Progress of Theoretical Physics, Vol. 119, No. 2, February 2008

# Sfermion Mass Relations in Orbifold Family Unification

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(Received September 18, 2007)

We derive relations among sfermion masses based on orbifold family unification models. Sfermion mass relations are specific to each model and can be useful for the selection of a realistic model.

## §1. Introduction

Supersymmetric grand unified theories (SUSY GUTs) on an orbifold have desirable features as realistic models beyond the minimal supersymmetric standard model (MSSM). The triplet-doublet splitting of Higgs multiplets is elegantly realized in the framework of SUSY $SU(5)$ GUT in five dimensions.[1),2)] Four-dimensional chiral fermions are generated through the dimensional reduction. These phenomena originate from the fact that some of the zero modes are projected out by orbifolding, i.e., by non-trivial boundary conditions (BCs) concerning the extra dimensions on bulk fields. There is a possibility that a (complete) family unification can be realized by eliminating all mirror particles from the low-energy spectrum. Mirror particles are particles with opposite quantum numbers in the standard model (SM) gauge group $G_{SM}$.

Recently, family unification has been studied in SUSY $SU(N)$ GUTs defined on the five-dimensional space-time $M^4\times(S^1/Z_2)$.[3),***)] Here, $M^4$ is the four-dimensional Minkowski space-time, and $S^1/Z_2$ is the one-dimensional orbifold. A great variety of models have been found in which zero modes from a single bulk field and a few brane fields compose three families. We refer to them as "orbifold family unification models". At present, it is necessary to make strong predictions in order to distinguish among models with experimental data. Many works concerning mass relations among scalar particles have been carried out with the motivation that relations specific to each model will provide information that is useful for understanding the structure of the MSSM and beyond in four-dimensional SUSY models.[7)–15),†)] Sum rules among sfermion masses have also been derived in two kinds of orbifold family unification models, and it has been pointed out that they can be useful probes of

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***) The possibility that one might realize the complete family unification utilizing an orbifold is also suggested in Ref. 4) in a different context. In Ref. 5), three families are derived from a combination of a bulk gauge multiplet and a few brane fields. In Ref. 6), they are realized as composite fields.

†) Scalar mass relations have been examined in four-dimensional superstring models.[16),17)]

each model.[18),*)]

In this paper, we study sfermion masses on the basis of orbifold family unification models, employing certain assumptions regarding the breakdown of SUSY and gauge symmetries, and then we derive relations among them.

This paper is organized as follows. In §2, we give an outline of orbifold family unification models. In §3, we present a generic mass formula for sfermions and derive specific relations among sfermion masses. Sfermion sum rules at the TeV scale are also derived from several models on the basis of the assumption that the MSSM holds below the compactification scale. Section 4 is devoted to conclusions and discussion.

## §2. Orbifold family unification

First, we review the arguments given in Ref. 3). We study $SU(N)$ gauge theory on $M^4 \times (S^1/Z_2)$, with the gauge symmetry breaking pattern $SU(N) \to SU(3) \times SU(2) \times SU(r) \times SU(s) \times U(1)^n$, which is realized with the $Z_2$ parity assignment

$$P_0 = \mathrm{diag}(+1, +1, +1, +1, +1, -1, \dots, -1, -1, \dots, -1), \tag{2·1}$$

$$P_1 = \mathrm{diag}(+1, +1, +1, -1, -1, \underbrace{+1, \dots, +1}_{r}, \underbrace{-1, \dots, -1}_{s}), \tag{2·2}$$

where $s = N - 5 - r$ and $N \geq 6$. The quantity $n$ is an integer that depends on the breaking pattern. The matrices $P_0$ and $P_1$ are the representation matrices (up to sign factors) of the fundamental representation of the $Z_2$ transformation $(y \to -y)$ and the $Z_2'$ transformation $(y \to 2\pi R - y)$, respectively. Here, $y$ is the coordinate of $S^1/Z_2$, and $R$ is the radius of $S^1$. After the breakdown of $SU(N)$, the rank-$k$ completely antisymmetric tensor representation $[N, k]$, whose dimension is ${}_N C_k$, is decomposed into a sum of multiplets of the subgroup $SU(3) \times SU(2) \times SU(r) \times SU(s)$ as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left( {}_3C_{l_1}, {}_2C_{l_2}, {}_rC_{l_3}, {}_sC_{l_4} \right), \tag{2·3}$$

where $l_1$, $l_2$ and $l_3$ are integers, we have the relation $l_4 = k - l_1 - l_2 - l_3$, and our notation is such that ${}_nC_l = 0$ for $l > n$ and $l < 0$. Here and hereafter we use ${}_nC_l$ instead of $[n, l]$ in many cases. We list the $U(1)$ charges for representations of the subgroups in Table I. The $U(1)$ charges are those in the subgroups

$$SU(5) \supset SU(3) \times SU(2) \times U(1)_1, \tag{2·4}$$

$$SU(N-5) \supset SU(r) \times SU(N-5-r) \times U(1)_2,\ SU(N-5-1) \times U(1)_2, \tag{2·5}$$

$$SU(N) \supset SU(5) \times SU(N-5) \times U(1)_3, \tag{2·6}$$

up to normalization. We assume that $G_{SM} = SU(3) \times SU(2) \times U(1)_1$, up to normalization of the hypercharge. Particle species are identified with the SM fermions

*) Sfermion masses have been studied from the viewpoint of flavor symmetry and its violation in $SU(5)$ SUSY orbifold GUT.[19)]

Table I. The $U(1)$ charges for various representations of fermions.

| species | representation | $U(1)_1$ | $U(1)_2$ | $U(1)_3$ |
|---|---|---|---|---|
| $(\nu_R)^c$, $\hat{\nu}_R$ | $({}_3C_0, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3})$ | 0 | $(N-5)l_3 - rk$ | $-5k$ |
| $(d'_R)^c$, $d_R$ | $({}_3C_1, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-1})$ | $-2$ | $(N-5)l_3 - r(k-1)$ | $N-5k$ |
| $l'_L$, $(l_L)^c$ | $({}_3C_0, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-1})$ | 3 | $(N-5)l_3 - r(k-1)$ | $N-5k$ |
| $(u_R)^c$, $u'_R$ | $({}_3C_2, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-2})$ | $-4$ | $(N-5)l_3 - r(k-2)$ | $2N-5k$ |
| $(e_R)^c$, $e'_R$ | $({}_3C_0, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-2})$ | 6 | $(N-5)l_3 - r(k-2)$ | $2N-5k$ |
| $q_L$, $(q'_L)^c$ | $({}_3C_1, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-2})$ | 1 | $(N-5)l_3 - r(k-2)$ | $2N-5k$ |
| $(e'_R)^c$, $e_R$ | $({}_3C_3, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-3})$ | $-6$ | $(N-5)l_3 - r(k-3)$ | $3N-5k$ |
| $(u'_R)^c$, $u_R$ | $({}_3C_1, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-3})$ | 4 | $(N-5)l_3 - r(k-3)$ | $3N-5k$ |
| $q'_L$, $(q_L)^c$ | $({}_3C_2, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-3})$ | $-1$ | $(N-5)l_3 - r(k-3)$ | $3N-5k$ |
| $l_L$, $(l'_L)^c$ | $({}_3C_3, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-4})$ | $-3$ | $(N-5)l_3 - r(k-4)$ | $4N-5k$ |
| $(d_R)^c$, $d'_R$ | $({}_3C_2, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-4})$ | 2 | $(N-5)l_3 - r(k-4)$ | $4N-5k$ |
| $(\hat{\nu}_R)^c$, $\nu_R$ | $({}_3C_3, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-5})$ | 0 | $(N-5)l_3 - r(k-5)$ | $5N-5k$ |

by the gauge quantum numbers. Here, we use $(d_R)^c$, $l_L$, $(u_R)^c$, $(e_R)^c$ and $q_L$ to represent down-type anti-quark singlets, lepton doublets, up-type anti-quark singlets, positron-type lepton singlets and quark doublets. The particles with primes are regarded as mirror particles and believed to have no zero modes. Each fermion has a definite chirality, e.g., $(d_R)^c$ is left-handed and $d_R$ is right-handed. Here, the subscript $L$ $(R)$ represents left-handedness (right-handedness) for Weyl fermions. Here, $(d_R)^c$ represents the charge conjugate of $d_R$ and transforms as a left-handed Weyl fermion under the four-dimensional Lorentz transformation.

A fermion with spin 1/2 in five dimensions is regarded as a Dirac fermion or a pair of Weyl fermions with opposite chiralities in four dimensions. The left-handed Weyl fermion and the corresponding right-handed one should have opposite $Z_2$ parities as implied by the requirement that the kinetic term is invariant under the $Z_2$ parity transformation. We define the $Z_2$ parity of the representation $({}_pC_{l_1}, {}_qC_{l_2}, {}_rC_{l_3}, {}_sC_{l_4})_L$ as

$$\mathcal{P}_0 = (-1)^{l_1+l_2}(-1)^k \eta_k, \quad \mathcal{P}_1 = (-1)^{l_1+l_3}(-1)^k \eta'_k, \tag{2·7}$$

where $\eta_k$ and $\eta'_k$ are the intrinsic $Z_2$ parities. By definition, $\eta_k$ and $\eta'_k$ each takes the value $+1$ or $-1$. We list the $Z_2$ parity assignment for species in Table II. Note that mirror particles have the $Z_2$ parity $\mathcal{P}_0 = -(-1)^k \eta_k$. Hence *all zero modes of mirror particles can be eliminated by the proper choice of the $Z_2$ parity when we take* $(-1)^k \eta_k = +1$. Hereafter, we consider such a case.

We write the flavor numbers of $(d_R)^c$, $l_L$, $(u_R)^c$, $(e_R)^c$, $q_L$ and the (heavy) neutrino singlets as $n_{\bar{d}}$, $n_l$, $n_{\bar{u}}$, $n_{\bar{e}}$, $n_q$ and $n_{\bar{\nu}}$. Both left-handed and right-handed Weyl fermions having even $Z_2$ parities, $\mathcal{P}_0 = \mathcal{P}_1 = +1$, compose chiral fermions in the SM. When we choose $(-1)^k \eta'_k = +1$, the flavor number of the chiral fermions are given by

$$n_{\bar{d}} = \sum_{i=1,4} \sum_{l_3=0,2,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3}, \tag{2·8}$$

$$n_l = \sum_{i=1,4} \sum_{l_3=1,3,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3}, \tag{2·9}$$

Table II. The $Z_2$ parity assignment for various representations of fermions.

| species | representation | $\mathcal{P}_0$ | $\mathcal{P}_1$ |
|---|---|---|---|
| $(\nu_R)^c$ | $({}_3C_0, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3})_L$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $\hat{\nu}_R$ | $({}_3C_0, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3})_R$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $(d'_R)^c$ | $({}_3C_1, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-1})_L$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $l'_L$ | $({}_3C_0, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-1})_L$ | $-(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $d_R$ | $({}_3C_1, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-1})_R$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $(l_L)^c$ | $({}_3C_0, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-1})_R$ | $(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $(u_R)^c$ | $({}_3C_2, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-2})_L$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $(e_R)^c$ | $({}_3C_0, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-2})_L$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $q_L$ | $({}_3C_1, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-2})_L$ | $(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $u'_R$ | $({}_3C_2, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-2})_R$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $e'_R$ | $({}_3C_0, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-2})_R$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $(q'_L)^c$ | $({}_3C_1, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-2})_R$ | $-(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $(e'_R)^c$ | $({}_3C_3, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-3})_L$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $(u'_R)^c$ | $({}_3C_1, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-3})_L$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $q'_L$ | $({}_3C_2, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-3})_L$ | $-(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $e_R$ | $({}_3C_3, {}_2C_0, {}_rC_{l_3}, {}_sC_{k-l_3-3})_R$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $u_R$ | $({}_3C_1, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-3})_R$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $(q_L)^c$ | $({}_3C_2, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-3})_R$ | $(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $l_L$ | $({}_3C_3, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-4})_L$ | $(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $(d_R)^c$ | $({}_3C_2, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-4})_L$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $(l'_L)^c$ | $({}_3C_3, {}_2C_1, {}_rC_{l_3}, {}_sC_{k-l_3-4})_R$ | $-(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |
| $d'_R$ | $({}_3C_2, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-4})_R$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $(\hat{\nu}_R)^c$ | $({}_3C_3, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-5})_L$ | $-(-1)^k\eta_k$ | $-(-1)^{l_3}(-1)^k\eta'_k$ |
| $\nu_R$ | $({}_3C_3, {}_2C_2, {}_rC_{l_3}, {}_sC_{k-l_3-5})_R$ | $(-1)^k\eta_k$ | $(-1)^{l_3}(-1)^k\eta'_k$ |

$$n_{\bar{u}} = n_{\bar{e}} = \sum_{i=2,3}\sum_{l_3=0,2,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3}, \qquad (2\cdot10)$$

$$n_q = \sum_{i=2,3}\sum_{l_3=1,3,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3}, \qquad (2\cdot11)$$

$$n_{\bar{\nu}} = \sum_{i=0,5}\sum_{l_3=0,2,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3}, \qquad (2\cdot12)$$

using the equivalence of the charge conjugation. When we choose $(-1)^k\eta'_k = -1$, we obtain formulae in which $n_l$ is replaced by $n_{\bar{d}}$ and $n_q$ by $n_{\bar{u}}$ $(= n_{\bar{e}})$ in Eqs. (2·8) – (2·11). The total number of (heavy) neutrino singlets is given by $n_{\bar{\nu},k}^{(+-)} = \sum_{i=0,5}\sum_{l_3=1,3,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3}$ for $(-1)^k\eta'_k = -1$.

For arbitrary $N$ $(\geq 6)$ and $r$, the flavor numbers from $[N, k]$ with $((-1)^k\eta_k, (-1)^k\eta'_k) = (a, b)$ are equal to those from $[N, N-k]$ with $((-1)^{N-k}\eta_{N-k}, (-1)^{N-k}\eta'_{N-k}) = (a, -b)$ if $r$ is odd, and the flavor numbers from $[N, k]$ with $((-1)^k\eta_k, (-1)^k\eta'_k) = (a, b)$ are equal to those from $[N, N-k]$ with $((-1)^{N-k}\eta_{N-k}, (-1)^{N-k}\eta'_{N-k}) = (a, b)$ if $r$ is even. We list the flavor number of each chiral fermion derived from $[N, k]$ ($N = 5, \cdots, 9$ and $k = 1, \cdots, [N/2]$ where $[*]$ stands for Gauss's symbol, i.e., $[N/2] = N/2$ if $N$ is even and $[N/2] = (N-1)/2$ if $N$ is odd) in Table III. In the 8-th column, the numbers in the parentheses are the flavor numbers of

Table III. The flavor number of each chiral fermion with $(-1)^k\eta_k = (-1)^k\eta'_k = +1$.

| representation | $(p,q,r,s)$ | $n_{\bar{d}}$ | $n_l$ | $n_{\bar{u}}$ | $n_{\bar{e}}$ | $n_q$ | $n_{\bar{\nu}}$ ($n_{\bar{\nu}}$ with $(-1)^k\eta'_k = -1$) |
|---|---|---|---|---|---|---|---|
| $[N,1]$ | $(3,2,r,s)$ | 1 | 0 | 0 | 0 | 0 | $s$ $(r)$ |
| $[N,2]$ | $(3,2,r,s)$ | $s$ | $r$ | 1 | 1 | 0 | ${}_rC_2 + {}_sC_2$ $(rs)$ |
| $[6,3]$ | (3,2,1,0) | 0 | 0 | 1 | 1 | 1 | 0 (0) |
| | (3,2,0,1) | 0 | 0 | 2 | 2 | 0 | 0 (0) |
| $[7,3]$ | (3,2,2,0) | 1 | 0 | 1 | 1 | 2 | 0 (0) |
| | (3,2,1,1) | 0 | 1 | 2 | 2 | 1 | 0 (0) |
| | (3,2,0,2) | 1 | 0 | 3 | 3 | 0 | 0 (0) |
| $[8,3]$ | (3,2,3,0) | 3 | 0 | 1 | 1 | 3 | 0 (1) |
| | (3,2,2,1) | 1 | 2 | 2 | 2 | 2 | 1 (0) |
| | (3,2,1,2) | 1 | 2 | 3 | 3 | 1 | 0 (1) |
| | (3,2,0,3) | 3 | 0 | 4 | 4 | 0 | 1 (0) |
| $[8,4]$ | (3,2,3,0) | 1 | 1 | 3 | 3 | 3 | 0 (0) |
| | (3,2,2,1) | 2 | 0 | 2 | 2 | 4 | 0 (0) |
| | (3,2,1,2) | 1 | 1 | 3 | 3 | 3 | 0 (0) |
| | (3,2,0,3) | 2 | 0 | 6 | 6 | 0 | 0 (0) |
| $[9,3]$ | (3,2,4,0) | 6 | 0 | 1 | 1 | 4 | 0 (4) |
| | (3,2,3,1) | 3 | 3 | 2 | 2 | 3 | 3 (1) |
| | (3,2,2,2) | 2 | 4 | 3 | 3 | 2 | 2 (2) |
| | (3,2,1,3) | 3 | 3 | 4 | 4 | 1 | 1 (3) |
| | (3,2,0,4) | 6 | 0 | 5 | 5 | 0 | 4 (0) |
| $[9,4]$ | (3,2,4,0) | 1 | 4 | 6 | 6 | 4 | 1 (0) |
| | (3,2,3,1) | 4 | 1 | 4 | 4 | 6 | 0 (1) |
| | (3,2,2,2) | 3 | 2 | 4 | 4 | 6 | 1 (0) |
| | (3,2,1,3) | 2 | 3 | 6 | 6 | 4 | 0 (1) |
| | (3,2,0,4) | 5 | 0 | 10 | 10 | 0 | 1 (0) |

the neutrino singlets for $(-1)^k\eta'_k = -1$.

Our four-dimensional world is assumed to be a boundary at one of the fixed points, on the basis of the 'brane world scenario'. There exist two kinds of four-dimensional fields in our low-energy theory. One is a brane field, which exists only at the boundary, and the other is the zero mode, which stems from the bulk field. The Kaluza-Klein (KK) modes do not appear in our low-energy world, because they have heavy masses of $O(1/R)$, which is the magnitude of the unification scale, $M_U$. There are many possibilities for deriving three families from the zero modes of (a few of) the bulk field(s) and suitable brane fields from the viewpoint of chiral anomaly cancellation. Chiral anomalies may arise at the boundaries with the appearance of chiral fermions. Such anomalies must be cancelled in the four-dimensional effective theory by the contribution of the brane chiral fermions and/or counterterms, such as the Chern-Simons term.[20)–22)]

## §3. Sfermion mass relations

Here we consider the SUSY version of $SU(N)$ models. In SUSY models, the hypermultiplet in the five-dimensional bulk is equivalent to a pair of chiral multiplets with opposite gauge quantum numbers in four dimensions. The chiral multiplet with the representation $[N, N-k]$, which is conjugate to $[N, k]$, contains a left-handed Weyl fermion with $[N, N-k]_L$. This Weyl fermion can be made a right-handed fermion with $[N, k]_R$ by applying charge conjugation. Hence, our analysis given in the previous section is effective for SUSY models.

We employ the following assumptions in our analysis.

1. Three families in the MSSM come from zero modes of the bulk field with the representation $[N, k]$ and some brane fields. Higgs fields originate from other multiplets.
2. We do not specify the mechanism by which the $N=1$ SUSY is broken in four dimensions.*) Soft SUSY breaking terms respect the gauge invariance.
3. Extra gauge symmetries are broken by the Higgs mechanism simultaneously with the orbifold breaking at the scale $M_U = O(1/R)$. Then the $D$-term contributions to the scalar masses can appear as a dominant source of scalar mass splitting.
4. The proper theory beyond the SM is the MSSM. The MSSM holds from the TeV scale to $M_U$.

### 3.1. *Sfermion mass formula*

We consider the case with the intrinsic $Z_2$ parity assignment $(-1)^k \eta_k = (-1)^k \eta'_k = +1$. In the case with $(-1)^k \eta'_k = -1$, similar relations can be derived with a suitable exchange of sfermion species. We list the sfermion species as zero modes of five-dimensional fields with even $Z_2$ parities, $\mathcal{P}_0 = \mathcal{P}_1 = +1$, in Table IV. In Table IV, $\tilde{f}$ represents the scalar partner of the fermion $f$, and the charge conjugation is performed for the fields with $l_1 + l_2 =$ odd. The asterisk indicates complex conjugation. Note that signs of $U(1)$ charges are changed by the charge conjugation. As sfermion species are labeled by the numbers $(l_1, l_2, l_3)$, we use this label in place of $\tilde{f}$.

Table IV. The assignment of sfermions and those $U(1)$ charges.

| species | $(l_1, l_2, l_3)$ | $l_1 + l_2$ | $U(1)_2$ | $U(1)_3$ |
|---|---|---|---|---|
| $\tilde{d}^*_R$ | $(1, 0, \text{even})$ | 1 | $-(N-5)l_3 + r(k-1)$ | $-N+5k$ |
| $\tilde{l}_L$ | $(0, 1, \text{odd})$ | 1 | $-(N-5)l_3 + r(k-1)$ | $-N+5k$ |
| $\tilde{u}^*_R$ | $(2, 0, \text{even})$ | 2 | $(N-5)l_3 - r(k-2)$ | $2N-5k$ |
| $\tilde{e}^*_R$ | $(0, 2, \text{even})$ | 2 | $(N-5)l_3 - r(k-2)$ | $2N-5k$ |
| $\tilde{q}_L$ | $(1, 1, \text{odd})$ | 2 | $(N-5)l_3 - r(k-2)$ | $2N-5k$ |
| $\tilde{e}^*_R$ | $(3, 0, \text{even})$ | 3 | $-(N-5)l_3 + r(k-3)$ | $-3N+5k$ |
| $\tilde{u}^*_R$ | $(1, 2, \text{even})$ | 3 | $-(N-5)l_3 + r(k-3)$ | $-3N+5k$ |
| $\tilde{q}_L$ | $(2, 1, \text{odd})$ | 3 | $-(N-5)l_3 + r(k-3)$ | $-3N+5k$ |
| $\tilde{l}_L$ | $(3, 1, \text{odd})$ | 4 | $(N-5)l_3 - r(k-4)$ | $4N-5k$ |
| $\tilde{d}^*_R$ | $(2, 2, \text{even})$ | 4 | $(N-5)l_3 - r(k-4)$ | $4N-5k$ |

*) The Scherk-Schwarz mechanism, in which SUSY is broken by the difference between the BCs of bosons and fermions, is typical.[23] This mechanism on $S^1/Z_2$ leads to a restricted type of soft SUSY breaking parameters, such as $M_i = \beta/R$ for bulk gauginos and $m^2_{\tilde{f}} = (\beta/R)^2$ for bulk scalar particles, where $\beta$ is a real parameter and $R$ is the radius of $S^1$.

Sfermion masses squared at $M_U$ are given by

$$
\begin{aligned}
{m^{(\alpha,\beta)}_{(l_1,l_2,l_3)}}^2(M_U) = m^2_{[N,k]} &+ (-1)^{l_1+l_2}\sum_{A=1}^{r-1} Q^A_\alpha D^A_{(r)} + (-1)^{l_1+l_2}\sum_{B=1}^{r-1} Q^B_\beta D^B_{(s)} \\
&+ (-1)^{l_1+l_2}\left[(N-5)l_3 - r(k-(l_1+l_2))\right] D_2 \\
&+ (-1)^{l_1+l_2}\left[(l_1+l_2)N - 5k\right] D_3,
\end{aligned} \tag{3·1}
$$

where $m^2_{[N,k]}$ is a common soft SUSY breaking mass parameter that respects the $SU(N)$ gauge symmetry, and the other terms on the right-hand side represent $D$-term contributions. The $D$-term contributions, in general, originate from $D$-terms related to broken gauge symmetries when the soft SUSY breaking parameters possess a non-universal structure and the rank of the gauge group decreases after the breakdown of gauge symmetry.[9),24)] In most cases, the magnitude of the $D$-term condensation is, at most, of order $\sim$TeV$^2$ and hence the $D$-term contributions can induce sizable effects on the sfermion spectrum. The indices $\alpha$ and $\beta$ indicate the members of the multiplets of $SU(r)$ and $SU(s)$, and run from 1 to ${}_rC_{l_3}$ and from 1 to ${}_sC_{l_4}$, respectively. The quantities $Q^A_\alpha$ are the broken diagonal charges of $[r, l_3]$, which form the Cartan sub-algebra of $SU(r)$, and are given by

$$
Q^A_\alpha = Q^A_\alpha([r, l_3]) = \sum_{a=a_1}^{a_{l_3}} Q^A_a, \tag{3·2}
$$

where $Q^A_a$ are the diagonal charges (up to normalization) of fields with the fundamental representation $[r, 1]$, defined by

$$
Q^A_a \equiv (1-a)\delta^A_{a-1} + \sum_{i=0}^{r-1-a} \delta^A_{a+i}. \tag{3·3}
$$

The numbering for $\alpha$ is defined by

$$
\begin{aligned}
(a_1, \cdots, a_{l_3}) &= (1, \cdots, l_3) && \text{for } \alpha = 1 \\
&= (1, \cdots, l_3 - 1, l_3 + 1) && \text{for } \alpha = 2 \\
&\cdots \\
&= (1, \cdots, l_3 - 1, r) && \text{for } \alpha = l_3 - r + 1 \\
&= (1, \cdots, l_3 - 2, l_3, l_3 + 1) && \text{for } \alpha = l_3 - r + 2 \\
&\cdots \\
&= (r + 1 - l_3, \cdots, r) && \text{for } \alpha = {}_rC_{l_3}.
\end{aligned} \tag{3·4}
$$

Using the formulae for diagonal charges given in (3·2) and (3·3) and the definition of the numbering given in (3·4), the broken diagonal charges of $[r, r-l_3]$ (the complex conjugate representation of $[r, l_3]$) are given by

$$
Q^A_\alpha([r, r-l_3]) = -Q^A_{{}_rC_{l_3}+1-\alpha}([r, l_3]). \tag{3·5}
$$

The same holds for the charges $Q^B_\beta$. The quantities $D^A_{(r)}$, $D^B_{(s)}$, $D_2$ and $D_3$ are parameters that include $D$-term condensations, and their magnitudes are model dependent.

### 3.2. *Sfermion mass relations*

We now derive relations among the sfermion masses at $M_U$ by eliminating the unknown parameters ($m^2_{[N,k]}$, $D^A_{(r)}$, $D^B_{(s)}$, $D_2$, $D_3$) in the mass formula (3·1).

First, we find the following relations from the mass formula (3·1) and Table IV:

$$m^{(\alpha,\beta)}_{(2,0,l_3)}{}^2 = m^{(\alpha,\beta)}_{(0,2,l_3)}{}^2, \ m^{(\alpha,\beta)}_{(3,0,l_3)}{}^2 = m^{(\alpha,\beta)}_{(1,2,l_3)}{}^2. \quad (3·6)$$

(Here and hereafter we abbreviate $m^{(\alpha,\beta)}_{(l_1,l_2,l_3)}{}^2(M_U)$ as $m^{(\alpha,\beta)}_{(l_1,l_2,l_3)}{}^2$.) This type of relation generally appears if an up-type anti-squark singlet exists, and the number of relations is $n_{\bar{u}}$ ($= n_{\bar{e}}$). Hereafter, we consider only up-type anti-squark singlets (in place of positron-type slepton singlets).

Before we derive other relations, we estimate the total number of independent relations. The number of each sfermion derived from the bulk field $[N,k]$ is equal to that of each fermion given in Eqs. (2·8) – (2·12). The total number of sfermions, excluding slepton singlets, is given by

$$N_{\text{tot}} = \sum_{i=1}^{4} \sum_{l_3=0,1,\ldots} {}_rC_{l_3} \cdot {}_{N-5-r}C_{k-i-l_3} = \sum_{i=1}^{4} {}_{N-5}C_{k-i}. \quad (3·7)$$

The number of unknown parameters is $N-4$, because the number of $D$-term condensations is equal to the difference between the ranks of $SU(N)$ and $G_{SM}$. Hence, the number of independent relations excluding (3·6) is $N_{\text{tot}} - N + 4$. We find that no relation is derived from $[N,1]$ and one relation, $m^{(\alpha,\beta)}_{(2,0,0)}{}^2 = m^{(\alpha,\beta)}_{(0,2,0)}{}^2$ [of the type (3·6)] is derived from $[N,2]$.

By carrying out the summation over all members in each multiplet of $SU(r)$ and $SU(s)$, the following formula is derived:

$$\begin{aligned}
&\sum_{\alpha=1}^{{}_rC_{l_3}} \sum_{\beta=1}^{{}_sC_{l_4}} m^{(\alpha,\beta)}_{(l_1,l_2,l_3)}{}^2 \\
&\quad = {}_rC_{l_3} \cdot {}_sC_{l_4} \Big( m^2_{[N,k]} + (-1)^{l_1+l_2} \left[ (N-5)l_3 - r(k-(l_1+l_2)) \right] D_2 \\
&\qquad + (-1)^{l_1+l_2} \left[ (l_1+l_2)N - 5k \right] D_3 \Big). \qquad (3·8)
\end{aligned}$$

Note that both $D^A_{(r)}$ and $D^B_{(s)}$ vanish, because of the traceless property of the diagonal generators. If the number of multiplets ($N_{\text{mul}}$) is greater than three, $N_{\text{mul}} - 3$ kinds of relations are derived by eliminating the unknown parameters ($m^2_{[N,k]}$, $D_2$, $D_3$).

The remaining relations are derived by carrying out a summation over multiplets with suitable coefficients (not a universal one), and they are formally written

$$\sum_{\alpha} c_\alpha m^{(\alpha,\beta)}_{(l_1,l_2,l_3)}{}^2 = \sum_{\alpha'} c'_{\alpha'} m^{(\alpha',\beta')}_{(l'_1,l'_2,l'_3)}{}^2, \ \sum_{\beta} d_\beta m^{(\alpha,\beta)}_{(l_1,l_2,l_3)}{}^2 = \sum_{\beta'} d'_{\beta'} m^{(\alpha',\beta')}_{(l'_1,l'_2,l'_3)}{}^2, \quad (3·9)$$

where $c_\alpha$, $c'_\alpha$, $d_\beta$ and $d'_\beta$ are coefficients that satisfy the following relations:

$$\sum_{\alpha} c_\alpha = \sum_{\alpha'} c'_{\alpha'}, \ \sum_{\alpha} c_\alpha Q^A_\alpha = \sum_{\alpha'} c'_{\alpha'} Q^A_{\alpha'}, \quad (3·10)$$

$$\sum_{\beta} d_\beta = \sum_{\beta'} d'_{\beta'}, \quad \sum_{\beta} d_\beta Q^A_\beta = \sum_{\beta'} d'_{\beta'} Q^A_{\beta'}. \tag{3·11}$$

Sfermion mass relations [excluding the type (3·6)] derived from $[6,3]$ - $[9,4]$ are listed in Table V. We have classified the mass relations into three types, but the forms of the mass relations are not unique. For example, we derive the second-

Table V. The sfermion mass relations derived from $[6,3]$ – $[9,4]$.

| rep. | $(p,q,r,s)$ | sfermion mass relations |
|---|---|---|
| $[6,3]$ | (3,2,1,0) | ${m^{(1,1)}_{(1,1,1)}}^2 = {m^{(1,1)}_{(1,2,0)}}^2$ |
| | (3,2,0,1) | ${m^{(1,1)}_{(2,0,0)}}^2 = {m^{(1,1)}_{(1,2,0)}}^2$ |
| $[7,3]$ | (3,2,2,0) | $5{m^{(1,1)}_{(1,0,2)}}^2 + 9{m^{(1,1)}_{(1,2,0)}}^2 = 7\sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(1,1,1)}}^2$ |
| | (3,2,1,1) | $5{m^{(1,1)}_{(0,1,1)}}^2 + 9{m^{(1,1)}_{(1,2,0)}}^2 = 7\left({m^{(1,1)}_{(1,1,1)}}^2 + {m^{(1,1)}_{(2,0,0)}}^2\right)$ |
| | (3,2,0,2) | $5{m^{(1,1)}_{(1,0,0)}}^2 + 9{m^{(1,1)}_{(1,2,0)}}^2 = 7\sum_{\beta=1}^{2}{m^{(1,\beta)}_{(2,0,0)}}^2$ |
| $[8,3]$ | (3,2,3,0) | $5\sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(1,0,2)}}^2 + 9{m^{(1,1)}_{(1,2,0)}}^2 = 8\sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(1,1,1)}}^2$,<br>${m^{(3,1)}_{(1,0,2)}}^2 - {m^{(1,1)}_{(1,1,1)}}^2 = {m^{(2,1)}_{(1,0,2)}}^2 - {m^{(2,1)}_{(1,1,1)}}^2 = {m^{(1,1)}_{(1,0,2)}}^2 - {m^{(3,1)}_{(1,1,1)}}^2$ |
| | (3,2,2,1) | $\sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(1,1,1)}}^2 + 2{m^{(1,1)}_{(1,0,2)}}^2 = \sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(0,1,1)}}^2 + 2{m^{(1,1)}_{(2,0,0)}}^2$,<br>$6{m^{(1,1)}_{(2,0,0)}}^2 + \sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(1,1,1)}}^2 = 5{m^{(1,1)}_{(1,0,2)}}^2 + 3{m^{(1,1)}_{(1,2,0)}}^2$,<br>${m^{(1,1)}_{(0,1,1)}}^2 - {m^{(2,1)}_{(0,1,1)}}^2 = {m^{(2,1)}_{(1,1,1)}}^2 - {m^{(1,1)}_{(1,1,1)}}^2$ |
| | (3,2,1,2) | $\sum_{\beta=1}^{2}{m^{(1,\beta)}_{(2,0,0)}}^2 + 2{m^{(1,1)}_{(1,0,0)}}^2 = \sum_{\beta=1}^{2}{m^{(1,\beta)}_{(0,1,1)}}^2 + 2{m^{(1,1)}_{(1,1,1)}}^2$,<br>$6{m^{(1,1)}_{(1,1,1)}}^2 + \sum_{\beta=1}^{2}{m^{(1,\beta)}_{(2,0,0)}}^2 = 5{m^{(1,1)}_{(1,0,0)}}^2 + 3{m^{(1,1)}_{(1,2,0)}}^2$,<br>$m^{(1,1)}_{(0,1,1)} - m^{(1,2)}_{(0,1,1)} = m^{(1,2)}_{(2,0,0)} - m^{(1,1)}_{(2,0,0)}$ |
| | (3,2,0,3) | $5\sum_{\beta=1}^{3}{m^{(1,\beta)}_{(1,0,0)}}^2 + 9{m^{(1,1)}_{(1,2,0)}}^2 = 8\sum_{\beta=1}^{3}{m^{(1,\beta)}_{(2,0,0)}}^2$,<br>${m^{(1,3)}_{(1,0,0)}}^2 - {m^{(1,1)}_{(2,0,0)}}^2 = {m^{(1,2)}_{(1,0,0)}}^2 - {m^{(1,2)}_{(2,0,0)}}^2 = {m^{(1,1)}_{(1,0,0)}}^2 - {m^{(1,3)}_{(2,0,0)}}^2$ |
| $[8,4]$ | (3,2,3,0) | ${m^{(1,1)}_{(0,1,3)}}^2 = {m^{(1,1)}_{(2,2,0)}}^2$,<br>${m^{(1,1)}_{(2,0,2)}}^2 = {m^{(3,1)}_{(2,1,1)}}^2$, ${m^{(2,1)}_{(2,0,2)}}^2 = {m^{(2,1)}_{(2,1,1)}}^2$, ${m^{(3,1)}_{(2,0,2)}}^2 = {m^{(1,1)}_{(2,1,1)}}^2$ |
| | (3,2,2,1) | ${m^{(1,1)}_{(1,0,2)}}^2 = {m^{(1,1)}_{(2,2,0)}}^2$, ${m^{(1,1)}_{(2,0,2)}}^2 = {m^{(1,1)}_{(1,2,0)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 = {m^{(2,1)}_{(2,1,1)}}^2$, ${m^{(2,1)}_{(1,1,1)}}^2 = {m^{(1,1)}_{(2,1,1)}}^2$ |
| | (3,2,1,2) | ${m^{(1,1)}_{(0,1,1)}}^2 = {m^{(1,1)}_{(2,2,0)}}^2$, ${m^{(1,1)}_{(2,0,0)}}^2 = {m^{(1,1)}_{(2,1,1)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 = {m^{(1,2)}_{(1,2,0)}}^2$, ${m^{(1,2)}_{(1,1,1)}}^2 = {m^{(1,1)}_{(1,2,0)}}^2$ |
| | (3,2,0,3) | ${m^{(1,1)}_{(1,0,0)}}^2 = {m^{(1,1)}_{(2,2,0)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 = {m^{(1,3)}_{(1,2,0)}}^2$, ${m^{(1,2)}_{(2,0,0)}}^2 = {m^{(1,2)}_{(1,2,0)}}^2$, ${m^{(1,3)}_{(2,0,0)}}^2 = {m^{(1,1)}_{(1,2,0)}}^2$ |

(continued)

Table V.

| rep. | $(p,q,r,s)$ | sfermion mass relations |
|---|---|---|
| | (3,2,4,0) | $9\sum_{\alpha=1}^{4} {m^{(\alpha,1)}_{(1,1,1)}}^2 = 5\sum_{\alpha=1}^{6} {m^{(\alpha,1)}_{(1,0,2)}}^2 + 6{m^{(1,1)}_{(1,2,0)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 - {m^{(2,1)}_{(1,1,1)}}^2 = {m^{(4,1)}_{(1,0,2)}}^2 - {m^{(2,1)}_{(1,0,2)}}^2 = {m^{(5,1)}_{(1,0,2)}}^2 - {m^{(3,1)}_{(1,0,2)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 - {m^{(3,1)}_{(1,1,1)}}^2 = {m^{(4,1)}_{(1,0,2)}}^2 - {m^{(1,1)}_{(1,0,2)}}^2 = {m^{(6,1)}_{(1,0,2)}}^2 - {m^{(3,1)}_{(1,0,2)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 - {m^{(4,1)}_{(1,1,1)}}^2 = {m^{(5,1)}_{(1,0,2)}}^2 - {m^{(1,1)}_{(1,0,2)}}^2$ |
| | (3,2,3,1) | $\sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(1,0,2)}}^2 + \sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(1,1,1)}}^2 = \sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(0,1,1)}}^2 + 3{m^{(1,1)}_{(2,0,0)}}^2$,<br>$\sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(1,0,2)}}^2 + 4\sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(0,1,1)}}^2 + 3{m^{(1,1)}_{(1,2,0)}}^2 = 6\sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(1,1,1)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 - {m^{(2,1)}_{(1,1,1)}}^2 = {m^{(3,1)}_{(1,0,2)}}^2 - {m^{(2,1)}_{(1,0,2)}}^2 = {m^{(2,1)}_{(0,1,1)}}^2 - {m^{(1,1)}_{(0,1,1)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 - {m^{(3,1)}_{(1,1,1)}}^2 = {m^{(3,1)}_{(1,0,2)}}^2 - {m^{(1,1)}_{(1,0,2)}}^2 = {m^{(3,1)}_{(0,1,1)}}^2 - {m^{(1,1)}_{(0,1,1)}}^2$ |
| $[9,3]$ | (3,2,2,2) | ${m^{(1,1)}_{(1,0,2)}}^2 + \sum_{\alpha=1}^{2} {m^{(\alpha,1)}_{(1,1,1)}}^2 = {m^{(1,1)}_{(1,0,0)}}^2 + \sum_{\beta=1}^{2} {m^{(1,\beta)}_{(2,0,0)}}^2$,<br>$3\left(\sum_{\alpha=1}^{2} {m^{(\alpha,1)}_{(1,1,1)}}^2 + \sum_{\beta=1}^{2} {m^{(1,\beta)}_{(2,0,0)}}^2\right) = 2{m^{(1,1)}_{(1,2,0)}}^2 + 5\left({m^{(1,1)}_{(1,0,0)}}^2 + {m^{(1,1)}_{(1,0,2)}}^2\right)$,<br>$2{m^{(1,1)}_{(1,0,0)}}^2 + 2{m^{(1,1)}_{(1,0,2)}}^2 = \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} {m^{(\alpha,\beta)}_{(0,1,1)}}^2$,<br>${m^{(1,1)}_{(1,1,1)}}^2 - {m^{(2,1)}_{(1,1,1)}}^2 = {m^{(2,1)}_{(0,1,1)}}^2 - {m^{(1,1)}_{(0,1,1)}}^2 = {m^{(2,2)}_{(0,1,1)}}^2 - {m^{(1,2)}_{(0,1,1)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,2)}_{(2,0,0)}}^2 = {m^{(1,2)}_{(0,1,1)}}^2 - {m^{(1,1)}_{(0,1,1)}}^2$ |
| | (3,2,1,3) | $\sum_{\beta=1}^{3} {m^{(1,\beta)}_{(1,0,0)}}^2 + \sum_{\beta=1}^{3} {m^{(1,\beta)}_{(2,0,0)}}^2 = \sum_{\beta=1}^{3} {m^{(1,\beta)}_{(0,1,1)}}^2 + 3{m^{(1,1)}_{(1,1,1)}}^2$,<br>$\sum_{\beta=1}^{3} {m^{(1,\beta)}_{(1,0,0)}}^2 + 4\sum_{\beta=1}^{3} {m^{(1,\beta)}_{(0,1,1)}}^2 + 3{m^{(1,1)}_{(1,2,0)}}^2 = 6\sum_{\beta=1}^{3} {m^{(1,\beta)}_{(2,0,0)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,2)}_{(2,0,0)}}^2 = {m^{(1,3)}_{(1,0,0)}}^2 - {m^{(1,2)}_{(1,0,0)}}^2 = {m^{(1,2)}_{(0,1,1)}}^2 - {m^{(1,1)}_{(0,1,1)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,3)}_{(2,0,0)}}^2 = {m^{(1,3)}_{(1,0,0)}}^2 - {m^{(1,1)}_{(1,0,0)}}^2 = {m^{(1,3)}_{(0,1,1)}}^2 - {m^{(1,1)}_{(0,1,1)}}^2$ |
| | (3,2,0,4) | $9\sum_{\beta=1}^{4} {m^{(1,\beta)}_{(2,0,0)}}^2 = 5\sum_{\beta=1}^{6} {m^{(1,\beta)}_{(1,0,0)}}^2 + 6{m^{(1,1)}_{(1,2,0)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,2)}_{(2,0,0)}}^2 = {m^{(1,4)}_{(1,0,0)}}^2 - {m^{(1,2)}_{(1,0,0)}}^2 = {m^{(1,5)}_{(1,0,0)}}^2 - {m^{(1,3)}_{(1,0,0)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,3)}_{(2,0,0)}}^2 = {m^{(1,4)}_{(1,0,0)}}^2 - {m^{(1,1)}_{(1,0,0)}}^2 = {m^{(1,6)}_{(1,0,0)}}^2 - {m^{(1,3)}_{(1,0,0)}}^2$,<br>${m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,4)}_{(2,0,0)}}^2 = {m^{(1,5)}_{(1,0,0)}}^2 - {m^{(1,1)}_{(1,0,0)}}^2$ |

(continued)

type relation as $\sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(2,0,2)}}^2 = \sum_{\alpha=1}^{3} {m^{(\alpha,1)}_{(2,1,1)}}^2$ and two third-type relations as ${m^{(1,1)}_{(2,0,2)}}^2 = {m^{(3,1)}_{(2,1,1)}}^2$ and ${m^{(2,1)}_{(2,0,2)}}^2 = {m^{(2,1)}_{(2,1,1)}}^2$ from $[8,4]$ for $(3,2,3,0)$. Using these, three third-type relations are given in Table V. The mass relations derived from $[9,4]$ for $(p,q,r,s) = (3,2,1,3)$ are obtained from those for $(p,q,r,s) = (3,2,3,1)$ through the following replacements:

$$ {m^{(\alpha,1)}_{(1,0,2)}}^2 \to {m^{(1,\beta)}_{(1,2,0)}}^2,\ {m^{(1,1)}_{(0,1,3)}}^2 \to {m^{(1,1)}_{(2,1,1)}}^2,\ {m^{(\alpha,1)}_{(2,0,2)}}^2 \to {m^{(1,\beta)}_{(1,1,1)}}^2, $$

Table V.

| rep. | $(p,q,r,s)$ | sfermion mass relations |
|---|---|---|
| $[9,4]$ | (3,2,4,0) | $12\sum_{\alpha=1}^{6}{m^{(\alpha,1)}_{(2,0,2)}}^2 = 5\sum_{\alpha=1}^{4}{m^{(\alpha,1)}_{(0,1,3)}}^2 + 13\sum_{\alpha=1}^{4}{m^{(\alpha,1)}_{(2,1,1)}}^2,$<br>$23\sum_{\alpha=1}^{6}{m^{(\alpha,1)}_{(2,0,2)}}^2 = 27\sum_{\alpha=1}^{4}{m^{(\alpha,1)}_{(2,1,1)}}^2 + 30{m^{(1,1)}_{(2,2,0)}}^2,$<br>${m^{(2,1)}_{(0,1,3)}}^2 - {m^{(1,1)}_{(0,1,3)}}^2 = {m^{(4,1)}_{(2,1,1)}}^2 - {m^{(3,1)}_{(2,1,1)}}^2$<br>$= {m^{(2,1)}_{(2,0,2)}}^2 - {m^{(4,1)}_{(2,0,2)}}^2 = {m^{(3,1)}_{(2,0,2)}}^2 - {m^{(5,1)}_{(2,0,2)}}^2,$<br>${m^{(3,1)}_{(0,1,3)}}^2 - {m^{(1,1)}_{(0,1,3)}}^2 = {m^{(4,1)}_{(2,1,1)}}^2 - {m^{(2,1)}_{(2,1,1)}}^2$<br>$= {m^{(1,1)}_{(2,0,2)}}^2 - {m^{(4,1)}_{(2,0,2)}}^2 = {m^{(3,1)}_{(2,0,2)}}^2 - {m^{(6,1)}_{(2,0,2)}}^2,$<br>${m^{(4,1)}_{(0,1,3)}}^2 - {m^{(1,1)}_{(0,1,3)}}^2 = {m^{(4,1)}_{(2,1,1)}}^2 - {m^{(1,1)}_{(2,1,1)}}^2$<br>$= {m^{(1,1)}_{(2,0,2)}}^2 - {m^{(5,1)}_{(2,0,2)}}^2$ |
| | (3,2,3,1) | $\sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(1,0,2)}}^2 - 3{m^{(1,1)}_{(0,1,3)}}^2 = \sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(2,0,2)}}^2 - \sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(1,1,1)}}^2$<br>$= 3{m^{(1,1)}_{(1,2,0)}}^2 - \sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(2,1,1)}}^2,$<br>$27\left({m^{(1,1)}_{(1,2,0)}}^2 + {m^{(1,1)}_{(0,1,3)}}^2\right) = 12{m^{(1,1)}_{(2,2,0)}}^2 + 7\left(\sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(2,0,2)}}^2 + \sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(1,1,1)}}^2\right),$<br>$8\sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(1,0,2)}}^2 + \sum_{\alpha=1}^{3}{m^{(\alpha,1)}_{(2,1,1)}}^2,$<br>$= {m^{(1,1)}_{(0,1,3)}}^2 + 8{m^{(1,1)}_{(1,2,0)}}^2 + 18{m^{(1,1)}_{(2,2,0)}}^2$<br>${m^{(1,1)}_{(1,0,2)}}^2 - {m^{(2,1)}_{(1,0,2)}}^2 = {m^{(3,1)}_{(1,1,1)}}^2 - {m^{(2,1)}_{(1,1,1)}}^2$<br>$= {m^{(2,1)}_{(2,1,1)}}^2 - {m^{(3,1)}_{(2,1,1)}}^2 = {m^{(2,1)}_{(2,0,2)}}^2 - {m^{(1,1)}_{(2,0,2)}}^2,$<br>${m^{(1,1)}_{(1,0,2)}}^2 - {m^{(3,1)}_{(1,0,2)}}^2 = {m^{(1,1)}_{(1,1,1)}}^2 - {m^{(3,1)}_{(1,1,1)}}^2$<br>$= {m^{(3,1)}_{(2,1,1)}}^2 - {m^{(1,1)}_{(2,1,1)}}^2 = {m^{(3,1)}_{(2,0,2)}}^2 - {m^{(1,1)}_{(2,0,2)}}^2$ |
| | (3,2,2,2) | $\sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(2,1,1)}}^2 - \sum_{\beta=1}^{2}m^{(1,\beta)}_{(1,2,0)} = \sum_{\beta=1}^{2}{m^{(1,\beta)}_{(1,0,2)}}^2 - \sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(0,1,1)}}^2$<br>$= {m^{(1,1)}_{(2,0,0)}}^2 - {m^{(1,1)}_{(2,0,2)}}^2,$<br>$\sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(2,1,1)}}^2 + \sum_{\alpha=1}^{2}{m^{(\alpha,1)}_{(0,1,1)}}^2 = \sum_{\beta=1}^{2}{m^{(1,\beta)}_{(1,0,2)}}^2 + \sum_{\beta=1}^{2}{m^{(1,\beta)}_{(1,2,0)}}^2,$<br>$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}{m^{(\alpha,\beta)}_{(1,1,1)}}^2 = 2{m^{(1,1)}_{(2,0,2)}}^2 + 2{m^{(1,1)}_{(2,0,0)}}^2,$<br>${m^{(1,1)}_{(1,2,0)}}^2 - {m^{(1,2)}_{(1,2,0)}}^2 = {m^{(1,1)}_{(1,0,2)}}^2 - {m^{(1,2)}_{(1,0,2)}}^2$<br>$= {m^{(1,1)}_{(1,1,1)}}^2 - {m^{(1,2)}_{(1,1,1)}}^2 = {m^{(2,1)}_{(1,1,1)}}^2 - {m^{(2,2)}_{(1,1,1)}}^2,$<br>${m^{(1,1)}_{(2,1,1)}}^2 - {m^{(2,1)}_{(2,1,1)}}^2 = {m^{(1,1)}_{(0,1,1)}}^2 - {m^{(2,1)}_{(0,1,1)}}^2$<br>$= {m^{(1,1)}_{(1,1,1)}}^2 - {m^{(2,1)}_{(1,1,1)}}^2 = {m^{(1,2)}_{(1,1,1)}}^2 - {m^{(2,2)}_{(1,1,1)}}^2$ |

$$
\begin{aligned}
&{m^{(\alpha,1)}_{(1,1,1)}}^2 \to {m^{(1,\beta)}_{(2,0,0)}}^2,\ {m^{(1,1)}_{(1,2,0)}}^2 \to {m^{(1,1)}_{(1,0,0)}}^2,\ {m^{(\alpha,1)}_{(2,1,1)}}^2 \to {m^{(1,\beta)}_{(0,1,1)}}^2,\\
&{m^{(1,1)}_{(2,2,0)}}^2 \to {m^{(1,1)}_{(2,2,0)}}^2.
\end{aligned}
\tag{3·12}
$$

In the same way, the mass relations derived from $[9,4]$ for $(p,q,r,s)=(3,2,0,4)$ are obtained from those for $(p, q, r, s) = (3, 2, 4, 0)$ through the following replacements:

$$m^{(\alpha,1)}_{(2,0,2)}{}^2 \to m^{(1,\beta)}_{(2,0,0)}{}^2,\ m^{(\alpha,1)}_{(0,1,3)}{}^2 \to m^{(1,\beta)}_{(1,0,0)}{}^2,$$
$$m^{(\alpha,1)}_{(2,1,1)}{}^2 \to m^{(1,\beta)}_{(1,2,0)}{}^2,\ m^{(1,1)}_{(2,2,0)}{}^2 \to m^{(1,1)}_{(2,2,0)}{}^2. \tag{3·13}$$

We have obtained mass relations among the sfermions which stem from the bulk field with $[N, k]$ (where $N \leq 9$). These relations are specific to each $[N, k]$ and the gauge symmetry breaking pattern, and they may be useful probes to select models.

The brane fields at $y = 0$ are $SU(5) \times SU(N-5)$ multiplets, and their soft masses satisfy the $SU(5)$ GUT relations,

$$m_{\tilde{q}_L}{}^2 = m_{\tilde{u}^*_R}{}^2 = m_{\tilde{e}^*_R}{}^2, \quad m_{\tilde{l}_L}{}^2 = m_{\tilde{d}^*_R}{}^2. \tag{3·14}$$

To this point, we have assumed that all zero modes survive after the breakdown of the extra gauge symmetries. In the case that particle mixing and/or decoupling occurs, some relations should be modified. We need further model-dependent analyses to derive specific relations in such a case.

### 3.3. *Sfermion sum rules*

We derive sum rules among the sfermion masses at the TeV scale in two kinds of models as examples, under the assumption that the MSSM holds from the TeV scale to $M_U$ and that the conventional renormalization group equations (RGEs) of soft SUSY breaking parameters are valid.[*),**)] We find that the sum rules can be powerful probes of orbifold family unification, because they depend on the $Z_2$ parity assignment and the particle identification.

(a) $SU(8) \to G_{SM} \times SU(3) \times U(1)_3$

Here we study the sum rules among the sfermion masses that come from the $[8, 3]$ of $SU(8)$ after the orbifold breaking $SU(8) \to G_{SM} \times SU(3) \times U(1)_3$, with $(p, q, r, s) = (3, 2, 3, 0)$. After the breakdown of $SU(8)$, the third antisymmetric representation, $[8, 3]$, with ${}_8C_3$ components is decomposed into a sum of multiplets of the subgroup $SU(3)_C \times SU(2)_L \times SU(3)$,

$$[8,3] = \sum_{l_1=0}^{3} \sum_{l_2=0}^{3-l_1} \left({}_3C_{l_1}, {}_2C_{l_2}, {}_3C_{3-l_1-l_2}\right), \tag{3·15}$$

where $l_1$ and $l_2$ are integers. The $Z_2$ parity of $({}_3C_{l_1}, {}_2C_{l_2}, {}_3C_{3-l_1-l_2})$ is given by

$$\mathcal{P}_0 = -(-1)^{l_1+l_2}\eta_3,\ \mathcal{P}_1 = (-1)^{l_2}\eta'_3, \tag{3·16}$$

where $\eta_3$ and $\eta'_3$ are the intrinsic $Z_2$ parities, each of which takes the value $+1$ or $-1$. We assume that the $Z_2$ parity (3·16) is assigned for the left-handed Weyl

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*) For detailed descriptions of the methods and derivations, see Ref. 18).

**) In Ref. 25), Dine et al. pointed out that hidden sector interactions can give rise to sizable effects on the RG evolution of soft SUSY breaking parameters if hidden sector fields are treated as dynamical. In Ref. 26), Cohen et al. derived mass relations among scalar fields by using RGEs modified by the hidden dynamics from the GUT scale to an intermediate scale, where auxiliary fields in the hidden sector freeze into their VEV.

Table VI. Sfermions with even $Z_2$ parity from $[8,3]$ with $(p,q,r,s)=(3,2,3,0)$.

| Rep. | Rep. for left-handed fermions | Sfermion species |
|---|---|---|
| $({}_3C_3, {}_2C_0, {}_3C_0)_R$ | $(\mathbf{1},\mathbf{1},\mathbf{1})_{-9}$ | $\tilde{e}_R^*$ |
| $({}_3C_1, {}_2C_2, {}_3C_0)_R$ | $(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{-9}$ | $\tilde{u}_R^*$ |
| $({}_3C_1, {}_2C_1, {}_3C_1)_L$ | $(\mathbf{3},\mathbf{2},\mathbf{3})_{1}$ | $\tilde{q}_{1L}$, $\tilde{q}_{2L}$, $\tilde{q}_{3L}$ |
| $({}_3C_1, {}_2C_0, {}_3C_2)_R$ | $(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{7}$ | $\tilde{b}_R^*$, $\tilde{s}_R^*$, $\tilde{d}_R^*$ |

fermions. The corresponding right-handed ones have opposite $Z_2$ parities. Let us take $\eta_3 = -1$ and $\eta_3' = -1$. In this case, particles with even $Z_2$ parities are given in Table VI. Each particle possesses a zero mode whose scalar component is identified with one of the MSSM particles in four dimensions. In the first column, the subscript $L$ $(R)$ represents the left-handedness (right-handedness) for Weyl fermions. In the second column, the quantum numbers after the charge conjugation are listed for the right-handed ones. The subscript indicates the $U(1)_3$ charge. In the last column, our particle identification is given for scalar partners. Note that the particle identification is not unique but can be fixed by experiments.

After the breakdown of $SU(3) \times U(1)_3$ gauge symmetry, we have the following mass formulae at $M_U$:

$$m^2_{\tilde{e}^*_R}(M_U) = m^2_{\tilde{u}^*_R}(M_U) = m^2_{[8,3]} - 9D', \tag{3·17}$$

$$m^2_{\tilde{q}_{1L}}(M_U) = m^2_{[8,3]} + D_1 + D_2 + D', \tag{3·18}$$

$$m^2_{\tilde{q}_{2L}}(M_U) = m^2_{[8,3]} - D_1 + D_2 + D', \tag{3·19}$$

$$m^2_{\tilde{q}_{3L}}(M_U) = m^2_{[8,3]} - 2D_2 + D', \tag{3·20}$$

$$m^2_{\tilde{b}^*_R}(M_U) = m^2_{[8,3]} + D_1 + D_2 + 7D', \tag{3·21}$$

$$m^2_{\tilde{s}^*_R}(M_U) = m^2_{[8,3]} - D_1 + D_2 + 7D', \tag{3·22}$$

$$m^2_{\tilde{d}^*_R}(M_U) = m^2_{[8,3]} - 2D_2 + 7D', \tag{3·23}$$

where $m_{[8,3]}$ is a soft SUSY breaking scalar mass parameter, $D_1$ and $D_2$ are parameters which represent $D$-term condensations related to the $SU(3)$ generator and $D'$ stands for the $D$-term contribution of $U(1)_3$. By eliminating these four unknown parameters, we obtain the relations

$$m^2_{\tilde{e}^*_R}(M_U) = m^2_{\tilde{u}^*_R}(M_U), \tag{3·24}$$

$$m^2_{\tilde{q}_{1L}}(M_U) - m^2_{\tilde{b}^*_R}(M_U) = m^2_{\tilde{q}_{2L}}(M_U) - m^2_{\tilde{s}^*_R}(M_U) = m^2_{\tilde{q}_{3L}}(M_U) - m^2_{\tilde{d}^*_R}(M_U), \tag{3·25}$$

$$\begin{aligned} &9m^2_{\tilde{u}^*_R}(M_U) + 5\left(m^2_{\tilde{b}^*_R}(M_U) + m^2_{\tilde{s}^*_R}(M_U) + m^2_{\tilde{d}^*_R}(M_U)\right) \\ &\qquad = 8\left(m^2_{\tilde{q}_{1L}}(M_U) + m^2_{\tilde{q}_{2L}}(M_U) + m^2_{\tilde{q}_{3L}}(M_U)\right). \end{aligned} \tag{3·26}$$

Then, using ordinary RGEs in the MSSM, we obtain the following sum rules among sfermion masses:

$$M^2_{\tilde{u}_R} - M^2_{\tilde{e}_R} = \zeta_3 M_3^2 - 20\zeta_1 M_1^2 + \left(-\frac{5}{3}M_W^2 + \frac{5}{3}M_Z^2\right)\cos 2\beta - 10\mathcal{S}, \tag{3·27}$$

$$M_{\tilde{u}_L}^2 - M_{\tilde{b}_R}^2 - 2F_b = M_{\tilde{c}_L}^2 - M_{\tilde{s}_R}^2 = M_{\tilde{t}_L}^2 - M_{\tilde{d}_R}^2 + F_t + F_b - m_t^2, \tag{3·28}$$

$$\begin{aligned} &9M_{\tilde{u}_R}^2 + 5\left(M_{\tilde{b}_R}^2 + M_{\tilde{s}_R}^2 + M_{\tilde{d}_R}^2\right) - 8\left(M_{\tilde{u}_L}^2 + M_{\tilde{c}_L}^2 + M_{\tilde{t}_L}^2\right) \\ &\quad = -24\zeta_2 M_2^2 + 180\zeta_1 M_1^2 + \left(-17M_W^2 + 5M_Z^2\right)\cos 2\beta \\ &\qquad + 8F_t - 2F_b - 8m_t^2 - 30\mathcal{S}, \end{aligned} \tag{3·29}$$

where $M_{\tilde{f}}^2$ represents the diagonal elements of the sfermion mass-squared matrices at the TeV scale, $M_i$ $(i = 1, 2, 3)$ are the gaugino masses at the TeV scale, $\beta$ is defined in terms of the ratio of the VEVs of neutral components of the Higgs bosons as $\tan\beta \equiv v_2/v_1$, and $F_t$ and $F_b$ stand for the effects of the top and bottom Yukawa interactions, respectively. The parameters $\zeta_i$ and $\mathcal{S}$ are defined by

$$\begin{aligned} &\zeta_3 \equiv -\frac{8}{9}\left(\left(\frac{\alpha_3(M_U)}{\alpha_3}\right)^2 - 1\right),\ \zeta_2 \equiv \frac{3}{2}\left(\left(\frac{\alpha_2(M_U)}{\alpha_2}\right)^2 - 1\right), \\ &\zeta_1 \equiv \frac{1}{198}\left(\left(\frac{\alpha_1(M_U)}{\alpha_1}\right)^2 - 1\right), \end{aligned} \tag{3·30}$$

$$\mathcal{S} \equiv \frac{1}{10b_1}\left(1 - \frac{\alpha_1(M_U)}{\alpha_1}\right)\sum_{\tilde{F}} Y(\tilde{F}) n_{\tilde{F}} m_{\tilde{F}}^2, \tag{3·31}$$

where the quantities $\alpha_i \equiv g_i^2/(4\pi)$ are the structure constants defined by the gauge couplings $g_i$ at the TeV scale, and $n_{\tilde{F}}$ represents the degrees of freedom of the sfermions and Higgs bosons $\tilde{F}$.

In the case with $\eta_3 = -1$ and $\eta_3' = +1$, the following sum rules are obtained

$$\begin{aligned} M_{\tilde{u}_R}^2 - M_{\tilde{e}_R}^2 &= M_{\tilde{c}_R}^2 - M_{\tilde{\mu}_R}^2 = M_{\tilde{t}_R}^2 - M_{\tilde{\tau}_R}^2 - m_t^2 + 2F_t - 2F_\tau \\ &= \zeta_3 M_3^2 - 20\zeta_1 M_1^2 + \left(-\frac{5}{3}M_W^2 + \frac{5}{3}M_Z^2\right)\cos 2\beta - 10\mathcal{S}, \end{aligned} \tag{3·32}$$

$$M_{\tilde{u}_R}^2 - M_{\tilde{\tau}_L}^2 - F_\tau = M_{\tilde{c}_R}^2 - M_{\tilde{\mu}_L}^2 = M_{\tilde{t}_R}^2 - M_{\tilde{e}_L}^2 + 2F_t - m_t^2, \tag{3·33}$$

$$\begin{aligned} &9M_{\tilde{u}_L}^2 + 5\left(M_{\tilde{e}_L}^2 + M_{\tilde{\mu}_L}^2 + M_{\tilde{\tau}_L}^2\right) - 8\left(M_{\tilde{u}_R}^2 + M_{\tilde{c}_R}^2 + M_{\tilde{t}_R}^2\right) \\ &\quad = -15\zeta_3 M_3^2 + 24\zeta_2 M_2^2 - 240\zeta_1 M_1^2 + \left(7M_W^2 - 10M_Z^2\right)\cos 2\beta \\ &\qquad - 5F_\tau + 16F_t - 8m_t^2 + 60\mathcal{S}. \end{aligned} \tag{3·34}$$

Here we have used the particle identification such that $({}_3C_2, {}_2C_1, {}_3C_0)_R^c = \tilde{q}_{1L}$, $({}_3C_2, {}_2C_0, {}_3C_1)_L = \tilde{u}_R^*, \tilde{c}_R^*, \tilde{t}_R^*$, $({}_3C_0, {}_2C_2, {}_3C_1)_L = \tilde{e}_R^*, \tilde{\mu}_R^*, \tilde{\tau}_R^*$ and $({}_3C_0, {}_2C_1, {}_3C_2)_R^c = \tilde{l}_{3L}, \tilde{l}_{2L}, \tilde{l}_{1L}$. Here, the superscript $(c)$ represents the complex conjugate.

(b) $SU(8) \to G_{SM} \times SU(2) \times U(1)^2$

Here we study the sum rules among the sfermion masses coming from the $[8, 3]$ of $SU(8)$ after the orbifold breaking $SU(8) \to G_{SM} \times SU(2) \times U(1)^2$ with $(p, q, r, s) = (3, 2, 1, 2)$. Let us take $\eta_3 = -1$ and $\eta_3' = -1$. In this case, the particles with even $Z_2$ parities are given in Table VII. The subscript in the second column indicates

Table VII. Sfermions with even $Z_2$ parity from [8, 3] with $(p, q, r, s) = (3, 2, 1, 2)$.

| Rep. | Rep. for left-handed fermions | Sfermion species |
|---|---|---|
| $({}_3C_1, {}_2C_0, {}_1C_0, {}_2C_2)_R$ | $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{(-2,7)}$ | $\tilde{d}_R^*$ |
| $({}_3C_0, {}_2C_1, {}_1C_1, {}_2C_1)_L$ | $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})_{(1,7)}$ | $\tilde{l}_{1L}$, $\tilde{l}_{2L}$ |
| $({}_3C_2, {}_2C_0, {}_1C_0, {}_2C_1)_L$ | $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{2})_{(1,1)}$ | $\tilde{u}_R^*$, $\tilde{c}_R^*$ |
| $({}_3C_0, {}_2C_2, {}_1C_0, {}_2C_1)_L$ | $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})_{(1,1)}$ | $\tilde{e}_R^*$, $\tilde{\mu}_R^*$ |
| $({}_3C_1, {}_2C_1, {}_1C_1, {}_2C_0)_L$ | $(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{(-2,1)}$ | $\tilde{q}_{1L}$ |
| $({}_3C_1, {}_2C_2, {}_1C_0, {}_2C_0)_R$ | $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{(0,-9)}$ | $\tilde{t}_R^*$ |
| $({}_3C_3, {}_2C_0, {}_1C_0, {}_2C_0)_R$ | $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{(0,-9)}$ | $\tilde{\tau}_R^*$ |

extra $U(1)$ charges. In the last column, our particle identification is given for scalar partners.

After the breakdown of the $SU(2) \times U(1)^2$ gauge symmetry, we obtain the following relations at $M_U$:

$$m^2_{\tilde{e}^*_R}(M_U) = m^2_{\tilde{u}^*_R}(M_U),\ m^2_{\tilde{\mu}^*_R}(M_U) = m^2_{\tilde{c}^*_R}(M_U),\ m^2_{\tilde{\tau}^*_R}(M_U) = m^2_{\tilde{t}^*_R}(M_U), \tag{3·35}$$

$$m^2_{\tilde{l}_{1L}}(M_U) + m^2_{\tilde{u}^*_R}(M_U) = m^2_{\tilde{l}_{2L}}(M_U) + m^2_{\tilde{c}^*_R}(M_U), \tag{3·36}$$

$$m^2_{\tilde{u}^*_R}(M_U) + m^2_{\tilde{c}^*_R}(M_U) + 2m^2_{\tilde{d}^*_R}(M_U) = m^2_{\tilde{l}_{1L}}(M_U) + m^2_{\tilde{l}_{2L}}(M_U) + 2m^2_{\tilde{q}_{1L}}(M_U), \tag{3·37}$$

$$6m^2_{\tilde{q}_{1L}}(M_U) + m^2_{\tilde{u}^*_R}(M_U) + m^2_{\tilde{c}^*_R}(M_U) = 5m^2_{\tilde{d}^*_R}(M_U) + 3m^2_{\tilde{t}^*_R}(M_U). \tag{3·38}$$

Then, using ordinary RGEs of the MSSM, we obtain the following sum rules among sfermion masses:

$$\begin{aligned} M^2_{\tilde{u}_R} - M^2_{\tilde{e}_R} &= M^2_{\tilde{c}_R} - M^2_{\tilde{\mu}_R} = M^2_{\tilde{t}_R} - M^2_{\tilde{\tau}_R} - m_t^2 + 2F_t - 2F_\tau \\ &= \zeta_3 M_3^2 - 20\zeta_1 M_1^2 + \left(-\frac{5}{3}M_W^2 + \frac{5}{3}M_Z^2\right)\cos 2\beta - 10\mathcal{S}, \end{aligned} \tag{3·39}$$

$$M^2_{\tilde{e}_L} + M^2_{\tilde{u}_R} = M^2_{\tilde{\mu}_L} + M^2_{\tilde{c}_R}, \tag{3·40}$$

$$M^2_{\tilde{u}_R} + M^2_{\tilde{c}_R} + 2M^2_{\tilde{d}_R} - M^2_{\tilde{e}_L} - M^2_{\tilde{\mu}_L} - 2M^2_{\tilde{u}_L} = 2\zeta_3 M_3^2 - 4\zeta_2 M_2^2 + 20\zeta_1 M_1^2, \tag{3·41}$$

$$\begin{aligned} &6M^2_{\tilde{u}_L} + M^2_{\tilde{u}_R} + M^2_{\tilde{c}_R} - 5M^2_{\tilde{d}_R} - 3M^2_{\tilde{t}_R} \\ &\quad = 6\zeta_2 M_2^2 - 30\zeta_1 M_1^2 + 3M_W^2 \cos 2\beta + 6F_t - 3m_t^2. \end{aligned} \tag{3·42}$$

In the case with $\eta_3 = -1$ and $\eta'_3 = +1$, the following sum rules are obtained:

$$M^2_{\tilde{u}_R} - M^2_{\tilde{e}_R} = \zeta_3 M_3^2 - 20\zeta_1 M_1^2 + \left(-\frac{5}{3}M_W^2 + \frac{5}{3}M_Z^2\right)\cos 2\beta - 10\mathcal{S}, \tag{3·43}$$

$$M^2_{\tilde{d}_R} + M^2_{\tilde{u}_L} = M^2_{\tilde{s}_R} + M^2_{\tilde{c}_L}, \tag{3·44}$$

$$M^2_{\tilde{u}_L} + M^2_{\tilde{c}_L} + 2M^2_{\tilde{e}_L} - M^2_{\tilde{d}_R} - M^2_{\tilde{s}_R} - 2M^2_{\tilde{u}_R} = -2\zeta_3 M_3^2 + 4\zeta_2 M_2^2 - 20\zeta_1 M_1^2, \tag{3·45}$$

$$\begin{aligned} &6M^2_{\tilde{u}_R} + M^2_{\tilde{u}_L} + M^2_{\tilde{c}_L} - 5M^2_{\tilde{e}_L} - 3M^2_{\tilde{t}_L} \\ &\quad = 5\zeta_3 M_3^2 - 6\zeta_2 M_2^2 + 50\zeta_1 M_1^2 + \left(\frac{1}{3}M_W^2 + \frac{5}{3}M_Z^2\right)\cos 2\beta \\ &\qquad + 3F_t + 3F_b - 3m_t^2 - 10\mathcal{S}. \end{aligned} \tag{3·46}$$

Here, we have used the particle identification such that $({}_3C_1, {}_2C_0, {}_1C_1, {}_2C_1)^c_R = \tilde{s}^*_R, \tilde{d}^*_R$, $({}_3C_0, {}_2C_1, {}_1C_1, {}_2C_2)^c_R = \tilde{l}_{1L}$, $({}_3C_2, {}_2C_0, {}_1C_1, {}_2C_0)_L = \tilde{u}^*_R$, $({}_3C_0, {}_2C_2, {}_1C_1, {}_2C_0)_L = \tilde{e}^*_R$ and $({}_3C_1, {}_2C_1, {}_1C_0, {}_2C_1)_L = \tilde{q}_{1L}, \tilde{q}_{2L}$.

## §4. Conclusions and discussion

We have studied sfermion masses on the basis of orbifold family unification models, under some assumptions regarding the breakdown of SUSY and gauge symmetries, and derived relations among them. The sfermion sum rules at the TeV scale have also been derived from several models under the assumption that the MSSM holds below the compactification scale. The sfermion mass relations and sum rules are specific to each model and can be useful for the selection of a realistic model.

We have assumed that the Higgs fields originate from other multiplets. Two kinds of weak Higgs doublets in the MSSM can be derived as zero modes of the hypermultiplets, whose representations are $[N, 1]$ with $(\eta_1, \eta_1') = (+1, -1)$ and $(-1, +1)$, respectively. Then, the triplet-doublet splitting is elegantly realized in the same way as the orbifold SUSY $SU(5)$ GUT.

In our model, a non-abelian subgroup such as $SU(r) \times SU(s)$ of $SU(N)$ plays the role of the family symmetry, and its $D$-term contributions lift the mass degeneracy. The mass degeneracy for each squark and slepton species in the first two families is favorable for suppressing flavor-changing neutral current (FCNC) processes. It is known that the dangerous FCNC processes can be avoided if the sfermion masses in the first two families are rather large or the fermion and its superpartner mass matrices are aligned. We have derived sfermion relations and sum rules under the assumption that FCNC processes are suppressed by a mechanism other than the mass degeneracy, without loss of generality. Conversely, the requirement of degenerate masses would yield a constraint on the $D$-term condensations and/or SUSY breaking mechanism. For example, if we consider the Scherk-Schwarz mechanism for $N = 1$ SUSY breaking, the $D$-term condensations vanish for the gauge symmetries broken at the orbifold breaking scale $M_U$, because of a universal structure of the soft SUSY breaking parameters. In this case, the sum rules would not be useful probes.

We have assumed that extra gauge symmetries are broken at the same scale as $M_U$. If, however, they are broken at different scales, soft SUSY breaking parameters are subject to extra renormalization effects and consequently possess a non-universal structure. As a result, $D$-term contributions can appear. In this case, our analysis should be modified by considering the renormalization group to be running for sfermion masses. In the case that effects such as $F$-term contributions and/or higher-dimensional operators are sizable, we should consider them.

The sum rules of sparticle masses at the TeV scale can be derived if the nature of the physics between the breaking scale $M_U$ and the weak scale is specified. In our analysis, we have assumed gravity-mediated SUSY breaking in the case that the dynamics in the hidden sector do not have sizable effects on the renormalization group evolution of soft SUSY breaking parameters.[18] It is also important to study the case with strong dynamics in the hidden sector.[26]

**Acknowledgements**

This work was supported in part by Scientific Grants from the Ministry of Education, Culture, Sports, Science and Technology of Japan [Nos. 18204024 and 18540259 (Y. K.)].

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