Nilpotent schemes and group-like schemes

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Abstract

We give a definition of nilpotent association schemes as a generalization of nilpotent groups and investigate their basic properties. Moreover, for a group-like scheme, we characterize the nilpotency by its character products.

1 Introduction

The notion of a finite association scheme generalizes the one of a finite group. So it is natural to ask which properties of finite groups hold for association schemes.

In [6], Takegahara defined nilpotent schemes as a generalization of finite groups, and he consider some of their basic properties. However, Takegahara restricted himself to commutative schemes. In this article, we suggest a definition of nilpotency for finite schemes without assuming them to be commutative. For each finite schemes, we define the upper central series and call a finite scheme nilpotent if the upper central series ends at the scheme itself. Referring to this definition we will be able to show that subschemes and factor schemes of nilpotent finite schemes are nilpotent (Proposition 2.7 and Proposition 2.8). But we do not know whether we can define the lower central series (Question 2.11).

In [6], Takegahara also gave a characterization of commutative nilpotent schemes by their Krein parameters. In general, non-commutative schemes do not have Krein parameters. So we consider group-like schemes [2]. A scheme is said to be a group-like scheme if any product of characters is a linear combination of irreducible characters. For a group-like scheme, we can define the lower central series and show that it characterizes nilpotency of the scheme (Theorem 4.12).

Also we will give some remarks on p-schemes.

2 Definitions and basic facts

We say that (X, S) is an *association scheme* or a *scheme* in the sense of a finite scheme in [7] or [8]. For $s, t, u \in S$, let p_{st}^{u} denote the *intersection number* or the *structure constant*,

n_s the *valency*, and σ_s the *adjacency matrix*. The diagonal relation will be denoted by 1. For $T \subset S$, we also use the notations $n_T = \sum_{t \in T} n_t$ and $\sigma_T = \sum_{t \in T} \sigma_t$. We call $n_S = |X|$ the *order* of (X, S). An element $s \in S$ is said to be *thin* if $n_s = 1$. A closed subset T is said to be *thin* if every element of T is thin. Let $\mathrm{Irr}(S)$ denote the complete set of complex irreducible characters of S. For the other notation and terminology, see [7] or [8], and [1].

Let (X, S) be an association scheme. We say that $s \in S$ is *central* in S if $\sigma_s \sigma_t = \sigma_t \sigma_s$ for all $t \in S$. A closed subset T of S is said to be *central* in S if every element of T is central in S. Note that the set of all central elements of S does not need to be a closed subset of S. Let $Z(S)$ denote the maximal central thin closed subset. Namely

$$Z(S) = \{s \in S \mid n_s = 1 \text{ and } \sigma_s \sigma_t = \sigma_t \sigma_s \text{ for all } t \in S\}.$$

Define

$$Z_0(S) = \{1\}$$

and define $Z_{i+1}(S)$ inductively by

$$Z_{i+1}(S) /\!\!/ Z_i(S) = Z(S /\!\!/ Z_i(S))$$

for $i \in \{0, 1, 2, \cdots\}$. Then we have a sequence of closed subsets :

$$\{1\} = Z_0(S) \subset Z_1(S) \subset \cdots .$$

We call this sequence the *upper central series* of S. Note that $Z_i(S)$ is a normal closed subset of S.

Definition 2.1. We say that (X, S) is a *nilpotent scheme* if $Z_\ell(S) = S$ for some non-negative integer ℓ. We also say that S is nilpotent. In this case, we call the smallest integer ℓ such that $Z_\ell(S) = S$ the *nilpotency class* of S.

A closed subset T of S is said to be *nilpotent* if a corresponding subscheme is nilpotent. This definition is independent of the choice of a subscheme though a subscheme is not uniquely determined.

Remark 2.2. If (X, S) is thin, then the upper central series is just the upper central series of the corresponding finite group.

We investigate basic properties of nilpotent schemes and upper central series. The next lemma is easy but very important in this section.

Lemma 2.3. *Let s be a central thin element of S, T a closed subset of S. Then s^T is a central thin element of $S /\!\!/ T$.*

Proof. We have $(TsT)(Ts^*T) = Tss^*T \subset T$. This means that s^T is thin.

Take any $t \in S$. Then $\sigma_s \sigma_t = \sigma_t \sigma_s = \sigma_u$ for some $u \in S$ since s is thin. Then $a_{s^T t^T u^T} \neq 0$ and $a_{t^T s^T u^T} \neq 0$. Since s^T is also thin, we have $\sigma_{s^T} \sigma_{t^T} = \sigma_{u^T} = \sigma_{t^T} \sigma_{s^T}$. □

Definition 2.4. We call a sequence

$$S = S_0 \supset S_1 \supset \cdots \supset S_r = \{1\}$$

of closed subsets of S a *central series* of S if $S_{i-1}/\!\!/S_i \subset Z(S/\!\!/S_i)$ for every $i \in \{1, 2, \cdots, r\}$.

If $Z_\ell(S) = S$ for some non-negative integer ℓ, then $S = Z_\ell(S) \supset \cdots \supset Z_0(S) = \{1\}$ is a central series of S.

Theorem 2.5. *Suppose S has a central series $S = S_0 \supset S_1 \supset \cdots \supset S_r = \{1\}$. Then $Z_i(S) \supset S_{r-i}$ for every $i \in \{0, 1, 2, \cdots, r\}$. Moreover, S is nilpotent and the nilpotency class of S is at most that of S.*

Proof. If $i = 0$, then the statement is clear by $Z_0(S) = \{1\} = S_r$.

We suppose $Z_i(S) \supset S_{r-i}$ and show that $Z_{i+1}(S) \supset S_{r-i-1}$. Since $S_{r-i-1}/\!\!/S_{r-i}$ is central thin in $S/\!\!/S_{r-i}$, so is $Z_i(S)S_{r-i-1}/\!\!/Z_i(S)$ in $S/\!\!/Z_i(S)$ by Lemma 2.3. This means that

$$Z_i(S)S_{r-i-1}/\!\!/Z_i(S) \subset Z(S/\!\!/Z_i(S)) = Z_{i+1}(S)/\!\!/Z_i(S)$$

and $S_{r-i-1} \subset Z_{i+1}(S)$. □

By this theorem, the next theorem holds.

Theorem 2.6. (1) *A scheme (X, S) is nilpotent if and only if it has a central series.*

(2) *Let T be a closed subset contained in $Z(S)$. Then S is nilpotent if and only if $S/\!\!/T$ is nilpotent.*

Proposition 2.7. *Let T be a closed subset of a nilpotent scheme (X, S). Then T is nilpotent and the nilpotency class of T is at most that of S.*

Proof. Let T be a closed subset of a nilpotent scheme S. Take a central series

$$S = S_0 \supset S_1 \supset \cdots \supset S_r = \{1\}$$

of S. Put $T_i = T \cap S_i$. We show that

$$T = T_0 \supset T_1 \supset \cdots \supset T_r = \{1\}$$

is a central series of T. We have

$$T/\!\!/T_{i+1} = T/\!\!/(T \cap S_{i+1}) \cong S_{i+1}T/\!\!/S_{i+1} \subset S/\!\!/S_{i+1}.$$

This induces $T_i/\!\!/T_{i+1} \cong S_{i+1}T_i/\!\!/S_{i+1} \subset S_i/\!\!/S_{i+1}$. This means that $T_i/\!\!/T_{i+1}$ is a central thin closed subset of $T/\!\!/T_{i+1}$. □

Proposition 2.8. *Suppose S is nilpotent and T is a closed subset of S. Then $S/\!\!/T$ is nilpotent and the nilpotency class of $S/\!\!/T$ is at most that of S.*

Proof. Take a central series

$$S = S_0 \supset S_1 \supset \cdots \supset S_r = \{1\}$$

of S. We show that

$$S/\!\!/T = S_0T/\!\!/T \supset S_1T/\!\!/T \supset \cdots \supset S_rT/\!\!/T = \{1\}$$

is a central series of $S/\!\!/T$.

Since $Z(S/\!\!/S_{i+1}) \supset S_i/\!\!/S_{i+1}$, we have $Z(S/\!\!/S_{i+1}T) \supset S_iT/\!\!/S_{i+1}T$ by Lemma 2.3. This shows that $Z((S/\!\!/T)/\!\!/(S_{i+1}T/\!\!/T)) \supset (S_iT/\!\!/T)/\!\!/(S_{i+1}T/\!\!/T)$. □

We note that T does not need to be normal in Proposition 2.8.

Let T and U be closed subsets of S such that $T \subset U$. Following [8], we define

$$K_U(T) = \{u \in U \mid u^*Tu \subset T\}.$$

Then T is strongly normal in $K_U(T)$.

Proposition 2.9. *Suppose S is nilpotent and T is a proper closed subset of S. Then $Z(S/\!\!/T)$ is non-trivial. Especially $K_S(T) \supsetneq T$.*

Proof. There exists a positive integer i such that $Z_{i-1}(S) \subset T$ and $Z_i(S) \not\subset T$. Then $Z_i(S)T$ is closed in S since $Z_i(S)$ is normal in S. Now $Z_i(S)T \supsetneq T \supset Z_{i-1}(S)$ and the natural surjection $S/\!\!/Z_{i-1}(S) \to S/\!\!/T$ induces a surjection $Z_i(S)/\!\!/Z_{i-1}(S) \to Z_i(S)T/\!\!/T$. By Lemma 2.3, $Z_i(S)T/\!\!/T$ is central thin in $S/\!\!/T$. Especially $Z_i(S)T \subset K_S(T)$ and $K_S(T) \supsetneq T$. □

Proposition 2.10. *Let (X, S) be the direct product of schemes (X_1, S_1) and (X_2, S_2). Then (X, S) is nilpotent if and only if both (X_1, S_1) and (X_2, S_2) are nilpotent.*

Proof. This is clear by the definition and the arguments above. □

Question 2.11. We want to define the *lower central series* for an association scheme. Usually, in group theory, the lower central series is defined by higher commutators. But, in scheme theory, they cannot be used. For any subset T of S, the commutator $[S, T]$ contains the thin residue $\mathbf{O}^{\vartheta}(S)$ of S by [8, Theorem 3.2.1 (ii)].

Put $L^0(S) = S$. Then it seems to be natural to define $L^{i+1}(S)$ by the following. For a fixed i, let $\mathcal{Z}$ be the set of normal closed subsets T of S contained in $L^i(S)$ such that $L^i(S)/\!\!/T \subset Z(S/\!\!/T)$. Define $L^{i+1}(S) = \bigcap_{T \in \mathcal{Z}} T$. If $L^{i+1}(S) \in \mathcal{Z}$, then the definition seems to be very nice. But we do not know whether $L^{i+1}(S) \in \mathcal{Z}$.

3 Character values and upper central series

In this section, we will determine the upper central series by character values. So we can determine whether a scheme is nilpotent by its characters.

First, we recall the following facts.

Proposition 3.1 ([2, Lemma 2.2 and Lemma 2.3]). *Let (X, S) be an association scheme, $s \in S$, and $\varphi \in \mathrm{Irr}(S)$. We fix a representation Φ which affords φ. Then the following statments hold.*

(1) *If ξ is an eigenvalue of $\Phi(\sigma_s)$, then $|\xi| \leq n_s$.*

(2) *We have $|\varphi(\sigma_s)| \leq n_s\varphi(1)$.*

(3) *The equlity $\varphi(\sigma_s) = n_s\varphi(1)$ holds if and only if $\Phi(\sigma_s) = n_sE$, where E is the identity matrix.*

We consider the case $|\varphi(\sigma_s)| = n_s\varphi(1)$ in the above proposition.

Proposition 3.2. *Let (X, S) be an association scheme, $s \in S$, and $\varphi \in \mathrm{Irr}(S)$. We fix a representation Φ which affords φ. If $|\varphi(\sigma_s)| = n_s\varphi(1)$ holds, then $\Phi(\sigma_s) = n_s\varepsilon E$ for some root of unity ε.*

Proof. Suppose $|\varphi(\sigma_s)| = n_s\varphi(1)$. By Proposition 3.1 (1), all eigenvalues of $\Phi(\sigma_s)$ are $n_s\varepsilon$ for some complex number ε such that $|\varepsilon| = 1$.

By [8, Lemma 3.1.1 (ii)], there exists a positive integer ℓ such that $1 \in s^\ell$. Put ${\sigma_s}^\ell = \sum_{t\in S}\alpha_t\sigma_t$. Note that α_t is non-negative integer for any $t \in S$ and $\alpha_1 \neq 0$. We have ${n_s}^\ell = \sum_{t\in S}\alpha_t n_t$. Then $\varphi({\sigma_s}^\ell) = {n_s}^\ell\varepsilon^\ell\varphi(1)$. On the other hand, $\varphi({\sigma_s}^\ell) = \alpha_1\varphi(1) + \sum_{t\neq 1}\alpha_t\varphi(\sigma_t)$. Now

$$ {n_s}^\ell\varphi(1) = |\varphi({\sigma_s}^\ell)| \leq \alpha_1\varphi(1) + \sum_{t\neq 1}\alpha_t|\varphi(\sigma_t)| \leq \alpha_1\varphi(1) + \sum_{t\neq 1}\alpha_t n_t\varphi(1) = {n_s}^\ell\varphi(1). $$

This shows that $\varphi(\sigma_t) = n_t\varphi(1)$ if $\alpha_t \neq 0$. So $\Phi(\sigma_t) = n_tE$ if $\alpha_t \neq 0$. Hence $\Phi({\sigma_s}^\ell) = \Phi(\sigma_s)^\ell = {n_s}^\ell E$. If $\Phi(\sigma_s)$ is not diagonalizable, then its power cannot be a diagonal matrix, since its eigenvalues are non-zero. So we have $\Phi(\sigma_s) = n_s\varepsilon E$. Also we can see that ε is a ℓ-th root of unity. □

For $\chi \in \mathrm{Irr}(S)$, we define

$$ K(\chi) = \{s \in S \mid \chi(\sigma_s) = n_s\chi(1)\}. $$

Then $K(\chi)$ is a closed subset of S [2, Theorem 3.2]. Also we define

$$ Z(\chi) = \{s \in S \mid |\chi(\sigma_s)| = n_s\chi(1)\}. $$

Then we have the following.

Proposition 3.3. *For $\chi \in \mathrm{Irr}(S)$, $Z(\chi)$ is a closed subset of S.*

Proof. Let Φ be a representation which affords χ. Suppose $s, t \in Z(\chi)$. Then $\Phi(\sigma_s)$ and $\Phi(\sigma_t)$ are scalar matrices and we have $\chi(\sigma_s\sigma_t) = \chi(\sigma_s)\chi(\sigma_t)/\chi(1)$. Now

$$ n_sn_t\chi(1) = |\chi(\sigma_s\sigma_t)| \leq \sum_{u\in S} p_{st}^u|\chi(\sigma_u)| \leq \sum_{u\in S} p_{st}^u n_u\chi(1) = n_sn_t\chi(1). $$

This shows that $|\chi(\sigma_u)| = n_u\chi(1)$ if $u \in st$ and this means that $Z(\chi)$ is closed. □

Remark 3.4. We note that $Z(\chi)$ does not need to be normal in S.

We consider a normal closed subset T of S. For $\chi \in \mathrm{Irr}(S/\!\!/T)$, we can define a character χ' of S by

$$\chi'(\sigma_s) = \frac{n_s}{n_{s^T}}\chi(\sigma_{s^T}).$$

Then $\chi' \in \mathrm{Irr}(S)$. We identify χ' with χ and regard $\mathrm{Irr}(S/\!\!/T)$ as a subset of $\mathrm{Irr}(S)$ (see [3]).

Theorem 3.5. *Let T be a normal closed subset of a scheme (X, S). Then*

$$Z(S/\!\!/T) = \left(\bigcap_{\chi \in \mathrm{Irr}(S/\!\!/T)} Z(\chi) \right) /\!\!/T.$$

Especially, $Z(S) = \bigcap_{\chi \in \mathrm{Irr}(S)} Z(\chi)$.

Proof. Suppose $s^T \in Z(S/\!\!/T)$. Since s^T is thin and central, easily we can see that $|\chi(\sigma_{s^T})| = n_{s^T}\chi(1)$ for $\chi \in \mathrm{Irr}(S/\!\!/T)$. We know that $\chi(\sigma_{s^T}) = n_{s^T}\chi(\sigma_s)/n_s$. This means that $s \in Z(\chi)$ and $s \in \bigcap_{\chi \in \mathrm{Irr}(S/\!\!/T)} Z(\chi)$.

Suppose $s \in \bigcap_{\chi \in \mathrm{Irr}(S/\!\!/T)} Z(\chi)$. Then we have $s^* \in \bigcap_{\chi \in \mathrm{Irr}(S/\!\!/T)} Z(\chi)$ by Proposition 3.3. Let $\varphi \in \mathrm{Irr}(S/\!\!/T)$ and Φ a representation affording φ. Since $\Phi(\sigma_s) = n_s \varepsilon E$ and $\Phi(\sigma_{s^*}) = n_s \varepsilon^{-1} E$, we have $\Phi(\sigma_s \sigma_s^*) = {n_s}^2 E$. This means that $ss^* \subset K(\varphi)$. Since $\bigcap_{\varphi \in \mathrm{Irr}(S/\!\!/T)} K(\varphi) = T$, we have $ss^* \subset T$. By the normality of T, s^T is thin. Also s^T is central in $S/\!\!/T$ since $\Phi(\sigma_s)$ is scalar matrix for every irreducible representation Φ of $S/\!\!/T$. □

By Theorem 3.5, we can determine $Z_1(S)$, $Z_2(S)$, and so on, by character values. So the upper central series is determined.

4 Group-like schemes

In [6], Takegahara gave an interesting characterization of nilpotent schemes for commutative schemes. He used Krein parameters to characterize nilpotent schemes, but Krein parameters are not defined for non-commutative schemes, in general. We will try to characterize nilpotent schemes by a similar way for group-like schemes.

Let (X, S) be an association scheme. We will define a binary relation $\sim$ on S as follows. For $s, t \in S$, we write $s \sim t$ if

$$\frac{1}{n_s}\chi(\sigma_s) = \frac{1}{n_t}\chi(\sigma_t)$$

for every $\chi \in \mathrm{Irr}(S)$. Then $\sim$ is an equivalence relation. For $s \in S$, put $\widetilde{s} = \bigcup_{t \sim s} t$ and $\widetilde{S} = \{\widetilde{s} \mid s \in S\}$. Then $\widetilde{S}$ is a partition of $X \times X$. Put $V = \bigoplus_{\widetilde{s} \in \widetilde{S}} \mathbb{C}\sigma_{\widetilde{s}}$. Then the center of the adjacency algebra $Z(\mathbb{C}S)$ is contained in V.

For two characters χ and φ of S, it seems to be natural to define the product $\chi\varphi$ by

$$(\chi\varphi)(\sigma_s) = \frac{1}{n_s}\chi(\sigma_s)\varphi(\sigma_s)$$

(see [4]). Note that the product does not need to be a character. Moreover, it does not need to be a linear combination of irreducible characters.

Theorem 4.1. *For an association scheme (X, S), the following statements are equivalent.*

(1) $V = Z(\mathbb{C}S)$.

(2) $\dim_{\mathbb{C}} Z(\mathbb{C}S) = |\widetilde{S}|$.

(3) *There exists a partition $S = \bigcup_{\lambda\in\Lambda} T_\lambda$ such that $\{\sigma_{T_\lambda} \mid \lambda \in \Lambda\}$ is a basis of $Z(\mathbb{C}S)$.*

(4) *$Z(\mathbb{C}S)$ is closed under the Hadamard product.*

(5) *For any $\chi, \varphi \in \mathrm{Irr}(S)$, $\chi\varphi$ is a linear combination of $\mathrm{Irr}(S)$.*

Proof. The equivalences of (1), (2), (3), and (4) are shown in [2, Theorem 4.1]. The equivalence of (4) and (5) are by a direct calculation. □

We call a scheme (X, S) with the property in the above theorem a *group-like scheme* [2]. If (X, S) is group-like, then $(X, \widetilde{S})$ becomes a commutative scheme. For a group-like scheme (X, S), we can define a bijection $\mathrm{Irr}(S) \to \mathrm{Irr}(\widetilde{S})$ $(\chi \mapsto \widetilde{\chi})$ by

$$\widetilde{\chi}(\sigma_{\widetilde{s}}) = \frac{\chi(\sigma_{\widetilde{s}})}{\chi(1)}.$$

Note that we write 1 for the identity element of the adjacency algebra $\mathbb{C}S$ here. Of course, $\sigma_1 = 1$ holds. Recall that the primitive central idempotent of $\mathbb{C}S$ corresponding to $\chi \in \mathrm{Irr}(S)$ is given by

$$e_\chi = \frac{m_\chi}{n_S}\sum_{s\in S}\frac{1}{n_s}\chi(\sigma_{s^*})\sigma_s.$$

It is easy to see that $e_\chi = e_{\widetilde{\chi}}$.

Since $(X, \widetilde{S})$ is commutative for a group-like scheme (X, S), we can define *Krein parameters* $q_{\widetilde{\chi}\widetilde{\varphi}}^{\widetilde{\xi}}$ by

$$e_{\widetilde{\chi}} \circ e_{\widetilde{\varphi}} = \frac{1}{n_S}\sum_{\widetilde{\xi}\in\mathrm{Irr}(\widetilde{S})} q_{\widetilde{\chi}\widetilde{\varphi}}^{\widetilde{\xi}} e_{\widetilde{\xi}}$$

for $\widetilde{\chi}, \widetilde{\varphi}, \widetilde{\xi} \in \mathrm{Irr}(\widetilde{S})$. Since $e_\chi = e_{\widetilde{\chi}}$, we also write $q_{\chi\varphi}^{\xi}$ instead of $q_{\widetilde{\chi}\widetilde{\varphi}}^{\widetilde{\xi}}$. It is known that the Krein parameters are non-negative real numbers (Krein condition [1, II, Theorem 3.8]).

Suppose (X, S) is group-like. Then, for any $\chi, \varphi \in \mathrm{Irr}(S)$, $\chi\varphi$ is a linear combination of $\mathrm{Irr}(S)$:

$$\chi\varphi = \sum_{\xi\in\mathrm{Irr}(S)} r_{\chi\varphi}^{\xi}\xi$$

and

$$r_{\chi\varphi}^{\xi} = \frac{m_\xi}{m_\chi m_\varphi} q_{\chi\varphi}^{\xi}$$

by direct calculations (or see [4]). Note that $r_{\chi\varphi}^{\xi}$ is also a non-negative real number for a group-like scheme.

Let T be a normal closed subset of S. For $\chi \in \mathrm{Irr}(S)$, $\chi \in \mathrm{Irr}(S/\!\!/T)$ if and only if $e_\chi e_T \neq 0$, where $e_T = {n_T}^{-1}\sigma_T$. Since T is normal, e_T is a central idempotent of $\mathbb{C}S$ and we have

$$e_T = \sum_{\chi \in \mathrm{Irr}(S/\!\!/T)} e_\chi$$

and

$$\chi(e_T) = \begin{cases} \chi(1) & \text{if } \chi \in \mathrm{Irr}(S/\!\!/T), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.2. *Let (X, S) be a group-like scheme, and T a normal closed subset of S. Then the factor scheme $(X/T, S/\!\!/T)$ is group-like.*

Proof. Let $\chi, \varphi \in \mathrm{Irr}(S/\!\!/T)$. Since (X, S) is group-like, we have $\chi\varphi = \sum_{\xi \in \mathrm{Irr}(S)} r_{\chi\varphi}^{\xi}\xi$. So we have

$$\begin{aligned} \sum_{\xi \in \mathrm{Irr}(S)} r_{\chi\varphi}^{\xi}\xi(e_T) &= (\chi\varphi)(e_T) = \frac{1}{n_T}\sum_{t \in T}(\chi\varphi)(\sigma_t) = \frac{1}{n_T}\sum_{t \in T}\frac{1}{n_t}\chi(\sigma_t)\varphi(\sigma_t) \\ &= \frac{1}{n_T}\sum_{t \in T}\frac{1}{n_t}n_t\chi(1)n_t\varphi(1) = \frac{1}{n_T}\chi(1)\varphi(1)\sum_{t \in T} n_t = \chi(1)\varphi(1) \\ &= (\chi\varphi)(1) = \sum_{\xi \in \mathrm{Irr}(S)} r_{\chi\varphi}^{\xi}\xi(1). \end{aligned}$$

Since $r_{\chi\varphi}^{\xi}$ is a non-negative real number, this shows that $r_{\chi\varphi}^{\xi} \neq 0$ implies $\xi(e_T) \neq 0$ and so $\xi \in \mathrm{Irr}(S/\!\!/T)$. Now $\chi\varphi$ is a linear combination of $\mathrm{Irr}(S/\!\!/T)$, and $S/\!\!/T$ is group-like. $\square$

Let T be a closed subset of (X, S). For $s, u \in S$, $s^T = u^T$ if and only if

$$\frac{1}{n_s}\sigma_T\sigma_s\sigma_T = \frac{1}{n_u}\sigma_T\sigma_u\sigma_T$$

by [8, Lemma2.3.2]. In the above condition, we can also replace σ_T by e_T.

Remark 4.3. Let (X, S) be a group-like scheme, and T a closed subset of S. The factor scheme $(X/T, S/\!\!/T)$ does not need to be group-like if T is not normal.

Proposition 4.4. *Let (X, S) be a group-like scheme, and T a normal closed subset of S. Put $\widetilde{T} = \{\widetilde{t} \mid t \in T\}$. Then $\widetilde{T}$ is a (normal) closed subset of $\widetilde{S}$ and $\widetilde{S/\!\!/T} \cong \widetilde{S}/\!\!/\widetilde{T}$.*

Proof. Since T is normal in S, σ_T is in the center of the adjacency algebra $\mathbb{C}S$. So, if $t \in T$, then $\widetilde{t} \subset \bigcup_{u \in T} u$. Hence $\sigma_T = \sigma_{\widetilde{T}}$ and this means that $\widetilde{T}$ is closed.

We show that $\widetilde{S /\!\!/ T}$ is a fusion of $\widetilde{S} /\!\!/ \widetilde{T}$ (for a *fusion*, see [7, §1.7]). Suppose $\widetilde{s}^{\widetilde{T}} = \widetilde{u}^{\widetilde{T}}$ for $s, u \in S$. For any $\chi \in \mathrm{Irr}(S /\!\!/ T)$, we have

$$\begin{aligned} \frac{1}{n_{s^T}} \chi(\sigma_{s^T}) &= \frac{1}{n_s} \chi(\sigma_s) = \frac{1}{n_{\widetilde{s}}} \chi(\sigma_{\widetilde{s}}) = \frac{1}{n_{\widetilde{s}}} \chi(e_T \sigma_{\widetilde{s}} e_T) \\ &= \frac{1}{n_{\widetilde{u}}} \chi(e_T \sigma_{\widetilde{u}} e_T) = \frac{1}{n_{\widetilde{u}}} \chi(\sigma_{\widetilde{u}}) = \frac{1}{n_u} \chi(\sigma_u) = \frac{1}{n_{u^T}} \chi(\sigma_{u^T}). \end{aligned}$$

So we have $\widetilde{s^T} = \widetilde{u^T}$. This means that $\widetilde{S /\!\!/ T}$ is a fusion of $\widetilde{S} /\!\!/ \widetilde{T}$.

Since $S /\!\!/ T$ and $\widetilde{S} /\!\!/ \widetilde{T}$ are group-like, we have $|\mathrm{Irr}(S /\!\!/ T)| = |\widetilde{S /\!\!/ T}|$ and $|\mathrm{Irr}(\widetilde{S} /\!\!/ \widetilde{T})| = |\widetilde{S} /\!\!/ \widetilde{T}|$. For $\chi \in \mathrm{Irr}(S)$, $\chi \in \mathrm{Irr}(S /\!\!/ T)$ if and only if $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{S} /\!\!/ \widetilde{T})$. So $|\mathrm{Irr}(S /\!\!/ T)| = |\mathrm{Irr}(\widetilde{S} /\!\!/ \widetilde{T})|$. Now we have that $|\widetilde{S /\!\!/ T}| = |\widetilde{S} /\!\!/ \widetilde{T}|$ and $\widetilde{S /\!\!/ T} \cong \widetilde{S} /\!\!/ \widetilde{T}$. □

Theorem 4.5. *Let (X, S) be a group-like scheme. Then (X, S) is nilpotent if and only if $(X, \widetilde{S})$ is nilpotent. Moreover, $S = S_0 \supset S_1 \supset \cdots \supset S_r = \{1\}$ is a central series of S if and only if $\widetilde{S} = \widetilde{S}_0 \supset \widetilde{S}_1 \supset \cdots \supset \widetilde{S}_r = \{1\}$ is a central series of $\widetilde{S}$.*

Proof. It is enough to show the second part of the theorem.

Suppose $S = S_0 \supset S_1 \supset \cdots \supset S_r = \{1\}$ is a central series of S. Then $\bigcup_{\widetilde{s} \in \widetilde{S_{r-1}}} \widetilde{s} = \bigcup_{s \in S_{r-1}} s$ and $S /\!\!/ S_{r-1} = S_0 /\!\!/ S_{r-1} \supset S_1 /\!\!/ S_{r-1} \supset \cdots \supset S_{r-1} /\!\!/ S_{r-1} = \{1\}$ is a central series of $S /\!\!/ S_{r-1}$. By the induction on the length r, we have that $\widetilde{S /\!\!/ S_{r-1}} = \widetilde{S_0 /\!\!/ S_{r-1}} \supset \widetilde{S_1 /\!\!/ S_{r-1}} \supset \cdots \supset \widetilde{S_{r-1} /\!\!/ S_{r-1}} = \{1\}$ is a central series of $S /\!\!/ S_{r-1}$. By Proposition 4.4, $\widetilde{S_i /\!\!/ S_{r-1}} \cong \widetilde{S_i} /\!\!/ \widetilde{S_{r-1}}$ for every $i \in \{0, 1, \cdots, r-1\}$. Now $\widetilde{S} = \widetilde{S}_0 \supset \widetilde{S}_1 \supset \cdots \supset \widetilde{S}_r = \{1\}$ is a central series of $\widetilde{S}$.

The converse is proved similarly. □

Corollary 4.6. *Let (X, S) be a group-like scheme. Suppose that (X, T) is a fusion of (X, S) and $(X, \widetilde{S})$ is a fusion of (X, T). Then (X, S) is nilpotent if and only if so is (X, T).*

Proof. In this case, (X, T) is group-like and $(X, \widetilde{T}) \cong (X, \widetilde{S})$. So the statement is clear by Theorem 4.5. □

Example 4.7. Suppose (X, S) is a thin scheme defined by a finite group G. Then (X, S) is nilpotent if and only if the group G is nilpotent by the definition (Remark 2.2). The relation $\sim$ is just the conjugation of G. So (X, S) is always group-like and $(X, \widetilde{S})$ is the *group association scheme* of G [1, II, Examples 2.1 (2)]. So the group association scheme is nilpotent if and only if the corresponding finite group is nilpotent.

In [6], Takegahara characterized commutative nilpotent schemes by their Krein parameters. Now we can apply his method to group-like schemes. But his argument did not mention the structure of schemes. Here we will consider his argument precisely.

Let (X,S) be a group-like scheme. For $\chi,\varphi\in\mathrm{Irr}(S)$, we define

$$\mathrm{Supp}(\chi\varphi)=\{\xi\in\mathrm{Irr}(S)\mid r_{\chi\varphi}^{\xi}\neq 0\}.$$

Of course, $r_{\chi\varphi}^{\xi}$ can be replaced by the Krein parameter $q_{\chi\varphi}^{\xi}$. Let 1_S be the trivial character of S, namely $1_S(\sigma_s)=n_s$, and put $I^0(S)=\{1_S\}$. Define $I^{i+1}(S)$ inductively by $I^{i+1}(S)=\{\chi\in\mathrm{Irr}(S)\mid\mathrm{Supp}(\chi\overline{\chi})\subset I^i(S)\}$, where $\overline{\chi}$ is the complex conjugate of χ. Note that $\chi(\sigma_{s^*})=\overline{\chi}(\sigma_s)$ for any $s\in S$. One of the main results in [6] shows that (X,S) is nilpotent if and only if $I^{\ell}(S)=\mathrm{Irr}(S)$ for some non-negative integer ℓ when (X,S) is commutative. Now we do not assume the commutativity of (X,S) and give a central series. Put $L^i(S)=\{s\in S\mid\chi(\sigma_s)=n_s\chi(1)\text{ for any }\chi\in I^i(S)\}$. Then $L^i(S)$ is a normal closed subset of S by [2, Theorem 4.3].

Lemma 4.8. *Let (X,S) be a group-like scheme. We use the above notations. Then $L^i(S)/\!\!/L^{i+1}(S)\subset Z(S/\!\!/L^{i+1}(S))$.*

Proof. First, we suppose that (X,S) is commutative. Let $s\in L^i(S)$ and let $\chi\in I^{i+1}(S)$. Then, since χ is linear, we have

$$\begin{aligned}\chi(\sigma_s\sigma_{s^*}) &= \chi(\sigma_s)\chi(\sigma_{s^*})=\chi(\sigma_s)\overline{\chi}(\sigma_s)=n_s(\chi\overline{\chi})(\sigma_s)=n_s\sum_{\xi\in I^i}r_{\chi\overline{\chi}}^{\xi}\xi(\sigma_s)\\ &= n_s\sum_{\xi\in I^i}r_{\chi\overline{\chi}}^{\xi}\xi(1)n_s={n_s}^2(\chi\overline{\chi})(1)={n_s}^2.\end{aligned}$$

Now $\chi(\sigma_s\sigma_{s^*})=\sum_{t\in S}p_{ss^*}^t\chi(\sigma_t)$, ${n_s}^2=\sum_{t\in S}p_{ss^*}^t n_t$, and $|\chi(\sigma_t)|\leq n_t$. So $p_{ss^*}^t\neq 0$ implies that $\chi(\sigma_t)=n_t$. Since this holds for every $\chi\in I^{i+1}(S)$, we have $ss^*\subset L^{i+1}(S)$. This means that $s^{L^{i+1}(S)}$ is thin. Since (X,S) is assumed to be commutative, we have $L^i(S)/\!\!/L^{i+1}(S)\subset Z(S/\!\!/L^{i+1}(S))$.

Next we suppose that (X,S) is group-like. We have the commutative scheme $(X,\widetilde{S})$. Then it is easy to see that $L^i(\widetilde{S})=\widetilde{L^i(S)}$ for any i. So we have $\widetilde{L^i(S)}/\!\!/\widetilde{L^{i+1}(S)}\subset Z(\widetilde{S}/\!\!/\widetilde{L^{i+1}(S)})$. The isomorphism $\widetilde{S}/\!\!/\widetilde{L^{i+1}(S)}\cong S/\!\!/\widetilde{L^{i+1}(S)}$ in Proposition 4.4 shows that $L^i(S)/\!\!/\widetilde{L^{i+1}(S)}\subset Z(S/\!\!/\widetilde{L^{i+1}(S)})$. Now we can conclude that $L^i(S)/\!\!/L^{i+1}(S)\subset Z(S/\!\!/L^{i+1}(S))$. □

This lemma shows the following.

Proposition 4.9. *If (X,S) is group-like and $L^{\ell}(S)=\{1\}$ for some non-negative integer ℓ. Then $S=L^0(S)\supset L^1(S)\supset\cdots\supset L^{\ell}(S)=\{1\}$ is a central series of S and (X,S) is nilpotent.*

Proposition 4.10. *Let (X,S) be a group-like scheme. Suppose a sequence of closed subset $S=S_0\supset S_1\supset\cdots$ satisfies $S_i/\!\!/S_{i+1}\subset Z(S/\!\!/S_{i+1})$ for any $i\in\{0,1,\cdots\}$. Then, for $\chi\in\mathrm{Irr}(S/\!\!/S_{i+1})$, $\mathrm{Supp}(\chi\overline{\chi})\subset\mathrm{Irr}(S/\!\!/S_i)$ holds. Especially, $I^i(S)\supset\mathrm{Irr}(S/\!\!/S_i)$ and $L^i(S)\subset S_i$ for any $i\in\{0,1,\cdots,r\}$.*

Proof. Let $\chi \in \mathrm{Irr}(S/\!\!/S_{i+1})$ and $s \in S_i$. Since $S_i/\!\!/S_{i+1} \subset Z(S/\!\!/S_{i+1})$, we have $\chi(\sigma_s) = \varepsilon n_s \chi(1)$ for some $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| = 1$. Now

$$\chi\overline{\chi}(\sigma_s) = \frac{1}{n_s}\chi(\sigma_s)\overline{\chi(\sigma_s)} = n_s\chi(1)^2 = n_s(\chi\overline{\chi})(1).$$

Now $\chi\overline{\chi}(\sigma_s) = \sum_{\xi\in\mathrm{Irr}(S)} r^{\xi}_{\chi\overline{\chi}}\xi(\sigma_s)$, $n_s(\chi\overline{\chi})(1) = \sum_{\xi\in\mathrm{Irr}(S)} r^{\xi}_{\chi\overline{\chi}} n_s\xi(1)$, and $|\xi(\sigma_s)| \leq n_s\xi(1)$. So, if $r^{\xi}_{\chi\overline{\chi}} \neq 0$, then $\xi(\sigma_s) = n_s\xi(1)$. Since this holds for any $s \in S_i$, we have $\mathrm{Supp}(\chi\overline{\chi}) \subset \mathrm{Irr}(S/\!\!/S_i)$.

We prove the last part by an induction on i. Clearly $I^0(S) = \{1_S\} = \mathrm{Irr}(S/\!\!/S_0)$. For $\chi \in \mathrm{Irr}(S/\!\!/S_{i+1})$, $\mathrm{Supp}(\chi\overline{\chi}) \subset \mathrm{Irr}(S/\!\!/S_i) \subset I^i(S)$. So $\chi \in I^{i+1}(S)$ and $\mathrm{Irr}(S/\!\!/S_{i+1}) \subset I^{i+1}(S)$. Then clearly $L^{i+1}(S) \subset S_{i+1}$. □

Corollary 4.11. *Let (X, S) be a group-like scheme. Then $I^i(S) = \mathrm{Irr}(S/\!\!/L_i)$ for any $i \in \{0, 1, \cdots\}$.*

Proof. By the definition, it is clear that $I^i(S) \subset \mathrm{Irr}(S/\!\!/L_i)$. The converse is by Lemma 4.8 and Proposition 4.10. □

By Proposition 4.10, we can see that $L^i(S)$ satisfies the property in Question 2.11. So we call the sequence

$$S = L^0(S) \supset L^1(S) \supset \cdots$$

the *lower central series* of a group-like scheme (X, S).

The following result is the main theorem in this section.

Theorem 4.12. *A group-like scheme (X, S) is nilpotent if and only if $L^{\ell}(S) = \{1\}$ for some non-negative integer ℓ.*

Proof. Clear by Proposition 4.9 and Proposition 4.10. □

Remark 4.13. We expected that every nilpotent scheme is group-like, but there are counter examples, for example, the schemes of order 16, No. 159, No. 177, No. 186, and so on, in [5].

5 p-Schemes

Let p be a prime number. Following the definition in [8], we call an association scheme (X, S) a *p-scheme* if n_S and n_s for all $s \in S$ are p-power numbers. We show that a nilpotent scheme is a p-scheme if the order of the scheme is p-power. To show this fact, we show the following proposition.

Proposition 5.1. *Let (X, S) be a nilpotent scheme. Then $n_s \mid n_S$ for every $s \in S$.*

Proof. We prove the proposition by the induction on n_S. By the definition of nilpotent schemes, there exists a central thin closed subset T. For $s \in S$, $n_s \mid n_{s^T} n_T$ by [8, Lemma 4.3.1]. Now n_{s^T} is a divisor of $n_{S/\!\!/T}$ by the inductive hyposesis and $n_S = n_T n_{S/\!\!/T}$. So n_s is a divisor of n_S. □

Proposition 5.2. *Let (X, S) be a nilpotent scheme of p-power order. Then (X, S) is a p-scheme.*

Proof. This is clear by Proposition 5.1 □

Remark 5.3. (1) We note that the converse of this proposition is not true. There are p-schemes which have no nontrivial central thin element, for example, schemes of order 32, No. 10851, No. 17336, and No. 17337 in the list [5] for $p = 2$.

(2) The association schemes of order 32, No. 17336 and No. 17337 in [5] are 2-schemes and group-like, but not nilpotent.

(3) Let (X, S) be a nilpotent scheme, and let $\{p_1, p_2, \cdots, p_r\}$ be the set of prime divisors of n_S. In this case, (X, S) does not need to be a direct product of its p_i-subschemes. For example, for distinct prime numbers p and q, the wreath product of the thin scheme of order p by the thin scheme of q is nilpotent but it is not a direct product of a p-scheme and a q-scheme.

Question 5.4. When is a p-scheme nilpotent ?

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