Clifford Theory for Commutative Association Schemes

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Abstract

In this paper, Clifford's well-known theorem on irreducible characters of finite groups is generalized to finite commutative association schemes. Our theorem relates irreducible characters of finite commutative schemes to those of their strongly normal closed subsets.

1 Introduction

Clifford's theorem is one of the most important theorems in the theory of characters of finite groups. In this paper, we consider Clifford's theorem for finite schemes. We shall adopt notation and terminology from Zieschang's book [6].

It is natural to consider Clifford's theorem for normal closed subsets. However, we have the following example.

Example 1.1. Let G be the association scheme defined by the following relation matrix.

$$\begin{pmatrix}
0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 \\
1 & 0 & 2 & 2 & 5 & 5 & 6 & 6 & 3 & 3 & 4 & 4 \\
2 & 2 & 0 & 1 & 4 & 6 & 3 & 5 & 4 & 6 & 3 & 5 \\
2 & 2 & 1 & 0 & 6 & 4 & 5 & 3 & 6 & 4 & 5 & 3 \\
3 & 6 & 4 & 5 & 0 & 3 & 2 & 5 & 4 & 2 & 6 & 1 \\
3 & 6 & 5 & 4 & 3 & 0 & 5 & 2 & 2 & 4 & 1 & 6 \\
4 & 5 & 3 & 6 & 2 & 6 & 0 & 4 & 5 & 1 & 3 & 2 \\
4 & 5 & 6 & 3 & 6 & 2 & 4 & 0 & 1 & 5 & 2 & 3 \\
6 & 3 & 4 & 5 & 4 & 2 & 6 & 1 & 0 & 3 & 2 & 5 \\
6 & 3 & 5 & 4 & 2 & 4 & 1 & 6 & 3 & 0 & 5 & 2 \\
5 & 4 & 3 & 6 & 5 & 1 & 3 & 2 & 2 & 6 & 0 & 4 \\
5 & 4 & 6 & 3 & 1 & 5 & 2 & 3 & 6 & 2 & 4 & 0
\end{pmatrix}$$

Then $H = \{g_0, g_1, g_2\}$ is a normal closed subset of G. The character tables of G and H

are as follows.

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	m_i
χ_1	1	1	2	2	2	2	2	1
χ_2	1	1	2	-1	-1	-1	-1	2
χ_3	1	-1	0	-1	-1	1	1	3
χ_4	2	0	-2	1	1	-1	-1	3

	g_0	g_1	g_2	m_i
φ_1	1	1	2	1
φ_2	1	1	-2	1
φ_3	1	-1	0	2

Now we can see that $(\chi_3)_H = \varphi_3$ and $(\chi_4)_H = \varphi_2 + \varphi_3$. If Clifford-type theorem holds for this example, then we expect that the sets of irreducible constituents of $(\chi_3)_H$ and $(\chi_4)_H$ coincide since they have a common constituent φ_3. So this example shows that Clifford-type theorem does not hold for this example.

The previous example shows that Clifford's theorem does not hold for normal closed subsets of finite schemes. However, for finite thin schemes (finite groups), the notion of a normal closed subset is equivalent to the one of a strongly normal closed subset. Therefore, we restrict ourselves to strongly normal closed subsets. So far, we do not know of a non-commutative scheme for which Clifford's theorem fails. Thus, there is hope that Clifford's theorem holds for arbitrary finite schemes, not only for commutative finite schemes. In order to prove our main result, we shall now first look at Clifford Theory for group-graded algebras.

2 Group-graded algebras and crossed products

In this section, we introduce the theory of group-graded algebras in Dade [2] or Curtis, Reiner [1, §11]. To simplify our argument, we always suppose that the coefficient field F is an algebraically closed field and F-algebras and F-modules are finite dimensional over F. Modules will be right modules.

Let S be a finite group, and let A be an F-algebra. Suppose A is a direct sum of F-subspaces A_s, $s \in S$. The algebra A is called *S-graded* (group-graded) if

(1) $A_sA_t \subseteq A_{st}$ for $s, t \in S$.

For an S-graded algebra A, A_1 is a subalgebra of A. Furthermore, if

(2) $A_sA_t = A_{st}$ for $s, t \in S$,

we say that A is *strongly S-graded.* If an S-graded algebra A satisfies

(3) for every $s \in S$, A_s contains a unit a_s in A,

then the condition (2) holds, and in this case, A is a crossed product of S over A_1 [2, Theorem 5.10]. For crossed products, it is known that Clifford's theorem holds.

Assume that the algebra A satisfies the condition (3). Then A is a free right and left A_1-module with a free basis $\{a_s \mid s \in S\}$. For an A-module M, the *restriction* of M to A_1 is denoted by M_{A_1}. For an A_1-module L, the *induced module* of L to A is

$$L^A := L \otimes_{A_1} A = \bigoplus_{s \in S} L \otimes a_s.$$

Now $L \otimes a_s$ is an A_1-submodule of L^A. Let L be a simple (irreducible) A_1-module. Put

$$T := \{s \in S \mid L \otimes a_s \cong L\},$$

then T is a subgroup of S. We write $\mathrm{Irr}(A \mid L)$ for the set of simple A-modules M such that M_{A_1} contains L as a simple submodule. Here we identify isomorphic modules. Now we can state Clifford's theorem for crossed products.

Theorem 2.1 ([1, Proposition 11.16])**.** *Let S be a finite group, A a finite dimensional S-graded algebra with the property (3) above, and let L be a simple A_1-module. Put $T := \{s \in S \mid L \otimes a_s \cong L\}$. Then we have the followings.*

(1) *If $M \in \mathrm{Irr}(A \mid L)$, then M_{A_1} is semisimple and $M_{A_1} = e\left(\bigoplus_{t \in T \setminus S} L \otimes a_t\right)$ for some positive integer e.*

(2) *Put $B = \sum_{t \in T} A_t$. Then the map $\mathrm{Irr}(B \mid L) \to \mathrm{Irr}(A \mid L)$ defined by $N \mapsto N^A$ is a bijection.*

3 The case that the adjacency algebra is a crossed product

Let (X, G) be an association scheme. In this section, we do not assume the commutativity of (X, G). Suppose that H is a strongly normal closed subset of G, namely, the factor $G//H$ is essentially a finite group. Consider the double coset decomposition of G,

$$G = \bigcup_{g^H \in G//H} HgH.$$

(Of course, $HgH = gH = Hg$ holds since H is normal in G.) Now we have a direct sum decomposition of the adjacency algebra:

$$\mathbb{C}G = \bigoplus_{g^H \in G//H} \mathbb{C}(HgH),$$

where $\mathbb{C}(HgH) = \bigoplus_{h \in HgH} \mathbb{C}\sigma_h$. By the definition, it is clear that $\mathbb{C}G$ is a $G//H$-graded algebra. If $\mathbb{C}G$ satisfies the condition (3) in Section 2, then the Clifford's theorem (Theorem 2.1) holds.

Example 3.1 (Semidirect products)**.** Let $(X, G)\Theta$ be a semidirect product defined in [6, Section 2]. Then $G = G \times 1$ is a strongly normal closed subset, and the condition (3) in Section 2 holds. So the Clifford's theorem holds for semidirect products. In this case, the adjacency algebra is a *skew group ring* of Θ over $\mathbb{C}G$.

Example 3.2. Let G be the association scheme defined by the following relation matrix.

$$\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 2 & 2 & 3 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 3 & 3 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
2 & 3 & 2 & 2 & 3 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 2 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 2 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
3 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
3 & 2 & 2 & 3 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
3 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

Now $H = \{g_0, g_1\}$ is a strongly normal closed subset of G. This is not a semidirect product, but the condition (3) in Section 2 holds. The character table of G is as follows.

	g_0	g_1	g_2	g_3	m_i
χ_1	1	6	3	4	1
χ_2	1	6	-3	-4	1
χ_3	1	-1	$\sqrt{2}$	$-\sqrt{2}$	6
χ_4	1	-1	$-\sqrt{2}$	$\sqrt{2}$	6

In Section 4, we will show that the condition (3) holds if G is commutative and $|H| = |HgH|$ for any $g \in G$. So we can construct similar examples from arbitrary symmetric designs. Actually, this example is constructed by $PG(2, 2)$.

4 Clifford's theorem for commutative schemes

In this section, we consider Clifford's theorem for commutative schemes and their strongly normal closed subsets. Let (X, G) be a commutative scheme, and let H be a strongly normal closed subset of G. We consider the decomposition of adjacency algebras

$$\mathbb{C}H = \bigoplus_{\varphi \in \mathrm{Irr}(H)} e_\varphi \mathbb{C}H, \quad \mathbb{C}G = \bigoplus_{\varphi \in \mathrm{Irr}(H)} e_\varphi \mathbb{C}G.$$

Obviously, $\mathrm{Irr}(e_\varphi \mathbb{C}H) = \{\varphi\}$ and $\mathrm{Irr}(e_\varphi \mathbb{C}G) = \mathrm{Irr}(G \mid \varphi)$, so we consider Clifford's theorem between $e_\varphi \mathbb{C}H$ and $e_\varphi \mathbb{C}G$. Here we denote $\mathrm{Irr}(G \mid \varphi)$ for the set of irreducible characters χ of G such that the restriction of χ to H contains φ as an irreducible constituent.

As in Section 3, we decompose $e_\varphi \mathbb{C}G$ as

$$e_\varphi \mathbb{C}G = \bigoplus_{g^H \in G//H} e_\varphi \mathbb{C}(HgH).$$

Then $e_\varphi \mathbb{C}G$ is $G//H$-graded since e_φ is in $\mathbb{C}H$. We note that $e_\varphi \mathbb{C}(HgH)$ can be zero. So we put

$$Z//H := \{g^H \in G//H \mid e_\varphi \mathbb{C}(HgH) \neq 0\}.$$

Then we have the crucial lemma in this paper.

Lemma 4.1. *If $e_\varphi \mathbb{C}(HgH) \neq 0$, then $e_\varphi \mathbb{C}(HgH)$ contains a unit in $e_\varphi \mathbb{C}G$.*

To prove this lemma, we need the next proposition. Let H be a normal closed subset of G. For a character τ of the factor scheme $G//H$ and $g \in G$, we define

$$\widetilde{\tau}(\sigma_g) := \frac{n_g}{n_{g^H}} \tau(\sigma_{g^H}).$$

Then $\widetilde{\tau}$ is a character of G [3, Theorem 3.5]. We identify τ and $\widetilde{\tau}$, and regard τ as a character of G.

Proposition 4.2 ([4, Theorem 3.3 and 3.4])**.** *Let (X, G) be a (not necessary commutative) association scheme, and H a strongly normal closed subset of G. If χ is a character of G and τ is a character of $G//H$, then the product $\chi\tau$ is a character of G, where*

$$\chi\tau(\sigma_g) = \chi(\sigma_g)\tau(\sigma_{g^H}) = \frac{1}{n_g}\chi(\sigma_g)\tau(\sigma_g).$$

Moreover, if $\chi \in \mathrm{Irr}(G)$ and $\tau(1) = 1$, then $\chi\tau \in \mathrm{Irr}(G)$ and the multiplicity $m_{\chi\tau}$ of $\chi\tau$ equals to m_χ.

Proof of Lemma 4.1. Suppose $e_\varphi \mathbb{C}(HgH) \neq 0$. Then there exists $f \in HgH$ such that $e_\varphi \sigma_f \neq 0$. Since $HgH = HfH$, we may assume that $e_\varphi \sigma_g \neq 0$. We will show that $e_\varphi \sigma_g$ is a unit in $e_\varphi \mathbb{C}G$. By the commutativity of $\mathbb{C}G$, there exists $\chi \in \mathrm{Irr}(G \mid \varphi)$ such that $\chi(\sigma_g) \neq 0$. If we show that $\eta(\sigma_g) \neq 0$ for any $\eta \in \mathrm{Irr}(G \mid \varphi)$, then $e_\varphi \sigma_g$ is a unit in $e_\varphi \mathbb{C}G$, since any eigenvalue of $e_\varphi \sigma_g$ acting on $e_\varphi \mathbb{C}G$ is of the form $\eta(\sigma_g)$, $\eta \in \mathrm{Irr}(G \mid \varphi)$.

For any $\tau \in \mathrm{Irr}(G//H)$, we have $\chi\tau \in \mathrm{Irr}(G \mid \varphi)$ by Proposition 4.2. By the commutativity, $G//H$ has a structure of an abelian group, and $\mathrm{Irr}(G//H)$ is also an abelian group. Now $\mathrm{Irr}(G//H)$ acts on $\mathrm{Irr}(G \mid \varphi)$. Let U be the $\mathrm{Irr}(G//H)$-orbit of

$\mathrm{Irr}(G \mid \varphi)$ containing χ, and let Stab_χ be the stabilizer of χ. Put $e_U := \sum_{\eta \in U} e_\eta$. Then

$$\begin{aligned} e_U &:= \sum_{\eta \in U} e_\eta = \frac{1}{|\mathrm{Stab}_\chi|} \sum_{\tau \in \mathrm{Irr}(G//H)} \frac{m_{\chi\tau}}{n_G} \sum_{f \in G} \frac{1}{n_f} \overline{\chi\tau(\sigma_f)} \sigma_f \\ &= \frac{1}{|\mathrm{Stab}_\chi|} \sum_{\tau \in \mathrm{Irr}(G//H)} \frac{m_\chi}{n_G} \sum_{f \in G} \frac{1}{n_f} \overline{\chi(\sigma_f)\tau(\sigma_{f^H})} \sigma_f \\ &= \frac{m_\chi}{n_G |\mathrm{Stab}_\chi|} \sum_{f \in G} \frac{1}{n_f} \overline{\chi(\sigma_f)} \left(\sum_{\tau \in \mathrm{Irr}(G//H)} \overline{\tau(\sigma_{f^H})} \right) \sigma_f \\ &= \frac{m_\chi |G//H|}{n_G |\mathrm{Stab}_\chi|} \sum_{f \in H} \frac{1}{n_f} \overline{\chi(\sigma_f)} \sigma_f \in \mathbb{C}H. \end{aligned}$$

Since e_φ is primitive in $\mathbb{C}H$, we have $\mathrm{Irr}(G \mid \varphi) = U = \{\chi\tau \mid \tau \in \mathrm{Irr}(G//H)\}$. Now $\chi\tau(\sigma_g) = \chi(\sigma_g)\tau(\sigma_{g^H}) \neq 0$, because τ is a linear character of an abelian group $G//H$. This shows that the assertion holds. □

Lemma 4.3. *$Z//H$ is a subgroup of $G//H$, and $e_\varphi \mathbb{C}G$ is a crossed product of $Z//H$ over $e_\varphi \mathbb{C}H$.*

Proof. This is clear by Lemma 4.1. □

Now we can show the main result in this paper.

Theorem 4.4 (Clifford's theorem for Commutative Schemes)**.** *Let (X, G) be a commutative scheme, H a strongly normal closed subset of G, and $\varphi \in \mathrm{Irr}(H)$. Put $Z//H := \{g^H \in G//H \mid e_\varphi \mathbb{C}(HgH) \neq 0\}$. Then $Z//H$ is a subgroup of $G//H$, and we have the followings.*

1. *Take any $\xi \in \mathrm{Irr}(Z \mid \varphi)$ and fix it. Then $\mathrm{Irr}(Z \mid \varphi) = \{\xi\tau \mid \tau \in \mathrm{Irr}(Z//H)\}$.*
2. *The map $\mathrm{Irr}(Z \mid \varphi) \to \mathrm{Irr}(G \mid \varphi)$ defined by $\eta \mapsto \eta^G$ is a bijection. Here $\eta^G(\sigma_g) = \eta(\sigma_g)$ for $g \in Z$, and 0 otherwise.*
3. *For $\chi \in \mathrm{Irr}(G \mid \varphi)$, $m_\chi = \frac{n_G}{n_Z} m_\varphi$.*

Proof. (1) and (2) are clear by Proposition 4.2 and Lemma 4.3.

The rank of $e_\varphi \in \mathbb{C}G$ (as a matrix) is $|G//H| m_\varphi$. The multiplicities are constant on $\mathrm{Irr}(G \mid \varphi)$ by Proposition 4.2, and $|\mathrm{Irr}(G \mid \varphi)| = |Z//H|$. So we have $m_\chi = \frac{n_G}{n_Z} m_\varphi$ for $\chi \in \mathrm{Irr}(G \mid \varphi)$. (3) holds. □

Let L be a $\mathbb{C}H$-module affording $\varphi \in \mathrm{Irr}(H)$. Then easily we have

$$Z//H = \{g^H \in G//H \mid L \otimes_{\mathbb{C}H} \mathbb{C}(HgH) \cong L \quad \text{as } \mathbb{C}H\text{-modules}\}.$$

So Theorem 4.4 is a natural generalization of Clifford's theorem for finite group characters.

The end of this paper, we show an easy corollary of our result.

Corollary 4.5. *Let (X, G) be a commutative association scheme, and H a strongly normal closed subset of G. Then*

$$|H| + |G//H| - 1 \leq |G| \leq |H| \cdot |G//H|.$$

Moreover, $|G| = |H| + |G//H| - 1$ if and only if (X, G) is the wreath product of $(X, G)_{xH}$ by $G//H$ for $x \in X$, and $|G| = |H| \cdot |G//H|$ if and only if $\mathbb{C}G$ is a crossed product of $G//H$ over $\mathbb{C}H$. (For the definition of wreath products of association schemes, see [5].)

Proof. By the definition of a factor scheme, the former inequality holds clearly, and by Theorem 4.4, the later inequality holds.

It is easy to show the rest of the assertions. □

By this result, if (X, G) is commutative, H strongly normal in G, and $|HgH| = |H|$ for any $g \in G$, then the adjacency algebra $\mathbb{C}G$ is a crossed product of $G//H$ over $\mathbb{C}H$.

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