_+^1 and Y_-^1 are defined by

$$Y_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y_-^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2·37}$$

Then we find the equivalence relation

$$(X, X, X, I, I) \sim (X, e^{iaY}X, e^{-iaY_\omega}X, e^{iaY}I, e^{iaY_{\bar{\omega}}}I), \tag{2·38}$$

where Y, Y_ω and $Y_{\bar{\omega}}$ are defined by

$$Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad Y_\omega = \begin{pmatrix} 0 & \omega & \omega^2 \\ \omega^2 & 0 & \omega \\ \omega & \omega^2 & 0 \end{pmatrix}, \quad Y_{\bar{\omega}} = \begin{pmatrix} 0 & \omega^2 & \omega \\ \omega & 0 & \omega^2 \\ \omega^2 & \omega & 0 \end{pmatrix}. \tag{2·39}$$

In particular, we have the equivalence relation

$$(X, X, X, I, I) \sim (X, X_\omega, X_{\bar{\omega}}, \omega I, \omega I), \tag{2·40}$$

for $a = 4\pi/3$, and the equivalence relation

$$(X, X, X, I, I) \sim (X, X_{\bar{\omega}}, X_\omega, \bar{\omega} I, \bar{\omega} I), \tag{2·41}$$

for $a = 2\pi/3$. Here, X_ω and $X_{\bar{\omega}}$ are defined by

$$X_\omega = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{\bar{\omega}} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{2·42}$$

In this way, BCs can change under gauge transformations.

The symmetry of BCs in one theory differs from that in the other, but two theories should describe the same physics and be equivalent if they are related to by gauge transformations. This equivalence is guaranteed in the Hosotani mechanism, as will be explained in the next subsection.

2.3. *Hosotani mechanism and physical symmetry*

The Hosotani mechanism[1] in gauge theories defined on T^2/Z_3 is summarized as follows.
(i) Wilson line phases are phase factors in $W_j \Xi_j$ $(j = 1, 2)$ defined by

$$W_1 \Xi_1 \equiv P \exp\left\{ig \int_{C_1} (A_z dz + A_{\bar{z}} d\bar{z})\right\} \Xi_1, \tag{2·43}$$

$$W_2 \Xi_2 \equiv P \exp\left\{ig \int_{C_2} (A_z dz + A_{\bar{z}} d\bar{z})\right\} \Xi_2, \tag{2·44}$$

where C_j are noncontractible loops on T^2. The eigenvalues of $W_j \Xi_j$ are gauge-invariant and become physical degrees of freedom. Hence, Wilson line phases cannot be gauged away and parametrize degenerate vacua at the tree level.
(ii) The degeneracy is, in general, lifted by quantum effects. The physical vacuum is given by the configuration of Wilson line phases that minimizes the effective potential $V_{\rm eff}$.
(iii) If the configuration of the Wilson line phases is nontrivial, the gauge symmetry is spontaneously broken or restored by radiative corrections. Nonvanishing expectation values of the Wilson line phases give masses to gauge fields related to broken symmetries. Extradimensional components of gauge fields and some matter fields also acquire masses.
(iv) Two physical systems are equivalent if they are connected by a gauge transformation, which is a symmetry of the Lagrangian

$$\mathcal{L}\left(\Phi(x, z, \bar{z})\right)\Big|_{(\langle A_z\rangle, \langle A_{\bar{z}}\rangle, \Theta_0, \Theta_1)} = \mathcal{L}(\Phi'(x, z, \bar{z}))\Big|_{\left(\langle A'_z\rangle, \langle A'_{\bar{z}}\rangle, \Theta'_0, \Theta'_1\right)} \tag{2·45}$$

and is also preserved in the effective potential

$$V_{\rm eff}\left(\langle A_z\rangle, \langle A_{\bar{z}}\rangle, \Theta_0, \Theta_1\right) = V_{\rm eff}\left(\langle A'_z\rangle, \langle A'_{\bar{z}}\rangle, \Theta'_0, \Theta'_1\right). \tag{2·46}$$

The physical symmetries, parameters and spectrum are determined by the combination of BCs and the expectation value of Wilson line phases.[*]

[*] The dynamical rearrangement of QCD theta parameter was studied in a five-dimensional gauge theory with a mixed Chern-Simons term.[13]

Let us explain the last part of the mechanism in detail and how physical symmetry is determined. Dynamical phases are associated with the zero modes (z-independent modes) of A_z and $A_{\bar{z}}$ given by

$$\left\{\sum_p A_z^p T^p + \sum_{\bar{p}} A_{\bar{z}}^{\bar{p}} T^{\bar{p}};\quad T^p, T^{\bar{p}} \in \mathcal{H}_W\right\}, \tag{2·47}$$

where $\mathcal{H}_W$ is a set of generators that satisfy:

$$\mathcal{H}_W = \left\{T^p, T^{\bar{p}}\ ;\ T^p\Theta_i = \bar{\omega}\Theta_i T^p,\quad T^{\bar{p}}\Theta_i = \omega\Theta_i T^{\bar{p}},\quad i=0,1,2\right\}. \tag{2·48}$$

The potential for $A_z(x)$ and $A_{\bar{z}}(x)$ at the tree level is given by

$$V_{\rm tree} = \frac{1}{2}{\rm tr}[D_z, D_{\bar{z}}]^2 = \frac{g^2}{2}{\rm tr}[A_z, A_{\bar{z}}]^2. \tag{2·49}$$

$V_{\rm tree}$ takes a minimum when the expectation value of field strength $F_{z\bar{z}}$ vanishes. Suppose that, for $(\Theta_0, \Theta_1, \Theta_2, \Xi_1, \Xi_2, \gamma)$, $V_{\rm eff}$ is minimized at $\langle A_z\rangle$ and $\langle A_{\bar{z}}\rangle$ such that $\langle F_{z\bar{z}}\rangle = 0$ and $W_1 \neq I$ and/or $W_2 \neq I$. Perform the gauge transformation given by $\Omega = \exp\{ig(\langle A_z\rangle z + \langle A_{\bar{z}}\rangle\bar{z})\}$. This transforms $\langle A_z\rangle$ and $\langle A_{\bar{z}}\rangle$ into $\langle A'_z\rangle = \langle A'_{\bar{z}}\rangle = 0$. With this transformation, BCs change to

$$\begin{aligned}(\Theta'_0, \Theta'_1, \Theta'_2, \Xi'_1, \Xi'_2, \gamma) &= (\Theta_0, \Omega(e_1)\Theta_1, \Omega(e_1+e_2)\Theta_2, \Omega(e_1)\Xi_1, \Omega(e_2)\Xi_2, \gamma)\\ &\equiv (\Theta_0^{\rm sym}, \Theta_1^{\rm sym}, \Theta_2^{\rm sym}, \Xi_1^{\rm sym}, \Xi_2^{\rm sym}, \gamma),\end{aligned} \tag{2·50}$$

where $\Omega(e_1)$, $\Omega(e_2)$ and $\Omega(e_1+e_2)$ are defined by

$$\begin{aligned}&\Omega(e_1) = \exp\{ig(\langle A_z\rangle + \langle A_{\bar{z}}\rangle)\},\quad \Omega(e_2) = \exp\{ig(\omega\langle A_z\rangle + \bar{\omega}\langle A_{\bar{z}}\rangle)\},\\ &\Omega(e_1+e_2) = \exp\{-ig(\bar{\omega}\langle A_z\rangle + \omega\langle A_{\bar{z}}\rangle)\}.\end{aligned} \tag{2·51}$$

Because the expectation values of A'_z and $A'_{\bar{z}}$ vanish in the new gauge, the physical symmetry is spanned by the generators that commute with $(\Theta_0^{\rm sym}, \Theta_1^{\rm sym})$:

$$\mathcal{H}^{\rm sym} = \{T^\alpha\ ;\ [T^\alpha, \Theta_0^{\rm sym}] = [T^\alpha, \Theta_1^{\rm sym}] = 0\}. \tag{2·52}$$

The group generated by $\mathcal{H}^{\rm sym}$ defines the unbroken physical symmetry of the theory.

§3. Mode expansions and effective potential

3.1. *Mode expansions of six-dimensional fields*

Fields are classified as either Z_3 singlets or Z_3 triplets on T^2/Z_3. There are nine kinds of Z_3 singlet fields denoted by $\phi^{(\theta_0\theta_1\theta_2)}(x, z, \bar{z})$ where θ_i are eigenvalues of Θ_i.*) The mode expansions of $\phi^{(\theta_0\theta_1\theta_2)}(x, z, \bar{z})$ are given by

*) For convenience, θ_2 is denoted though it is not an independent parameter. Note that the relation $\theta_0\theta_1\theta_2 = 1$ stems from $T_\Phi[\Theta_0]T_\Phi[\Theta_1]T_\Phi[\Theta_2] = I$.

$$\phi^{(111)}(x,z,\bar{z}) = \phi_{0,0}(x) + {\sum_{n,m}}' \phi_{n,m}(x) f^{(0)}_{n,m}(z,\bar{z}),$$

$$\phi^{(1\omega\bar{\omega})}(x,z,\bar{z}) = \sum_{n,m} \phi_{n,m}(x) f^{(0)}_{n+\frac{1}{3},m+\frac{1}{3}}(z,\bar{z}),$$

$$\phi^{(1\bar{\omega}\omega)}(x,z,\bar{z}) = \sum_{n,m} \phi_{n,m}(x) f^{(0)}_{n+\frac{2}{3},m+\frac{2}{3}}(z,\bar{z}), \tag{3·1}$$

$$\phi^{(\omega\omega\omega)}(x,z,\bar{z}) = {\sum_{n,m}}' \phi_{n,m}(x) f^{(1)}_{n,m}(z,\bar{z}),$$

$$\phi^{(\omega\bar{\omega}1)}(x,z,\bar{z}) = \sum_{n,m} \phi_{n,m}(x) f^{(1)}_{n+\frac{1}{3},m+\frac{1}{3}}(z,\bar{z}),$$

$$\phi^{(\omega 1\bar{\omega})}(x,z,\bar{z}) = \sum_{n,m} \phi_{n,m}(x) f^{(1)}_{n+\frac{2}{3},m+\frac{2}{3}}(z,\bar{z}), \tag{3·2}$$

$$\phi^{(\bar{\omega}\bar{\omega}\bar{\omega})}(x,z,\bar{z}) = {\sum_{n,m}}' \phi_{n,m}(x) f^{(2)}_{n,m}(z,\bar{z}),$$

$$\phi^{(\bar{\omega}1\omega)}(x,z,\bar{z}) = \sum_{n,m} \phi_{n,m}(x) f^{(2)}_{n+\frac{1}{3},m+\frac{1}{3}}(z,\bar{z}),$$

$$\phi^{(\bar{\omega}\omega 1)}(x,z,\bar{z}) = \sum_{n,m} \phi_{n,m}(x) f^{(2)}_{n+\frac{2}{3},m+\frac{2}{3}}(z,\bar{z}), \tag{3·3}$$

where ${\sum}'_{n,m}$ means the summation over integers (n,m) excluding $n=m=0$ and normalization factors are absorbed by the four-dimensional fields $\phi_{n,m}(x)$. Note that only $\phi^{(111)}(x,z,\bar{z})$ has a zero mode. Here, $f^{(i)}_{n+\alpha,m+\beta}(z,\bar{z})$ are defined by

$$f^{(0)}_{n+\alpha,m+\beta}(z,\bar{z}) \equiv f_{n+\alpha,m+\beta}(z,\bar{z}) + f_{n+\alpha,m+\beta}(\omega z,\bar{\omega}\bar{z}) + f_{n+\alpha,m+\beta}(\bar{\omega}z,\omega\bar{z}), \tag{3·4}$$

$$f^{(1)}_{n+\alpha,m+\beta}(z,\bar{z}) \equiv \bar{\omega} f_{n+\alpha,m+\beta}(z,\bar{z}) + \omega f_{n+\alpha,m+\beta}(\omega z,\bar{\omega}\bar{z}) + f_{n+\alpha,m+\beta}(\bar{\omega}z,\omega\bar{z}), \tag{3·5}$$

$$f^{(2)}_{n+\alpha,m+\beta}(z,\bar{z}) \equiv \omega f_{n+\alpha,m+\beta}(z,\bar{z}) + \bar{\omega} f_{n+\alpha,m+\beta}(\omega z,\bar{\omega}\bar{z}) + f_{n+\alpha,m+\beta}(\bar{\omega}z,\omega\bar{z}), \tag{3·6}$$

where $f_{n+\alpha,m+\beta}(z,\bar{z})$ is defined by

$$f_{n+\alpha,m+\beta}(z,\bar{z}) \equiv \exp\left[\pi i\left\{\left(n+\alpha-\frac{n+\alpha+2(m+\beta)}{\sqrt{3}}i\right)z + \left(n+\alpha+\frac{n+\alpha+2(m+\beta)}{\sqrt{3}}i\right)\bar{z}\right\}\right]. \tag{3·7}$$

In the case of vanishing Wilson line phases, the mass squared of $\phi_{n,m}(x)$ is derived from the kinetic terms after compactification such that

$$M^2_{n,m}(\alpha,\beta) = \pi^2\left((n+\alpha)^2 + \frac{1}{3}(n+\alpha+2(m+\beta))^2\right)$$

$$= \frac{4}{3}\pi^2\left[(n+\alpha)^2+(n+\alpha)(m+\beta)+(m+\beta)^2\right]$$
$$= \frac{1}{3}\left[\left(\frac{n+\alpha}{R}\right)^2+\left(\frac{n+\alpha}{R}\right)\left(\frac{m+\beta}{R}\right)+\left(\frac{m+\beta}{R}\right)^2\right], \qquad (3\cdot8)$$

where α, $\beta=0, 1/3, 2/3$ and the physical size $|e_1|=|e_2|=2\pi R$ is used in the final expression.

In gauge theories on T^2/Z_3, there is another important representation, a Z_3 triplet. The Z_3 triplet field $\phi^A(x,z,\bar{z})$ $(A=1,2,3)$ satisfies BCs such that

$$\begin{aligned}
\phi^A(x,\omega z,\bar{\omega}\bar{z}) &= \sum_{B=1}^{3}(\mathcal{X})^{\mathcal{A}}{}_B\phi^B(x,z,\bar{z}),\\
\phi^A(x,\omega z+1,\bar{\omega}\bar{z}+1) &= \sum_{B=1}^{3}\left(e^{-2\pi i\gamma\mathcal{Y}}\mathcal{X}\right)^A{}_B\phi^B(x,z,\bar{z}),\\
\phi^A(x,\omega z+1+\omega,\bar{\omega}\bar{z}+1+\bar{\omega}) &= \sum_{B=1}^{3}\left(e^{2\pi i\gamma\mathcal{Y}_\omega}\mathcal{X}\right)^A{}_B\phi^B(x,z,\bar{z}),\\
\phi^A(x,z+1,\bar{z}+1) &= \sum_{B=1}^{3}\left(e^{-2\pi i\gamma\mathcal{Y}}\right)^A{}_B\phi^B(x,z,\bar{z}),\\
\phi^A(x,z+\omega,\bar{z}+\bar{\omega}) &= \sum_{B=1}^{3}\left(e^{-2\pi i\gamma\mathcal{Y}_{\bar{\omega}}}\right)^A{}_B\phi^B(x,z,\bar{z}), \qquad (3\cdot9)
\end{aligned}$$

where we take $T_{\phi^A}[\Theta_i]=I$ and γ is a real number. When we take $(\mathcal{X},\mathcal{Y},\mathcal{Y}_\omega,\mathcal{Y}_{\bar{\omega}}) = (X,Y,Y_\omega,Y_{\bar{\omega}})$, the mode expansion of $\phi^A(x,z,\bar{z})$ is given by

$$\phi^A(x,z,\bar{z})=\sum_{n,m}\phi_{n,m}(x)\begin{pmatrix} f^{(0)}_{n+\gamma,m+\gamma}(z,\bar{z})\\ f^{(1)}_{n+\gamma,m+\gamma}(z,\bar{z})\\ f^{(2)}_{n+\gamma,m+\gamma}(z,\bar{z})\end{pmatrix}, \qquad (3\cdot10)$$

where γ can take an arbitrary value. In the case of vanishing Wilson line phases, the mass squared of $\phi_{n,m}(x)$ is given by

$$\begin{aligned}
M^2_{n,m}(\gamma,\gamma) &= \pi^2\left((n+\gamma)^2+\frac{1}{3}(n+\gamma+2(m+\gamma))^2\right)\\
&= \frac{1}{3}\left[\left(\frac{n+\gamma}{R}\right)^2+\left(\frac{n+\gamma}{R}\right)\left(\frac{m+\gamma}{R}\right)+\left(\frac{m+\gamma}{R}\right)^2\right], \qquad (3\cdot11)
\end{aligned}$$

where the physical size $|e_1|=|e_2|=2\pi R$ is used in the final expression. There are no massless modes from the Z_3 triplet in the case that γ is not an integer.

3.2. *Classification of equivalence classes*

The classification of equivalence classes of BCs is reduced to the classification of (Θ_0,Θ_1). We classify equivalence classes, which contain a set of diagonal representation matrices (Θ_0, Θ_1). The diagonal matrices (Θ_0, Θ_1) are specified by nine non-negative integers $(l_p, m_p, n_p,\ l_q, m_q, n_q, l_r, m_r, n_r)$ such that

$$
\begin{aligned}
\Theta_0 &= \mathrm{diag}(\overbrace{1,\cdots,\ 1, 1,\cdots,\ 1, 1,\cdots,1}^{p}, \overbrace{\omega,\cdots,\omega,\omega,\cdots,\omega,\omega,\cdots,\omega}^{q} \\
&\qquad\qquad\qquad\qquad\qquad \overbrace{\bar{\omega},\cdots,\bar{\omega},\bar{\omega},\cdots,\bar{\omega},\bar{\omega},\cdots,\bar{\omega}}^{r=N-p-q}), \\
\Theta_1 &= \mathrm{diag}(1,\cdots,1,\omega,\cdots,\omega,\bar{\omega},\cdots,\bar{\omega},1,\cdots,1,\omega,\cdots,\omega,\bar{\omega},\cdots,\bar{\omega}, \\
&\qquad\qquad\qquad\qquad\qquad 1,\cdots,1,\omega,\cdots,\omega,\bar{\omega},\cdots,\bar{\omega}), \\
\Theta_2 &= \mathrm{diag}(\underbrace{1,\cdots,1}_{l_p},\underbrace{\bar{\omega},\cdots,\bar{\omega}}_{m_p},\underbrace{\omega,\cdots,\omega}_{n_p},\underbrace{\bar{\omega},\cdots,\bar{\omega}}_{l_q},\underbrace{\omega,\cdots,\omega}_{m_q},\underbrace{1,\cdots,1}_{n_q}, \\
&\qquad\qquad\qquad\qquad\qquad \underbrace{\omega,\cdots,\omega}_{l_r},\underbrace{1,\cdots,1}_{m_r},\underbrace{\bar{\omega},\cdots,\bar{\omega}}_{n_r}), \qquad (3\cdot12)
\end{aligned}
$$

where Θ_2 is denoted, for convenience, $N \geq l_p, m_p, n_p, l_q, m_q, n_q, l_r, m_r, n_r \geq 0$, $p = l_p + m_p + n_p$, $q = l_q + m_q + n_q$ and $r = l_r + m_r + n_r$. We denote each BC specified by $(l_p, m_p, n_p, l_q, m_q, n_q, l_r, m_r, n_r)$ (or a theory with such BCs) as $[l_p, m_p, n_p;\ l_q, m_q, n_q;\ l_r, m_r, n_r]$.

The matrix Θ_1 is interchanged with Θ_2 by the following interchange among entries such that

$$[l_p, m_p, n_p; l_q, m_q, n_q; l_r, m_r, n_r] \leftrightarrow [l_p, n_p, m_p; n_q, m_q, l_q; m_r, l_r, n_r]. \qquad (3\cdot13)$$

The matrix Θ_0 is interchanged with Θ_1 by the following interchange among entries such that

$$[l_p, m_p, n_p; l_q, m_q, n_q; l_r, m_r, n_r] \leftrightarrow [l_p, l_q, l_r; m_p, m_q, m_r; n_p, n_q, n_r]. \qquad (3\cdot14)$$

The matrix Θ_0 is interchanged with Θ_2 by the following interchange among entries such that

$$[l_p, m_p, n_p; l_q, m_q, n_q; l_r, m_r, n_r] \leftrightarrow [l_p, m_r, n_q; l_r, m_q, n_p; l_q, m_p, n_r]. \qquad (3\cdot15)$$

Using the equivalence relations (2·40) and (2·41), we can derive the following equivalence relations in the $SU(N)$ gauge theory:

$$
\begin{aligned}
&[l_p, m_p, n_p; l_q, m_q, n_q; l_r, m_r, n_r] \\
&\sim [l_p - 1, m_p + 1, n_p; l_q, m_q - 1, n_q + 1; l_r + 1, m_r, n_r - 1], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ l_p, m_q, n_r \geq 1, \\
&\sim [l_p + 1, m_p - 1, n_p; l_q, m_q + 1, n_q - 1; l_r - 1, m_r, n_r + 1], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ m_p, n_q, l_r \geq 1, \\
&\sim [l_p - 1, m_p, n_p + 1; l_q + 1, m_q - 1, n_q; l_r, m_r + 1, n_r - 1], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ l_p, m_q, n_r \geq 1, \\
&\sim [l_p + 1, m_p, n_p - 1; l_q - 1, m_q + 1, n_q; l_r, m_r - 1, n_r + 1], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ m_p, n_q, l_r \geq 1,
\end{aligned}
$$

$$\sim [l_p, m_p - 1, n_p + 1; l_q + 1, m_q, n_q - 1; l_r - 1, m_r + 1, n_r], \quad \text{for } m_p, n_q, l_r \geq 1,$$
$$\sim [l_p, m_p + 1, n_p - 1; l_q - 1, m_q, n_q + 1; l_r + 1, m_r - 1, n_r], \quad \text{for } n_p, l_q, m_r \geq 1. \quad (3\cdot16)$$

One can show that the number of equivalence classes of BCs including diagonal representations is ${}_{N+8}C_8 - 2 \cdot {}_{N+5}C_8$ for the $SU(N)$ gauge group.

3.3. *Effective potential*

We study the effective potential for extradimensional components of the gauge field in an $SU(3)$ gauge theory on $M^4 \times (T^2/Z_3)$. Let us adopt the representation matrices such that

$$\Theta_0 = \Theta_1 = \Theta_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}. \quad (3\cdot17)$$

With this assignment, the eigenvalues $(\theta_0, \theta_1, \theta_2)$ for gauge fields are determined from the transformation properties under Z_3 transformation (2·12) – (2·14), and are given in Table I. Here $(A_M^{1+}, \cdots, A_M^{6-})$ are defined by

Table I. $(\theta_0, \theta_1, \theta_2)$ for gauge fields.

	θ_0	θ_1	θ_2
A_μ^{1+}, A_μ^{4-}, A_μ^{6+}	$\bar{\omega}$	$\bar{\omega}$	$\bar{\omega}$
A_μ^{1-}, A_μ^{4+}, A_μ^{6-}	ω	ω	ω
A_μ^3, A_μ^8	1	1	1
A_z^{1+}, A_z^{4-}, A_z^{6+}	ω	ω	ω
A_z^{1-}, A_z^{4+}, A_z^{6-}	1	1	1
A_z^3, A_z^8	$\bar{\omega}$	$\bar{\omega}$	$\bar{\omega}$
$A_{\bar{z}}^{1+}$, $A_{\bar{z}}^{4-}$, $A_{\bar{z}}^{6+}$	1	1	1
$A_{\bar{z}}^{1-}$, $A_{\bar{z}}^{4+}$, $A_{\bar{z}}^{6-}$	$\bar{\omega}$	$\bar{\omega}$	$\bar{\omega}$
$A_{\bar{z}}^3$, $A_{\bar{z}}^8$	ω	ω	ω

$$A_M^{1+} \equiv \frac{1}{\sqrt{2}} \left(A_M^1 - i A_M^2 \right), \quad (3\cdot18)$$
$$A_M^{1-} \equiv \frac{1}{\sqrt{2}} \left(A_M^1 + i A_M^2 \right), \quad (3\cdot19)$$
$$A_M^{4+} \equiv \frac{1}{\sqrt{2}} \left(A_M^4 - i A_M^5 \right), \quad (3\cdot20)$$
$$A_M^{4-} \equiv \frac{1}{\sqrt{2}} \left(A_M^4 + i A_M^5 \right), \quad (3\cdot21)$$
$$A_M^{6+} \equiv \frac{1}{\sqrt{2}} \left(A_M^6 - i A_M^7 \right), \quad (3\cdot22)$$
$$A_M^{6-} \equiv \frac{1}{\sqrt{2}} \left(A_M^6 + i A_M^7 \right). \quad (3\cdot23)$$

We find that zero modes appear in A_μ^3, A_μ^8, A_z^{1-}, A_z^{4+}, A_z^{6-}, $A_{\bar{z}}^{1+}$, $A_{\bar{z}}^{4-}$ and $A_{\bar{z}}^{6+}$. From the vanishing field strength condition, $A_z(x)$ and $A_{\bar{z}}(x)$ are parametrized as

$$A_z(x) = \frac{\sqrt{2}\pi a}{g} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{\bar{z}}(x) = \frac{\sqrt{2}\pi a}{g} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3\cdot24)$$

Here, we set the zero mode a to be real using the residual $U(1)$ gauge symmetries.

Now we consider the effective potential $V_{\rm eff}$ for a by treating it as a background field. The effective potential is derived by writing $A_M = A_M^0 + A_M^q$, taking a suitable gauge fixing and integrating over the quantum part A_M^q and every quantum flactuation of other fields. Here A_M^0 is a background configuration of the gauge field A_M. The $V_{\rm eff}$ depends not only on A_M^0 but also on BCs, i.e., $V_{\rm eff} = V_{\rm eff}[A_M^0; \Theta_0, \Theta_1, \gamma]$. If the gauge fixing term is also invariant under the gauge transformation, i.e.,

$$D^M(A^0)A_M = 0 \to D^M(A'^0)A'_M = \Omega D^M(A^0)A_M\Omega^\dagger = 0, \tag{3·25}$$

it is shown that $V_{\rm eff}$ satisfies

$$V_{\rm eff}[A_M^0; \Theta_0, \Theta_1, \gamma] = V_{\rm eff}[A_M'^0; \Theta'_0, \Theta'_1, \gamma]. \tag{3·26}$$

This property implies that the minimum $V_{\rm eff}$ corresponds to the same symmetry as that of $(\Theta_0^{\rm sym}, \Theta_1^{\rm sym})$.

The one-loop effective potential is given by

$$V_{\rm eff}[A_M^0; \Theta_0, \Theta_1, \gamma] = \sum \mp \frac{i}{2} {\rm Tr}\ {\rm ln} D_M(A^0)D^M(A^0), \tag{3·27}$$

$$= \sum \mp \frac{1}{2} \int \frac{d^4p_E}{(2\pi)^4} \sum_{n,m} {\rm ln}(p_E^2 + \hat{M}_{n,m}^2 - i\varepsilon), \tag{3·28}$$

where p_E is a four-dimensional Euclidean momentum and the Wick rotation is applied. Here, we consider that $F_{MN}^0 = 0$ and every field has no mass term on six-dimensional space-time. The sums extend over all degrees of freedom of fields in the bulk in Eq. (3·27) and over all degrees of freedom of four-dimensional fields whose masses are $\hat{M}_{n,m}$ in Eq. (3·28). The sign is negative (positive) for bosons (FP ghosts and fermions). $D_M(A^0)$ denotes an appropriate covariant derivative with respect to A_M^0. For later convenience, we write down the formula of one-loop effective potential for $\hat{M}_{n,m}^2 = M_{n,m}^2(\alpha, \beta)$ as[14)]

$$\begin{aligned} V_{\rm eff}[A_M^0; \Theta_0, \Theta_1, \gamma] &= \sum_{(\alpha,\beta)} \mp \frac{1}{2} I(\alpha, \beta), \\ I(\alpha, \beta) &\equiv \int \frac{d^4p_E}{(2\pi)^4} \sum_{n,m} {\rm ln}(p_E^2 + M_{n,m}^2(\alpha,\beta) - i\varepsilon) \\ &= \frac{\sqrt{3}}{256\pi^7 R^4} {\sum_{n,m}}' \frac{1}{(n^2+m^2-nm)^3} \cos 2\pi(\alpha n + \beta m) \\ &\quad + (\alpha, \beta\text{-independent terms}). \end{aligned} \tag{3·29}$$

Our task now is to obtain mass squareds $\hat{M}_{n,m}^2$ for every field that couples to gauge fields. For simplicity, we consider an $SU(3)$ triplet scalar field $\phi = (\phi^1, \phi^2, \phi^3)$ whose eigenvalues $(\theta_0, \theta_1, \theta_2)$ are given in Table II. The

Table II. $(\theta_0, \theta_1, \theta_2)$ for ϕ.

	θ_0	θ_1	θ_2
ϕ^1	1	1	1
ϕ^2	ω	ω	ω
ϕ^3	$\bar{\omega}$	$\bar{\omega}$	$\bar{\omega}$

point is to consider ϕ as a Z_3 triplet, i.e., $\phi^1_{n,m}(x) = \phi^2_{n,m}(x) = \phi^3_{n,m}(x) \equiv \phi_{n,m}(x)$. Then the covariant derivative for $\phi(x, z, \bar{z})$ is calculated as

$$D_z\phi = (\partial_z + igA_z)\phi = \sum_{n,m}\phi_{n,m}(x)\begin{pmatrix} \partial_z & 0 & i\sqrt{2}\pi a \\ i\sqrt{2}\pi a & \partial_z & 0 \\ 0 & i\sqrt{2}\pi a & \partial_z \end{pmatrix}\begin{pmatrix} f^{(0)}_{n,m} \\ f^{(1)}_{n,m} \\ f^{(2)}_{n,m} \end{pmatrix}$$

$$= \sum_{n,m}\phi_{n,m}(x)\begin{pmatrix} i\pi\left(n - \frac{a}{\sqrt{2}} + \frac{n-\frac{a}{\sqrt{2}}+2\left(m-\frac{a}{\sqrt{2}}\right)}{\sqrt{3}}i\right)\bar{\omega}f^{(2)}_{n,m} \\ i\pi\left(n - \frac{a}{\sqrt{2}} + \frac{n-\frac{a}{\sqrt{2}}+2\left(m-\frac{a}{\sqrt{2}}\right)}{\sqrt{3}}i\right)\bar{\omega}f^{(0)}_{n,m} \\ i\pi\left(n - \frac{a}{\sqrt{2}} + \frac{n-\frac{a}{\sqrt{2}}+2\left(m-\frac{a}{\sqrt{2}}\right)}{\sqrt{3}}i\right)\bar{\omega}f^{(1)}_{n,m} \end{pmatrix}. \qquad (3{\cdot}30)$$

Hence the mass squareds for $\phi_{n,m}(x)$ are three $M^2_{n,m}(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$'s. In the same way, those of gauge fields are calculated from the covariant derivative $D_zA_M = \partial_zA_M + ig[A_z, A_M]$ and are $M^2_{n,m}(-a, -a)$, $M^2_{n,m}(\frac{1+\sqrt{3}}{2}a, \frac{1+\sqrt{3}}{2}a)$, $M^2_{n,m}(\frac{1-\sqrt{3}}{2}a, \frac{1-\sqrt{3}}{2}a)$ and five $M^2_{n,m}(0, 0)$'s. The same result holds for FP ghosts.

Using mass squareds and Eq. (3·29), we obtain the one-loop effective potential for a as

$$V_{\text{eff}} = -2I(-a, -a) - 2I\left(\frac{1+\sqrt{3}}{2}a, \frac{1+\sqrt{3}}{2}a\right) - 2I\left(\frac{1-\sqrt{3}}{2}a, \frac{1-\sqrt{3}}{2}a\right) - \frac{3}{2}I\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right). \qquad (3{\cdot}31)$$

The minimum V_{eff} is given at $a = 0$. When fermions are introduced, the non-vanishing expectation value of a can be obtained and the breakdown of $U(1)$ gauge symmetries can occur.

§4. Conclusions

We have studied equivalence classes of BCs in a gauge theory on the orbifold T^2/Z_3. General arguments have been given for BCs in gauge theories on T^2/Z_3 including various relations of BCs, and equivalence classes of BCs have been defined by the invariance under gauge transformation. Mode expansions have been given for six-dimensional Z_3 singlet fields and the Z_3 triplet field, and the classification of BCs for the $SU(N)$ gauge group has been carried out with the aid of equivalence relations. The one-loop effective potential for Wilson line phases has been calculated using the $SU(3)$ gauge theory. It is crucial to study dynamical gauge symmetry breaking and mass generation in a realistic model including fermions. It is also important to construct a phenomenologically viable model realizing gauge-Higgs unification[15)] and/or family unification[16)] based on them. The local grand unification can be realized by taking nontrivial Θ_i's.[*)] It is interesting to study the phenomenological aspects of such models. We hope to further study these subjects in the near future.

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Appendix A
—— *Useful Formulae* ——

For Y_+^k, Y_-^k $(k=1,2,3)$ and X defined by

$$\begin{aligned}
&Y_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y_+^2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ \omega & 0 & 0 \end{pmatrix}, \quad Y_+^3 = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \\ \omega^2 & 0 & 0 \end{pmatrix}, \\
&Y_-^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_-^2 = \begin{pmatrix} 0 & 0 & \omega^2 \\ \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad Y_-^3 = \begin{pmatrix} 0 & 0 & \omega \\ \omega & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \\
&X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\end{aligned} \tag{A·1}$$

the following relation holds:

$$X \exp\left[-i\sum_{k=1}^{3}\left(a^k Y_+^k + \bar{a}^k Y_-^k\right)\right] = \exp\left[-i\sum_{k=1}^{3}\left(\bar{\omega} a^k Y_+^k + \omega \bar{a}^k Y_-^k\right)\right] X. \tag{A·2}$$

For Y, Y_ω and $Y_{\bar{\omega}}$ defined by

$$Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad Y_\omega = \begin{pmatrix} 0 & \omega & \omega^2 \\ \omega^2 & 0 & \omega \\ \omega & \omega^2 & 0 \end{pmatrix}, \quad Y_{\bar{\omega}} = \begin{pmatrix} 0 & \omega^2 & \omega \\ \omega & 0 & \omega^2 \\ \omega^2 & \omega & 0 \end{pmatrix}, \tag{A·3}$$

the n-th powers of Y, Y_ω and $Y_{\bar{\omega}}$ are calculated as

$$\begin{aligned}
Y^n &= \frac{1}{3}\left(2^n - (-1)^n\right) Y + \frac{1}{3}\left(2^n + 2(-1)^n\right) I, \\
Y_\omega^n &= \frac{1}{3}\left(2^n - (-1)^n\right) Y_\omega + \frac{1}{3}\left(2^n + 2(-1)^n\right) I, \\
Y_{\bar{\omega}}^n &= \frac{1}{3}\left(2^n - (-1)^n\right) Y_{\bar{\omega}} + \frac{1}{3}\left(2^n + 2(-1)^n\right) I.
\end{aligned} \tag{A·4}$$

---

*) The 'local' gauge groups at fixed points were realized on T^2/Z_2 in Ref. 17). The string-derived orbifold grand unification theories were studied in Refs. 18) and 19).

Then e^{iaY}, e^{iaY_ω} and $e^{iaY_{\bar\omega}}$ are calculated as

$$
\begin{aligned}
e^{iaY} &= \frac{1}{3}\left(e^{2ai} - e^{-ai}\right) Y + \frac{1}{3}\left(e^{2ai} + 2e^{-ai}\right) I, \\
e^{iaY_\omega} &= \frac{1}{3}\left(e^{2ai} - e^{-ai}\right) Y_\omega + \frac{1}{3}\left(e^{2ai} + 2e^{-ai}\right) I, \\
e^{iaY_{\bar\omega}} &= \frac{1}{3}\left(e^{2ai} - e^{-ai}\right) Y_{\bar\omega} + \frac{1}{3}\left(e^{2ai} + 2e^{-ai}\right) I.
\end{aligned}
\tag{A·5}
$$

For the function $f_{n+\alpha,m+\beta}(z,\bar z)$ defined by

$$
\begin{aligned}
f_{n+\alpha,m+\beta}(z,\bar z) \equiv \exp\Bigg[\pi i \Bigg\{ &\left(n+\alpha - \frac{n+\alpha+2(m+\beta)}{\sqrt{3}} i\right) z \\
&+ \left(n+\alpha + \frac{n+\alpha+2(m+\beta)}{\sqrt{3}} i\right) \bar z \Bigg\}\Bigg],
\end{aligned}
\tag{A·6}
$$

the following transformation properties are derived:

$$
\begin{aligned}
&f_{n+\alpha,m+\beta}(z+1,\bar z+1) = \omega^{3\alpha} f_{n+\alpha,m+\beta}(z,\bar z), \\
&f_{n+\alpha,m+\beta}(z+\omega,\bar z+\bar\omega) = \omega^{3\beta} f_{n+\alpha,m+\beta}(z,\bar z), \\
&f_{n+\alpha,m+\beta}(z+\bar\omega,\bar z+\omega) = \bar\omega^{3(\alpha+\beta)} f_{n+\alpha,m+\beta}(z,\bar z), \\
&f_{n+\alpha,m+\beta}(\omega z,\bar\omega \bar z) = f_{m+\beta,-m-n-\alpha-\beta}(z,\bar z), \\
&f_{n+\alpha,m+\beta}(\bar\omega z,\omega \bar z) = f_{-m-n-\alpha-\beta,n+\alpha}(z,\bar z).
\end{aligned}
\tag{A·7}
$$

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